

Numerical time integration methods for nonsmooth systems.

Part II. Nonsmooth mechanical systems with impacts

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INRIA Rhône-Alpes, Grenoble.

SDS2010. Capitolo, Monopoli, Italy. June 8–11, 2010.

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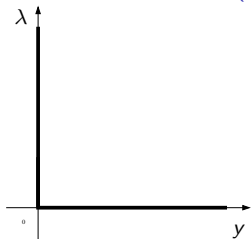
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Scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q)\dot{v} = F(t, q, v) + G(q)\lambda \\ \dot{q} = v \\ y = g(q) \\ 0 \leq y \perp \lambda \geq 0 \\ \dot{v}^+ = -e\dot{v}^- \end{cases} \quad (1)$$

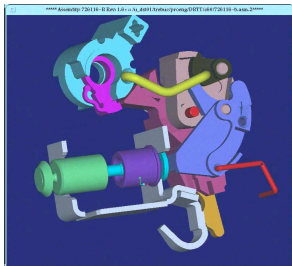
where $G(q) = \nabla g(q)$

Unilateral constraints (unilateral contact)



Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



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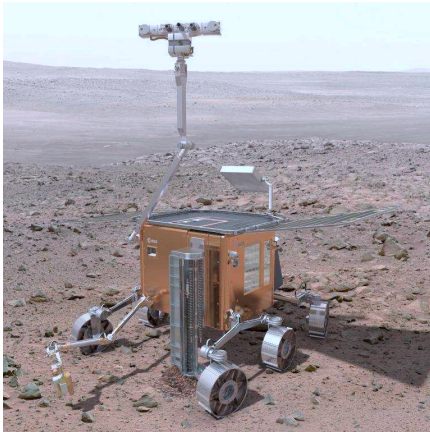
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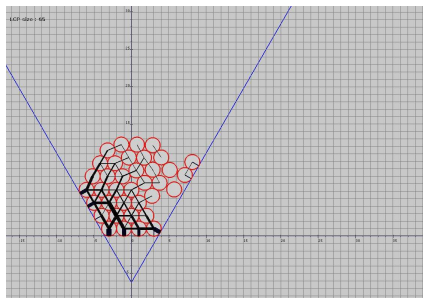
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Stack of beads with perturbation



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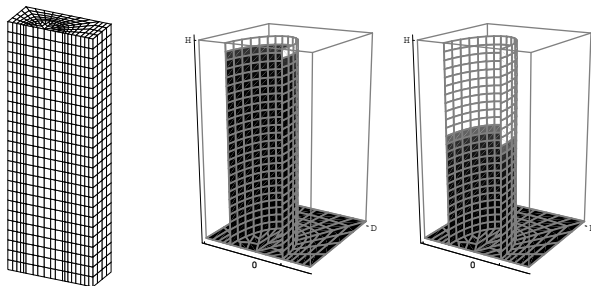
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FEM models with contact, friction cohesion, etc...



Joint work with Y. Monerie, IRSN.

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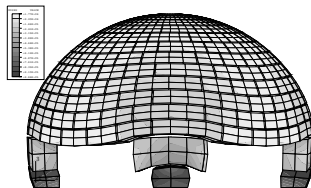
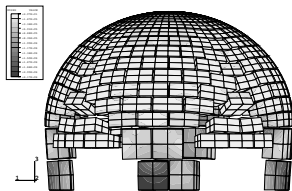
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Divided Materials and Masonry



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Definition (Smooth multibody dynamics)

$$\begin{cases} M(q) \frac{dv}{dt} + F(t, q, v) = 0, \\ v = \dot{q} \end{cases} \quad (2)$$

where

$$\blacktriangleright F(t, q, v) = N(q, v) + F_{int}(t, q, v) - F_{ext}(t)$$

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Definition (Boundary conditions)

- ▶ Initial Value Problem (IVP):

$$t_0 \in \mathbb{R}, \quad q(t_0) = q_0 \in \mathbb{R}^n, \quad v(t_0) = v_0 \in \mathbb{R}^n, \quad (3)$$

- ▶ Boundary Value Problem (BVP):

$$(t_0, T) \in \mathbb{R} \times \mathbb{R}, \quad \Gamma(q(t_0), v(t_0), q(T), v(T)) = 0 \quad (4)$$

Unilateral constraints

- ▶ Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q, t) = [g_\alpha(q, t) \geq 0, \quad \alpha \in \{1 \dots \nu\}]^T. \quad (5)$$

- ▶ Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t), g_\alpha(q, t) \geq 0, \alpha \in \{1 \dots \nu\}\}. \quad (6)$$

Normal cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n, y = - \sum_{\alpha} \lambda_{\alpha} \nabla g_{\alpha}(q, t), \lambda_{\alpha} \geq 0, \lambda_{\alpha} g_{\alpha}(q, t) = 0 \right\} \quad (7)$$

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

Introduction of the multipliers $\mu \in \mathbb{R}^m$

$$\begin{cases} M(q) \frac{dv}{dt} + F(t, q, v) = r = \nabla_q g(q, t) \lambda \\ -r \in N_{C(t)}(q(t)) \end{cases} \quad (8)$$

where $r = \nabla_q g(q, t) \lambda$ is the generalized force or generalized reactions due to the constraints.

Remark

- ▶ The unilateral constraints are said to be perfect due to the normality condition.
- ▶ Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; Mordukhovich, 1994)
- ▶ The right hand side is neither bounded (and then nor compact).
- ▶ The inclusion and the constraints concern the second order time derivative of q .

→ Standard Analysis of DI does no longer apply.

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Fundamental assumptions.

- ▶ The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
→ The equation are written in terms of a right continuous B.V.
(R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \quad (9)$$

- ▶ q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^t v^+(t) dt \quad (10)$$

- ▶ The acceleration, (\ddot{q} in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a, b]) = \int_{]a, b]} dv = v^+(b) - v^+(a) \quad (11)$$

Definition (Non Smooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \end{cases} \quad (12)$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- ▶ The non smooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- ▶ The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- ▶ This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Decomposition of measure

$$\begin{cases} dv = \gamma dt + (v^+ - v^-) d\nu + dv_s \\ di = f dt + p d\nu + di_s \end{cases} \quad (13)$$

where

- ▶ $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- ▶ f is the Lebesgue measurable force,
- ▶ $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- ▶ $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v , i.e. where $(v^+ - v^-) \neq 0$, i.e. $d\nu = \sum_i \delta_{t_i}$,
- ▶ p is the purely atomic impact percussions such that $p d\nu = \sum_i p_i \delta_{t_i}$,
- ▶ dv_s and di_s are singular measures with the respect to $dt + d\eta$.

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Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the non smooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = p d\nu, \quad (14)$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \quad (15)$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = f dt \quad (16)$$

or

$$M(q)\gamma^+ + F(t, q, v^+) = f^+ \quad [dt - a.e.] \quad (17)$$

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Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (8) is “replaced” by the following inclusion

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \quad (18)$$

Comments

This formulation provides a common framework for the non smooth dynamics containing inelastic impacts without decomposition.

→ Foundation for the time-stepping approaches.

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Comments

- ▶ *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- ▶ *The inclusion in terms of velocity* v^+ rather than of the coordinates q .

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Interpretation

- ▶ Inclusion of measure, $-di \in K$

- ▶ Case $di = r' dt = f dt$.

$$-f \in K \quad (19)$$

- ▶ Case $di = p_i \delta_i$.

$$-p_i \in K \quad (20)$$

- ▶ Inclusion in terms of the velocity. Viability Lemma

If $q(t_0) \in C(t_0)$, then

$$v^+ \in T_C(q), t \geq t_0 \Rightarrow q(t) \in C(t), t \geq t_0$$

→ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

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The Newton-Moreau impact rule

$$-di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \quad (21)$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{aligned} 0 \leq y \perp \lambda \geq 0 \\ \Updownarrow \\ -\lambda \in N_{\mathbb{R}^+}(y) \\ \Updownarrow \\ -\lambda \in N_{T_{\mathbb{R}^+}}(\dot{y}) \\ \Updownarrow \\ \text{if } y \leq 0 \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \end{aligned} \quad (22)$$

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The case of C is finitely represented

$$C = \{q \in \mathcal{M}(t), g_\alpha(q) \geq 0, \alpha \in \{1 \dots \nu\}\}. \quad (23)$$

Decomposition of di and v^+ onto the tangent and the normal cone.

$$di = \sum_{\alpha} \nabla_q^T g_\alpha(q) d\lambda_\alpha \quad (24)$$

$$U_\alpha^+ = \nabla_q g_\alpha(q) v^+, \alpha \in \{1 \dots \nu\} \quad (25)$$

Complementarity formulation (under constraints qualification condition)

$$-d\lambda_\alpha \in N_{T_{\mathbb{R}_+}(g_\alpha)}(U_\alpha^+) \Leftrightarrow \text{if } g_\alpha(q) \leq 0, \text{ then } 0 \leq U_\alpha^+ \perp d\lambda_\alpha \geq 0 \quad (26)$$

The case of C is \mathbb{R}_+

$$-di \in N_C(q) \Leftrightarrow 0 \leq q \perp di \geq 0 \quad (27)$$

is replaced by

$$-di \in N_{T_C(q)}(v^+) \Leftrightarrow \text{if } q \leq 0, \text{ then } 0 \leq v^+ \perp di \geq 0 \quad (28)$$

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Numerical time-integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time-stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event-driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Objectives & means

Objectives

Design nonsmooth event capturing methods with

- ▶ same properties as standard methods (robustness, accumulation, ...)
- ▶ Higher resolution (ratio error/computational cost)
- ▶ Higher order of accuracy

Means

1. Adaptive time-step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

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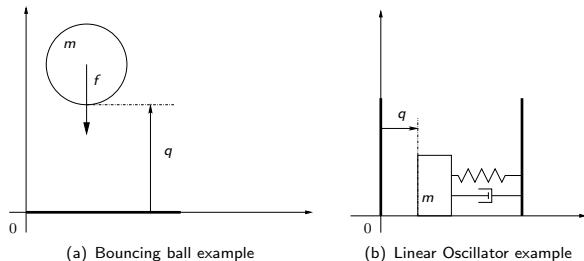


Figure: Academic test examples with analytical solutions

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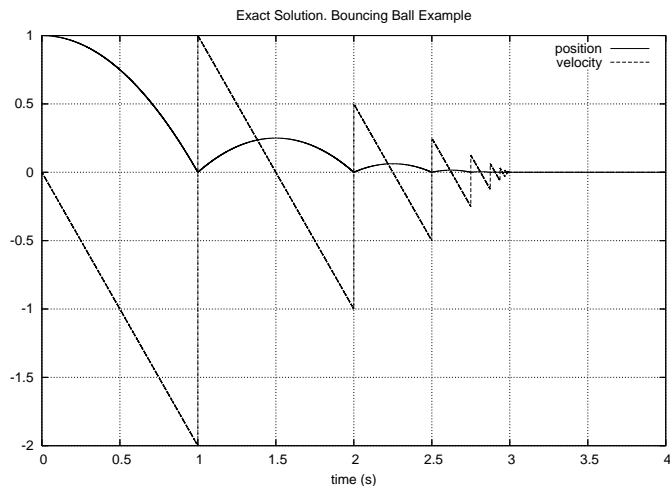


Figure: Analytical solutions. Bouncing ball example]

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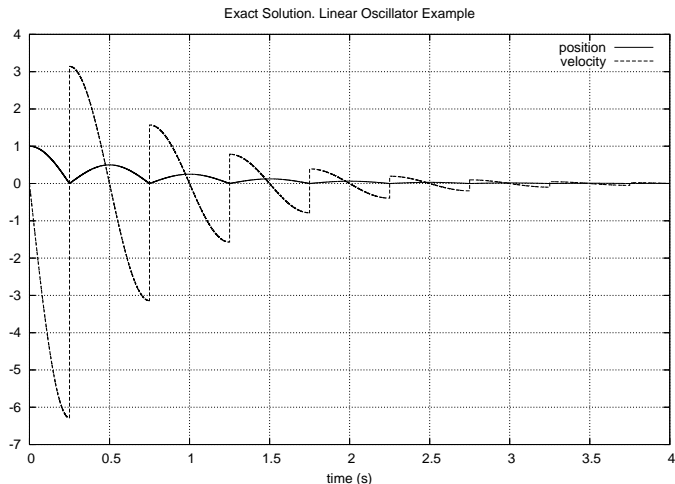


Figure: Analytical solutions. Linear Oscillator

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Moreau's Time stepping scheme (Moreau, 1988)

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Principle

$$\left\{ \begin{array}{l} M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, \\ q_{k+1} = q_k + hv_{k+\theta}, \\ U_{k+1} = G^T(q_{k+\theta})v_{k+1} \\ -P_{k+1} \in N_{T_{\mathbb{R}_+^m}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), \\ \tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0, 1]. \end{array} \right. \quad \begin{array}{l} (29a) \\ (29b) \\ (29c) \\ (29d) \\ (29e) \end{array}$$

with $\theta \in [0, 1], \gamma \geq 0$ and $x_{k+\alpha} = (1 - \alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

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Schatzman's Time stepping scheme (Paoli and Schatzman, 2002)

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Principle

$$\left\{ \begin{array}{l} M(q_k + 1)(q_{k+1} - 2q_k + q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1} \quad (30a) \\ v_{k+1} = \frac{q_{k+1} - q_{k-1}}{2h}, \quad (30b) \\ -p_{k+1} \in N_K \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right), \quad (30c) \end{array} \right.$$

where N_K defined the normal cone to K .

For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right) \perp \nabla g \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right) p_{k+1} \geq 0 \quad (31)$$

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Shared mathematical properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proof of order

Mechanical properties

- ▶ Position vs. velocity constraints
- ▶ Respect of the impact in one step (Moreau) vs. Two-steps(Schatzman)
- ▶ Linearized constraints rather than nonlinear.

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Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^*(f) = \{(t, x) \in [0, T] \times \mathbb{R}^n, 0 \leq t \leq T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (32)$$

Such graphs are closed bounded subsets of $[0, T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t, x), (s, y)) = \max\{|t - s|, \|x - y\|\}. \quad (33)$$

Defining the excess of separation between two graphs by

$$e(gr^*(f), gr^*(g)) = \sup_{(t,x) \in gr^*(f)} \inf_{(s,y) \in gr^*(g)} d((t, x), (s, y)), \quad (34)$$

the Hausdorff distance between two filled-in graphs h^* is defined by

$$h^*(gr^*(f), gr^*(g)) = \max\{e(gr^*(f), gr^*(g)), e(gr^*(g), gr^*(f))\}. \quad (35)$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in \mathcal{L}_1

$$\|e\|_1 = h \sum_{i=0}^N |f_i - f(t_i)| \quad (36)$$

In the same way, the p -norm can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^N |f_i - f(t_i)|^p \right)^{1/p} \quad (37)$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

Global order of convergence.

Definition

A one-step time-integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \quad (38)$$

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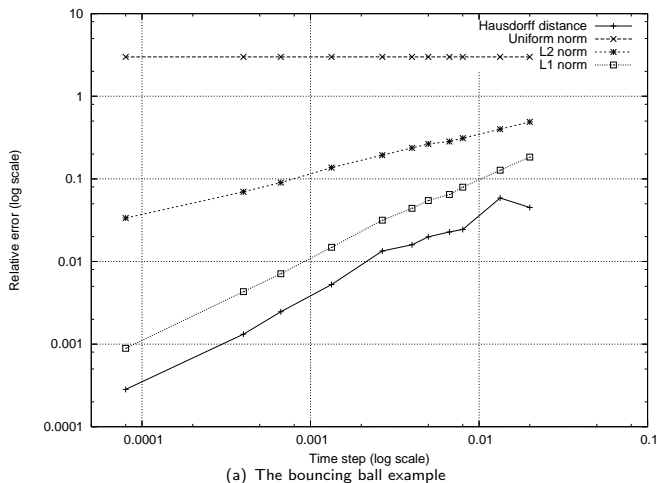


Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

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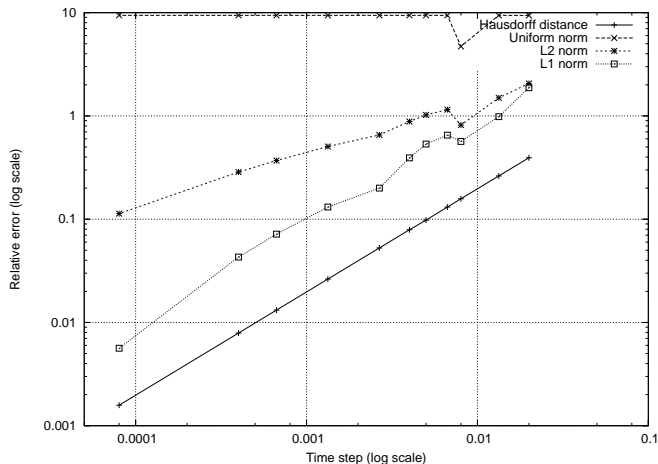
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Empirical order of convergence. Moreau's time-stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

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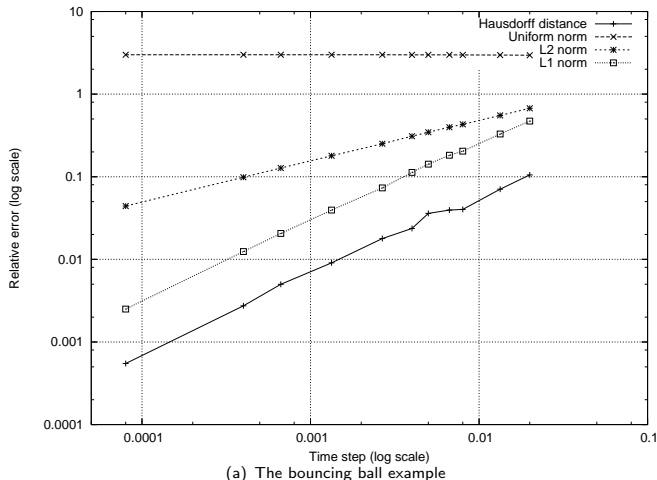


Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time-stepping scheme

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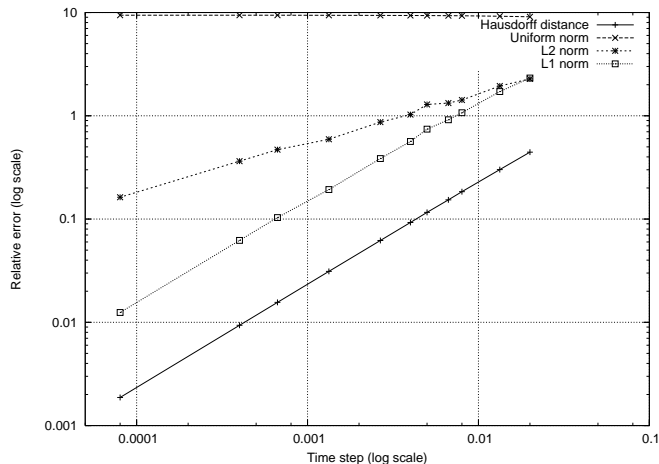


Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

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Notation

$$e = x(t_k + h) - x_{k+1} = \begin{bmatrix} e_v \\ e_q \end{bmatrix} = \begin{bmatrix} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{bmatrix} \quad (39)$$

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Assumption 1 : Existence and uniqueness

A unique global solution over $[0, T]$ for Moreau's sweeping process is assumed such that $q(\cdot)$ is absolutely continuous and admits a right velocity $v^+(\cdot)$ at every instant t of $[0, T]$ and such that the function $v^+ \in LBV([0, T], \mathbb{R}^n)$.

→ Assumption 2 is ensured in the framework introduced by Ballard (Ballard, 2000) who proves the existence and uniqueness of a solution in a general framework mainly based on the analyticity of data.

Assumption 2 : Smoothness of data

The following smoothness on the data will be assumed: a) the inertia operator $M(q)$ is assumed to be of class C^p and definite positive, b) the force mapping $F(t, q, v)$ is assumed to be of class C^p , c) the constraint functions $g(q)$ are assumed to be of class C^{p+1} and d) the Jacobian matrix $G(q) = \nabla_q^T g(q)$ is assumed to have full-row rank.

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Lemma

Let $I = [t_k, t_{k+1}]$. Let us assume that the function $f \in BV(I, \mathbb{R}^n)$. Then we have the following inequality for the θ -method, $\theta \in [0, 1]$,

$$\left\| \int_{t_k}^{t_{k+1}} f(s) ds - h(\theta f(t_{k+1}) + (1 - \theta)f(t_k)) \right\| \leq C(\theta)(t_{k+1} - t_k) \text{var}(f, I), \quad (40)$$

where $\text{var}(f, I) \in \mathbb{R}$ is the variation of f on I and $C(\theta) = \theta$ if $\theta \geq 1/2$ and $C(\theta) = 1 - \theta$ otherwise. Furthermore, the value of $C(\theta)$ yields a sharp bound in (40).

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Proposition

Under Assumptions 1 and 2, the local order of consistency of the Moreau time-stepping scheme for the generalized coordinates is

$$e_q = \mathcal{O}(h)$$

and at least for the velocities

$$e_v = \mathcal{O}(1)$$

Comments

The bounds are reached if an impact is located within the time-step and the activation of the constraint is not correct.

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One impact at time $t_* \in (t_k, t_{k+1})$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dl = P\delta_{t_*}, \text{ with } P = G(t_*)p. \quad (41)$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \quad (42)$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geq 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \quad (43)$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P_{k+1}^\alpha \geq 0, U_{k+1}^\alpha - U_k^\alpha = -(1+e)U_k^\alpha\} \quad (44)$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1})$ without persistent contact, i.e. , $di = p\delta_{t_*}$.

1. If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by

$$e_v = K_v h + \mathcal{O}(h^2) \quad (45)$$

2. If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by

$$e_v = K_v + \mathcal{O}(h) \quad (46)$$

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Other cases are treated in the same way

- ▶ *One impact and smooth Lagrange multiplier* The same result holds as in the first Lemma.
- ▶ *losing contact event (take-off) without impact* The order of the time-integration scheme depends on the regularity of the contact forces (at least continuous).
- ▶ *Finite accumulation* The order of the time-integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

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Order “0” case

Standard error estimates do not apply for Order 0.

We propose to extend it to the order 0 of consistency by assuming that the the local error estimate is given by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \quad (47)$$

where x_1 is the result of the time integration with one time-step of length h and $x_{1/2}$ with two time-steps of length $h/2$.

The adaptive time-step control used for smooth ODE is then apply directly Hairer et al. (1993).

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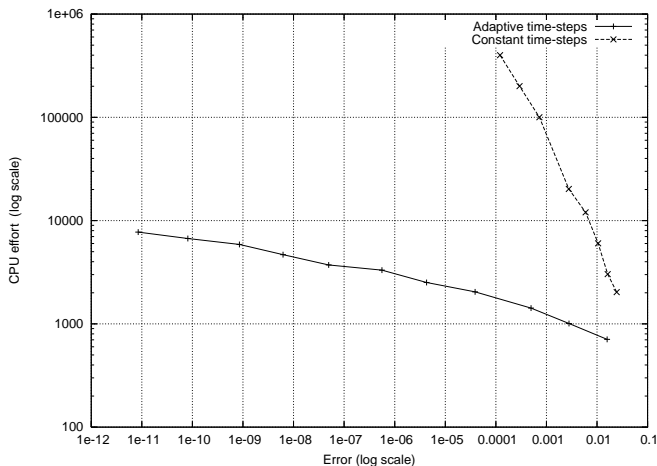
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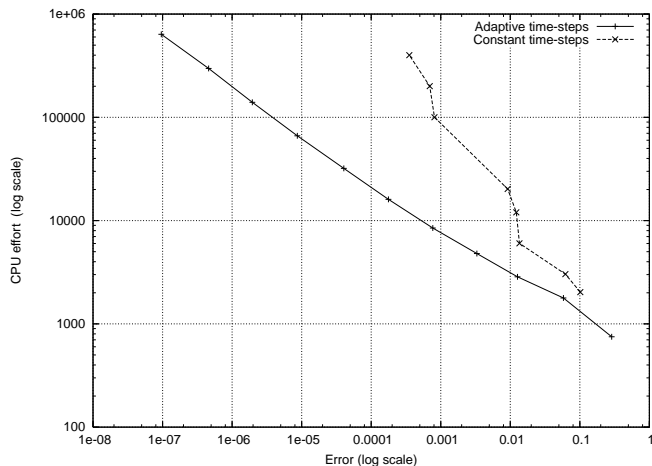
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(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

Order “0” time-step adjustment for the Moreau’s time-stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

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Background

Work of Mannshardt (1978) on time-integration schemes of any order for ODE/DAEs with discontinuities (with transversality assumption)

Principle

- ▶ Let us assume only one event per time-step at instants t_* .
- ▶ Choose any ODE/DAE solvers of order p
- ▶ Perform a rough location of the event inside the time step of length h
Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \quad (48)$$

Dichotomy, Newton, Local Interpolants, Dense output,...

- ▶ Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- ▶ Perform an integration on $[t_a, t_b]$ with Moreau's time-stepping scheme
- ▶ Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

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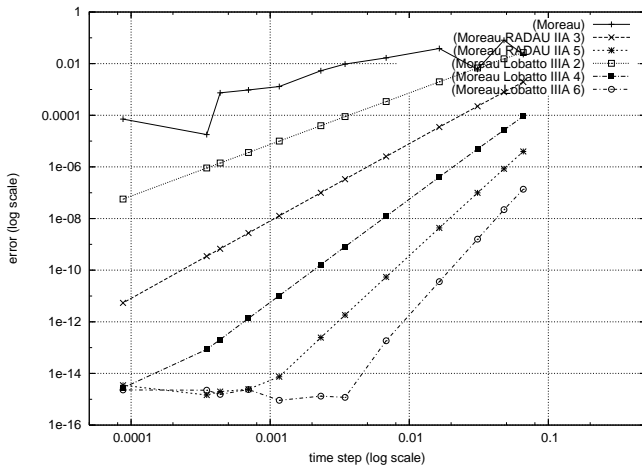
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Results on the linear oscillator



(a) The linear oscillator example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge-Kutta method.

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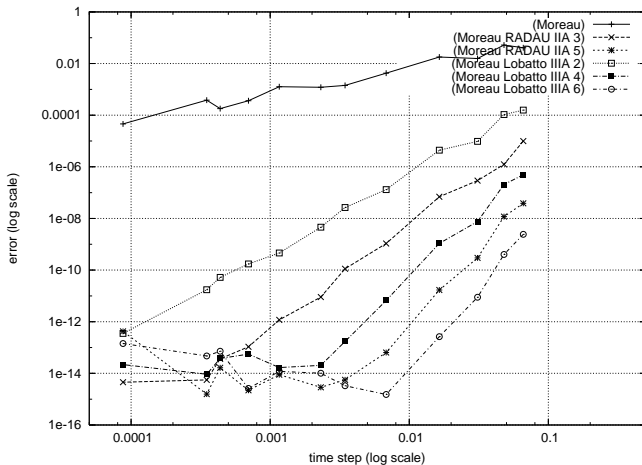
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Finite accumulation

- ▶ Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- ▶ By induction, repeat this process up to the end of the original time step.

Results on the Bouncing Ball



(a) The Bouncing Ball example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Splitting-based methods.

Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x, t) + g(x, t) \quad (49)$$

A example of splitting-based method is given by the following procedure

1. Perform the integration of f on $[t_k, t_{k+1}]$ to obtain $\tilde{x}(t_{k+1})$ that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x, t) dt \quad (50)$$

2. Perform the integration of g on $[t_k, t_{k+1}]$ with initial value $\tilde{x}(t_{k+1})$ to obtain $\hat{x}(t_{k+1})$ that is

$$\hat{x}(t_{k+1}) = \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x, t) dt \quad (51)$$

Properties

- ▶ $x(t_k + 1) \neq \hat{x}(t_{k+1})$ is the general case. (except special linear case, constant dynamics, ...)
- ▶ $\hat{x}(t_{k+1}) \rightarrow x(t_{k+1})$ when $t_{k+1} \rightarrow t_k$

Splitting-based for Moreau scheme without continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v), \\ \dot{q} = v, \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (52)$$

yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{\mathbb{T}_{\mathbb{R}^+}}(y)(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (53)$$

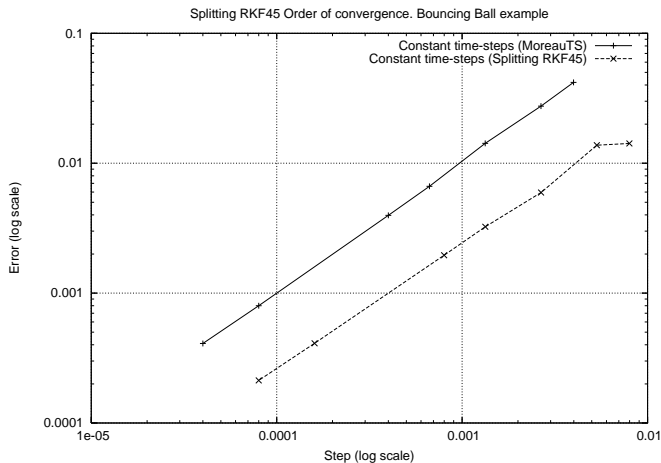
and leads to the approximation $q_{k+1} = q(t_{k+1})$ and $v_{k+1} = v(t_{k+1})$.

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Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

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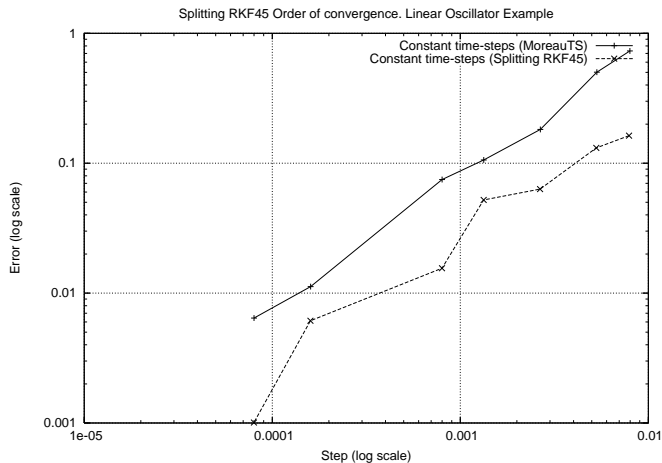
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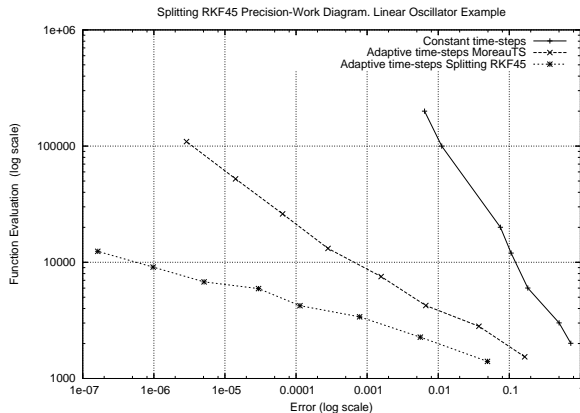
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Splitting-based for Moreau scheme with continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}^+}(y)}(\dot{y}(t)) \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (54)$$

yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}^+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (55)$$

and leads to the approximation $q_{k+1} = q(t_{k+1})$ and $v_{k+1} = v(t_{k+1})$.

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- ▶ Higher resolution schemes
- ▶ Work with finite accumulation of events

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- ▶ Theoretical works on orders and practical error estimations
Collaborations with people the ODE/DAE communities are welcome.
- ▶ Adaptive time-step strategies on the higher order time-stepping schemes.
- ▶ Improve the pre-detection process of the event and the order of discontinuity
- ▶ Test on nonlinear mechanical systems.
- ▶ Other types of time-stepping schemes (Splitting and time discontinuous Galerkin methods)

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Thank you for your attention.

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