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# Analysis of a new variational model to restore point-like and curve-like singularities in imaging<sup>1</sup>

GILLES AUBERT

LABORATOIRE J.A. DIEUDONNÉ

Université de Nice SOPHIA ANTIPOLIS, Parc Valrose 06108 Nice CEDEX 2, FRANCE

e.mail: gaubert@unice.fr

LAURE BLANC-FÉRAUD

ARIANA CNRS/INRIA/UNSA SOPHIA ANTIPOLIS

Inria, 2004 route des Lucioles BP 93 06902 SOPHIA ANTIPOLIS CEDEX, FRANCE

e.mail: Laure.Blanc-Feraud@inria.fr

DANIELE GRAZIANI<sup>2</sup>

ARIANA CNRS/INRIA/UNSA SOPHIA ANTIPOLIS

Inria, 2004 route des Lucioles BP 93 06902 SOPHIA ANTIPOLIS CEDEX, FRANCE

e.mail: Daniele.Graziani@inria.fr

e.mail:daniele.graziani@gmail.com

**Abstract.** The paper is concerned with the analysis of a new variational model to restore point-like and curve-like singularities in biological images. To this aim we investigate the variational properties of a suitable energy which governs these pathologies. Finally in order to realize numerical experiments we minimize, in the discrete setting, a regularized version of this functional by fast descent gradient scheme.

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## 1. INTRODUCTION

In biological image processing one might be interested in restoring an image, possibly corrupted by noise, characterized by an high intensity value on sets of low dimension, such as points or curves in 2-D and 3-D images and which decreases to 0 in a neighborhood of the singularities. Then across these sets there is no jump of the gradient of the image. This phenomenon requires a new approach, which is different from the classical denoising model based on the gradient operator. From a general point of

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<sup>2</sup>Corresponding author.

view in order to restore the given data  $u_0$ , one would like to minimize an energy with an  $L^2$ -fidelity term and a proper regularization criterion  $F$ . To this end one has to solve the following minimum problem:

$$(1.1) \quad \min_{u \in X} F(u) + \frac{\lambda}{2} \|u - u_0\|_2^2.$$

$\lambda > 0$  is a positive weight, the functional space  $X$  and the criterion  $F$  must be chosen according to the singularities to be restored. In our case it means that  $F$  must be given by a proper differential operator which is singular in a suitable sense on points and curves.

Hence, from a variational point of view, we have to find a functional space whose elements are able to produce Radon measures, whose support contains the singularities we would like to preserve. Then, in line with recent works for detecting point-like target problem (see [2, 11]), we consider the divergence as an operator able of producing Radon measures concentrated on points and curves. We deal with the space  $\Delta\mathcal{M}^p(\Omega)$  of  $W_0^{1,p}$ -functions whose gradient is an  $L^p$ -vector field with distributional divergence given by a Radon measure, where  $1 < p < \frac{N}{N-1}$ ,  $\Omega \subset \mathbb{R}^N$  is the image domain and  $N$  is the space dimension (see Section 2 below). The restriction on  $p$  is due to the fact that when  $p \geq \frac{N}{N-1}$  the distributional divergence  $\text{Div}\nabla u$  of  $\nabla u$  cannot be a measure concentrated on points (see [2, 11]). Then to restore point-like and curve-like singularities in 2D or 3D, we consider general criterion of type

$$(1.2) \quad F(u) := \begin{cases} \int_{\Omega} f(\Delta u) dx + \int_{\Omega} g(\nabla u) dx & \text{on } W_0^{1,p,1}(\text{Div}; \Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\text{Div}; \Omega), \end{cases}$$

where  $f, g$  are convex functions with linear and  $p$ -growth from above respectively.  $W_0^{1,p,1}(\text{Div}; \Omega)$  is the space of  $W_0^{1,p}$ -functions whose gradient is an  $L^p$ -vector field whose distributional divergence is an  $L^1$ -function (see Section 2 for definition). Further, since functional (1.2) is not lower semicontinuous with respect to the  $W_0^{1,p}$ -weak convergence, we deal with its lower semicontinuous envelope (or relaxed functional) on the space  $\Delta\mathcal{M}^p(\Omega)$ . So the first goal of the paper is to provide an integral representation result for the relaxed functional of functional (1.2), which can guarantee that the regularization criterion we have chosen, matches with the singularities we aim to preserve. When  $1 < p < \frac{N}{N-1}$  we provide such a representation by using the notion of  $p$ -capacity which is a useful tool to distinguish sets of 0-Lebesgue measure. In fact it turns out that, to extend functional (1.2) to the whole space  $\Delta\mathcal{M}^p(\Omega)$ , one has to measure the absolutely continuous part of the distributional divergence of  $\nabla u$  and, at the same time, control its singular parts, encoded in two mutually singular measures  $\mu^a$  concentrated on sets of positive  $p$ -capacity such as 2D curves or 3D surfaces, and  $\mu^0$  concentrated on sets with 0  $p$ -capacity, such as points in 2D or curves in 3D (see for instance [10, 13, 18] for general properties of  $p$ -capacity). Indeed by applying the  $p$ -capacitary decomposition, proven in [7], to the singular part, given by the Radon-Nykodim decomposition, of the measure  $\text{Div}\nabla u$ , we have

$$\text{Div}\nabla u = \Delta u dx + \text{Div}^s \nabla u = \Delta u dx + \mu^a + \mu^0,$$

(see subsection 2.2 below). Then we prove that according to previous decomposition, the lower semi-continuous envelope of  $F$  is given by:

$$(1.3) \quad \mathcal{F}(u) := \int_{\Omega} f(\Delta u) dx + \int_{\Omega} g(\nabla u) dx + \int_{\Omega} f^{\infty}\left(\frac{d\mu^a}{d|\mu^a|}\right) d|\mu^a| + \int_{\Omega} f^{\infty}\left(\frac{d\mu^0}{d|\mu^0|}\right) d|\mu^0|,$$

where  $u \in \Delta\mathcal{M}^p(\Omega)$  and  $f^{\infty}$  is the recession function given by  $\lim_{s \rightarrow +\infty} \frac{f(st)}{s}$ , with  $t \in \mathbb{R}$ .  $\frac{d\mu^a}{d|\mu^a|}$ ,  $\frac{d\mu^0}{d|\mu^0|}$  are the Radon-Nikodym derivatives of the measures  $\mu^a$  and  $\mu^0$  with respect to their total variation (see Theorem 3.1).

As a consequence of the relaxation result, we derive, under suitable growth assumptions on the integrands  $f$  and  $g$ , the existence of a minimizer of  $\mathcal{F}$  on the space  $\Delta\mathcal{M}^p(\Omega)$ . Moreover we also obtain the  $\Gamma$ -convergence to  $\mathcal{F}$  of a sequence of more regular integral functionals (see Theorem 3.3).

In the second part of this work we focus on concrete applications in biological image processing in dimension 2. In practice we consider the following functional

$$\mathcal{J}(u) := \int_{\Omega} |\Delta u| dx + \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + |\mu^a|(\Omega) + |\mu^0|(\Omega) + \frac{\lambda}{2} \int_{\Omega} |u - u_0|^2 dx,$$

which corresponds to the particular case  $f = |\cdot|$  and  $g = |\cdot|^p$ . Its smoother version is given by:

$$\mathcal{J}_h(u) := \begin{cases} \int_{\Omega} w_{\epsilon_h}(|\Delta u|) dx + \frac{1}{p} \int_{\Omega} w_{\epsilon_h}(|\nabla u|^p) dx + \frac{\lambda}{2} \int_{\Omega} |u - u_0|^2 dx & \text{on } W_0^{1,p,1}(\text{Div}; \Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\text{Div}; \Omega). \end{cases}$$

where  $w_{\epsilon_h}$  is the Huber function (see Section 4 for definition) and  $\{\epsilon_h\}_h \rightarrow 0$  as  $h \rightarrow 0$ . Then, by adapting some ideas of [14, 16] to our setting, we minimize the discrete version (see Subsection 4.1 for notation and definition in discrete setting) of  $\mathcal{J}_h$  by a fast descent gradient algorithm.

Finally we conclude the paper by testing the proposed model to both synthetic and biological real images provided for us by "Institut Pasteur de Paris".

The paper is organized as follows. Section 2 is devoted to notations, preliminary definitions and results. In Section 3 we address the relaxation result and its consequences. In Section 4 we illustrate our model by showing experimental results on both synthetic and biological images.

## 2. DEFINITION AND MAIN PROPERTIES

**2.1. Distributional divergence and classical spaces.** In the next subsection we recall the definition of the distributional space  $L^{p,q}(\text{Div}; \Omega)$  and  $\mathcal{DM}^p(\Omega)$ ,  $1 \leq p, q \leq +\infty$ , (see [1, 5]). In all the paper  $\Omega \subset \mathbb{R}^N$  is an open bounded set with lipschitz boundary.  $N \geq 2$  is the space dimension.

**Definition 2.1.** *We say that  $U \in L^{p,q}(\text{Div}; \Omega)$  if  $U \in L^p(\Omega; \mathbb{R}^N)$  and if its distributional divergence  $\text{Div}U \in L^q(\Omega)$ . If  $p = q$  the space  $L^{p,q}(\text{Div}; \Omega)$  will be denoted by  $L^p(\text{Div}; \Omega)$ .*

*We say that a function  $u \in W^{1,p}(\Omega)$  belongs to  $W^{1,p,q}(\text{Div}; \Omega)$  if  $\nabla u \in L^{p,q}(\text{Div}; \Omega)$ . We say that a function  $u \in W_0^{1,p}(\Omega)$  belongs to  $W_0^{1,p,q}(\text{Div}; \Omega)$  if  $\nabla u \in L^{p,q}(\text{Div}; \Omega)$ .*

**Definition 2.2.** *For  $U \in L^p(\Omega; \mathbb{R}^N)$ ,  $1 \leq p \leq +\infty$ , set*

$$|\text{Div}U|(\Omega) := \sup\{ \langle U, \nabla \varphi \rangle : \varphi \in C_0^\infty(\Omega), |\varphi| \leq 1 \}.$$

We say that  $U$  is an  $L^p$ -divergence measure field, i.e.  $U \in \mathcal{DM}^p(\Omega)$ , if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^N)} + |\text{Div}U|(\Omega) < +\infty.$$

We recall that  $U \in L^p(\Omega; \mathbb{R}^N)$  belongs to  $\mathcal{DM}^p(\Omega)$  if and only if there exists a Radon measure denoted by  $\text{Div}U$  such that

$$\langle U, \nabla \varphi \rangle = - \int_{\Omega} \text{Div}U \varphi \quad \forall \varphi \in C_0^\infty(\Omega),$$

and the total variation of the measure  $\text{Div}U$  is given by  $|\text{Div}U|(\Omega)$ .

Let us recall the following classical result (see [6] Proposition 3.1).

**Theorem 2.1.** *Let  $\{U_h\}_h \subset \mathcal{DM}^p(\Omega)$  be such that*

$$(2.1) \quad U_h \rightharpoonup U \quad \text{in } L^p(\Omega; \mathbb{R}^N), \quad \text{as } h \rightarrow +\infty \text{ for } 1 \leq p < +\infty.$$

*Then*

$$\|U\|_{L^p(\Omega; \mathbb{R}^N)} \leq \liminf_{h \rightarrow +\infty} \|U_h\|_{L^p(\Omega; \mathbb{R}^N)}, \quad |\text{Div}U|(\Omega) \leq \liminf_{h \rightarrow +\infty} |\text{Div}U_h|(\Omega).$$

Finally we define the following space

$$(2.2) \quad \Delta \mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \nabla u \in \mathcal{DM}^p(\Omega)\}.$$

**2.2.  $p$ -capacity.** If  $\mathcal{K} \subset \mathbb{R}^N$  is a compact set and  $\chi_{\mathcal{K}}$  denotes its characteristic function, we define:

$$\text{Cap}_p(\mathcal{K}, \Omega) = \inf \left\{ \int_{\Omega} |\nabla f|^p dx, f \in C_0^\infty(\Omega), f \geq \chi_{\mathcal{K}} \right\}.$$

If  $U \subset \Omega$  is an open set and  $\mathcal{K} \subset U$  is a compact set, its  $p$ -capacity is given by

$$\text{Cap}_p(U, \Omega) = \sup_{\mathcal{K} \subset U} \text{Cap}_p(\mathcal{K}, \Omega).$$

Finally if  $A \subset U \subset \Omega$  with  $A$  Borel set and  $U$  open, then

$$\text{Cap}_p(A, \Omega) = \inf_{A \subset U \subset \Omega} \text{Cap}_p(U, \Omega).$$

We recall the following result (see for instance [13], Theorem 2.27) that explains the relationship between  $p$ -capacity and Hausdorff measures. Such a result is crucial to have geometric information on null  $p$ -capacity sets.

**Theorem 2.2.** *Assume  $1 < p < N$ . If  $\mathcal{H}^{N-p}(A) < \infty$  then  $\text{Cap}_p(A, \Omega) = 0$ .*

For general properties we refer the reader to [10, 13, 18].

It is known (see [7]) that given a Radon measure  $\mu$  the following decomposition holds

$$(2.3) \quad \mu = \mu^a + \mu^0,$$

where the measure  $\mu^a$  is absolutely continuous with respect to the  $p$ -capacity and  $\mu^0$  is singular with respect to the  $p$ -capacity, that is concentrated on sets with zero  $p$ -capacity. Besides it is also known (see [7]) that every measure which is absolutely continuous with respect to the  $p$ -capacity can be characterized as an element of  $L^1 + W^{-1,p'}$ , leading to the finer decomposition:

$$(2.4) \quad \mu = f - \text{Div}G + \mu^0,$$

where  $G \in L^{p'}(\Omega; \mathbb{R}^2)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $f \in L^1(\Omega)$ . In particular if  $u \in \Delta\mathcal{M}^p(\Omega)$  by applying the classical Radon-Nikodym decomposition together with (2.4) to the measure  $\text{Div}\nabla u$  we get:

$$(2.5) \quad \text{Div}\nabla u = \Delta u dx + \mu_a + \mu_0 = \Delta u dx + f - \text{Div}G + \mu^0.$$

**2.3. Functionals and their properties.** Let  $f : \mathbb{R} \rightarrow [0, +\infty]$  and  $g : \mathbb{R}^N \rightarrow [0, +\infty]$  be convex functions such that

$$(2.6) \quad f(t) \leq C_1(1 + |t|) \quad \forall t \in \mathbb{R},$$

where  $0 < C_1 < +\infty$  is a constant;

$$(2.7) \quad g(\xi) \leq C_2(1 + |\xi|^p) \quad \forall \xi \in \mathbb{R}^N,$$

where  $0 < C_2 < +\infty$  is a constant.

We shall consider the following functionals defined on  $\Delta\mathcal{M}^p(\Omega)$ :

$$(2.8) \quad F(u) := \begin{cases} \int_{\Omega} f(\Delta u) dx + \int_{\Omega} g(\nabla u) dx & \text{on } W_0^{1,p,1}(\text{Div}; \Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\text{Div}; \Omega); \end{cases}$$

$$(2.9) \quad \mathcal{F}(u) := \int_{\Omega} f(\Delta u) dx + \int_{\Omega} g(\nabla u) dx + \int_{\Omega} f^{\infty}\left(\frac{d\mu^a}{d|\mu^a|}\right) d|\mu^a| + \int_{\Omega} f^{\infty}\left(\frac{d\mu^0}{d|\mu^0|}\right) d|\mu^0|,$$

where  $f^{\infty}$  is the recession function given by  $\lim_{s \rightarrow +\infty} \frac{f(st)}{s}$ , with  $t \in \mathbb{R}$  and the measure  $\mu^a$  and  $\mu^0$  are given by decomposition (2.5).  $\frac{d\mu^a}{d|\mu^a|}$ ,  $\frac{d\mu^0}{d|\mu^0|}$  are the Radon-Nikodym derivatives of the measures  $\mu^a$  and  $\mu^0$  with respect their total variation.

Finally we will always assume  $1 < p < \frac{N}{N-1}$ . This restriction on  $p$  is due to the fact that when  $p \geq \frac{N}{N-1}$  the distributional divergence of  $U$ , and therefore the distributional Laplacian of a Sobolev function, cannot be a measure concentrated on sets with zero  $p$ -capacity (see [2]). We do this in order to allow concentration on sets of  $N - 2$  dimension such as points in 2D and curves in 3D. Indeed in this case the support of  $\mu^a = f - \text{Div}G$  contains sets of dimension  $N - 1$  such as  $N - 1$  manifolds with or without boundary inside  $\Omega$  (see [18], Section 4.7, for a detailed discussion on the space  $W^{-1,p'}(\Omega)$ ), while the support of the term  $\mu^0$  contains lower dimensional sets such as points for  $N \geq 2$  or curves for  $N \geq 3$ .

**2.4. Relaxation and  $\Gamma$ -convergence.** Let  $F$  be the functional defined in (2.8). For every  $u \in \Delta\mathcal{M}^p(\Omega)$ , we define the lower semicontinuous envelope or relaxed functional with respect to the  $W_0^{1,p}$ -weak convergence of  $F$  given by:

$$(2.10) \quad SC^-F(u) := \inf_{u_h \subset W_0^{1,p,1}(\text{Div}; \Omega)} \left\{ \liminf_{h \rightarrow +\infty} F(u_h) \quad u_h \rightharpoonup u \right\}.$$

Since we deal with  $W_0^{1,p}$ -weak convergence, functional (2.10) is characterized by the two following inequalities:

- for every  $u \in \Delta\mathcal{M}^p(\Omega)$  and every  $\{u_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$ , such that  $u_h \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$ ,

$$(2.11) \quad SC^-F(u) \leq \liminf_{h \rightarrow +\infty} F(u_h),$$

- for every  $u \in \Delta\mathcal{M}^p(\Omega)$  there exists  $\{\bar{u}_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$ , such that  $\bar{u}_h \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$ ,

$$(2.12) \quad SC^- F(u) \geq \limsup_{h \rightarrow +\infty} F(\bar{u}_h).$$

We give also the definition of  $\Gamma$ -convergence for a sequence  $\{F_h\}_h$  of functional defined on  $\Delta\mathcal{M}^p(\Omega)$  of type (2.8), with respect to the  $W_0^{1,p}(\Omega)$ -weak convergence. Functional  $F^\Gamma := \Gamma - \lim F_h$ , if it exists, is characterized by the following two inequalities:

- for every  $u \in \Delta\mathcal{M}^p(\Omega)$  and every  $\{u_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$ , such that  $u_h \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$ ,

$$(2.13) \quad F^\Gamma(u) \leq \liminf_{h \rightarrow +\infty} F_h(u_h),$$

- for every  $u \in \Delta\mathcal{M}^p(\Omega)$  there exists  $\{\bar{u}_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$ , such that  $\bar{u}_h \rightarrow u$  weakly  $W_0^{1,p}(\Omega)$

$$(2.14) \quad F^\Gamma(u) = \lim_{h \rightarrow +\infty} F_h(\bar{u}_h).$$

$F^\Gamma$  is  $W_0^{1,p}$ -weak lower semicontinuous on  $\Delta\mathcal{M}^p(\Omega)$ . Moreover if  $G : \Delta\mathcal{M}^p(\Omega) \rightarrow \mathbb{R}$  is a continuous operator with respect to the  $W_0^{1,p}(\Omega)$ -weak convergence we have:

$$(2.15) \quad \Gamma - \lim(F_h + G) = \Gamma - \lim F_h + G.$$

For general properties of the relaxation and  $\Gamma$ -convergence we refer to [3, 4, 8, 9].

### 3. RELAXATION THEOREM AND EXISTENCE OF MINIMIZERS

In this section we state and prove the relaxation formula. The result is close, in the spirit, to the the classical theorem of Goffmann-Serrin (see [12]) in the  $BV$ -framework. Indeed, in a similar way, to extend functional (2.8) to the whole space  $\Delta\mathcal{M}^p(\Omega)$ , the singular parts of the distributional divergence of  $\nabla u$ , have to be taken into account. To prove the relaxation argument we adapt the proof of ([12]) to our variational framework.

**Theorem 3.1.** *Let  $f : \mathbb{R} \rightarrow [0, +\infty)$ ,  $g : \mathbb{R}^N \rightarrow [0, +\infty)$  be convex functions satisfying (2.6) and (2.7). Then the following formula holds:*

$$(3.1) \quad SC^- F(u) = \mathcal{F}(u) \quad \forall u \in \Delta\mathcal{M}^p(\Omega).$$

**Proof. Step one:** We prove first inequality (2.11).

Let  $\{u_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$  such that  $u_h \rightarrow u \in \Delta\mathcal{M}^p(\Omega)$ . Since  $f$  is convex we have

$$(3.2) \quad f(t) = \sup_k (a_k + b_k t), \quad f^\infty(t) = \sup_k (b_k t) \quad t \in \mathbb{R},$$

Then for every test function  $\varphi \in C_0^\infty(\Omega)$  such that  $0 \leq \varphi \leq 1$  we get

$$F(u_h) \geq \int_\Omega \varphi f(\Delta u_h) dx + \int_\Omega g(\nabla u_h) dx \geq \int_\Omega \varphi (a_k + b_k \Delta u_h) dx + \int_\Omega g(\nabla u_h) dx.$$

By integrating by parts we obtain

$$F(u_h) \geq \int_\Omega \varphi a_k dx - \int_\Omega b_k \nabla \varphi \nabla u_h dx + \int_\Omega g(\nabla u_h) dx.$$

Then taking the weak limit and integrating by parts we get

$$\liminf_{h \rightarrow +\infty} F(u_h) \geq \int_\Omega a_k \varphi dx + \int_\Omega b_k \varphi d\text{Div} \nabla u + \int_\Omega g(\nabla u) dx;$$

hence by taking into account decomposition (2.5) we infer

$$\liminf_{h \rightarrow +\infty} F(u_h) \geq \int_{\Omega} \varphi(a_k + b_k \Delta u) dx + \int_{\Omega} \varphi b_k \frac{d\mu^a}{|d\mu^a|} |d\mu^a| + \int_{\Omega} \varphi b_k \frac{d\mu^0}{|d\mu^0|} |d\mu^0| + \int_{\Omega} g(\nabla u) dx.$$

Finally, since the measures  $dx$ ,  $\mu_a$  and  $\mu_0$  are mutually singular, to achieve inequality (2.11) it suffices to take the supremum over  $\varphi$  and  $k$  first, and then infimum over all the sequences  $\{u_h\}_h \subset W_0^{1,p,1}(\text{Div}; \Omega)$ . So we get (2.11).

**Step two :** We now prove inequality (2.12).

For  $u \in \Delta \mathcal{M}^p(\Omega)$ , let us consider the regularized sequence  $\{u_h\}_h \subset C_0^\infty(\Omega)$  of  $u$  defined as  $u_h := u * \rho_h$  with  $\{\rho_h\}_h$  standard sequence of mollifiers. By standard properties of the mollifying sequence, decomposition (2.5) writes as

$$\begin{aligned} a_k + b_k \Delta u_h &:= a_k + b_k \int_{\Omega} \rho_h(x-y) d\text{Div} \nabla u \\ &= \int_{\Omega} \rho_h(x-y) (a_k + b_k) \Delta u(y) dy + \int_{\Omega} \rho_h(x-y) b_k \frac{d\mu^a}{|d\mu^a|} |d\mu^a| + \int_{\Omega} \rho_h(x-y) b_k \frac{d\mu^0}{|d\mu^0|} |d\mu^0|; \end{aligned}$$

and by taking into account (3.2) we have

$$a_k + b_k \Delta u_h \leq \int_{\Omega} \rho_h(x-y) f(\Delta(u(y))) dy + \int_{\Omega} \rho_h(x-y) f^\infty\left(\frac{d\mu^a}{|d\mu^a|}\right) |d\mu^a| + \int_{\Omega} \rho_h(x-y) f^\infty\left(\frac{d\mu^0}{|d\mu^0|}\right) |d\mu^0|.$$

Taking the supremum on  $k$  on the left hand side of (3.3) and integrating over  $\Omega$  with respect to  $x$  we get:

$$\begin{aligned} \int_{\Omega} f(\Delta u_h) dx &\leq \int_{\Omega} \left( \int_{\Omega} \rho_h(x-y) f(\Delta(u(y))) dy \right) dx + \int_{\Omega} \left( \int_{\Omega} \rho_h(x-y) f^\infty\left(\frac{d\mu^a}{|d\mu^a|}\right) |d\mu^a| \right) dx \\ (3.3) \quad &+ \int_{\Omega} \left( \int_{\Omega} \rho_h(x-y) f^\infty\left(\frac{d\mu^0}{|d\mu^0|}\right) |d\mu^0| \right) dx. \end{aligned}$$

By applying, thanks to (2.6), Fubini's Theorem to the right hand side of (3.3)

$$(3.4) \quad \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\Delta u_h) dx \leq \int_{\Omega} f(\Delta u) dx + \int_{\Omega} f^\infty\left(\frac{d\mu^a}{|d\mu^a|}\right) |d\mu^a| + \int_{\Omega} f^\infty\left(\frac{d\mu^0}{|d\mu^0|}\right) |d\mu^0|,$$

where we have used the fact that  $\int_{\Omega} \rho_h(x-y) dx = 1$ , for  $h$  large enough. Finally, since  $\nabla u_h$  converges in measure to  $\nabla u$ , from the convexity of  $g$  together with (2.7), and Vitali's convergence theorem it follows that:

$$(3.5) \quad \lim_{h \rightarrow +\infty} \int_{\Omega} g(\nabla u_h) dx = \int_{\Omega} g(\nabla u) dx.$$

Then (3.4) and (3.5) imply

$$\limsup_{h \rightarrow +\infty} F(u_h) \leq \mathcal{F}(u).$$

So (2.11) and (2.12) are proven.  $\square$

### 3.1. Existence of minimizers.

**Theorem 3.2.** *Let  $f : \mathbb{R} \rightarrow [0, +\infty)$  be a convex function satisfying (2.6) such that*

$$(3.6) \quad f(t) \geq c_1 |t| \quad \forall t \in \mathbb{R},$$



where  $0 < c_1 < +\infty$  is a constant. Let  $g : \mathbb{R}^N \rightarrow [0, +\infty)$  be a convex function satisfying (2.7) such that

$$(3.7) \quad g(\xi) \geq c_2 |\xi|^p \quad \forall \xi \in \mathbb{R}^N,$$

where  $0 < c_2 < +\infty$  is a constant.

Then there exists a minimum  $u \in \Delta\mathcal{M}^p(\Omega)$  of functional  $\mathcal{F}$  defined in (2.9). Moreover the following equality holds

$$(3.8) \quad \inf_{u \in \Delta\mathcal{M}^p(\Omega)} F(u) = \min_{u \in \Delta\mathcal{M}^p(\Omega)} \mathcal{F}(u).$$

**Proof.** Let  $\{u_h\}_h \subset \Delta\mathcal{M}^p(\Omega)$  be a minimizing sequence. Then from (3.7) it follows that

$$c_2 \|\nabla u_h\|_p^p \leq \mathcal{F}(u_h) \leq M.$$

Therefore there exists a subsequence, still denoted by  $\{u_h\}_h \subset \Delta\mathcal{M}^p(\Omega)$ , and a function  $u \in W_0^{1,p}$  such that  $u_h \rightarrow u$  with respect to the  $W_0^{1,p}(\Omega)$ -weak convergence. From Theorem 2.1 and (3.6) we have

$$|\operatorname{Div} \nabla u|(\Omega) \leq \liminf_{h \rightarrow +\infty} \mathcal{F}(u_h) \leq M,$$

which implies  $u \in \Delta\mathcal{M}^p(\Omega)$ . As immediate consequence of Theorem 3.1, functional  $\mathcal{F}$  is lower semi-continuous with respect to the  $W_0^{1,p}$ -weak convergence. Then the existence of a minimum follows via the direct methods of the calculus of variations.

Finally thanks to growth conditions (2.6) and (3.6) the infimum is finite. Then property (3.8) can be achieved by standard arguments (see for instance [4]).  $\square$

**3.2.  $\Gamma$ -convergence.** In this subsection we state a  $\Gamma$ -convergence result for a sequence of integral functional of type (2.8). We also address the classical variational property which ensures the convergence of minimizing sequences of the approximating functional to minimizers of the limit functional.

**Theorem 3.3.** *Let  $f_h : \mathbb{R} \rightarrow [0, +\infty)$ ,  $g_h : \mathbb{R}^N \rightarrow [0, +\infty)$  be sequences of Borel functions. Assume that  $f_h$ ,  $g_h$  converge pointwise monotonically, as  $h \rightarrow +\infty$ , to  $f$  and  $g$  satisfying all the hypotheses of Theorem 3.1.*

For every  $u \in \Delta\mathcal{M}(\Omega)$ , we define

$$(3.9) \quad F_h(u) = \begin{cases} \int_{\Omega} f_h(\Delta u) dx + \int_{\Omega} g_h(\nabla u) dx & \text{if } u \in W_0^{1,p,1}(\operatorname{Div}; \Omega) \\ +\infty & \text{if } u \in \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\operatorname{Div}; \Omega). \end{cases}$$

Then for every  $u \in \Delta\mathcal{M}^p(\Omega)$ , we have

$$F^\Gamma = \Gamma - \lim F_h(u) = \mathcal{F}(u).$$

If in addition we have

$$(3.10) \quad c_3 |t| \leq f_h(t) \quad \text{for every } t \in \mathbb{R}, \quad c_4 |\xi|^p \leq g_h(\xi) \quad \text{for every } \xi \in \mathbb{R}^N,$$

where  $0 < c_3, c_4 < +\infty$ , then for every minimizing sequence  $\{u_h\}_h \subset \Delta\mathcal{M}^p(\Omega)$  of  $F_h$ , there exists a subsequence  $\{u_{h_k}\}_k \subset \Delta\mathcal{M}^p(\Omega)$  and a minimizer  $\bar{u} \in \Delta\mathcal{M}^p(\Omega)$  of  $\mathcal{F}$  such that  $\{u_{h_k}\}_k$  converges, with respect to  $W_0^{1,p}$ -weak convergence, to  $\bar{u}$ , and

$$(3.11) \quad \min_{\Delta\mathcal{M}^p(\Omega)} \mathcal{F}(u) = \lim_{h \rightarrow +\infty} \inf_{W_0^{1,p,1}(\text{Div}; \Omega)} F_h.$$

**Proof.** From Proposition 5.7 of [4] and Theorem 3.1 we have that  $F_h$   $\Gamma$ -converges to  $\mathcal{F}$ .

By (3.10) and by Theorem 2.1, for every minimizing sequences  $\{u_h\}_h \subset \Delta\mathcal{M}^p(\Omega)$  of  $F_h$  the following compactness property holds:

$$(3.12) \quad F_h(u_h) \leq M \Rightarrow \exists \{u_{h_k}\}_k, u \in \Delta\mathcal{M}^p(\Omega) \quad \text{with } u_{h_k} \rightharpoonup u$$

Then the second assertion follows directly from the fundamental theorem of  $\Gamma$ -convergence (see for instance section 1.5 of [3]).  $\square$

#### 4. NUMERICAL APPLICATIONS

In view of numerical applications we will always consider  $N = 2$ ,  $f(t) = |t|$  and  $g(\xi) = \frac{1}{p}|\xi|^p$  (with  $1 < p < 2$ ), so that the recession function  $f^\infty(t)$  is equal to  $|t|$ . Moreover we add an  $L^2$ -discrepancy term, and we consider the following functional

$$(4.1) \quad \mathcal{J}(u) := \int_{\Omega} |\Delta u| dx + \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + |\mu^a|(\Omega) + |\mu^0|(\Omega) + \frac{\lambda}{2} \int_{\Omega} |u - u_0|^2 dx,$$

where  $u_0 \in L^2(\Omega)$  is the observed data, and  $\lambda$  is a positive weight. We note that Theorem 3.2, together with the strong convexity of  $J$  gives immediately the existence and uniqueness of a minimizer of  $\mathcal{J}$  on  $\Delta\mathcal{M}^p(\Omega)$ .

We also introduce a smoother version of functional (4.1). To this end let  $\{\epsilon_h\}_h$  be an infinitesimal sequence of positive numbers and  $w_{\epsilon_h}$  the Huber function defined by:

$$(4.2) \quad w_{\epsilon_h}(x) = \begin{cases} |x| & \text{if } |x| \geq \epsilon_h \\ \frac{x^2}{2\epsilon_h} + \frac{\epsilon_h}{2} & \text{otherwise.} \end{cases}$$

We define the following sequences of functionals on  $\Delta\mathcal{M}^p(\Omega)$

$$(4.3) \quad \mathcal{J}_h(u) := \begin{cases} \int_{\Omega} w_{\epsilon_h}(|\Delta u|) dx + \frac{1}{p} \int_{\Omega} w_{\epsilon_h}(|\nabla u|^p) dx + \frac{\lambda}{2} \int_{\Omega} |u - u_0|^2 dx & \text{on } W_0^{1,p,1}(\text{Div}; \Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\text{Div}; \Omega). \end{cases}$$

By Theorem 3.1, 3.3 and (2.15), we infer that variational property (3.11) holds for  $\mathcal{J}_h$  and  $\mathcal{J}$ . Hence for numerical purposes it makes sense to look at the discrete version of functionals  $J_h$  and  $J$ . However we precise that the analogy between continuous and discrete setting will be only formal, since we do not let the mesh size step  $\delta x$  of the discrete domain go to 0 (see next section).

**4.1. Discrete setting.** We define the discrete rectangular domain  $\Omega$  of step size  $\delta x = 1$  and dimension  $d_1 d_2$ , where  $\Omega = \{1, \dots, d_1\} \times \{1, \dots, d_2\} \subset \mathbb{Z}^2$ . In order to simplify the notations we set  $X = \mathbb{R}^{d_1 \times d_2}$  and  $Y = X \times X$ .  $u \in X$  denotes a matrix of size  $d_1 \times d_2$ . For  $u \in X$ ,  $u_{i,j}$  denotes its (i,j)-th component with

$(i, j) \in \{1, \dots, d_1\} \times \{1, \dots, d_2\}$ . For  $g \in Y$ ,  $g_{i,j}$  denotes the  $i,j$ -th component of  $g$  with  $g_{i,j} = (g_{i,j}^1, g_{i,j}^2)$  and  $(i, j) \in \{1, \dots, d_1\} \times \{1, \dots, d_2\}$

We endowed the space  $X$  and  $Y$  with standard scalar product and standard norm. For  $u, v \in X$  we have:

$$\langle u, v \rangle_X = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} u_{i,j} v_{i,j}.$$

For  $g, h \in Y$  we have:

$$\langle g, h \rangle_Y = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \sum_{l=1}^2 g_{i,j}^l h_{i,j}^l.$$

For  $u \in X$  and  $p \in [1, +\infty)$  we set:

$$\|u\|_p := \left( \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |u_{i,j}|^p \right)^{\frac{1}{p}}.$$

For  $g \in Y$  and  $p \in [1, +\infty)$ :

$$\|g\|_p := \left( \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |g_{i,j}|_2^p \right)^{\frac{1}{p}}.$$

If  $G, F$  are two vector spaces and  $H : G \rightarrow F$  is a linear operator, then the norm of  $H$  is defined by

$$\|H\| := \max_{\|u\|_G \leq 1} (\|Hu\|_F).$$

**Definition 4.1.** A function  $F : X \rightarrow \mathbb{R}$  is said to be  $L$ -lipschitz differentiable if it is differentiable and

$$|\nabla F(u) - \nabla F(v)|_2 \leq L|u - v|_2,$$

for every  $u, v \in X$ .

**Definition 4.2.** Let  $\psi : X \rightarrow \mathbb{R}$  be a convex function. The operator

$$\text{prox}_\psi : X \rightarrow X \quad x \mapsto \arg \min_{y \in X} \left\{ \psi(y) + \frac{1}{2}|y - x|_2 \right\}$$

is called proximal operator associated to  $\psi$ .

If  $\text{prox}_{\lambda\psi}$  can be computed exactly for every  $\lambda \geq 0$  and every  $x \in X$ , the function  $\psi$  is said to be simple.

If  $u \in X$  the gradient  $\nabla u \in Y$  is given by:

$$(\nabla u)_{i,j} = ((\nabla u)_{i,j}^1, (\nabla u)_{i,j}^2)$$

where

$$(\nabla u)_{i,j}^1 = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < d_1 \\ 0 & \text{if } i = d_1, \end{cases}$$

$$(\nabla u)_{i,j}^2 = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < d_2 \\ 0 & \text{if } j = d_2. \end{cases}$$

We also introduce the discrete version of the divergence operator defined as the adjoint operator of the gradient:  $\operatorname{div} = -\nabla^*$ . If  $p \in Y$ , we have

$$(\operatorname{div} p)_{i,j} = \begin{cases} p_{i,j}^1 + p_{i,j}^2 & \text{if } i, j = 1 \\ p_{i,j}^1 + p_{i,j}^2 - p_{i-1,j}^2 & \text{if } i = 1, 1 < j < d_2 \\ p_{i,j}^1 - p_{i-1,j}^1 + p_{i,j}^2 - p_{i-1,j}^2 & \text{if } 1 < i < d_1, 1 < j < d_2 \\ -p_{i-1,j}^1 + p_{i,j}^2 - p_{i-1,j}^2 & \text{if } i = d_1, 1 < j < d_2 \\ p_{i,j}^1 - p_{i-1,j}^1 + p_{i,j}^2 & \text{if } 1 < i < d_1, j = 1 \\ p_{i,j}^1 - p_{i-1,j}^1 - p_{i-1,j}^2 & \text{if } 1 < i < d_1, j = d_2 \\ -(p_{i-1,j}^1 + p_{i-1,j}^2) & \text{if } i = d_1, j = d_2. \end{cases}$$

Then we can define the discrete version of the Laplacian operator as  $\Delta u = \operatorname{div}(\nabla u)$ .

**4.2. Nesterov scheme.** Here we briefly recall the fast descent gradient Nesterov's algorithm (see [14, 15]). We state it in the formulation proposed in [16, 17]. For further details and general statements we refer the reader to [16, 17] and references therein.

**Proposition 4.1.** *Let  $F : X \rightarrow \mathbb{R}$  be given by:*

$$F(u) = F_1(u) + F_2(u) \quad \text{for } u \in X,$$

where  $F_1$  is a convex  $L$ -Lipschitz differentiable function and  $F_2$  a simple function. Then the following algorithm<sup>3</sup>:

$$(4.4) \quad \begin{cases} u_0 \in X & A_0 = 0 & g = 0 & u = 0 \\ \text{do for } k : 1, \dots, K \\ t = \frac{2}{L} \\ a = t + \sqrt{t^2 + 4tA} \\ v = \operatorname{prox}_{AF_2}(u_0 - g) \\ y = \frac{Au + Av}{A+a} \\ u = \operatorname{prox}_{\frac{1}{L}F_2}(y - \frac{1}{L}\nabla F_1(y)) \\ g = g + a\nabla F_1(u) \\ A = A + a \end{cases}$$

ensures that:

$$(4.5) \quad 0 \leq F(u_k) - F(u^*) \leq L \frac{|u^* - u_0|_2}{k^2},$$

where  $u^* \in X$  is a minimum point of  $F$  and  $u_0 \in X$  is an initial data.

**4.3. The discrete functionals.** For  $u \in X$  such that  $u = 0$  on  $\partial\Omega$ , we define the discrete analogous of functional (4.1) as

$$(4.6) \quad J(u) = \|\Delta u\|_1 + \frac{1}{p} \|\nabla u\|_p^p + \frac{\lambda}{2} |u - u_0|_2^2.$$

We also introduce the discrete counterpart of functional (4.3) as:

$$(4.7) \quad J_\epsilon(u) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} w_\epsilon (|(\Delta u)_{i,j}|) + \frac{1}{p} \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} w_\epsilon (|(\nabla u)_{i,j}|^p) + \frac{\lambda}{2} |u - u_0|_2^2,$$

<sup>3</sup>we omit the dependance on  $k$  to simplify the notation

where  $\epsilon > 0$  here is a small fixed parameter.

**Remark 4.1.** *Let us stress very clearly that by no means we intend to claim that if the mesh size step  $\delta x \rightarrow 0$ , then the discrete functional (4.6)  $\Gamma$ -converges to functional (4.1) defined in continuous setting. Indeed we do not provide such a result here. Therefore here the analogy between continuous and discrete is formal.*

We shall consider the minimization problems:

$$(4.8) \quad \min_{u \in X} J(u),$$

$$(4.9) \quad \min_{u \in X} J_\epsilon(u).$$

As in [17] we define the notion of  $\delta$ -solution associated to problem (4.8), which will be used to give an estimation of the number of iterations of the minimization algorithm.

**Definition 4.3.** *A  $\delta$ -solution of (4.8) is an element  $u_\delta \in X$  such that*

$$J(u_\delta) - J(\bar{u}) \leq \delta,$$

where  $\bar{u}$  is a solution of problem (4.8).

**4.4. The proposed algorithm.** We minimize  $J_\epsilon$  by applying algorithm (4.4) with

$$F_1(u) = J_\epsilon(u) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} w_\epsilon (|\Delta u_{i,j}|) + \frac{1}{p} \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} w_\epsilon (|\nabla u_{i,j}|^p) + \frac{\lambda}{2} |u - u_0|_2 \quad F_2(u) = 0.$$

Indeed it is not difficult to check that for  $u \in X$  with  $u = 0$  on the boundary we have

$$(4.10) \quad \nabla F_1(u) = \Delta(\Psi) - \operatorname{div}(\Phi) + \lambda(u - u_0)$$

where

$$\Psi_{i,j} = \begin{cases} \frac{(\Delta u)_{i,j}}{|\Delta u_{i,j}|} & \text{if } |(\Delta u)_{i,j}| \geq \epsilon \\ \frac{(\Delta u)_{i,j}}{\epsilon} & \text{otherwise} \end{cases} \quad \Phi_{i,j} = \begin{cases} \frac{(\nabla u)_{i,j}}{|\nabla u_{i,j}|^{2-p}} & \text{if } |\nabla u_{i,j}|^p \geq \epsilon \\ \frac{(\nabla u)_{i,j}}{\epsilon} & \text{otherwise.} \end{cases}$$

Then, taking into account that  $\Psi$  and  $\Phi$  are Lipschitz functions with constant  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon^p}$  respectively, we infer

$$|\nabla F_1(u) - \nabla F_1(v)|_2 \leq \left( \frac{\|\Delta\|_2^2}{\epsilon} + \frac{\|\operatorname{div}\|_2^2}{\epsilon^{2-p}} + \lambda \right) |u - v|_2.$$

Therefore, by recalling that, if  $N = 2$ ,  $\|\Delta\|_2 \leq 8$  and  $\|\operatorname{div}\|_2 \leq 2\sqrt{2}$ , we conclude that

$$(4.11) \quad |\nabla F_1(u) - \nabla F_1(v)|_2 \leq \left( \frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda \right) |u - v|_2.$$

Thanks to inequality (4.11) we are in position of applying algorithm (4.4). Then in our case algorithm (4.4) ensures that:

$$(4.12) \quad 0 \leq J_\epsilon(u_k) - J_\epsilon(u_\epsilon^*) \leq \left( \frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda \right) \frac{|u_\epsilon^* - u_0|_2}{k^2},$$

where  $u_\epsilon^*$  is a minimum of  $J_\epsilon$ .

**4.5. Computer examples.** Before running our algorithm all the parameters have to be fixed. It is easy to see that for every  $u \in X$  we have

$$(4.13) \quad 0 \leq J_\epsilon(u) - J(u) \leq d_1 d_2 \epsilon.$$

Then by using (4.13) (4.12) and the fact that  $u_\epsilon^*$  is a minimum of  $J_\epsilon$  we have

$$J(u_k) \leq J_\epsilon(u_k) \leq J_\epsilon(u_\epsilon^*) + \left(\frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda\right) \frac{|u_\epsilon^* - u_0|_2}{k^2} \leq J_\epsilon(\bar{u}) + \left(\frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda\right) \frac{|u_\epsilon^* - u_0|_2}{k^2},$$

where  $\bar{u}$  is a minimum of  $J$ . By applying again bound (4.13) we deduce

$$J(u_k) \leq J(\bar{u}) + d_1 d_2 \epsilon + \left(\frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda\right) \frac{|u_\epsilon^* - u_0|_2}{k^2}.$$

Therefore the worst case precision to get a  $\delta$ -solution of (4.8) is:

$$J(u_k) - J(\bar{u}) = \left(\frac{64}{\epsilon} + \frac{8}{\epsilon^{2-p}} + \lambda\right) \frac{|\bar{u} - u_0|_2}{k^2} + d_1 d_2 \epsilon;$$

then the optimal choices are

$$\epsilon = \frac{\delta}{d_1 d_2}, \quad K = \left\lceil \sqrt{\left(\frac{64 d_1 d_2}{\delta} + 8 \frac{d_1 d_2}{\delta^{2-p}} + \lambda\right) C} \right\rceil + 1,$$

where  $C := \max_X |\bar{u} - u_0|_2$  and  $K$  the total number of iterations. For images rescaled in  $[0, 1]$ , in the worst case problem we deal with (see Figure 4), the number of iterations  $K$  needed to get a  $\delta$ -solution of order 1 does not exceed the value 8000. In all numerical tests we let run the algorithm for no more than 2000 iterations. The parameter  $\epsilon$  is always fixed in order to get a  $\delta$ -solution of order 1. This choice seems to lead to good restoration results.

The parameter  $\lambda$  is tuned according to the level noise and its value is specified on each numerical test.

Finally as exponent  $p$  we always take  $p = 1.5$ .

In Figure 1, 2 and 3 we test the algorithm against noise on synthetic image containing open curve. A Gaussian Noise is added to the original image. The image domain is of size  $d_1 \times d_2 = 128 \times 128$ . CPU time is about 50 s running on an Intel (R) Xeon(R) CPU 5120 at 1.86GHz.

Figure 4 shows an application of the algorithm on a biological image of cell spots provided for us by "Institut Pasteur de Paris". The image domain is of size  $d_1 \times d_2 = 757 \times 510$ . We fix  $\epsilon = 2.5e - 6$ . We let run the algorithm until 2000 iterations to obtain a  $\delta$ -solution of order 1 shown in Figure 5. CPU time is about 20 mn running on an Intel (R) Xeon(R) CPU 5120 at 1.86GHz.

Figure 7, provided for us by "Institut Pasteur de Paris", is an image of biological cell containing DNA filaments. The image domain is of size  $d_1 \times d_2 = 323 \times 323$ .

In Figure 8 we show the result of the restoration algorithm after 2000 iterations with  $\epsilon = 9.5e - 5$ . CPU time is about 9 mn running on an Intel (R) Xeon(R) CPU 5120 at 1.86GHz.

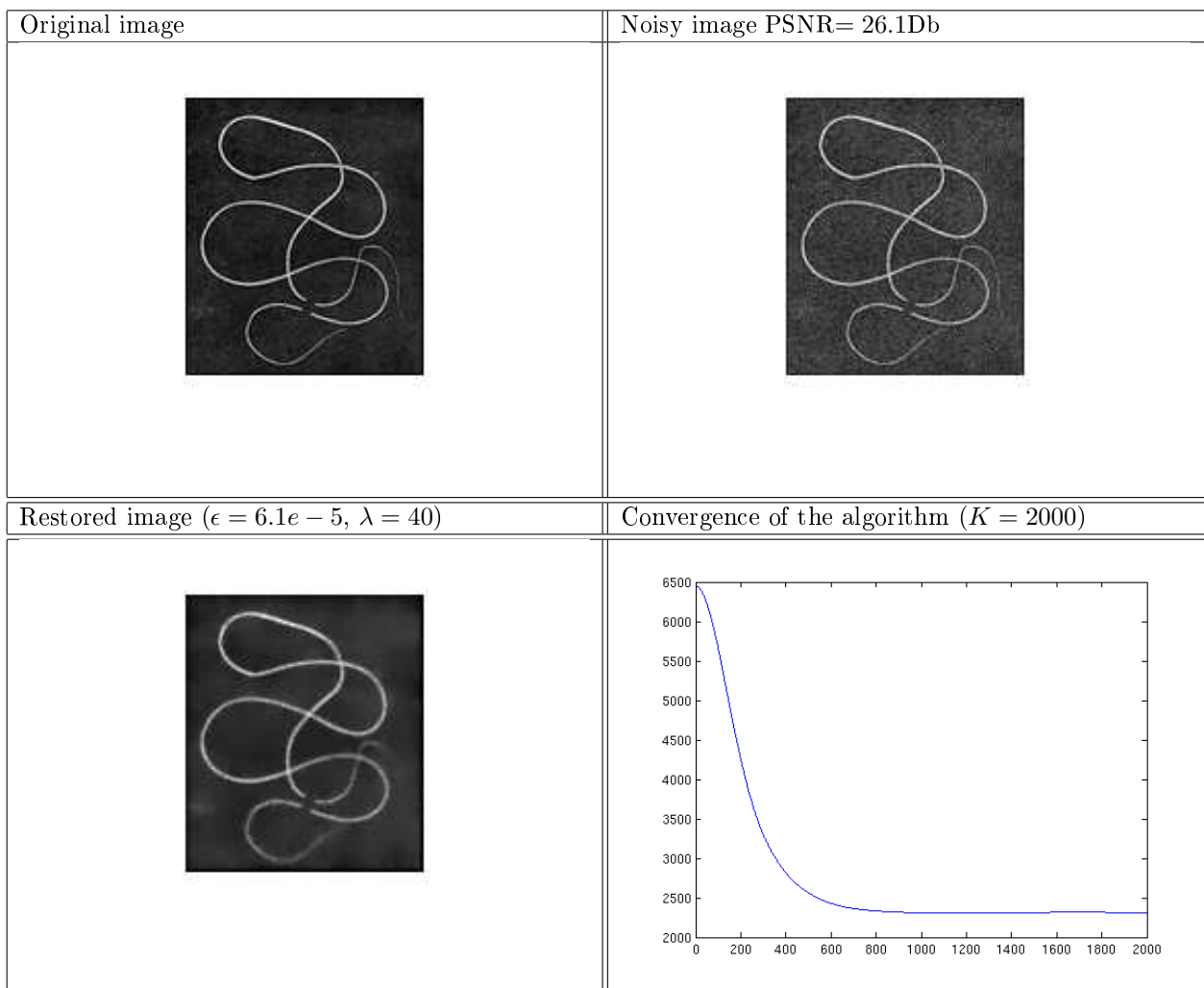


FIGURE 1. Synthetic images. We test the proposed algorithm on noisy data. Top left: Original images. Top right: noisy image. Down left: Restored image given by a  $\delta$ -solution of order 1. Down right: Convergence on the algorithm. On the  $y$ -axis the value of  $J(u_k)$ . On the  $x$ -axis the number of iterations.

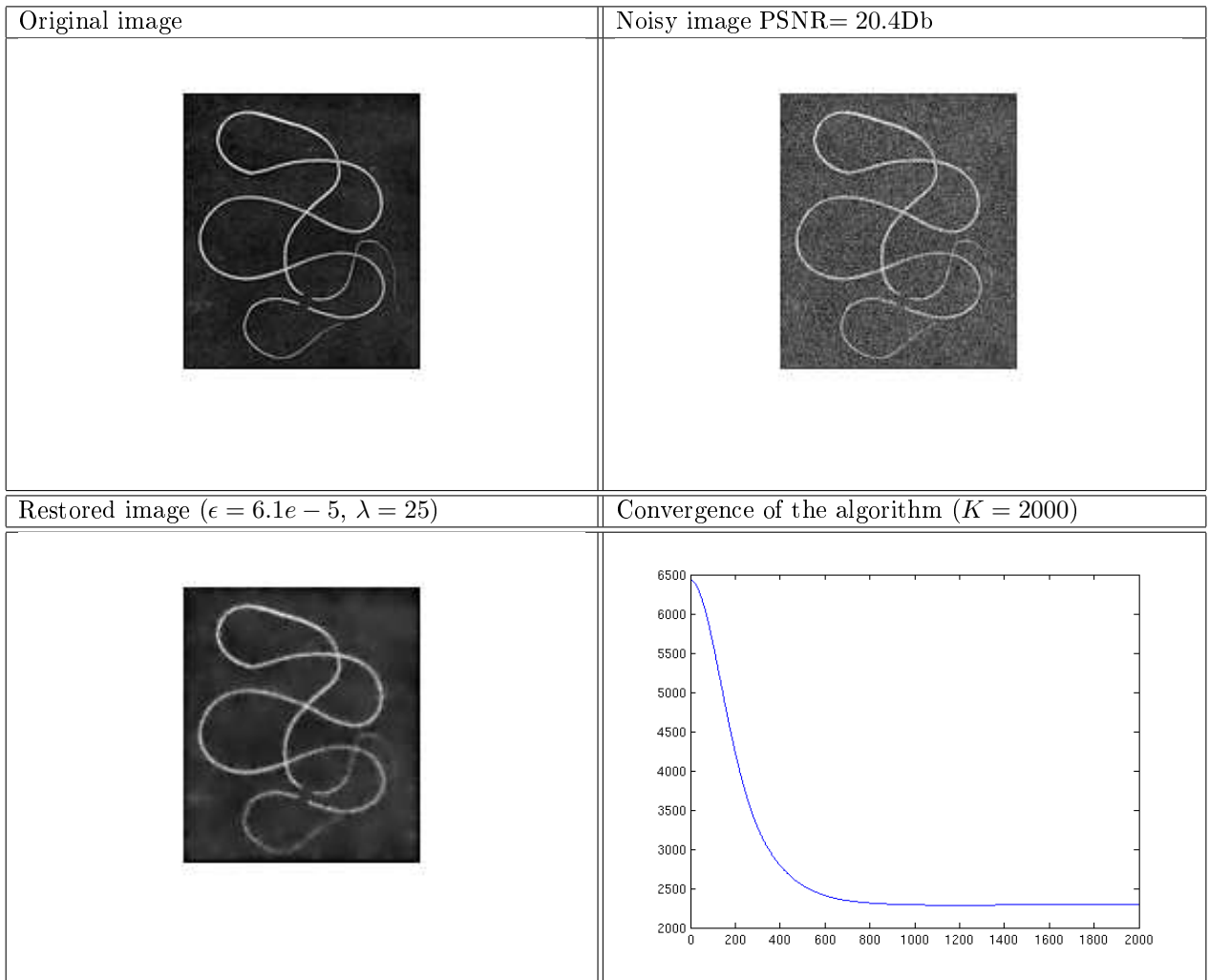


FIGURE 2. Synthetic images. Top left: Original images. Top right: noisy image. Down left: Restored image given by a  $\delta$ -solution of order 1. Down right: Convergence on the algorithm. On the  $y$ -axis the value of  $J(u_k)$ . On the  $x$ -axis the number of iterations.



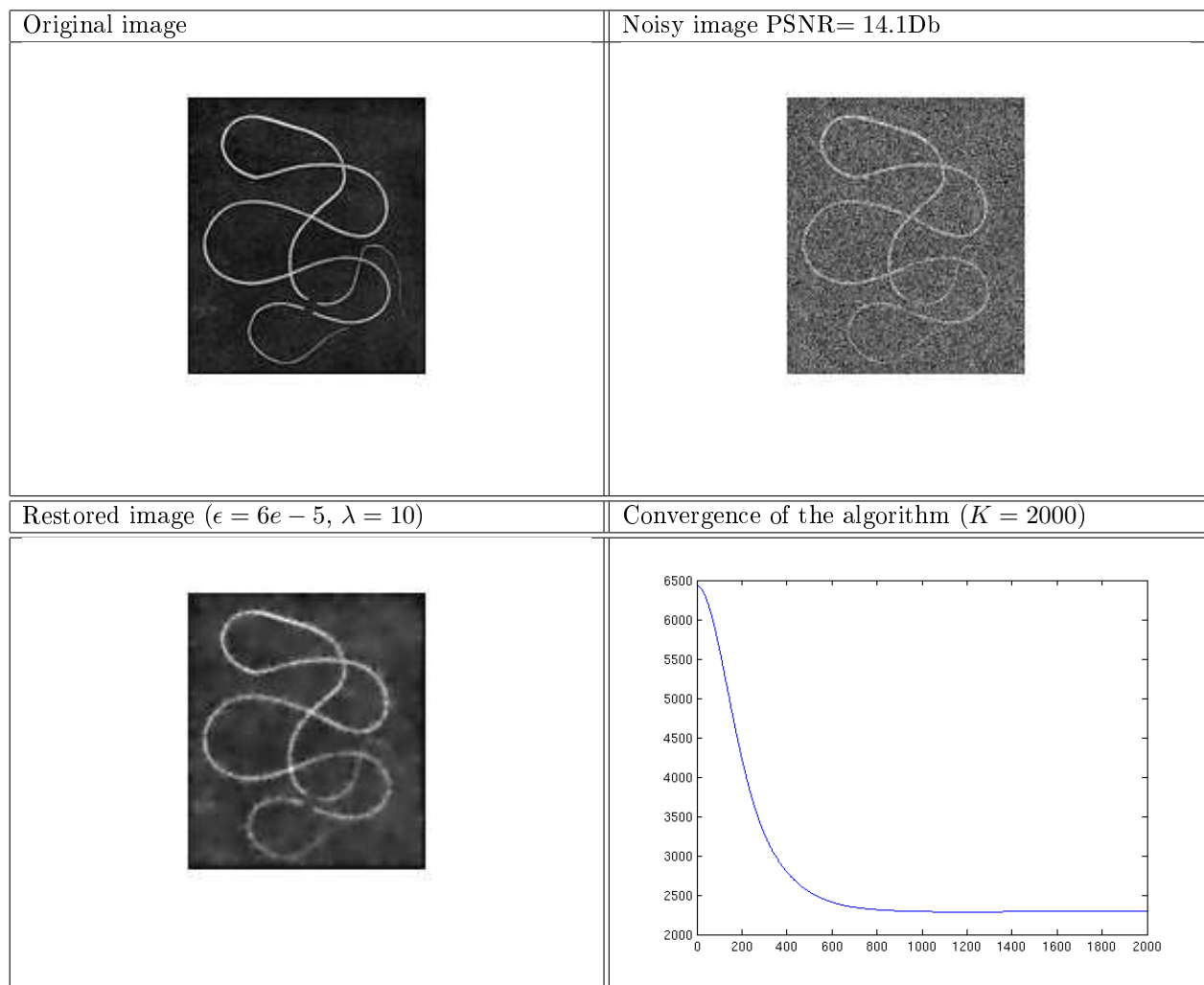


FIGURE 3. Synthetic images. We test the proposed algorithm on noisy data. Top left: Original image. Top right: noisy image. Down left: Restored image given by a  $\delta$ -solution of order 1. Down right: Convergence on the algorithm. On the  $y$ -axis the value of  $J(u_k)$ . On the  $x$ -axis the number of iterations.

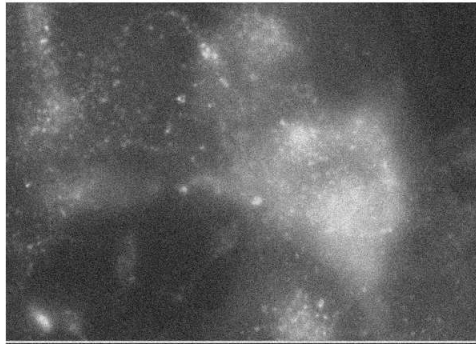


FIGURE 4. Noisy image of cell spots

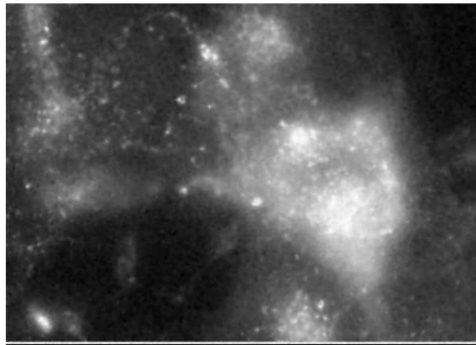


FIGURE 5. Restored image given by a  $\delta$ -solution of order 1. Number of iterations  $K = 2000$ ,  $\epsilon = 2.5e - 6$ ,  $\lambda = 20$ .

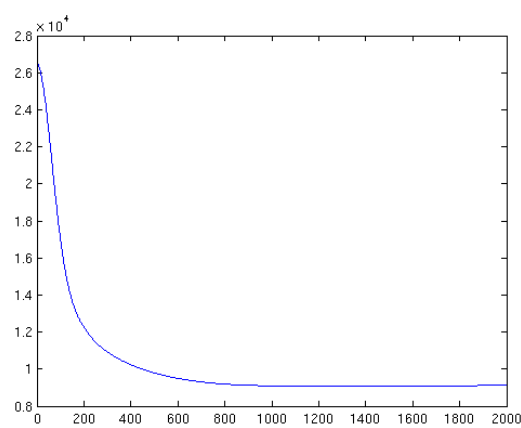


FIGURE 6. Convergence of the algorithm. On the  $y$ -axis the value of  $J(u_k)$ . On the  $x$ -axis the number of iterations.

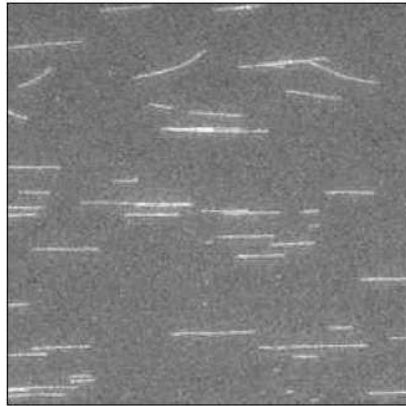


FIGURE 7. Noisy image of biological filaments provided by "Institut Pasteur de Paris"

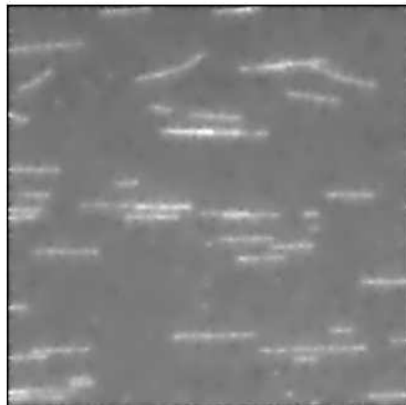


FIGURE 8. Restored image given by a  $\delta$ -solution of order 1. Number of iterations  $K = 2000$ ,  $\epsilon = 9.5e - 6$ ,  $\lambda = 20$

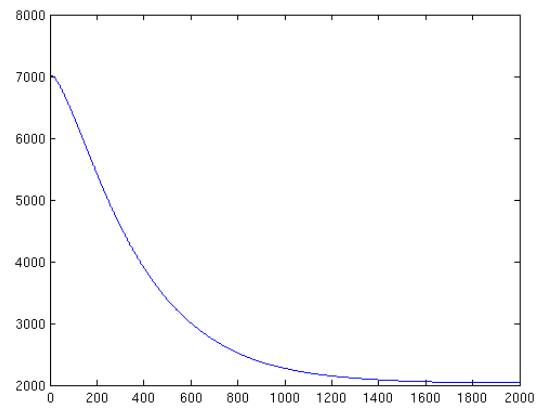


FIGURE 9. Convergence of the algorithm. On the  $y$ -axis the value of  $J(u_k)$ . On the  $x$ -axis the number of iterations

## 5. FINAL REMARKS

In this work a new variational model for restoring point and curve-like singularities in images has been developed both from a theoretical and an experimental point of view. We emphasize that, according to our knowledge, this approach seems to be new in the literature. Moreover we stress out that, despite some extra smoothing effect, mainly due to the fact we work with a mesh step size equal to 1, we obtain good experimental results.

Certainly there are many rooms for improvement both from a theoretical and numerical point of view, such as analysis of functional with more general discrepancy term of type:

$$\mathcal{F}(u) + \int_{\Omega} |Hu - u_0|^q,$$

where  $H$  is a linear operator and  $q \geq 1$ ; as well as performing 3D-numerical simulations on real data. These are subject of our current investigation.

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