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Numerical time integration methods for nonsmooth systems. Part I. Low relative degree and electrical networks applications Part II. Nonsmooth mechanical systems with impacts

Vincent Acary

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Numerical time integration methods for nonsmooth systems.

Part I. Low relative degree and electrical networks applications

Vincent Acary
INRIA Rhône-Alpes, Grenoble.

SDS2010. Capitolo, Monopoli, Italy. June 8–11, 2010.

vincent.acary@inrialpes.fr

Joint work with Olivier Bonnefon & Bernard Brogliato

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The RLC circuit with a diode

Example

A LC oscillator supplying a load resistor through a half-wave rectifier (see figure 1).

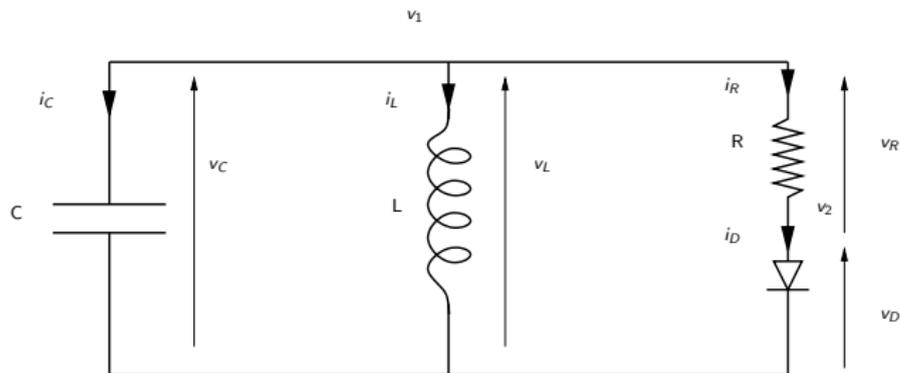


Figure: Electrical oscillator with half-wave rectifier

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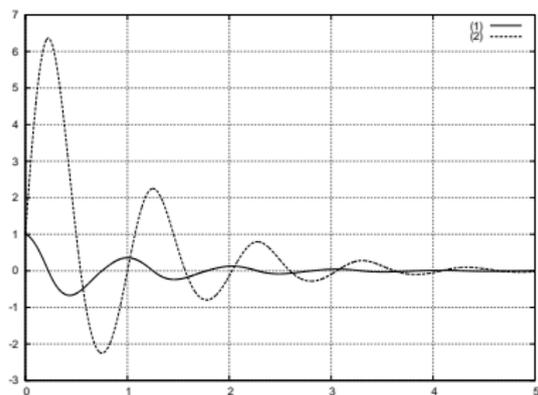
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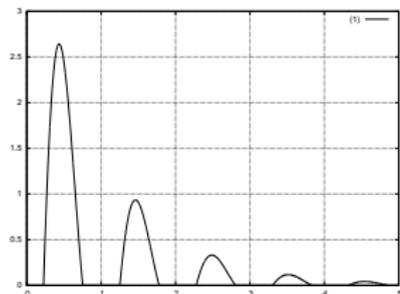
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The RLC circuit with a diode

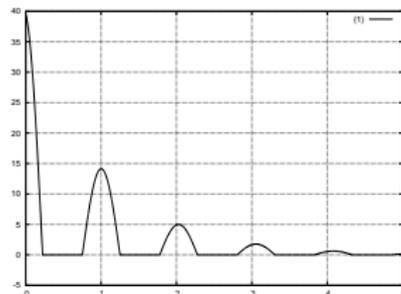
Example



(a) state versus time v_L and i_L



(b) Diode current i_D



(c) Diode voltage v_D

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Example

- ▶ Kirchhoff laws :

$$\begin{aligned}v_L &= v_C \\v_R + v_D &= v_C \\i_C + i_L + i_R &= 0 \\i_R &= i_D\end{aligned}$$

- ▶ Branch constitutive equations for linear devices are :

$$\begin{aligned}i_C &= C\dot{v}_C \\v_L &= L\dot{i}_L \\v_R &= Ri_R\end{aligned}$$

- ▶ "branch constitutive equation" of the diode

$$0 \in \mathcal{F}(i_D, v_D)$$

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Example

The following dynamical system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

$$v_D = v_L - Ri_D$$

$$0 \in \mathcal{F}(v_D, i_D)$$

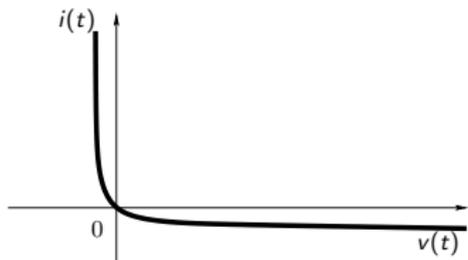
with the state variable $x \triangleq \begin{pmatrix} v_L \\ i_L \end{pmatrix}$ and $\lambda \triangleq i_D$, $y \triangleq v_D$, we get

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \in \mathcal{F}(y, \lambda) \end{cases} \quad (1)$$

Diode behavior

A modeling choice

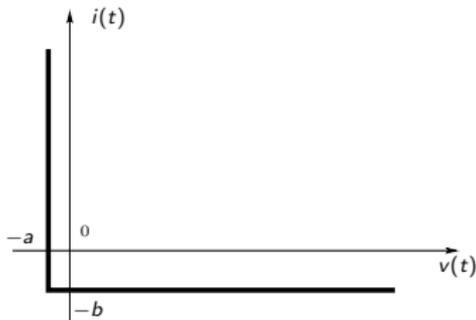
smooth modeling



(a)

$$i(t) = i_s \exp\left(-\frac{v(t)}{\alpha} - 1\right)$$

nonsmooth modeling



(b)

$$0 \leq i(t) + b \perp v(t) + a \geq 0$$

Figure: Two models of diodes.

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Why a nonsmooth modeling ?

- ▶ To avoid stiff nonlinear models by using ideal constraints.
- ▶ To model the ideal behavior of switched components without artificial regularization

The diode-bridge rectifier

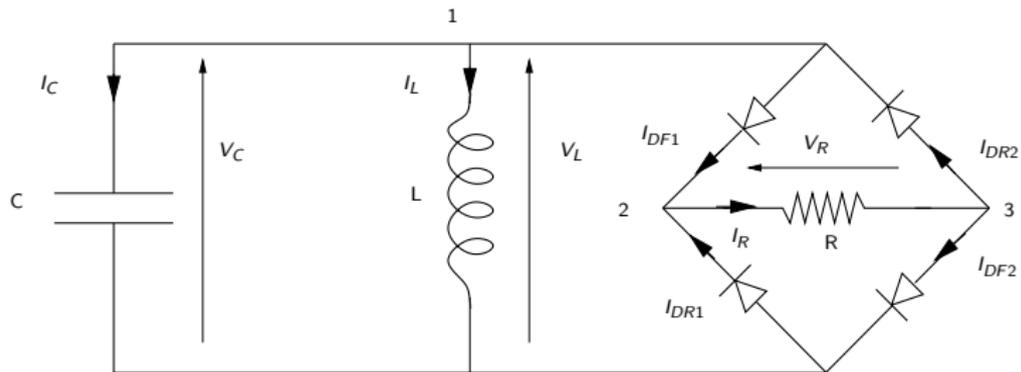


Figure: The Diode-bridge rectifier

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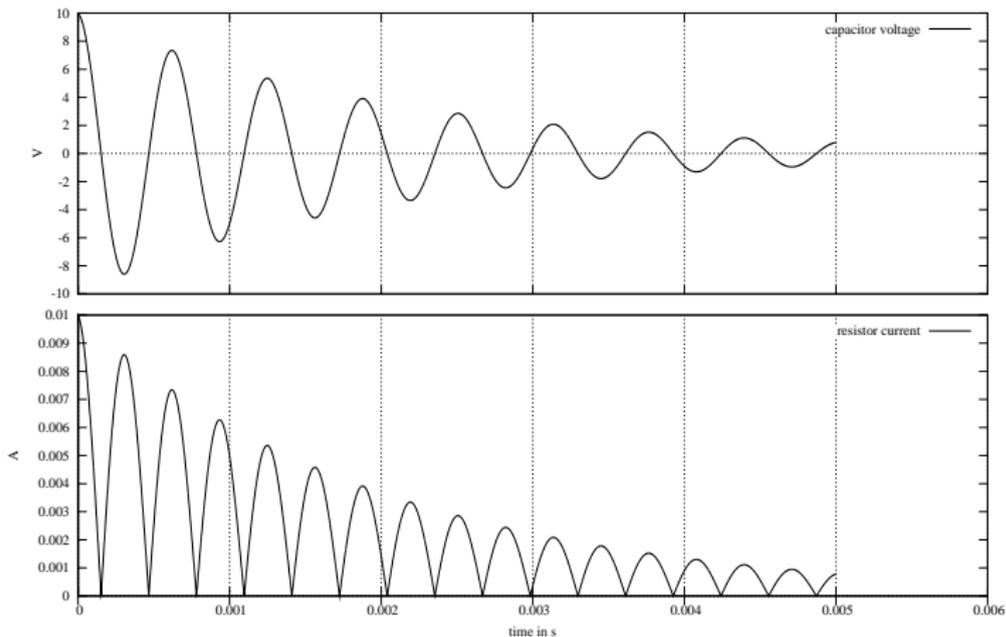


Figure: The Diode-bridge rectifier. Standard results

The diode-bridge rectifier

Differential systems

The dynamical equations are formulated as

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (2)$$

choosing :

$$x = \begin{bmatrix} V_L \\ I_L \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} I_{DR1} \\ I_{DF2} \\ V_2 - V_1 \\ V_1 - V_3 \end{bmatrix}, \quad \lambda = \begin{bmatrix} V_2 \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \quad (3)$$

and with

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1/C & 1/C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (4)$$

A typical example of nonsmooth systems

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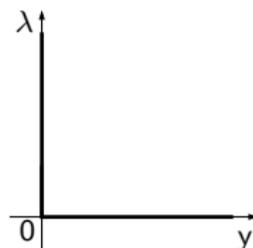
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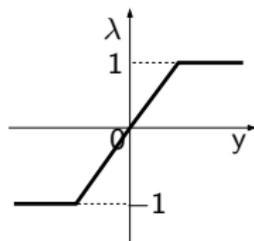
Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (5)$$

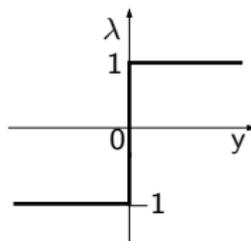
with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
 $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, for m constraints.



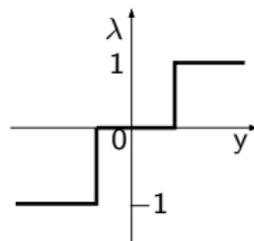
Piecewise linear systems



Saturation



Relay



Relay with dead zone

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A slightly more general class of nonsmooth systems

Differential inclusion into normal cones

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ -y \in N_K(\lambda) \end{cases} \quad (6)$$

where K is a convex set and $N_K(\lambda)$ stands for the normal cone to K taken at λ

Usual examples for K

- ▶ $K = \mathbb{R}^m$, then we obtain linear time invariant DAE

$$-y \in N_{\mathbb{R}^m}(\lambda) \iff y = 0, \quad \lambda \in \mathbb{R}^m \quad (7)$$

- ▶ $K = \mathbb{R}_+^m$, then we obtain Linear Complementarity Systems (LCS)

$$-y \in N_{\mathbb{R}_+^m}(\lambda) \iff 0 \leq y \perp \lambda \geq 0 \quad (8)$$

- ▶ $K = [-1, 1]^m$, then we obtain linear relay systems (related to Filippov's DI and sliding mode control).

$$-y \in N_{[-1,1]^m}(\lambda) \iff \lambda \in \text{sgn}(y) \quad (9)$$

Objectives of my talk

- ▶ Understand what can be the nature of the solutions (uniqueness, smoothness).
- ▶ How perform the numerical time–integration ?
- ▶ Open issues for the time–integration of large dynamical systems arising in electrical network applications.

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Definition (Linear Complementarity Systems (LCS))

A Linear Complementarity System (LCS) is defined by

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (10)$$

for the quadruplet $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times m}$.

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Definition (Relative degree in the SISO case ($m = 1$))

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{m \times n} \times \mathbb{R}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \quad (11)$$

In the Single Input/ Single Output (SISO) case ($m = 1$), the relative degree r is defined by the first non zero Markov parameter :

$$D, CB, CAB, CA^2B, \dots, CA^{r-1}B, \dots \quad (12)$$

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Definition (Uniform relative degree in the MIMO case ($m > 1$))

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times m}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \quad (11)$$

In the multiple input/multiple output (MIMO) case ($m > 1$), an *uniform* relative degree is defined as follows:

- ▶ If D is non singular, the relative degree is equal to 0.
- ▶ If $D = 0$, it is assumed to be the first positive integer r such that

$$CA^i B = 0, \quad i = 0 \dots r - 2 \quad (12)$$

while

$$CA^{r-1} B \text{ is non singular.} \quad (13)$$

- ▶ Other cases yield a nonuniform relative degree.

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Interpretation with the Markov parameters

The Markov parameters arise naturally when we derive with respect to time the output y ,

$$y = Cx + D\lambda$$

$$\dot{y} = CAx + CB\lambda, \text{ if } D = 0$$

$$\ddot{y} = CA^2x + CAB\lambda, \text{ if } D = 0, CB = 0$$

...

$$y^{(r)} = CA^r x + CA^{r-1} B \lambda, \text{ if } D = 0, CB = 0, CA^{r-2} B = 0, r = 1 \dots r - 2$$

...

and the first non zero Markov parameter allows us to define the output y directly in terms of the input λ .

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Interpretation in terms of differential index of DAE

Let us consider the differential index ν of the following linear time-invariant DAE

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ y = 0 \end{cases} \quad (11)$$

If the uniform relative degree of the quadruplet (A, B, C, D) is r , then the differential index $\nu = r + 1$.

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Example (Third relative degree LCS)

$$\begin{cases} \ddot{x}(t) = \lambda(t) - 1, \\ y(t) = x(t) \\ 0 \leq y(t) \perp \lambda(t) \geq 0 \end{cases} \quad (12)$$

with the initial conditions, $x(0) = x_0 \geq 0, \dot{x}(t_0) = \dot{x}_0, \ddot{x}(t_0) = \ddot{x}_0$.

- ▶ the relative degree r is obviously 3, since $y^{(3)} = \ddot{x} = \lambda - 1$
- ▶ since $x_0 \geq 0$ satisfies the constraint, the function $x : [0, T] \rightarrow \mathbb{R}$ is usually assumed to be an absolutely continuous function of time.

The notion of relative degree. Well-posedness

Example (Third relative degree LCS)

Let us consider the dynamics at t_* when the constraint $y = x \geq 0$ becomes active, i.e., $x(t_*) = 0$,

- ▶ If $\dot{x}(t_*^-) > 0$, the system will instantaneously leaves the constraints with $\dot{x}(t_*^+) = \dot{x}(t_*^-) > 0$.
- ▶ If $\dot{x}(t_*^-) < 0$, the velocity \dot{x} needs to jump to respect the constraint in t_*^+ . (B.V. function)
- ▶ If $\dot{x}(t_*^-) < 0, \ddot{x}(t_*^-) < 0$, the velocity and the acceleration need to jump to respect the constraint in t_*^+ . (Dirac + B.V. function)

→ In the latter case, \ddot{x} and λ must be considered as derivative of Dirac distribution.

Well-posedness for $r \geq 1$

What is the meaning of $\lambda \geq 0$ when $\lambda = \delta^{(r-1)}$

- ▶ If the initial conditions do not satisfy the constraints, the relative degree $r \geq 1$ needs a rigorous definition of the sign of a distribution. (see [Acary et al., 2008] for details)

→ In this talk, we will focus on LCS of relative degree $r \leq 1$. The passive linear systems are of relative degree ≥ 1 [Camlibel, 2001, Heemels and Brogliato, 2003]).

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Example (The relative degree not sufficient [Heemels and Brogliato, 2003])

Let us consider the following LCS

$$\begin{cases} \dot{x} = -x + \lambda \\ y = x - \lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (12)$$

This system is strictly equivalent to

$$\dot{x} = \begin{cases} -x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (13)$$

which leads to non existence of solutions for $x(0) < 0$ and to non uniqueness for $x(0) > 0$.

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Possible further assumptions for existence and uniqueness

- ▶ The Rational Complementarity problem [Heemels, 1999, Camlibel, 2001, Camlibel et al., 2002]. The P-matrix property plays henceforth a fundamental role and provides the existence of global solution of the LCS in the sense of Caratheodory.
- ▶ For the relative degree $r = 1$ case, $\exists P > 0$, such that $PB = C^T$. A standard monotone differential inclusion is retrieved such that

$$- [\dot{z} + f(z, t)] \in A(z) \quad (12)$$

where A is a maximal monotone operator and f Lipschitz continuous.

→ Existence and uniqueness of a solution $u \in C^0$ and $\dot{u} \in L^\infty$.

[Bastien and Schatzman, 2002]

To summarize

C^1 solutions

Mainly when $B\lambda$ singleton

- ▶ D positive definite (relative degree 0)
- ▶ D is a P-matrix
- ▶ D is a co-positive matrix

Absolutely continuous solutions

- ▶ relative degree 1 with $CB > 0$ or $PB = C^T$ and $P > 0$
- ▶ consistent initial conditions

Solution of Bounded Variations

- ▶ relative degree 1 with $CB > 0$ or $PB = C^T$ and $P > 0$
- ▶ inconsistent initial conditions

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Open issues of the well-posedness

Cases that are taken into account by the uniform relative degree (MIMO case)

For instance, D or CB singular but non zero

- ▶ Generalize the differential index ?

Assumption of positive definiteness of the Markov parameters.

- ▶ General assumption equivalent to $B.SOL(Cx, D)$ for the relative degree 0.

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Targeted system

$$\dot{x} = Ax + u(t) + r$$

$$y = Cx + D\lambda + a(t)$$

$$R = B\lambda$$

$$0 \in y + N_{\mathbb{R}_+^m}(\lambda)$$

$$x(t_0) = x_0$$

] Differential Equations

] Input/output relations
(nonsmooth components)

] Generalized equation

] Initial conditions

(13)

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Time-stepping schemes. Design principles.

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First principle

The fully implicit evaluation of the generalized equation in (13), that is on $[t_k, t_{k+1}]$,

$$0 \in y_{k+1} + N_{\mathbb{R}_+^m}(\lambda_{k+1}) \quad (14)$$

Second principle

A consistent evaluation of the unknown variables and their derivatives according to their smoothness.

For instance, time-stepping schemes must not approximate high order time-derivatives of functions which are not sufficiently smooth or must not try to point-wisely evaluate distributions.

Time-stepping schemes for a solution for class C^1

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Required assumptions

- ▶ $B\lambda$ Lipschitz continuous function of x .
- ▶ In particular,

$$0 \in Cx + D\lambda + a + N_K(\lambda) \quad (15)$$

possesses a unique solution for all x .

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Proposed scheme

$$\left\{ \begin{array}{l} x_{k+1} - x_k = h(Ax_{k+\theta} + u_{k+\theta} + r_{k+\gamma}), \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1} + a_{k+1}, \\ r_{k+1} = B\lambda_{k+1}, \\ 0 \in y_{k+1} + N_K(\lambda_{k+1}), \end{array} \right. \quad (16)$$

with $\theta \in [0, 1]$ and $\gamma \in [0, 1]$. $x_{k+\theta} = (1 - \theta)x_k + \theta x_{k+1}$
The initial value of $\lambda_0 = \lambda(t_0)$ is given by the solution of (15).

Time-stepping schemes for a solution for class C^1

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One-step LCP

$$\begin{cases} y_{k+1} = M\lambda_{k+1} + q \\ 0 \leq y_{k+1} \perp \lambda_{k+1} \geq 0, \end{cases} \quad (17)$$

with

$$M = D + h\gamma C(I - h\theta A)^{-1}B, \quad (18)$$

and

$$q = a_{k+1} + C(I - h\theta A)^{-1} [(I + h(1 - \theta)A)x_k + hu_{k+\theta} + h(1 - \gamma)B\lambda_k]. \quad (19)$$

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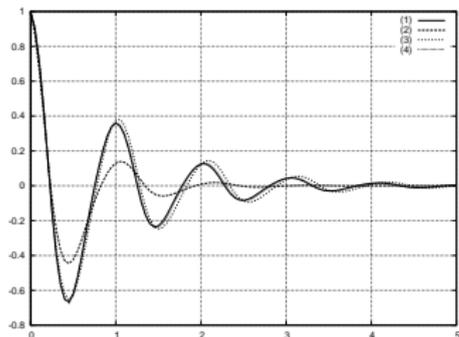
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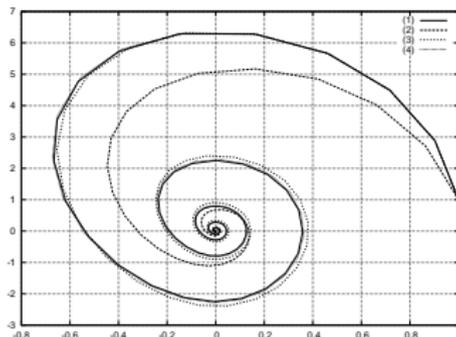
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(a) State x_1 vs. time.



(b) Phase portrait. x_1 vs. x_2 .

Figure: Solution of the RLC circuit with the time-stepping scheme (16). (1) Exact solution $x(t_k)$. (2) x_k with $\theta = 1, \gamma = 1$. (3) x_k with $\theta = 1/2, \gamma = 1$. (4) x_k with $\theta = 1/2, \gamma = 1/2$.

Influence of θ and γ

- ▶ Numerical parameters allows us to control numerical damping and order
- ▶ Second order accuracy can be achieved with $\theta = \gamma = 1/2$.
- ▶ Higher order approximations are useless and generate instabilities.

Time-stepping schemes for an absolutely continuous solution

Required assumptions

- ▶ Relative degree equal to 1 with consistent initial conditions

$$Cx_0 + a(t_0) \in \mathbb{R}_+^m. \quad (20)$$

and smooth function $a(\cdot)$

- ▶ Monotony, positive definiteness, passivity assumption,

Proposed scheme

$$\left\{ \begin{array}{l} x_{k+1} - x_k = h(Ax_{k+\theta} + u_{k+\theta} + r_{k+1}), \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1} + a_{k+1}, \\ r_{k+1} = B\lambda_{k+1}, \\ 0 \in y_{k+1} + N_K(\lambda_{k+1}), \end{array} \right. \quad (21)$$

with $\theta \in [0, 1]$.

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Properties

- ▶ Order 1 is achieved for $\theta \in [0, 1]$
- ▶ θ controls the numerical damping and the stability

Why do not use $\gamma = 1/2$?

- ▶ No improvements for the order of accuracy
- ▶ Severe instabilities and numerical artifacts on r and λ .

With $\gamma = 1/2$, we attempt a second order approximation of a function of bounded variations λ .

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Required assumptions

- ▶ Relative degree equal to 1 with inconsistent initial conditions

$$Cx_0 + a(t_0) \notin \mathbb{R}_+^m. \quad (22)$$

and/or nonsmooth function $a(\cdot)$.

- ▶ Monotony, positive definiteness, passivity assumption,
- ▶ Relative degree 2 with consistent initial conditions.

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Measure differential equations (in a nutshell)

$$dx = Ax(t)dt + u(t)dt + Bdi, \quad (23)$$

- ▶ dx is the differential measure associated with the RCBV function $\dot{x}(t)$ and di is also a measure
- ▶ The absolutely continuous function $\lambda(t)$ is the Radon-Nikodym derivative of di with respect to the Lebesgue measure, *i.e.* :

$$\frac{di}{dt} = \lambda(t). \quad (24)$$

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Measure decomposition

$$di = \lambda(t)dt + \sum_i \sigma_i \delta_{t_i} \quad (25)$$

where δ_{t_i} is the Dirac measure at time of discontinuities t_i and σ_i the amplitude.

- ▶ Smooth dynamics :

$$\dot{x}(t) = Ax(t) + u(t) + B\lambda(t), \quad dt - \text{almost everywhere}, \quad (26)$$

- ▶ Jump dynamics at t_i :

$$x(t_i^+) - x(t_i^-) = B\sigma_i. \quad (27)$$

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Design of a consistent scheme

Only the measure of the time-intervals $(t_k, t_{k+1}]$ are considered such that :

$$dx((t_k, t_{k+1}]) = \int_{t_k}^{t_{k+1}} Ax(t) + u(t) dt + Bdi((t_k, t_{k+1}]). \quad (28)$$

By definition of the differential measure, we get

$$dx((t_k, t_{k+1}]) = x(t_{k+1}^+) - x(t_k^+). \quad (29)$$

The measure of the time-interval by di is kept as an unknown variable denoted by

$$\sigma_{k+1} = di((t_k, t_{k+1}]). \quad (30)$$

Finally, the remaining Lebesgue integral in (28) is approximated by an implicit Euler scheme

$$\int_{t_k}^{t_{k+1}} Ax(t) + u(t) dt \approx hAx_{k+1} + u_{k+1}. \quad (31)$$

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Proposed scheme

$$\left\{ \begin{array}{l} x_{k+1} - x_k = h(Ax_{k+1} + u_{k+1}) + \sigma_{k+1}, \\ y_{k+1} = Cx_{k+1} + a_{k+1}, \\ r_{k+1} = B\sigma_{k+1}, \\ 0 \in y_{k+1} + N_K(\sigma_{k+1}). \end{array} \right. \quad (32)$$

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Properties

- ▶ At best order 1 is achieved. (no rigorous proof for a finite accumulation of jumps)
- ▶ Implicitly implements a restitution rule with “no rebounds”

Time-stepping schemes for solution of bounded variations

Example (An RLD circuit)

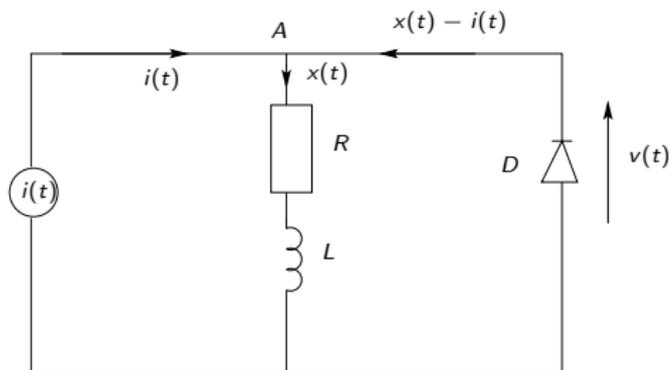


Figure: A circuit with an ideal diode, a resistor, an inductor and a current source.

$$\begin{cases} \dot{x}(t) = -\frac{R}{L}x(t) + v(t) \\ 0 \leq w(t) = x(t) - i(t) \perp v(t) \geq 0 \\ x(t^+) = i(t^+) + \max[0, x(t^-) - i(t^+)] \text{ at jump times,} \end{cases} \quad (33)$$

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Let us choose $x(0^-) = -2$ and

$$i(t) = \begin{cases} 0 & \text{for all } t \in [0, 5) \\ 2 & \text{for all } t \in (5, 10) \\ -2 & \text{for all } t \geq 10 \end{cases}$$

The analytical solution with this value for the current source is:

$$x(t) = \begin{cases} x(0^+) = 0, & x(t) = 0, & v(t) = 0 & \text{on } t \in (0, 5) \\ x(5^+) = 2, & x(t) = 2, & v(t) = 2 & \text{on } t \in [5, 10) \\ x(10^+) = 2, & x(t) = 2e^{(10-t)} & v(t) = 0 & \text{on } t \in [10, +\infty). \end{cases} \quad (34)$$

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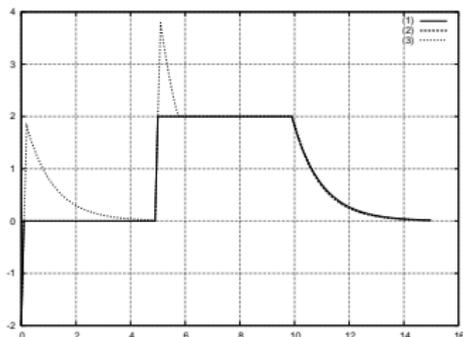
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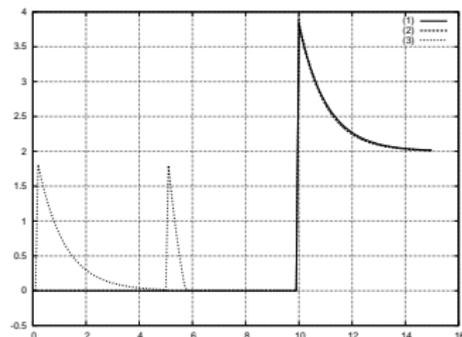
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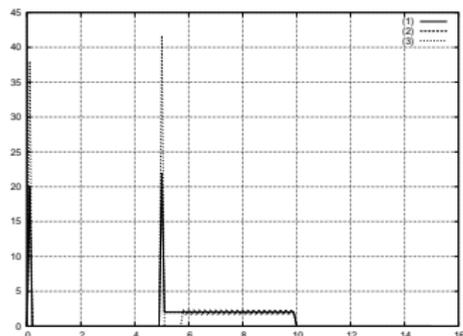
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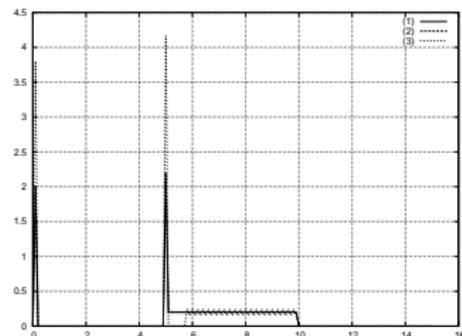
(a) state x_k vs t_k



(b) output w_k vs t_k



(c) variable λ_k vs t_k



(d) variable σ_k vs t_k

Figure: Simulation of system (33). (1) scheme (30). (2) scheme (21) with $\theta = 1/2$. (3) scheme (16) with $\theta = 1/2, \gamma = 1/2$.

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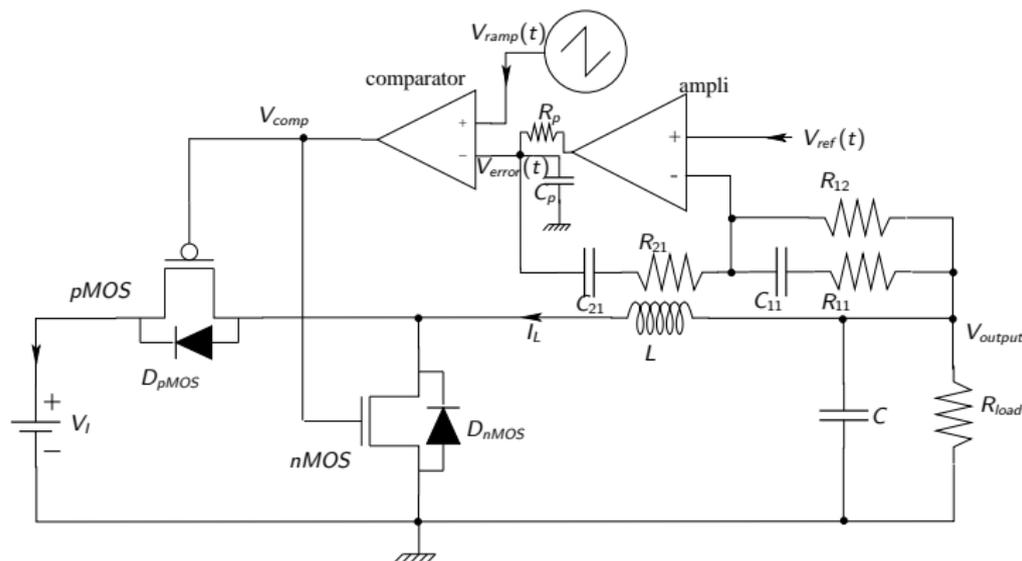


Figure: Buck converter.

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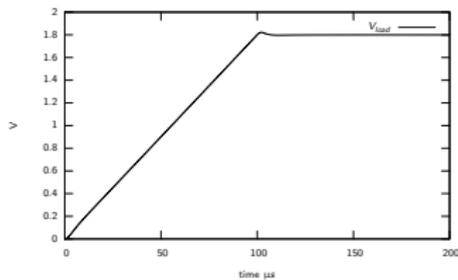
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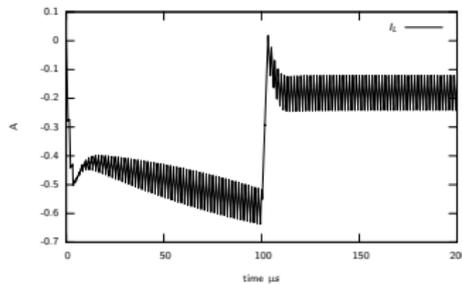
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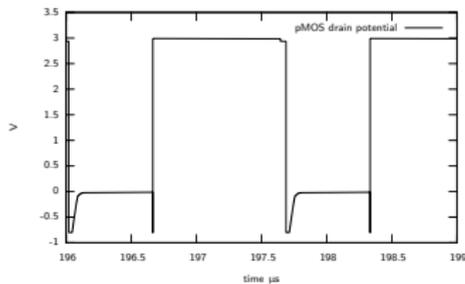
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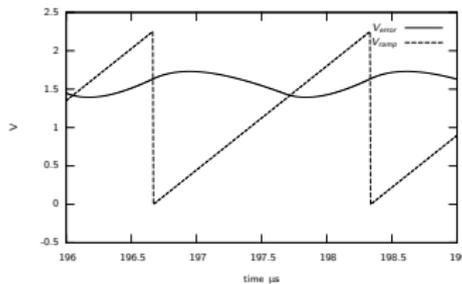
(a) V_{load}



(b) I_L



(c) pMOS drain potential



(d) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using standard parameters.

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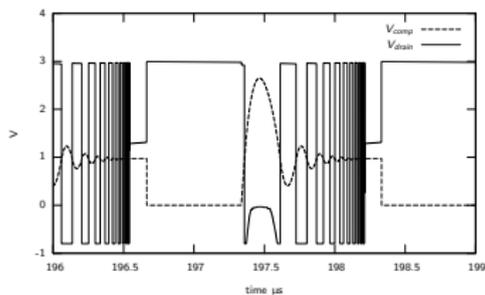
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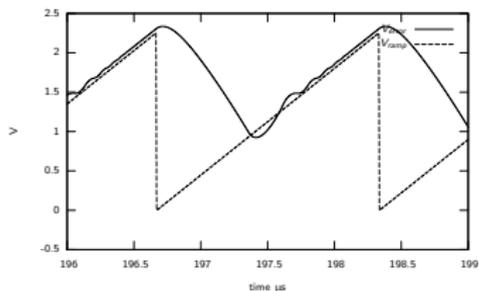
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(a) V_{comp} and V_{drain}



(b) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using sliding mode parameters.

For more general formulations and more complex systems, are we able to infer the nature of the solutions? That is to say,

- ▶ Define and predict an equivalent notion to index and relative degree for instance, for a matrix D semi-definite positive.
- ▶ Given passive components, are we able to forecast the nature of the solutions from some topological considerations ? (as for the DAE case.)
- ▶ Adapt the time-stepping schemes in an hierarchical way in taking into account the "index" of each variable.

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Thank you for your attention.

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- V. Acary, B. Brogliato, and D. Goeleven. Higher order Moreau's sweeping process: mathematical formulation and numerical simulation. *Mathematical Programming Ser. A*, 113(1):133–217, 2008.
- J. Bastien and M. Schatzman. Numerical precision for differential inclusions with uniqueness. *ESAIM M2AN: Mathematical Modelling and Numerical Analysis*, 36(3):427–460, 2002.
- M. K. Camlibel. *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*. PhD thesis, Katholieke Universiteit Brabant, 2001. ISBN: 90 5668 073X.
- M. K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Consistency of a time-stepping method for a class of piecewise-linear networks. *IEEE Trans. Circuits and systems I*, 49:349–357, 2002.
- W.P.M.H. Heemels. *Linear Complementarity Systems. A Study in Hybrid Dynamics*. PhD thesis, Technical University of Eindhoven, 1999. ISBN 90-386-1690-2.
- W.P.M.H. Heemels and B. Brogliato. The complementarity class of hybrid dynamical systems. *European Journal of Control*, 9:311–349, 2003.

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