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General coupling of porous flows and hyperelastic formulations – From thermodynamics principles to energy balance

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General coupling of porous flows and hyperelastic formulations – From thermodynamics principles to energy balance

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Abstract: We formulate a general poromechanics model – within the framework of a two-phase mixture theory – compatible with large strains and without any simplification in the momentum expressions, in particular concerning the fluid flows. The only specific assumptions made are fluid incompressibility and isothermal conditions. Our formulation is based on fundamental physical principles – namely, essential conservation and thermodynamics laws – and we thus obtain a Clausius-Duhem inequality which is crucial for devising compatible constitutive laws. We then propose to model the solid behavior based on a generalized hyperelastic free energy potential – with additional viscous effects – which allows to represent a wide range of mechanical behaviors. The resulting formulation takes the form of a coupled system similar to a fluid-structure interaction problem written in an Arbitrary Lagrangian-Eulerian formalism, with additional volume-distributed interaction forces. We achieve another important objective by identifying the essential energy balance prevailing in the model, and this paves the way for further works on mathematical analyses, and time and space discretizations of the formulation.

Key-words: poromechanics, mixtures, hyperelasticity, thermomechanics, thermodynamics, biomechanics

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Interaction fluide-structure entre un écoulement poreux et un solide hyperélastique – Des principes thermodynamiques aux estimations d'énergie

Résumé : Nous présentons, dans le cadre de la théorie des mélanges, un modèle poromécanique général valide en grands déplacements et sans simplification sur le bilan de conservation des moments, en particulier pour le système fluide. Les seules hypothèses faites sont l'incompressibilité du fluide et des conditions isothermes. Notre formulation s'appuie sur les principes de la thermodynamique et nous obtenons une inégalité de Clausius-Duhem fondamentale pour l'obtention de lois de comportement adaptées. Nous proposons alors de modéliser le solide avec un potentiel d'énergie libre hyperélastique généralisé auquel s'ajoute un potentiel visqueux, permettant ainsi de représenter une large gamme de comportements mécaniques. La formulation résultante prend la forme d'un système couplé similaire à ceux rencontrés en interaction fluide-structure de type ALE comprenant un couplage volumique supplémentaire. Nous sommes alors en mesure d'écrire sur le modèle complet des estimations d'énergie qui seront à l'origine de travaux futurs, que ce soit pour l'analyse mathématique du système ou la formulation de discrétisations en temps et en espace adaptées.

Mots-clés : poromécanique, mélanges, hyperélasticité, thermomécanique, thermodynamique, biomécanique

1 Introduction

Poromechanics has been a very active subject of research for some decades – as shown e.g. in the survey [12] – originally mostly motivated by civil engineering applications, see [35, 3], and well-established theories rigorously grounded in fundamental physical and thermodynamical principles are available [10, 26, 5]. Such approaches are frequently based on the so-called “mixture theory”, a purely macroscopic framework in which the porous medium is considered to be made of a superposition of solid and fluid phases at each point – for a saturated medium to which we restrict our attention. We note in passing that the detailed microstructure can be considered in alternative approaches, e.g. with homogenization procedures applied to relate the microscopic and macroscopic behaviors [21, 30], but we do not dwell on these approaches which are much more difficult to translate into effective computational modeling tools.

More recently, some novel applications have brought renewed modeling and computing challenges in the field, hence spurred further research to circumvent various limitations of the existing theories. In particular, potential challenging applications abound in biomechanics, for instance to model the blood perfusion of living (passive or active) tissues [27, 33], the circulation of gases in the lungs [34, 22, 2], or the effect of wind on plants [13], to cite just a few noteworthy examples in this emerging context.

In this context, in [7] a poromechanics formulation was proposed with the primary motivation of describing perfusion in the cardiac muscle. As the heart typically undergoes strains of 10 to 20%, the focus of this work was accordingly placed on the ability of the model to represent large displacements and finite strains, with general constitutive laws compatible with living tissue behavior [18, 20]. In addition, special care was exercised to make the proposed formulation consistent with – partial or complete – incompressibility of the solid and fluid constituents. However, a significant – although quite widespread – simplification was made in this work by totally neglecting the fluid inertia. As the coronary arteries are known to be host to rapid blood flows with high variations during the cardiac cycle, such a simplification is clearly quite drastic, and furthermore does not allow the formulation of discretization procedures which would preserve physical energy balances – with the kinetic energy as one of the major contributions.

Some other poromechanics formulations taking into account finite strains – and with or without fluid inertia – have already been proposed, see e.g. [14, 1, 25]. However, these formulations are mostly directly focused on constitutive assumptions – and indeed pertain to specific types of constitutive behaviors – rather than explicitly derived from the general setting of conservation and thermodynamics principles. Hence, it is difficult to see whether these essential principles are all satisfied, and how more general constitutive behaviors can be adequately modeled within the proper corresponding physical framework, in particular as regards energy considerations.

The objective of the present work is thus twofold. We first aim at presenting a general poromechanics theory compatible with large strains and porous fluid flows without any simplification in the momentum expressions, based on fundamental physical principles – namely, essential conservation and thermodynamics laws. This allows us to meet a second objective, namely, to identify the essential energy balances which must be satisfied in such formulations. This is

crucial in order to allow further mathematical analyses and the derivation of consistently stable time and space discretization procedures. Our motivation is – indeed – strongly oriented towards computational modeling, and we will show in the forthcoming paper [28] how such time schemes can be obtained, as the first important step in the numerical simulation workflow. We note in passing that – when applied to cardiac perfusion – this formulation should allow to comparatively assess the impact of the simplifications made in earlier works.

We would like to finally emphasize that our approach owes much to the work of O. Coussy and collaborators, see e.g. [15, 10, 11] and references therein, and we essentially follow the same major construction steps as in [10]. Our main contributions thus lie in the detailed generalization of this framework to nonlinear constitutive behaviors modeled by hyperelastic potentials and viscous effects – both in the solid and in the fluid – and in the incorporation of fluid mass source terms in the formulation, the latter being motivated in particular by the modeling of muscle tissue in its interaction with various blood compartments [7].

The outline of the paper is as follows. In Section 2 we introduce the notation and kinematical description, and we derive the mass conservation laws. We then establish in Section 3 the adequate principle of virtual work by invoking the conservation of momentum. In Section 4 we obtain the energy conservation law based on the first fundamental principle of thermodynamics, before proceeding to apply the second principle in Section 5, which – combined with the previous results – leads to a Clausius-Duhem inequality allowing to formulate consistent constitutive laws. Finally, in Section 6 we summarize and further analyze the governing equations of the complete model – both in strong and weak forms – and we establish a fundamental energy balance, before giving some concluding remarks in Section 7.

2 Notation, kinematical description and mass conservation

In this section we start by introducing the notation and kinematical description, and proceed to derive the mass conservation laws.

2.1 Classical definitions and kinematical relations

We consider a deformable solid which occupies at time t the space domain $\Omega(t)$ – denoted by Ω when there is no ambiguity – with boundary $\partial\Omega(t)$. The total Lagrangian formulation consists in describing the position history of each material point with respect to a reference configuration $(\Omega_0, \partial\Omega_0)$ – not necessarily equal to $(\Omega(0), \partial\Omega(0))$. Thus, the *deformation* is a one-to-one mapping $\underline{\varphi}$ from the reference configuration to the current configuration providing the position of each material point in time:

$$\underline{\varphi} : \begin{cases} \bar{\Omega}_0 & \rightarrow \mathbb{R}^3 \\ \underline{\xi} & \mapsto \underline{\mathbf{x}} = \underline{\varphi}(\underline{\xi}, t) \end{cases}$$

We denote by \underline{y} the displacement field

$$\underline{y}(\underline{\xi}, t) = \underline{\mathbf{x}} - \underline{\xi} = \underline{\varphi}(\underline{\xi}, t) - \underline{\xi},$$

and $\underline{\underline{F}}$ is the deformation gradient

$$\underline{\underline{F}}(\underline{\xi}, t) = \underline{\underline{\nabla}}_{\underline{\xi}} \underline{\varphi} = \underline{\underline{\mathbb{1}}} + \underline{\underline{\nabla}}_{\underline{\xi}} \underline{y},$$

such that the deformed volume is given by $Jd\Omega$ where $J = \det \underline{\underline{F}}$ and $d\Omega$ is the volume measure (here in the reference configuration), while the deformed area vector is given by $J\underline{\underline{F}}^{-T} \cdot d\underline{S}$. Furthermore, we introduce the right Cauchy-Green deformation tensor $\underline{\underline{C}} = \underline{\underline{F}}^T \cdot \underline{\underline{F}}$. We finally recall that the local changes of geometry are conveniently described by the Green-Lagrange strain tensor denoted by $\underline{\underline{\varepsilon}}$ and defined by

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{\mathbb{1}}}) = \frac{1}{2} \left(\underline{\underline{\nabla}}_{\underline{\xi}} \underline{y} + (\underline{\underline{\nabla}}_{\underline{\xi}} \underline{y})^T + (\underline{\underline{\nabla}}_{\underline{\xi}} \underline{y})^T \cdot \underline{\underline{\nabla}}_{\underline{\xi}} \underline{y} \right),$$

with linearized expression $\underline{\underline{\varepsilon}}$ given by

$$\underline{\underline{\varepsilon}}(\underline{y}) = \frac{1}{2} \left(\underline{\underline{\nabla}}_{\underline{\xi}} \underline{y} + (\underline{\underline{\nabla}}_{\underline{\xi}} \underline{y})^T \right).$$

2.2 Mixture and material derivatives

Classical two-phase poromechanics is a mixture theory in which the material contains at each point a volume fraction ϕ of fluid phase, and a remaining fraction $(1 - \phi)$ of solid phase called the “skeleton”. The fluid volume fraction ϕ is also referred to as the “porosity”. Unless otherwise specified, we henceforth denote the quantities specifically associated with the fluid and the skeleton with “f” and “s” subscripts, respectively. As an exception, we will retain all the notation introduced in the above Section 2.1 to denote all kinematical quantities associated with the skeleton – without the “s” subscript, since this also describes the motion of the fluid domain. Note that J then only represents an *apparent* change of volume of the skeleton, since it can be accompanied by a variation of the porosity.

For a field $u_s(\underline{x}, t)$ associated with the solid skeleton, we recall the classical concept of material derivative defined by

$$\frac{du_s}{dt} = \frac{\partial(u_s \circ \underline{\varphi})}{\partial t} \Big|_{\underline{\xi}} = \frac{\partial u_s}{\partial t} \Big|_{\underline{x}} + \underline{\nabla}_{\underline{x}} u_s \cdot \underline{v}_s, \quad (1)$$

where the velocity \underline{v}_s is defined by

$$\underline{v}_s = \frac{\partial \underline{x}}{\partial t} \Big|_{\underline{\xi}}.$$

Hence, when $u_s(\underline{x}, t)$ denotes the volume density associated with an extensive quantity $\mathcal{U} = \int_{\Omega'_t} u_s(\underline{x}, t) d\Omega$ defined on the solid skeleton in a Eulerian framework – for an arbitrary subdomain $\Omega'_t \subset \Omega_t$ associated with $\Omega'_0 \subset \Omega_0$ in the reference configuration – the material time derivative of the integrated density gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega'_t} u_s d\Omega &= \frac{d}{dt} \int_{\Omega'_0} u_s(\underline{x}(\underline{\xi}, t), t) J(\underline{\xi}, t) d\Omega \\ &= \int_{\Omega'_0} \left(J \frac{\partial u_s}{\partial t} \Big|_{\underline{x}} + J \underline{\nabla}_{\underline{x}} u_s \cdot \underline{v}_s + u_s \frac{\partial J}{\partial t} \Big|_{\underline{\xi}} \right) d\Omega. \end{aligned}$$

Using the identity

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t} \det \underline{\underline{F}} = \text{tr}(J \underline{\underline{F}}^{-1} \cdot \frac{\partial \underline{\underline{F}}}{\partial t}) = J \text{tr}(\underline{\underline{F}}^{-1} \cdot \underline{\underline{\nabla}}_{\underline{\underline{x}}} v_s) = J \text{tr}(\underline{\underline{\nabla}}_{\underline{\underline{x}}} v_s) = J \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot v_s,$$

we get

$$\frac{d}{dt} \int_{\Omega'_t} u_s d\Omega = \int_{\Omega'_t} \left(\frac{\partial u_s}{\partial t} \Big|_{\underline{\underline{x}}} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (u_s \otimes v_s) \right) d\Omega, \quad (2)$$

which also holds when u_s is a tensor of arbitrary order.

We can also introduce material derivatives for quantities associated with the fluid

$$\frac{d_f u_f}{dt} = \frac{\partial u_f}{\partial t} \Big|_{\underline{\underline{x}}} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} u_f \cdot v_f. \quad (3)$$

Remark 1 – We point out that we could of course also introduce a Lagrangian formalism for the fluid giving when differentiating an integral quantity on Ω'_t

$$\frac{d_f}{dt} \int_{\Omega'_t} u_f d\Omega = \int_{\Omega'_t} \left(\frac{\partial u_f}{\partial t} \Big|_{\underline{\underline{x}}} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (u_f \otimes v_f) \right) d\Omega. \quad (4)$$

Let us now consider an integral quantity $\mathcal{U} = \int_{\Omega'_t} u(\underline{\underline{x}}, t) d\Omega$ defined as the sum of two terms

$$\mathcal{U} = \int_{\Omega'_t} (u_s + u_f) d\Omega.$$

As above, the first term u_s is associated with the solid skeleton, i.e. its velocity is that of the skeleton, namely, v_s . The second term follows the fluid with the velocity v_f .

DEFINITION 1 (TOTAL TIME DERIVATIVE OF A MIXTURE QUANTITY)

We define the total time derivative of \mathcal{U} as

$$\frac{D\mathcal{U}}{Dt} = \frac{d}{dt} \int_{\Omega'_t} u_s d\Omega + \frac{d_f}{dt} \int_{\Omega'_t} u_f d\Omega.$$

Introducing $\underline{\underline{w}}$ the mass weighted relative velocity of the fluid defined by

$$\underline{\underline{w}} = \rho_f \phi (v_f - v_s) = \rho_f \phi v_r,$$

and u_m the proportion of u_f per unit mass

$$u_m = \frac{u_f}{\rho_f \phi},$$

then we have the following rule of differentiation, directly inferred from (2) and (4).

LEMMA 2 (DIFFERENTIATION LEMMA)

For a tensor g of arbitrary order, we have

$$\frac{D\mathcal{U}}{Dt} = \int_{\Omega'_t} \left(\frac{\partial u}{\partial t} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (u \otimes v_s) + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (u_m \otimes \underline{\underline{w}}) \right) d\Omega. \quad (5)$$

These relations have equivalent forms when considering Lagrangian coordinates attached to the skeleton and the corresponding derivatives. Defining

$$U(\underline{\xi}, t) = u(\underline{x}(\underline{\xi}, t))J(\underline{\xi}, t)$$

as the Lagrangian volume density, the first two terms of (5) correspond to the classical partial time derivative of a function defined in the skeleton reference configuration. Therefore, defining \underline{W} as

$$\underline{W} = J\underline{F}^{-1}\underline{w}$$

such that for any infinitesimal area $d\underline{S}_0$ transported into $d\underline{S}$ we have

$$\underline{W} \cdot d\underline{S}_0 = \underline{w} \cdot d\underline{S},$$

the Stokes formula gives

$$\int_{\Omega'_0} \underline{\nabla}_{\underline{\xi}} \cdot (u_m \otimes \underline{W}) d\Omega = \int_{\Omega'_t} \underline{\nabla}_{\underline{x}} \cdot (u_m \otimes \underline{w}) d\Omega. \quad (6)$$

This directly implies

$$\frac{DU}{Dt} = \int_{\Omega_0} \left(\frac{\partial U}{\partial t} + \underline{\nabla}_{\underline{\xi}} \cdot (u_m \otimes \underline{W}) \right) d\Omega. \quad (7)$$

Note that in the above formulae we use the same notation for a function of $\underline{\xi}$ and of \underline{x} – in this case for u_m . We will systematically commit this slight abuse of notation for all fields in the sequel.

2.3 Mass conservation laws

Lemma 2 can be used to obtain mass conservation laws for the entire system or for the fluid subsystem alone. Starting with the fluid, we consider a distributed mass source term θ – which represents a source *per se* if $\theta \geq 0$ and a sink otherwise [7]. We have for any subdomain Ω'_t

$$\frac{D}{Dt} \int_{\Omega'_t} \rho_f \phi d\Omega = \int_{\Omega'_t} \theta d\Omega. \quad (8)$$

Therefore, invoking Lemma 2 with $u = \rho_f \phi$, i.e. $u_m = 1$, we get

$$\frac{\partial}{\partial t} (\rho_f \phi) + \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_s + \underline{w}) = \theta. \quad (9)$$

We point out that since $\rho_f \phi$ is a fluid quantity we can also express the conservation law by the simple identity

$$\frac{d_f}{dt} \int_{\Omega'_t} (\rho_f \phi) d\Omega = \int_{\Omega'_t} \theta d\Omega. \quad (10)$$

Let us now consider the total mass conservation. Introducing the total density

$$\rho = \rho_s(1 - \phi) + \rho_f \phi, \quad (11)$$

we have as in (8),

$$\frac{D}{Dt} \int_{\Omega'_t} \rho \, d\Omega = \int_{\Omega'_t} \theta \, d\Omega. \quad (12)$$

Then, taking $u = \rho$ in Lemma 2 – namely, still $u_m = 1$ – we get

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}_{\underline{x}} \cdot (\rho \underline{v}_s + \underline{w}) = \theta. \quad (13)$$

Hence, we obtain by subtraction the conservation law for the skeleton

$$\frac{\partial}{\partial t} (\rho_s(1 - \phi)) + \underline{\nabla}_{\underline{x}} \cdot (\rho_s(1 - \phi) \underline{v}_s) = 0. \quad (14)$$

Remark 2 – In the three relations (9), (13) and (14), the time derivative identity (1) allows to extract the density from the divergence term by simply substituting for $\frac{\partial}{\partial t}$ the time derivative with respect to the skeleton $\frac{d}{dt}$. For example, (9) gives

$$\frac{d}{dt} (\rho_f \phi) + \rho_f \phi \underline{\nabla}_{\underline{x}} \cdot \underline{v}_s + \underline{\nabla}_{\underline{x}} \cdot \underline{w} = \theta.$$

We conclude this section by giving the corresponding mass conservation law in the Lagrangian framework attached to the skeleton. Let us denote by $\rho_0 = \rho_{s0}(1 - \phi_0) + \rho_{f0}\phi_0$ the density in the reference configuration Ω_0 . We define by m the Lagrangian density of added mass that represents the local change of fluid mass at every point $\underline{\xi}$ in the domain

$$\rho(\underline{\xi}, t) J(\underline{\xi}, t) = \rho_0(\underline{\xi}) + m(\underline{\xi}, t). \quad (15)$$

Since the solid mass is conserved, we directly have

$$m = \rho_f J \phi - \rho_{f0} \phi_0. \quad (16)$$

Using the differentiation formula in Lagrangian framework (7) with $U = \rho_0 + m$ and $u_m = 1$ we get

$$\frac{dm}{dt} + \underline{\nabla}_{\underline{\xi}} \cdot \underline{W} = \Theta, \quad (17)$$

where $\Theta = J\theta$ is the Lagrangian source density associated with θ .

3 Principle of virtual work for the mixture

In this section, we derive the principle of virtual work for the mixture, mainly based on the conservation of momentum.

3.1 Expression of accelerations

For the skeleton, the definition of the acceleration is straightforward and corresponds to

$$\underline{\gamma}_s = \frac{d\underline{v}_s}{dt} = \frac{\partial \underline{v}_s}{\partial t} + \underline{\nabla}_{\underline{x}} \underline{v}_s \cdot \underline{v}_s, \quad (18)$$

but for the fluid we need to use the material derivative with respect to the fluid particles, hence we have

$$\underline{\gamma}_f = \frac{d_f \underline{v}_f}{dt} = \frac{\partial \underline{v}_f}{\partial t} + \underline{\nabla}_{\underline{x}} \underline{v}_f \cdot \underline{v}_f = \underline{\gamma}_s + \frac{d\underline{v}_r}{dt} + \underline{\nabla}_{\underline{x}} (\underline{v}_s + \underline{v}_r) \cdot \underline{v}_r. \quad (19)$$

3.2 Momentum conservation laws

We now focus on the extension of the momentum conservation law to poromechanics systems. The system considered is open, hence we have for any sub-domain Ω'_t

$$\frac{D}{Dt} \int_{\Omega'_t} (\rho \underline{v}_s + \rho_f \phi \underline{v}_r) d\Omega = \int_{\Omega'_t} \theta \underline{v}_f d\Omega + \int_{\Omega'_t} \rho \underline{f} d\Omega + \int_{\partial\Omega'_t} \underline{t} dS, \quad (20)$$

where $\theta \underline{v}_f$ is the variation of momentum due to the fluid source term. Considering the left-hand side, we use Lemma 2 with $\underline{g} = \rho \underline{v}_s + \rho_f \phi \underline{v}_r$ and $\underline{g}_m = \underline{v}_f = \underline{v}_s + \underline{v}_r$ to obtain

$$\begin{aligned} \frac{D}{Dt} \int_{\Omega'_t} (\rho \underline{v}_s + \rho_f \phi \underline{v}_r) d\Omega &= \int_{\Omega'_t} \frac{\partial}{\partial t} (\rho \underline{v}_s + \rho_f \phi \underline{v}_r) d\Omega \\ &+ \int_{\Omega'_t} \left(\underline{\nabla}_{\underline{x}} \cdot ((\rho \underline{v}_s + \rho_f \phi \underline{v}_r) \otimes \underline{v}_s) + \underline{\nabla}_{\underline{x}} \cdot ((\underline{v}_s + \underline{v}_r) \otimes \underline{w}) \right) d\Omega. \end{aligned}$$

Developing the right hand side we get

$$\begin{aligned} &\frac{\partial}{\partial t} (\rho \underline{v}_s + \rho_f \phi \underline{v}_r) + \underline{\nabla}_{\underline{x}} \cdot ((\rho \underline{v}_s + \rho_f \phi \underline{v}_r) \otimes \underline{v}_s) + \underline{\nabla}_{\underline{x}} \cdot ((\underline{v}_s + \underline{v}_r) \otimes \underline{w}) = \\ &\rho \left(\frac{\partial \underline{v}_s}{\partial t} + \underline{\nabla}_{\underline{x}} \underline{v}_s \cdot \underline{v}_s \right) + \rho_f \phi \left(\frac{\partial \underline{v}_r}{\partial t} + \underline{\nabla}_{\underline{x}} \underline{v}_r \cdot \underline{v}_s + \underline{\nabla}_{\underline{x}} (\underline{v}_s + \underline{v}_r) \cdot \underline{v}_r \right) \\ &+ \left(\frac{\partial \rho}{\partial t} + \underline{\nabla}_{\underline{x}} \cdot (\rho \underline{v}_s + \underline{w}) \right) \underline{v}_s + \left(\frac{\partial}{\partial t} (\rho_f \phi) + \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_s + \underline{w}) \right) \underline{v}_r. \end{aligned}$$

Therefore, using the mass conservation laws and the acceleration definitions, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} (\rho \underline{v}_s + \rho_f \phi \underline{v}_r) + \underline{\nabla}_{\underline{x}} \cdot ((\rho \underline{v}_s + \rho_f \phi \underline{v}_r) \otimes \underline{v}_s) + \underline{\nabla}_{\underline{x}} \cdot ((\underline{v}_s + \underline{v}_r) \otimes \underline{w}) \\ &= \rho_s (1 - \phi) \underline{\gamma}_s + \rho_f \phi \underline{\gamma}_f + \theta (\underline{v}_s + \underline{v}_r), \end{aligned}$$

and the (first) momentum conservation law yields

$$\int_{\Omega'_t} (\rho_s (1 - \phi) \underline{\gamma}_s + \rho_f \phi \underline{\gamma}_f) d\Omega = \int_{\Omega'_t} \rho \underline{f} d\Omega + \int_{\partial\Omega'_t} \underline{t} dS. \quad (21)$$

3.3 Principle of virtual work

The fundamental law of dynamics is deduced from the momentum conservation law when classically introducing the Cauchy stress tensor $\underline{\sigma}$, which gives

$$\underline{\nabla}_{\underline{x}} \cdot \underline{\sigma} + \rho \underline{f} - (\rho_s (1 - \phi) \underline{\gamma}_s + \rho_f \phi \underline{\gamma}_f) = 0, \text{ in } \Omega_t.$$

Remark 3 – We recall that $\underline{\sigma}$ can be proven to be symmetric using the second momentum conservation law on the cross product between the position and the momentum (see e.g. [10]).

In order to simplify the writing of the principle of virtual work, let us define $\underline{\gamma} = \frac{1}{\rho} (\rho_s (1 - \phi) \underline{\gamma}_s + \rho_f \phi \underline{\gamma}_f)$. The weak formulation of the above fundamental law of dynamics is given on the space of admissible displacements $V(\Omega_t)$ by

$$\forall \underline{v}^* \in V(\Omega_t), \quad \int_{\Omega_t} \underline{\sigma} : \underline{\nabla}_{\underline{x}} \underline{v}^* d\Omega = \int_{\Omega_t} \rho (\underline{f} - \underline{\gamma}) \cdot \underline{v}^* d\Omega + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v}^* dS.$$

Considering the symmetry of the Cauchy stress tensor, we can symmetrize $\underline{\underline{\nabla}}_{\underline{\underline{x}}}$ by using the linearized deformation tensor in the deformed configuration $\underline{\underline{\varepsilon}}(\underline{\underline{v}}^*)$, so that

$$\forall \underline{\underline{v}}^* \in V(\Omega_t), \quad \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}^*) d\Omega = \int_{\Omega_t} \rho(\underline{\underline{f}} - \underline{\underline{\gamma}}) \cdot \underline{\underline{v}}^* d\Omega + \int_{\partial\Omega_t} \underline{\underline{t}} \cdot \underline{\underline{v}}^* dS.$$

The change of variable $\underline{\underline{x}} \rightarrow \underline{\underline{\xi}}$ allows to convert the last expression in the configuration attached to the skeleton. In fact,

$$\underline{\underline{\nabla}}_{\underline{\underline{x}}} \underline{\underline{v}}^* = \underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^* \cdot \underline{\underline{\nabla}}_{\underline{\underline{x}}} \underline{\underline{\xi}} = \underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^* \cdot (\underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{x}})^{-1} = \underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^* \cdot \underline{\underline{F}}^{-1},$$

hence, we get

$$\begin{aligned} \forall \underline{\underline{v}}^* \in V(\Omega_0), \quad \int_{\Omega_0} \underline{\underline{\sigma}} : \underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^* \cdot \underline{\underline{F}}^{-1} J d\Omega &= \int_{\Omega_0} (\rho_0 + m)(\underline{\underline{f}} - \underline{\underline{\gamma}}) \cdot \underline{\underline{v}}^* d\Omega \\ &+ \int_{\partial\Omega_0} J \|\underline{\underline{F}}^{-T} \cdot \underline{\underline{n}}_0\| \underline{\underline{t}} \cdot \underline{\underline{v}}^* dS. \end{aligned}$$

Considering the *second Piola-Kirchhoff stress tensor* in order to keep a symmetric tensor

$$\underline{\underline{\Sigma}} = J \underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot \underline{\underline{F}}^{-T},$$

and introducing $\underline{\underline{t}}_0 = J \|\underline{\underline{F}}^{-T} \cdot \underline{\underline{n}}_0\| \underline{\underline{t}}$ the transported traction on the boundary, standard manipulations yield the corresponding Lagrangian form for the weak formulation of the equation of motion

$$\begin{aligned} \forall \underline{\underline{v}}^* \in V(\Omega_0), \quad \int_{\Omega_0} (\rho_0 + m) \underline{\underline{\gamma}} \cdot \underline{\underline{v}}^* d\Omega + \int_{\Omega_0} \underline{\underline{\Sigma}} : d\underline{\underline{y}} \underline{\underline{e}} \cdot \underline{\underline{v}}^* d\Omega \\ = \int_{\Omega_0} (\rho_0 + m) \underline{\underline{f}} \cdot \underline{\underline{v}}^* d\Omega + \int_{\partial\Omega_0} \underline{\underline{t}}_0 \cdot \underline{\underline{v}}^* dS, \quad (22) \end{aligned}$$

with $d\underline{\underline{y}} \underline{\underline{e}}$ the differential of $\underline{\underline{e}}$ given by

$$d\underline{\underline{y}} \underline{\underline{e}} \cdot \underline{\underline{v}}^* = \frac{1}{2} (\underline{\underline{F}}^T \cdot \underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^* + (\underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \underline{\underline{v}}^*)^T \cdot \underline{\underline{F}}),$$

and the associated strong formulation is

$$\underline{\underline{\nabla}}_{\underline{\underline{\xi}}} \cdot (\underline{\underline{F}} \cdot \underline{\underline{\Sigma}}) + (\rho_0 + m)(\underline{\underline{f}} - \underline{\underline{\gamma}}) = 0, \quad \text{in } \Omega_0.$$

The weak formulation, also called principle of virtual work, can be finally summarized as

$$\forall \underline{\underline{v}}^* \in V(\Omega), \quad \mathcal{P}_i(\underline{\underline{v}}^*) + \mathcal{P}_{\text{int}}(\underline{\underline{v}}^*) = \mathcal{P}_{\text{ext}}(\underline{\underline{v}}^*), \quad (23)$$

with

$$\left\{ \begin{aligned} \mathcal{P}_i(\underline{\underline{v}}^*) &= \int_{\Omega_t} (\rho_s(1 - \phi)\underline{\underline{\gamma}}_s + \rho_f\phi\underline{\underline{\gamma}}_f) \cdot \underline{\underline{v}}^* d\Omega \\ &= \int_{\Omega_0} (\rho_{s0}(1 - \phi_0)\underline{\underline{\gamma}}_s + (\rho_{f0}\phi_0 + m)\underline{\underline{\gamma}}_f) \cdot \underline{\underline{v}}^* d\Omega \\ \mathcal{P}_{\text{int}}(\underline{\underline{v}}^*) &= \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}^*) d\Omega = \int_{\Omega_0} \underline{\underline{\Sigma}} : d\underline{\underline{y}} \underline{\underline{e}} \cdot \underline{\underline{v}}^* d\Omega \\ \mathcal{P}_{\text{ext}}(\underline{\underline{v}}^*) &= \int_{\Omega_t} \rho \underline{\underline{f}} \cdot \underline{\underline{v}}^* d\Omega + \int_{\partial\Omega_t} \underline{\underline{t}} \cdot \underline{\underline{v}}^* dS \\ &= \int_{\Omega_0} (\rho_0 + m) \underline{\underline{f}} \cdot \underline{\underline{v}}^* d\Omega + \int_{\partial\Omega_0} \underline{\underline{t}}_0 \cdot \underline{\underline{v}}^* dS \end{aligned} \right. \quad (24)$$

4 Kinetic energy and first law of thermodynamics

The objective of this section is to express the first law of thermodynamics applied to the poromechanics system considered.

4.1 Kinetic energy theorem

By definition, the *kinetic energy* of the mixture is

$$\begin{aligned} \mathcal{K} &= \mathcal{K}_s + \mathcal{K}_f = \frac{1}{2} \int_{\Omega_t} (\rho_s(1-\phi)\underline{v}_s^2 + \rho_f\phi\underline{v}_f^2) d\Omega \\ &= \frac{1}{2} \int_{\Omega_0} (\rho_0(1-\phi_0)\underline{v}_s^2 + (\rho_{f0}\phi_0 + m)\underline{v}_f^2) d\Omega. \end{aligned} \quad (25)$$

We then have the following identity.

LEMMA 3

When considering the total derivative of the mixture kinetic energy, we have

$$\begin{aligned} \frac{D\mathcal{K}}{Dt} &= \int_{\Omega_t} \left((\rho_s(1-\phi)\underline{\gamma}_s + \rho_f\phi\underline{\gamma}_f) \cdot \underline{v}_s + \rho_f\phi\underline{\gamma}_f \cdot \underline{v}_r + \frac{1}{2}\theta\underline{v}_f^2 \right) d\Omega \\ &= \int_{\Omega_0} \left((\rho_{s0}(1-\phi_0)\underline{\gamma}_s + \rho_{f0}\phi_0\underline{\gamma}_f) \cdot \underline{v}_s + (\rho_{f0}\phi_0 + m)\underline{\gamma}_f \cdot \underline{v}_r + \frac{1}{2}\Theta\underline{v}_f^2 \right) d\Omega_0. \end{aligned} \quad (26)$$

Proof. Straightforward by noting that

$$\begin{aligned} \frac{d_f}{dt} \int_{\Omega_t} \frac{1}{2} \rho_f \phi \underline{v}_f^2 d\Omega &= \int_{\Omega_t} \frac{1}{2} (\underline{v}_f^2 \frac{d_f}{dt} (\rho_f \phi d\Omega) + \rho_f \phi \frac{d_f \underline{v}_f^2}{dt} d\Omega) \\ &= \int_{\Omega_t} \left(\frac{1}{2} \theta \underline{v}_f^2 + \rho_f \phi \underline{\gamma}_f \cdot \underline{v}_f \right) d\Omega \end{aligned}$$

using the fluid mass conservation (10). ■

Therefore, introducing the relative contribution of the fluid in the inertial virtual work

$$\mathcal{P}_i^f(\underline{v}^*) = \int_{\Omega_t} \rho_f \phi \underline{\gamma}_f \cdot \underline{v}^* d\Omega, \quad (27)$$

and the kinetic energy variation due to the source term

$$\mathcal{J}_{\mathcal{K}\theta} = \frac{1}{2} \int_{\Omega_t} \theta \underline{v}_f^2 d\Omega, \quad (28)$$

we have the following result directly obtained by combining (23) applied with $\underline{v}^* = \underline{v}_s$ and (26).

THEOREM 4 (KINETIC ENERGY THEOREM)

$$\frac{D\mathcal{K}}{Dt} - \mathcal{P}_i^f(\underline{v}_r) + \mathcal{P}_{int}(\underline{v}_s) = \mathcal{J}_{\mathcal{K}\theta} + \mathcal{P}_{ext}(\underline{v}_s). \quad (29)$$

4.2 First law of thermodynamics

Let us introduce the internal energy and its corresponding volume-distributed form

$$\mathcal{E} = \int_{\Omega_t} e \, d\Omega.$$

Part of the internal energy is associated with the fluid phase, which is represented by a fluid energy per unit mass e_m . As is standard in fluid thermodynamics we also introduce the fluid enthalpy per unit mass, defined by

$$h_m = e_m + \frac{p}{\rho_f}, \quad (30)$$

and

$$\mathcal{J}_{\mathcal{H}_\theta} = \int_{\Omega_t} \theta h_m \, d\Omega$$

corresponds to the variation of enthalpy due to the source term [26]. Finally, we denote the external work specifically applied on the fluid by

$$\mathcal{P}_{\text{ext}}^f(\underline{v}^*) = \int_{\Omega_t} \rho_f \phi \underline{f} \cdot \underline{v}^* \, d\Omega + \int_{\partial\Omega_t} \phi (\underline{\sigma}_f \cdot \underline{n}) \cdot \underline{v}^* \, dS, \quad (31)$$

where $\underline{\sigma}_f$ is the fluid Cauchy stress tensor, with the usual decomposition into pressure and viscous parts, namely,

$$\underline{\sigma}_f = \underline{\sigma}_{\text{vis}}(\underline{v}_f) - p \underline{\mathbb{1}},$$

such that when applied to the relative velocity, we obtain the following power

$$\begin{aligned} \mathcal{P}_{\text{ext}}^f(\underline{v}_r) &= \int_{\Omega_t} \underline{f} \cdot \underline{w} \, d\Omega - \int_{\partial\Omega_t} \frac{p}{\rho_f} \underline{n} \cdot \underline{w} \, dS + \int_{\partial\Omega_t} \frac{1}{\rho_f} (\underline{\sigma}_{\text{vis}} \cdot \underline{n}) \cdot \underline{w} \, dS \\ &= \int_{\Omega_t} \underline{f} \cdot \underline{w} \, d\Omega + \int_{\Omega_t} \underline{\nabla}_{\underline{x}} \cdot \left(\frac{1}{\rho_f} \underline{\sigma}_f \cdot \underline{w} \right) \, d\Omega. \end{aligned}$$

PRINCIPLE 1 (FIRST LAW OF THERMODYNAMICS)

For an open system, the variation of the total energy of the system – which is the sum of the internal energy and the kinetic energy – equals the power of the external loads added to the heat rate \mathcal{Q} , and to the rates of kinetic energy and enthalpy due to the source terms, namely,

$$\frac{D\mathcal{E}_T}{Dt} = \frac{D\mathcal{K}}{Dt} + \frac{D\mathcal{E}}{Dt} = \mathcal{J}_{\mathcal{K}_\theta} + \mathcal{J}_{\mathcal{H}_\theta} + \mathcal{P}_{\text{ext}}(\underline{v}_s) + \mathcal{P}_{\text{ext}}^f(\underline{v}_r) + \mathcal{Q}. \quad (32)$$

Remark 4 – As already seen for several integrated quantities, we can decompose the power of external forces into two contributions, the first one given here by the power of forces working on a global velocity field \underline{v}_s considered on the whole mixture, and the second one for forces acting only on the fluid phase and working on the relative velocity \underline{v}_r , namely,

$$\mathcal{P}_{\text{ext}}^{\text{total}} = \mathcal{P}_{\text{ext}}(\underline{v}_s) + \mathcal{P}_{\text{ext}}^f(\underline{v}_r).$$

The heat rate can be decomposed into the volume-distributed heat source term and the heat flux at the boundaries as follows

$$\begin{aligned}\mathcal{Q} &= \int_{\Omega_t} q \, d\Omega - \int_{\partial\Omega_t} \underline{j}_q \cdot \underline{n} \, dS \\ &= \int_{\Omega_0} Q \, d\Omega - \int_{\partial\Omega_0} \underline{J}_Q \cdot \underline{n}_0 \, dS,\end{aligned}$$

with $\underline{J}_Q = J \underline{F}^{-1} \underline{j}_q$. We can then apply the differentiation lemma to the volume-distributed internal energy e , and define e_m such that

$$\frac{D\mathcal{E}}{Dt} = \int_{\Omega_t} \left(\frac{\partial e}{\partial t} + \nabla_{\underline{x}} \cdot (e \underline{v}_s) + \nabla_{\underline{x}} \cdot (e_m \underline{w}) \right) d\Omega,$$

and – as in fact the first principle can be applied to any subsystem associated with a subdomain $\Omega'_t \subset \Omega_t$ – we infer the corresponding local formulation using (29)

$$\begin{aligned}\frac{\partial e}{\partial t} + \nabla_{\underline{x}} \cdot (e \underline{v}_s) &= \theta h_m - \nabla_{\underline{x}} \cdot (h_m \underline{w}) + \underline{\sigma} : \underline{\varepsilon}(\underline{v}_s) + \nabla_{\underline{x}} \cdot \left(\frac{1}{\rho_f} \underline{\sigma}_{\text{vis}} \cdot \underline{w} \right) \\ &\quad + \underline{w} \cdot (\underline{f} - \underline{\gamma}_f) + q - \nabla_{\underline{x}} \cdot \underline{j}_q,\end{aligned}\quad (33)$$

or equivalently

$$\begin{aligned}\frac{de}{dt} + e \nabla_{\underline{x}} \cdot \underline{v}_s &= \theta h_m - \nabla_{\underline{x}} \cdot (h_m \underline{w}) + \underline{\sigma} : \underline{\varepsilon}(\underline{v}_s) + \nabla_{\underline{x}} \cdot \left(\frac{1}{\rho_f} \underline{\sigma}_{\text{vis}} \cdot \underline{w} \right) \\ &\quad + \underline{w} \cdot (\underline{f} - \underline{\gamma}_f) + q - \nabla_{\underline{x}} \cdot \underline{j}_q,\end{aligned}\quad (34)$$

which also gives in the total Lagrangian framework

$$\begin{aligned}\frac{dE}{dt} &= \Theta h_m - \nabla_{\underline{\xi}} \cdot (h_m \underline{W}) + \underline{\Sigma} : (d_{\underline{y}} \underline{\varepsilon} \cdot \underline{v}_s) + \nabla_{\underline{\xi}} \cdot \left(\frac{1}{\rho_f J} \underline{\Pi}_{\text{vis}}^T \cdot \underline{F} \cdot \underline{W} \right) \\ &\quad + (\underline{F} \cdot \underline{W}) \cdot (\underline{f} - \underline{\gamma}_f) + Q - \nabla_{\underline{\xi}} \cdot \underline{J}_Q,\end{aligned}\quad (35)$$

with $\underline{\Pi}_{\text{vis}}$ the first Piola-Kirchhoff stress tensor with respect to the skeleton configuration for the fluid viscous stresses, namely,

$$\underline{\Pi}_{\text{vis}} = J \underline{\sigma}_{\text{vis}} \cdot \underline{F}^{-T}.$$

5 Constitutive laws derived from the second principle

In this section, we proceed to apply the second law of thermodynamics, which will provide Clausius-Duhem inequalities allowing to infer adequate constitutive laws.

5.1 Second law of thermodynamics

As we did for the internal energy, we introduce the entropy and its corresponding volume-distributed form

$$\mathcal{S} = \int_{\Omega_t} s \, d\Omega,$$

and the entropy rate due to the source term

$$\mathcal{J}_{\mathcal{S}_\theta} = \int_{\Omega_t} \theta s_m \, d\Omega.$$

PRINCIPLE 2 (SECOND LAW OF THERMODYNAMICS)

Let us consider an open system within a temperature field T , and submitted to a heat source q and a heat flux \underline{j}_q , then we have

$$\frac{d\mathcal{S}}{Dt} \geq \mathcal{J}_{\mathcal{S}_\theta} + \int_{\Omega_t} \frac{q}{T} \, d\Omega - \int_{\partial\Omega_t} \frac{\underline{j}_q \cdot \underline{n}}{T} \, dS = \int_{\Omega_t} \left(\theta s_m + \frac{q}{T} - \underline{\nabla}_{\underline{x}} \cdot \left(\frac{\underline{j}_q}{T} \right) \right) \, d\Omega.$$

As we have done in the first principle, we can convert this into a space-distributed inequality, namely,

$$\frac{ds}{dt} + s \underline{\nabla}_{\underline{x}} \cdot \underline{v}_s \geq \theta s_m - \underline{\nabla}_{\underline{x}} \cdot (s_m \underline{w}) + \frac{q}{T} - \frac{1}{T} \underline{\nabla}_{\underline{x}} \cdot \underline{j}_q + \frac{1}{T^2} \underline{j}_q \cdot \underline{\nabla}_{\underline{x}} T. \quad (36)$$

In order to deal with thermodynamics functions of T and other variables – \underline{e} and m for the fluid-solid mixture and p for the fluid – we classically introduce

- for the fluid-solid mixture, the Helmholtz free energy

$$\psi(\underline{e}, m, T) = e(\underline{e}, m, s) - Ts,$$

- for the fluid, the Gibbs free energy (sometimes also called free enthalpy)

$$g_m(p, T) = h_m(p, s_m) - Ts_m.$$

As usual, we will also consider the Lagrangian counterparts of such quantities, namely densities per unit volume in the reference configuration of the skeleton, denoted by upper case letters such as S or Ψ . We also recall the fundamental identities (see e.g. [10])

$$\frac{\partial g_m}{\partial p} = \frac{1}{\rho_f}, \quad \frac{\partial g_m}{\partial T} = -s_m, \quad (37)$$

the first of which being the *state equation* of the fluid.

Remark 5 – It is very interesting – and by no means straightforward – that the internal energy and the Helmholtz free energy of the mixture only depend on the above compact sets of variables. Indeed, for example we have for the Helmholtz free energy in Lagrangian form

$$\Psi = \Psi_s(\underline{e}, J(1 - \phi), T) + m_f \psi_m\left(\frac{1}{\rho_f}, T\right), \quad (38)$$

where Ψ_s is the free energy of the skeleton (per unit volume of the reference configuration) which satisfies

$$d\Psi_s = \underline{\underline{\Sigma}} : d\underline{\underline{e}} - p d(J(1 - \phi)) - S_s dT,$$

with $\underline{\underline{\Sigma}} = \frac{\partial \Psi_s}{\partial \underline{\underline{e}}}$, while $m_f = m + \rho_{f0}\phi_0$ and ψ_m is the Helmholtz free energy of the fluid per unit mass with

$$d\psi_m = -p d\left(\frac{1}{\rho_f}\right) - s_f dT.$$

We infer, recalling the identity $\psi_m = g_m - \frac{p}{\rho_f}$,

$$\begin{aligned} d\Psi &= d\Psi_s + m_f d\psi_m + \psi_m dm \\ &= \underline{\underline{\Sigma}} : d\underline{\underline{e}} - p dJ + p d(J\phi) - m_f p d\left(\frac{1}{\rho_f}\right) + \left(g_m - \frac{p}{\rho_f}\right) dm - S dT \\ &= \underline{\underline{\Sigma}}^{tot} : d\underline{\underline{e}} + g_m dm - S dT, \end{aligned}$$

with $\underline{\underline{\Sigma}}^{tot} = \underline{\underline{\Sigma}} - pJ\underline{\underline{C}}^{-1}$, and where the last simplification directly follows from the identity $m_f \frac{1}{\rho_f} = J\phi$, which shows the expected result, and further establishes the fundamental identities

$$\frac{\partial \Psi}{\partial T} = -S, \quad \frac{\partial \Psi}{\partial m} = g_m. \quad (39)$$

We can now combine (34) and (36) to eliminate $(q - \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \underline{\underline{j}}_q)$, which yields

$$\begin{aligned} &\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}_s) + \underline{\underline{w}}(\underline{\underline{f}} - \underline{\underline{\gamma}}_f) + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \left(\frac{1}{\rho_f} \underline{\underline{\sigma}}_{\underline{\underline{vis}}} \cdot \underline{\underline{w}}\right) \\ &+ \theta g_m - \frac{d\psi}{dt} - \psi \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \underline{\underline{v}}_s - \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (h_m \underline{\underline{w}}) + T \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (s_m \underline{\underline{w}}) - s \frac{dT}{dt} - \frac{\underline{\underline{j}}_q}{T} \cdot \underline{\underline{\nabla}}_{\underline{\underline{x}}} T \geq 0. \end{aligned}$$

Then, using the Gibbs free energy g_m that satisfies from (37)

$$\underline{\underline{\nabla}}_{\underline{\underline{x}}} g_m = \frac{\partial g_m}{\partial p} \underline{\underline{\nabla}}_{\underline{\underline{x}}} p + \frac{\partial g_m}{\partial T} \underline{\underline{\nabla}}_{\underline{\underline{x}}} T = \frac{1}{\rho_f} \underline{\underline{\nabla}}_{\underline{\underline{x}}} p - s_m \underline{\underline{\nabla}}_{\underline{\underline{x}}} T,$$

we get, also substituting θ using the mass conservation (9),

$$\begin{aligned} &\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}_s) + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \left(\frac{1}{\rho_f} \underline{\underline{\sigma}}_{\underline{\underline{vis}}} \cdot \underline{\underline{w}}\right) + g_m \left(\frac{\partial \rho_f \phi}{\partial t} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (\rho_f \phi \underline{\underline{v}}_s)\right) \\ &- \frac{d\psi}{dt} - \psi \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \underline{\underline{v}}_s - s \frac{dT}{dt} + \frac{\underline{\underline{w}}}{\rho_f} \left(-\underline{\underline{\nabla}}_{\underline{\underline{x}}} p + \rho_f (\underline{\underline{f}} - \underline{\underline{\gamma}}_f)\right) - \frac{\underline{\underline{j}}_q}{T} \cdot \underline{\underline{\nabla}}_{\underline{\underline{x}}} T \geq 0, \quad (40) \end{aligned}$$

which can be rearranged into

$$\begin{aligned} &\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}_s) + \phi \underline{\underline{\sigma}}_{\underline{\underline{vis}}} : \underline{\underline{\varepsilon}}(\underline{\underline{v}}_r) + g_m \left(\frac{\partial \rho_f \phi}{\partial t} + \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (\rho_f \phi \underline{\underline{v}}_s)\right) - \frac{d\psi}{dt} - \psi \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot \underline{\underline{v}}_s \\ &- s \frac{dT}{dt} + \frac{\underline{\underline{w}}}{\rho_f} \left(-\underline{\underline{\nabla}}_{\underline{\underline{x}}} p + \rho_f (\underline{\underline{f}} - \underline{\underline{\gamma}}_f) + \frac{1}{\phi} \underline{\underline{\nabla}}_{\underline{\underline{x}}} \cdot (\phi \underline{\underline{\sigma}}_{\underline{\underline{vis}}})\right) - \frac{\underline{\underline{j}}_q}{T} \cdot \underline{\underline{\nabla}}_{\underline{\underline{x}}} T \geq 0, \quad (41) \end{aligned}$$

or in a Lagrangian framework using the Lagrangian counterpart of each quantity

$$\begin{aligned} \underline{\underline{\Sigma}} : (\underline{d}_y \underline{e} \cdot \underline{v}_s) + \phi \underline{\underline{\Sigma}}_{\text{vis}} : (\underline{d}_y \underline{e} \cdot \underline{v}_r) + g_m \frac{dm}{dt} - \frac{d\Psi}{dt} - S \frac{dT}{dt} \\ + \frac{W}{\rho_f} \cdot \left(-\underline{\nabla}_{\underline{\xi}} p + \rho_f \underline{\underline{F}}^T \cdot (\underline{f} - \underline{\gamma}_f) + \frac{1}{\phi} \underline{\underline{F}}^T \cdot \underline{\nabla}_{\underline{\xi}} \cdot (\phi \underline{\underline{\Pi}}_{\text{vis}}) \right) \\ - \frac{J_Q}{T} \cdot \underline{\nabla}_{\underline{\xi}} T \geq 0. \quad (42) \end{aligned}$$

These inequalities are known as ‘‘Clausius-Duhem inequalities’’, another version of which can be obtained by noting that, here for the Lagrangian form,

$$\frac{d\Psi}{dt} = \frac{\partial \Psi}{\partial \underline{\underline{e}}} : \underline{d}_y \underline{e} \cdot \underline{v}_s + \frac{\partial \Psi}{\partial m} \frac{dm}{dt} + \frac{\partial \Psi}{\partial T} \frac{dT}{dt} = \frac{\partial \Psi}{\partial \underline{\underline{e}}} : \underline{d}_y \underline{e} \cdot \underline{v}_s + g_m \frac{dm}{dt} - S \frac{dT}{dt},$$

recalling (39). Hence, we can rewrite the inequality (42) in the following manner.

THEOREM 5 (CLAUSIUS-DUHEM INEQUALITY)

$$\begin{aligned} \left(\underline{\underline{\Sigma}} - \frac{\partial \Psi}{\partial \underline{\underline{e}}} \right) : (\underline{d}_y \underline{e} \cdot \underline{v}_s) + \phi \underline{\underline{\Sigma}}_{\text{vis}} : (\underline{d}_y \underline{e} \cdot \underline{v}_r) \\ + \frac{W}{\rho_f} \cdot \left(-\underline{\nabla}_{\underline{\xi}} p + \rho_f \underline{\underline{F}}^T \cdot (\underline{f} - \underline{\gamma}_f) + \frac{1}{\phi} \underline{\underline{F}}^T \cdot \underline{\nabla}_{\underline{\xi}} \cdot (\phi \underline{\underline{\Pi}}_{\text{vis}}) \right) \\ - \frac{J_Q}{T} \cdot \underline{\nabla}_{\underline{\xi}} T \geq 0. \quad (43) \end{aligned}$$

5.2 Constitutive laws

Clausius-Duhem inequalities can be seen as providing the crucial conditions for devising constitutive laws, for which the second principle of thermodynamics should be satisfied. We therefore use (43) to infer the following result.

THEOREM 6

The second principle is verified with the following conditions:

- For the heat rate, we suppose that Fourier’s law is satisfied, namely,

$$J_Q = -\lambda_Q \underline{\nabla}_{\underline{\xi}} T.$$

- For the stress tensor, we assume it is the sum of the three contributions

$$\underline{\underline{\Sigma}} = \phi \underline{\underline{\Sigma}}_{\text{vis}} + \frac{\partial \Psi}{\partial \underline{\underline{e}}} + \frac{\partial \Psi_{\text{damp}}}{\partial \underline{\underline{\dot{e}}}},$$

with $\Psi^{\text{damp}}(\underline{\underline{e}}, \underline{\underline{\dot{e}}})$ a viscous pseudo-potential convex in the variable $\underline{\underline{\dot{e}}}$ = $\frac{d\underline{\underline{e}}}{dt} = \underline{d}_y \underline{e} \cdot \underline{v}_s$ and satisfying $\frac{\partial \Psi_{\text{damp}}}{\partial \underline{\underline{\dot{e}}}}(\underline{\underline{e}}, \underline{\underline{0}}) = \underline{\underline{0}}$, which entails

$$\frac{\partial \Psi_{\text{damp}}}{\partial \underline{\underline{\dot{e}}}} : \underline{\underline{\dot{e}}} \geq 0.$$

- For the fluid viscous term, we assume we always have the dissipation inequality

$$\underline{\underline{\sigma}}_{\text{vis}} : \underline{\underline{\varepsilon}}(v_f) \geq 0.$$

- For the relative velocity of the fluid, there exists a positive definite second order tensor $\underline{\underline{K}}_f$ such that

$$\frac{W}{\rho_f} = \underline{\underline{K}}_f \cdot (-\nabla_{\underline{\underline{x}}} p + \rho_f \underline{\underline{F}}^T \cdot (\underline{\underline{f}} - \underline{\underline{\gamma}}_f) + \frac{1}{\phi} \underline{\underline{F}}^T \cdot \nabla_{\underline{\underline{x}}} \cdot (\phi \underline{\underline{\Pi}}_{\text{vis}})),$$

meaning in the Eulerian framework that

$$\frac{w}{\rho_f} = \underline{\underline{k}}_f \cdot (-\nabla_{\underline{\underline{x}}} p + \rho_f (\underline{\underline{f}} - \underline{\underline{\gamma}}_f) + \frac{1}{\phi} \nabla_{\underline{\underline{x}}} \cdot (\phi \underline{\underline{\sigma}}_{\text{vis}})), \quad (44)$$

with $\underline{\underline{k}}_f = \frac{1}{J} \underline{\underline{F}} \cdot \underline{\underline{K}}_f \cdot \underline{\underline{F}}^T$.

Proof. We easily verify that, under the assumptions made,

$$\begin{aligned} & (\underline{\underline{\Sigma}} - \frac{\partial \Psi}{\partial \underline{\underline{e}}}) : d_{\underline{\underline{y}} \underline{\underline{e}}} \cdot v_s + \phi \underline{\underline{\Sigma}}_{\text{vis}} : d_{\underline{\underline{y}} \underline{\underline{e}}} \cdot v_r \\ & + \frac{W}{\rho_f} \cdot \left(-\nabla_{\underline{\underline{x}}} p + \rho_f \underline{\underline{F}}^T \cdot (\underline{\underline{f}} - \underline{\underline{\gamma}}_f) + \frac{1}{\phi} \underline{\underline{F}}^T \cdot \nabla_{\underline{\underline{x}}} \cdot (\phi \underline{\underline{\Pi}}_{\text{vis}}) \right) - \frac{J_Q}{T} \cdot \nabla_{\underline{\underline{x}}} T \\ & = \frac{1}{J} \underline{\underline{\sigma}}_{\text{vis}} : \underline{\underline{\varepsilon}}(v_f) + \frac{\partial \Psi_{\text{damp}}}{\partial \underline{\underline{\dot{e}}}} : \underline{\underline{\dot{e}}} + \frac{W}{\rho_f} \cdot \underline{\underline{K}}_f^{-1} \cdot \frac{W}{\rho_f} + \lambda_Q \nabla_{\underline{\underline{x}}} T \cdot \nabla_{\underline{\underline{x}}} T \geq 0. \end{aligned} \quad (45)$$

■

Note that (44) can be seen as a generalized form of Darcy's law, which would correspond to the pressure gradient term alone. The contribution of the viscous stresses is often referred to as the Brinkman term, but the second principle leads to this most general form including also inertia and external force effects, see also [36] for a survey of such laws. The tensors $\underline{\underline{k}}_f$ and $\underline{\underline{K}}_f$ are called permeability tensors, and the positive definite property requirement stated above leaves much room for complex modeling taking into account the specificities of the material considered – anisotropy, in particular – and the effect of the porosity variations on this tensor, see e.g. [10] and references therein. We further point out that, considering that the privileged directions of the permeability tensor should follow the underlying deformations of the material principal directions, the form $\underline{\underline{K}}_f$ is more intrinsic in nature than $\underline{\underline{k}}_f$.

5.3 Further modeling assumptions

First, we suppose that the temperature is constant and uniform, as is e.g. well-justified when modeling living tissues. Furthermore, we emphasize that we have not – so far – made any particular assumption on the fluid state equation, namely, on the expression of $g_m(p, T)$. From now on, we will focus on incompressible fluid behavior, namely

$$\rho_f = \rho_{f0},$$

which is not a restriction for most of the applications that we envision. In particular blood perfusion undoubtedly falls into that category, and it is also

the case of pulmonary air flows given the range of applicable Reynolds number [2]. This directly entails

$$g_m = \frac{p - p_0}{\rho_{f0}}, \quad (46)$$

where p_0 represents a reference pressure. Note that in this case we have

$$\psi_m = g_m - \frac{p}{\rho_f} = -\frac{p_0}{\rho_{f0}}, \quad (47)$$

hence ψ_m is constant, which means that the fluid does not store any usable free energy, as can be expected with an incompressible behavior, indeed.

5.4 Construction of an admissible Helmholtz free energy functional

With the assumptions made on the fluid constitutive behavior and considering isothermal conditions, the Helmholtz free energy expression (38) reduces to

$$\Psi = \Psi_s(\underline{e}, J_s) - p_0 J \phi, \quad (48)$$

with $J_s = J(1 - \phi)$. Note that this makes Ψ a function of \underline{e} and m only, since $J_s = J - J\phi$ and $J\phi = \frac{m}{\rho_{f0}} + \phi_0$. This leads to the following constitutive equations providing the stress tensor in reversible transformations, namely, disregarding dissipation effects,

$$\underline{\underline{\Sigma}} = \frac{\partial \Psi(\underline{e}, m)}{\partial \underline{e}} = \frac{\partial \Psi_s}{\partial \underline{e}} + \frac{\partial \Psi_s}{\partial J_s} \frac{\partial J}{\partial \underline{e}} = \frac{\partial \Psi_s}{\partial \underline{e}} - p J \underline{C}^{-1}, \quad (49)$$

with

$$p = -\frac{\partial \Psi_s}{\partial J_s}, \quad (50)$$

as discussed in Remark 5.

We are thus led to modeling the solid Helmholtz free energy Ψ_s . Abundant work has already been devoted to this matter, although the literature is much more scarce on finite strain formulations, see e.g. [12] and references therein. Here, our goal is not to give detailed formulations focused on specific types of porous materials, but rather to propose a general strategy inspired from hyperelastic concepts and well-adapted to computational purposes. The specific types of hyperelastic potentials to be selected will then depend on the materials considered in each application, and the corresponding constitutive parameters should be calibrated based on appropriate experimental trials, which the simplicity of the formulation should facilitate.

We propose to construct the free energy functional in the following – indeed, most natural – manner

$$\Psi_s = W^{skel}(\underline{e}) + W^{bulk}(J_s), \quad (51)$$

where W^{skel} is a hyperelastic potential representing the constitutive behavior of the skeleton as a *structure*, while W^{bulk} describes how the energy depends on the solid phase volume changes, against which the fluid interstitial pressure

exerts mechanical work. For the bulk term, we can for example use the form classically employed in hyperelastic formulations, see e.g. [9, 23],

$$W^{bulk}(J_s) = \kappa \left(\frac{J_s}{1 - \phi_0} - 1 - \ln \left(\frac{J_s}{1 - \phi_0} \right) \right),$$

with κ a bulk modulus which can be chosen to be large in order to enforce nearly-incompressible behavior in the solid, hence total (approximate) incompressibility since the fluid was already assumed to be incompressible. We further emphasize that W^{skel} should also incorporate some global bulk deformation energy, namely, provide a stress contribution which produces work on deformations associated with changes of volumes. In other words, we should have “ $\partial W^{skel} / \partial J \neq 0$ ”. In fact, when considering an experiment in which the porous medium is “inflated” by increasing the fluid pressure while maintaining zero total stresses, this gives from (49)

$$\frac{\partial W^{skel}}{\partial J} = \frac{\partial W^{skel}}{\partial \underline{\underline{e}}} : \frac{\underline{\underline{C}}}{3J} = p, \quad (52)$$

which shows how this bulk energy provides resistance to internal pressure, as desired.

Note that the expression (51) gives a material which exactly satisfies the so-called “Terzaghi effective stress principle” – see e.g. [10] and references therein – since we then have

$$\underline{\underline{\Sigma}} + pJ\underline{\underline{C}}^{-1} = \frac{\partial W^{skel}}{\partial \underline{\underline{e}}}(\underline{\underline{e}}),$$

where the right-hand side only depends on the macroscopic strain tensor $\underline{\underline{e}}$, which means that the stress quantity $\underline{\underline{\Sigma}} + pJ\underline{\underline{C}}^{-1}$ entirely determines the (static) deformations of the skeleton. If we want to specifically characterize a porous material that departs from this principle, we can instead consider

$$\Psi_s = W^{skel}(\underline{\underline{e}}) + W^{bulk} \left(J_s \frac{1 - \phi_0}{\bar{J}_s(J)} \right), \quad (53)$$

where $\bar{J}_s(J)$ is a function representing the variations of solid volume directly due to macroscopic volume changes in the absence of pore pressure, namely, the Poisson effect occurring “across the thickness” of the skeleton. For example, we could reasonably assume a linear ratio in the changes of volume, i.e.

$$\frac{\bar{J}_s(J)}{1 - \phi_0} - 1 = \alpha(J - 1) \quad \Leftrightarrow \quad \bar{J}_s(J) = (1 - \phi_0)(1 - \alpha + \alpha J), \quad (54)$$

where the ratio α should be less than one, and small when the compressibility of the solid material is low. Assuming (53), the stress tensor expression is now

$$\underline{\underline{\Sigma}} = \frac{\partial W^{skel}}{\partial \underline{\underline{e}}} - pJ\underline{\underline{C}}^{-1} - (W^{bulk})' \frac{(1 - \phi_0)J_s(\bar{J}_s)'}{(\bar{J}_s)^2} J\underline{\underline{C}}^{-1},$$

where the last term shows that the formulation no longer satisfies the effective stress principle. An alternative expression can be obtained by incorporating (50), leading to

$$\underline{\underline{\Sigma}} = \frac{\partial W^{skel}}{\partial \underline{\underline{e}}} - \left(1 - \frac{J_s(\bar{J}_s)'}{\bar{J}_s} \right) pJ\underline{\underline{C}}^{-1}.$$

The Biot modulus b being defined as the coefficient of the pressure term in the stress expression for infinitesimal transformations, we then have with the linear law (54) and upon linearization of the constitutive relation

$$b = 1 - \alpha(1 - \phi_0).$$

We also recall that the Biot modulus should tend to one when approaching solid incompressible behavior – namely, in which case we recover the effective stress principle – as is well-known and quite clear from our above discussion since α should vanish for incompressible materials. This justifies the use of the simpler construction (51) when considering nearly-incompressible behavior in the skeleton material.

Furthermore, as argued in [7], when large strains are considered special care must be exercised to maintain positive porosity. This can be achieved by including an additional penalization term in Ψ_s , such as

$$W^{por} = -\eta \ln\left(\frac{m}{\rho_{f0}} + \phi_0\right),$$

which makes the fluid pressure tend to minus infinity through (50) when the porosity approaches zero, as is most natural. Accordingly, with this additional contribution the “inflated skeleton” equilibrium identity (52) is transformed into

$$p = \frac{\partial W^{skel}}{\partial \underline{\underline{e}}} : \underline{\underline{C}} - \eta \left(\frac{m}{\rho_{f0}} + \phi_0\right)^{-1}, \quad (55)$$

which can also be considered for negative values of the pressure – namely, deflation.

Remark 6 – *The additional porosity constraint $\phi \leq 1$ is naturally satisfied with the type of bulk potential proposed above, since J_s approaching zero then makes the free energy tend to infinity.*

We point out that an alternative approach for formulating a poromechanics Helmholtz free energy functional is proposed and discussed in detail in [19] based on a decomposition of the strain tensor which, however, requires an isotropy assumption, see also [4] for related ideas. Our construction involves no such isotropy assumption and we – indeed – have among our chief objectives the modeling of living tissues, which are frequently made of highly anisotropic materials [18, 29]. Moreover, in our approach, a natural extension can also be formulated to take into account an active contribution, in order to model muscle contraction – e.g. in the heart, see [31, 7, 8] – which will be the subject of some further work.

Remark 7 – *In this discussion, we have significantly departed from the approach taken in [7], in which the free energy functional was constructed by extending the linear theory of poromechanics. Note that the strategy proposed in the present paper is more general, and makes the enforcement of incompressible conditions more natural, in particular.*

6 Governing equations and energy balance of the porous model

In this section, we summarize and further analyze the governing equations of the complete poromechanics model constructed according to the guidelines discussed in the previous sections. We recall that we have obtained the following coupled system

$$\begin{cases} (\rho_0 + m)\underline{\gamma} - \underline{\nabla}_{\underline{\xi}} \cdot (\underline{F} \cdot \underline{\Sigma}) = (\rho_0 + m)\underline{f}, & \text{in } \Omega_0 \\ \rho_f \underline{\gamma}_f + \underline{k}_f^{-1} \cdot \frac{\underline{w}}{\rho_f} + \underline{\nabla}_{\underline{x}} p - \frac{1}{\phi} \underline{\nabla}_{\underline{x}} \cdot (\phi \underline{\sigma}_{\text{vis}}) = \rho_f \underline{f}, & \text{in } \Omega_t \\ \frac{1}{J} \frac{dm}{dt} + \underline{\nabla}_{\underline{x}} \cdot \underline{w} = \theta, & \text{in } \Omega_t \end{cases} \quad (56)$$

6.1 Strong formulation

As a first important remark, it is quite straightforward to see that we can transform the first equation of this system to express the conservation of momentum for the skeleton alone by subtracting the Lagrangian form of the second equation multiplied by ϕ . This gives

$$\rho_{s0}(1 - \phi_0)\underline{\gamma}_s - \underline{\nabla}_{\underline{\xi}} \cdot (\underline{F} \cdot \underline{\Sigma}_s) - J\phi^2 \underline{k}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s) = \rho_{s0}(1 - \phi_0)\underline{f},$$

where $\underline{\Sigma}_s$ denotes the skeleton contribution in the stress tensor, namely,

$$\underline{\Sigma}_s = \underline{\Sigma} - \phi \underline{\Sigma}_{\text{vis}} + \phi p J C^{-1}.$$

Next, we proceed to reformulate the fluid inertia to make the second equation of System (56) explicit in the fluid velocity. We note that due to the conservation of fluid mass on an arbitrary subdomain Ω'_t ,

$$\frac{d_f}{dt} \int_{\Omega'_t} \rho_f \phi \, d\Omega = \int_{\Omega'_t} \theta \, d\Omega,$$

we have

$$\int_{\Omega'_t} \rho_f \phi \underline{\gamma}_f \, d\Omega = \frac{d_f}{dt} \int_{\Omega'_t} \rho_f \phi \underline{v}_f \, d\Omega - \int_{\Omega'_t} \theta \underline{v}_f \, d\Omega.$$

Using (5) we have

$$\begin{aligned} \frac{d_f}{dt} \int_{\Omega'_t} \rho_f \phi \underline{v}_f \, d\Omega &= \int_{\Omega'_t} \left(\frac{\partial}{\partial t} (\rho_f \phi \underline{v}_f) \Big|_{\underline{x}} + \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes \underline{v}_f) \right) d\Omega \\ &= \int_{\Omega'_t} \left(\frac{\partial}{\partial t} (\rho_f \phi \underline{v}_f) \Big|_{\underline{x}} + \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes \underline{v}_s) \right) d\Omega \\ &\quad + \int_{\Omega'_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \, d\Omega \\ &= \frac{d}{dt} \left(\int_{\Omega'_t} \rho_f \phi \underline{v}_f \, d\Omega \right) + \int_{\Omega'_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \, d\Omega \\ &= \frac{d}{dt} \left(\int_{\Omega'_0} J \rho_f \phi \underline{v}_f \, d\Omega \right) + \int_{\Omega'_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \, d\Omega \\ &= \int_{\Omega'_0} \frac{\partial}{\partial t} (J \rho_f \phi \underline{v}_f) \Big|_{\underline{\xi}} \, d\Omega + \int_{\Omega'_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \, d\Omega, \end{aligned}$$

which provides the identity

$$\rho_f \phi \underline{\gamma}_f = \frac{1}{J} \frac{d}{dt} (J \rho_f \phi \underline{v}_f) + \nabla_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) - \theta \underline{v}_f. \quad (57)$$

Gathering the above equations, we thus obtain the following summary.

System of porous medium equations – Strong formulation

$$\begin{cases} \rho_{s0}(1-\phi_0) \frac{d\underline{v}_s}{dt} - \nabla_{\underline{x}} \cdot (\underline{F} \cdot \underline{\Sigma}_s) - J \phi^2 \underline{k}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad = \rho_{s0}(1-\phi_0) \underline{f}, \quad \text{in } \Omega_0 \\ \frac{1}{J} \frac{d}{dt} (J \rho_f \phi \underline{v}_f) + \nabla_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) - \theta \underline{v}_f \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \phi^2 \underline{k}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s) + \phi \nabla_{\underline{x}} p - \nabla_{\underline{x}} \cdot (\phi \underline{\sigma}_{vis}) = \rho_f \phi \underline{f}, \quad \text{in } \Omega_t \\ \frac{1}{J} \frac{d}{dt} (J \rho_f \phi) + \nabla_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) = \theta, \quad \text{in } \Omega_t \end{cases} \quad (58)$$

where each of the two phases in the porous medium can have Dirichlet or Neumann boundary conditions on various parts of the boundary, the most common combinations being as follows:

- Dirichlet boundary conditions for both phases, namely, prescribed skeleton displacements and fluid velocities:

$$\underline{y} = \underline{y}^{pr}, \quad \underline{v}_f = \underline{v}_f^{pr},$$

on the subpart of the boundary which we denote by Γ_0^D in the reference configuration and Γ_t^D in the current configuration;

- Neumann boundary conditions for both phases, with proportional repartition of boundary traction:

$$\underline{\sigma} \cdot \underline{n} = \underline{t} \Leftrightarrow \underline{F} \cdot \underline{\Sigma} \cdot \underline{n}_0 = \underline{t}_0, \quad \underline{\sigma}_f \cdot \underline{n} = \underline{t},$$

on Γ_0^N (or Γ_t^N);

- Neumann boundary condition for the global mixture, but vanishing fluid flux and proportional repartition of *tangential* boundary traction:

$$\underline{\sigma} \cdot \underline{n} = \underline{t} \Leftrightarrow \underline{F} \cdot \underline{\Sigma} \cdot \underline{n}_0 = \underline{t}_0, \quad \Pi_\tau(\underline{\sigma}_f \cdot \underline{n}) = \Pi_\tau(\underline{t}), \quad (\underline{v}_f - \underline{v}_s) \cdot \underline{n} = 0,$$

on Γ_0^{Nnof} (or Γ_t^{Nnof}), where $\Pi_\tau = \underline{\mathbb{1}} - \underline{n} \otimes \underline{n}$ denotes the projection onto the tangential plane;

- Neumann boundary condition for the global mixture with fluid velocity coinciding with the solid velocity (no sliding):

$$\underline{\sigma} \cdot \underline{n} = \underline{t} \Leftrightarrow \underline{F} \cdot \underline{\Sigma} \cdot \underline{n}_0 = \underline{t}_0, \quad \underline{v}_f = \underline{v}_s,$$

on Γ_0^{Nnos} (or Γ_t^{Nnos}).

Remark 8 – We recognize in (58)₂ an equation very similar to the conservative form of the so-called Arbitrary Lagrangian Eulerian (ALE) formulation of the compressible Navier-Stokes equations [17, 24], where here the domain velocity is given by the skeleton physical velocity and the product $\rho_f \phi$ plays the role of a varying fluid mass density. However, compared with standard fluid-structure interaction problems, we have the additional distributed coupling term $\underline{k}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s)$ representing the interaction between the two phases.

6.2 Weak formulation

From the above system written in strong form, we can obtain an equivalent weak form – in other words, a variational formulation – by multiplying each equation by a corresponding test function. This gives the following system.

System of porous medium equations – Weak formulation

$$\left\{ \begin{array}{l} \int_{\Omega_0} \rho_{s0}(1-\phi_0) \frac{d\underline{v}_s}{dt} \cdot \underline{v}_s^* d\Omega + \int_{\Omega_0} \underline{\Sigma}_s : d\underline{v}_s \cdot \underline{v}_s^* d\Omega \\ - \int_{\Omega_t} (\underline{v}_f - \underline{v}_s) \cdot \phi^2 \underline{k}_f^{-1} \cdot \underline{v}_s^* d\Omega \\ = \int_{\Omega_0} \rho_{s0}(1-\phi_0) \underline{f} \cdot \underline{v}_s^* d\Omega + \int_{\Gamma_0^N} (1-\phi) \underline{t}_0 \cdot \underline{v}_s^* dS \\ + \int_{\Gamma_0^{Nnof} \cup \Gamma_0^{Nnos}} \underline{t}_0 \cdot \underline{v}_s^* dS - \int_{\Gamma_t^{Nnof}} \phi (\Pi_\tau \underline{t}) \cdot \underline{v}_s^* dS - \mathcal{R}_f(\underline{v}_s^*) \\ \mathcal{P}_i^f(\underline{v}_f^*) + \int_{\Omega_t} (\underline{v}_f - \underline{v}_s) \cdot \phi^2 \underline{k}_f^{-1} \cdot \underline{v}_f^* d\Omega \\ + \int_{\Omega_t} \left(-\frac{p}{\rho_f} \underline{\nabla}_x \cdot (\rho_f \phi \underline{v}_f^*) + \phi \underline{\sigma}_{vis} : \underline{\underline{\varepsilon}}(\underline{v}_f^*) \right) d\Omega \\ = \int_{\Omega_t} \rho_f \phi \underline{f} \cdot \underline{v}_f^* d\Omega + \int_{\Gamma_t^N \cup \Gamma_t^{Nnof}} \phi \underline{t} \cdot \underline{v}_f^* dS \\ \int_{\Omega_0} \frac{dm}{dt} q^* d\Omega + \int_{\Omega_t} \underline{\nabla}_x \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) q^* d\Omega = \int_{\Omega_t} \theta q^* d\Omega \end{array} \right. \quad (59)$$

to be verified for any admissible test functions \underline{v}_s^* , \underline{v}_f^* and q^* , and with the virtual power of fluid inertia forces given by

$$\begin{aligned} \mathcal{P}_i^f(\underline{v}_f^*) = & \int_{\Omega_0} \frac{d}{dt} (J \rho_f \phi \underline{v}_f) \cdot \underline{v}_f^* d\Omega + \int_{\Omega_t} \underline{\nabla}_x \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \cdot \underline{v}_f^* d\Omega \\ & - \int_{\Omega_t} \theta \underline{v}_f \cdot \underline{v}_f^* d\Omega, \quad (60) \end{aligned}$$

while

$$\begin{aligned}
 \mathcal{R}_f(\underline{v}_s^*) &= \int_{\Gamma_t^{Nnos}} \phi(\underline{\sigma}_f \cdot \underline{n}) \cdot \underline{v}_s^* dS + \int_{\Gamma_t^{Nnof}} \phi(\underline{\sigma}_f \cdot \underline{n}) \cdot \underline{n} (\underline{v}_s^* \cdot \underline{n}) dS \\
 &= \mathcal{P}_i^f(\underline{v}_s^*) + \int_{\Omega_t} (\underline{v}_f - \underline{v}_s) \cdot \phi^2 \underline{k}_f^{-1} \cdot \underline{v}_s^* d\Omega \\
 &\quad + \int_{\Omega_t} \left(-\frac{p}{\rho_f} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_s^*) + \phi \underline{\sigma}_{vis} : \underline{\underline{\varepsilon}}(\underline{v}_s^*) \right) d\Omega \\
 &\quad - \int_{\Omega_t} \rho_f \phi \underline{f} \cdot \underline{v}_s^* d\Omega - \int_{\Gamma_t^N} \phi \underline{t} \cdot \underline{v}_s^* dS - \int_{\Gamma_t^{Nnof}} \phi (\Pi_\tau \underline{t}) \cdot \underline{v}_s^* dS
 \end{aligned}$$

is the residual of the second equation (namely, the virtual work of fluid reaction forces) with respect to the fluid-related Dirichlet boundary conditions prescribed on the boundaries Γ_t^{Nnos} and Γ_t^{Nnof} . The complete Dirichlet boundary conditions for this system are

$$\begin{cases} \underline{y} = \underline{y}^{pr}, & \underline{v}_f = \underline{v}_f^{pr}, & \text{on } \Gamma_0^D \\ \underline{v}_f \cdot \underline{n} = \underline{v}_s \cdot \underline{n}, & & \text{on } \Gamma_t^{Nnof} \\ \underline{v}_f = \underline{v}_s, & & \text{on } \Gamma_t^{Nnos} \end{cases}$$

and the respective test functions must accordingly satisfy the corresponding homogeneous conditions. Note that in this variational formulation we have used a combination of Lagrangian and Eulerian terms according to which is the most effective form for computational purposes. In addition, the conservation of mass equation is now written with the added mass variable m , which is quite natural since the Helmholtz free energy functional is a function of this variable.

6.3 Variants of equations

Of course, we have an equivalent system when substituting (59)₁ with the total conservation of momentum in a weak form – namely, we simply substitute the expression of the residual term $\mathcal{R}_f(\underline{v}_s^*)$ – which gives the following alternative formulation.

System of porous medium equations – Second weak formulation

$$\begin{cases} \int_{\Omega_0} \rho_{s0}(1 - \phi_0) \frac{d\underline{v}_s}{dt} \cdot \underline{v}_s^* d\Omega + \mathcal{P}_i^f(\underline{v}_s^*) + \int_{\Omega_0} \underline{\Sigma} : d\underline{y}_{\underline{\underline{e}}} \cdot \underline{v}_s^* d\Omega \\ \quad = \int_{\Omega_0} (\rho_0 + m) \underline{f} \cdot \underline{v}_s^* d\Omega + \int_{\Gamma_0^N \cup \Gamma_0^{Nnof} \cup \Gamma_0^{Nnos}} \underline{t}_0 \cdot \underline{v}_s^* dS \\ (59)_2 \\ (59)_3 \end{cases} \quad (61)$$

Note that this form substantially simplifies the expression of the virtual work of external forces.

Furthermore, we can obtain alternative expressions of the fluid inertia power, as is usually done in standard fluid-structure interaction formulations. For example, the form employed in [23] is obtained by noting that

$$\left. \frac{\partial}{\partial t} (J \rho_f \phi \underline{v}_f) \right|_{\underline{\xi}} = \left. \frac{\partial}{\partial t} (J \rho_f \phi) \right|_{\underline{\xi}} \underline{v}_f + J \rho_f \phi \left. \frac{\partial \underline{v}_f}{\partial t} \right|_{\underline{\xi}},$$

while

$$J\rho_f\phi\frac{\partial\underline{v}_f}{\partial t}\Big|_{\underline{\xi}} = \sqrt{J\rho_f\phi}\left(\frac{\partial}{\partial t}(\sqrt{J\rho_f\phi}\underline{v}_f)\Big|_{\underline{\xi}} - \frac{1}{2\sqrt{J\rho_f\phi}}\frac{\partial}{\partial t}(J\rho_f\phi)\Big|_{\underline{\xi}}\underline{v}_f\right).$$

We then infer

$$\begin{aligned} \frac{\partial}{\partial t}(J\rho_f\phi\underline{v}_f)\Big|_{\underline{\xi}} &= \sqrt{J\rho_f\phi}\frac{\partial}{\partial t}(\sqrt{J\rho_f\phi}\underline{v}_f)\Big|_{\underline{\xi}} + \frac{1}{2}\frac{\partial}{\partial t}(J\rho_f\phi)\Big|_{\underline{\xi}}\underline{v}_f \\ &= \sqrt{J\rho_f\phi}\frac{\partial}{\partial t}(\sqrt{J\rho_f\phi}\underline{v}_f)\Big|_{\underline{\xi}} + \frac{J}{2}(\theta - \underline{\nabla}_{\underline{x}} \cdot (\rho_f\phi(\underline{v}_f - \underline{v}_s)))\underline{v}_f, \end{aligned}$$

where we have used (16) and the Eulerian form of (17) in the last substitution. Hence, we can transform (57) into the second fluid inertia identity

$$\begin{aligned} \rho_f\phi\gamma_f &= \frac{1}{J}\sqrt{J\rho_f\phi}\frac{d}{dt}(\sqrt{J\rho_f\phi}\underline{v}_f) - \frac{1}{2}\underline{\nabla}_{\underline{x}} \cdot (\rho_f\phi(\underline{v}_f - \underline{v}_s))\underline{v}_f \\ &\quad + \underline{\nabla}_{\underline{x}} \cdot (\rho_f\phi\underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) - \frac{1}{2}\theta\underline{v}_f, \quad (62) \end{aligned}$$

or equivalently in a weak form

$$\begin{aligned} \mathcal{P}_i^f(\underline{v}^*) &= \int_{\Omega_0} \frac{d}{dt}(\sqrt{J\rho_f\phi}\underline{v}_f) \cdot \sqrt{J\rho_f\phi}\underline{v}^* d\Omega - \frac{1}{2} \int_{\Omega_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f\phi(\underline{v}_f - \underline{v}_s)) \underline{v}_f \cdot \underline{v}^* d\Omega \\ &\quad + \int_{\Omega_t} \underline{\nabla}_{\underline{x}} \cdot (\rho_f\phi\underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \cdot \underline{v}^* d\Omega - \frac{1}{2} \int_{\Omega_t} \theta\underline{v}_f \cdot \underline{v}^* d\Omega. \quad (63) \end{aligned}$$

As we will see in the next section, this form is particularly useful when dealing with energy balances. Hence, we also expect this expression to be instrumental when devising energy-consistent discretization schemes.

6.4 Energy balance

We now make use of the above-derived system equations to reformulate directly the energy principle for the porous medium, in a form providing explicit estimates for the solution of the system (58) – or the associated weak forms (59)-(61). This type of estimate is crucial in particular for analyzing the mathematical properties of the solutions and their discretizations. Defining the total Helmholtz free energy

$$\mathcal{W} = \int_{\Omega_0} \Psi d\Omega,$$

we have the following result.

THEOREM 7

The solution of the general poromechanics problem written in (58) and (59)-(61) in strong and weak forms, respectively, satisfies the following energy balance

$$\begin{aligned} \frac{d\mathcal{K}}{dt} + \frac{d\mathcal{W}}{dt} &= - \int_{\Omega_0} \frac{\partial\Psi_{\text{damp}}}{\partial\dot{\underline{\underline{\epsilon}}}} : \dot{\underline{\underline{\epsilon}}} d\Omega - \int_{\Omega_t} \phi \underline{\underline{\sigma}}_{\text{vis}} : \underline{\underline{\epsilon}}(\underline{v}_f) d\Omega \\ &\quad - \int_{\Omega_t} (\underline{v}_f - \underline{v}_s) \cdot \phi^2 \underline{\underline{k}}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s) d\Omega + \mathcal{P}_{\text{ext}}^{\text{total}} + \mathcal{J}_{\mathcal{K}b} + \mathcal{J}_{\mathcal{K}\theta} + \mathcal{J}_{\mathcal{W}b} + \mathcal{J}_{\mathcal{G}\theta}, \quad (64) \end{aligned}$$

with

$$\mathcal{J}_{\mathcal{K}b} = -\frac{1}{2} \int_{\partial\Omega_t} \rho_f \phi \underline{v}_f^2 (\underline{v}_f - \underline{v}_s) \cdot \underline{n} \, dS, \quad \mathcal{J}_{\mathcal{K}\theta} = \frac{1}{2} \int_{\Omega_t} \underline{v}_f^2 \theta \, d\Omega,$$

the incoming rates of fluid kinetic energy due to the boundary flow and source term, respectively, and

$$\mathcal{J}_{\mathcal{W}b} = - \int_{\partial\Omega_t} \rho_f \phi \psi_m (\underline{v}_f - \underline{v}_s) \cdot \underline{n} \, dS, \quad \mathcal{J}_{\mathcal{G}\theta} = \int_{\Omega_t} g_m \theta \, d\Omega,$$

similar incoming rates of Helmholtz and Gibbs free energies.

Proof. First we consider the fluid inertia power in the form (63) applied to the fluid velocity. We start with the following identities

$$\underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f^2 (\underline{v}_f - \underline{v}_s)) = \underline{v}_f^2 \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) + 2(\underline{v}_f \cdot \underline{\nabla}_{\underline{x}} \underline{v}_f) \rho_f \phi (\underline{v}_f - \underline{v}_s),$$

and

$$\underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \cdot \underline{v}_f = \underline{v}_f^2 \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) + (\underline{v}_f \cdot \underline{\nabla}_{\underline{x}} \underline{v}_f) \rho_f \phi (\underline{v}_f - \underline{v}_s).$$

Hence,

$$\underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f \otimes (\underline{v}_f - \underline{v}_s)) \cdot \underline{v}_f = \frac{1}{2} \underline{v}_f^2 \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) + \frac{1}{2} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f^2 (\underline{v}_f - \underline{v}_s)),$$

which gives

$$\begin{aligned} \mathcal{P}_i^f(\underline{v}_f) &= \int_{\Omega_0} \frac{d}{dt} (\sqrt{J} \rho_f \phi \underline{v}_f) \cdot \sqrt{J} \rho_f \phi \underline{v}_f \, d\Omega \\ &\quad + \int_{\Omega_t} \frac{1}{2} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi \underline{v}_f^2 (\underline{v}_f - \underline{v}_s)) \, d\Omega - \frac{1}{2} \int_{\Omega_t} \theta \underline{v}_f^2 \, d\Omega \\ &= \frac{d\mathcal{K}_f}{dt} + \int_{\partial\Omega_t} \frac{1}{2} \rho_f \phi \underline{v}_f^2 (\underline{v}_f - \underline{v}_s) \cdot \underline{n} \, dS - \frac{1}{2} \int_{\Omega_t} \theta \underline{v}_f^2 \, d\Omega \\ &= \frac{d\mathcal{K}_f}{dt} - \mathcal{J}_{\mathcal{K}b} - \mathcal{J}_{\mathcal{K}\theta}. \end{aligned}$$

Therefore, adding the weak form (61)₁ tested with \underline{v}_s to the weak form (59)₂ tested with $(\underline{v}_f - \underline{v}_s)$ we obtain

$$\begin{aligned} &\frac{d\mathcal{K}_s}{dt} + \frac{d\mathcal{K}_f}{dt} + \int_{\Omega_0} \frac{\partial \Psi}{\partial \underline{e}} : d_{\underline{y}} \underline{e} \cdot \underline{v}_s \, d\Omega + \int_{\Omega_t} \left(-\frac{p}{\rho_f} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) + \phi \underline{\sigma}_{\text{vis}} : \underline{\underline{e}}(\underline{v}_f) \right) \, d\Omega \\ &= - \int_{\Omega_0} \frac{\partial \Psi_{\text{damp}}}{\partial \underline{\dot{e}}} : d_{\underline{y}} \underline{e} \cdot \underline{v}_s \, d\Omega - \int_{\Omega_t} (\underline{v}_f - \underline{v}_s) \cdot \phi^2 \underline{k}_f^{-1} \cdot (\underline{v}_f - \underline{v}_s) \, d\Omega + \mathcal{P}_{\text{ext}}^{\text{total}} + \mathcal{J}_{\mathcal{K}b} + \mathcal{J}_{\mathcal{K}\theta}. \end{aligned}$$

Note that this identity also holds when prescribing non-homogeneous Dirichlet boundary conditions – even though the test functions in the variational system are assumed to satisfy homogeneous conditions – since then the remaining residuals are exactly compensated by adequate terms in the power of external forces, by definition of these residuals. Finally, testing (59)₃ with $\frac{p-p_0}{\rho_f}$ we have

$$\int_{\Omega_t} \frac{p-p_0}{\rho_f} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) \, d\Omega + \int_{\Omega_0} \frac{p-p_0}{\rho_f} \frac{dm}{dt} \, d\Omega = \int_{\Omega_t} \frac{p-p_0}{\rho_f} \theta \, d\Omega,$$

which by the Stokes formula gives

$$\begin{aligned} - \int_{\Omega_t} \frac{p}{\rho_f} \underline{\nabla}_{\underline{x}} \cdot (\rho_f \phi (\underline{v}_f - \underline{v}_s)) d\Omega &= \int_{\Omega_0} \frac{p - p_0}{\rho_f} \frac{dm}{dt} d\Omega \\ &\quad - \int_{\partial\Omega_t} \rho_f \phi \frac{p_0}{\rho_f} (\underline{v}_f - \underline{v}_s) \cdot \underline{n} dS \\ &\quad - \int_{\Omega_t} \frac{p - p_0}{\rho_f} \theta d\Omega \\ &= \int_{\Omega_0} g_m \frac{dm}{dt} d\Omega - \mathcal{J}_{\mathcal{W}b} - \mathcal{J}_{G\theta}. \end{aligned}$$

Therefore, using $\dot{\underline{\underline{\epsilon}}} = d_{\underline{y}} \underline{\underline{\epsilon}} \cdot \underline{v}_s$, we have

$$\int_{\Omega_0} \left(\frac{\partial \Psi}{\partial \underline{\underline{\epsilon}}} : d_{\underline{y}} \underline{\underline{\epsilon}} \cdot \underline{v}_s + g_m \frac{dm}{dt} \right) d\Omega = \int_{\Omega_0} \left(\frac{\partial \Psi}{\partial \underline{\underline{\epsilon}}} : \dot{\underline{\underline{\epsilon}}} + \frac{\partial \Psi}{\partial m} \dot{m} \right) d\Omega = \frac{d\mathcal{W}}{dt},$$

which concludes the proof. \blacksquare

In Theorem 7, the first three contributions appearing in the right-hand side of the balance (64) are negative and respectively correspond to dissipation effects in the solid, in the fluid and in the interaction between the two components. The last four contributions correspond to source terms and external forces, hence they vanish for an isolated system.

Remark 9 – An incoming rate of Gibbs free energy can be naturally interpreted as the sum of external power performed on the corresponding added mass and increase of fluid Helmholtz free energy associated with this mass, as e.g.

$$\mathcal{J}_{G\theta} = \int_{\Omega_t} \left(p \frac{\theta}{\rho_f} + \psi_m \theta \right) d\Omega.$$

The reason why it is the Helmholtz free energy which appears in $\mathcal{J}_{\mathcal{W}b}$ is that the external work exerted on the incoming flow on the boundary is already included in $\mathcal{P}_{\text{ext}}^{\text{total}}$.

Remark 10 – When restricting the model to the case of small perturbations, a straightforward linearized form of (58) can be considered leading to equations equivalent to the Biot system [3, 37], which has been mathematically studied using Galerkin approximations [32] or semi-group theory [16] in the light of an energy balance similar to (64). More precisely, in order to recover the Biot system from (58) we just need to remark that, in the linearized case for an incompressible fluid, p and m are linearly linked to the skeleton deformation tensor trace. Indeed, we get from (50) after linearization

$$p = -K \left(\text{tr} \underline{\underline{\epsilon}}(\underline{y}) - \frac{m}{\rho_f} \right),$$

with K a positive constant involving the skeleton bulk modulus. Thus we can rewrite the mass conservation as

$$\frac{1}{K} \frac{dp}{dt} + \underline{\nabla} \cdot \underline{v}_s + \underline{\nabla} \cdot (\phi (\underline{v}_f - \underline{v}_s)) = \theta,$$

similar to the classical form of the mass conservation in the Biot system.

Note finally that, even in large displacement cases, when existence results exist they usually make use of energy estimate like (64), see for instance [6] for comparable systems.

7 Concluding remarks

We have proposed a complete framework for modeling finite strain poromechanics in a very general setting, the only specific assumptions made being fluid incompressibility and isothermal conditions. Our construction entirely relies on the crucial conservation and thermodynamics principles, and as a consequence we very naturally established a global energy balance for the resulting coupled system, directly from the weak formulation.

This fundamental energy balance is very important for further mathematical analyses of the system. Moreover, it also provides a milestone for discretizing the formulation, as fulfilling similar balances for the discretized solution is a key step in the stability analysis. This type of approach is for example already very effectively used in the numerical analysis of fluid-structure interaction problems – see e.g. [24] – to which we showed that our proposed formulation bears some resemblance. In this respect, adequate time discretization strategies will be presented in a follow-up to this work [28].

Finally, we emphasize that the proposed formulation leaves much room for modeling very general constitutive behaviors. This holds both for the fluid component – within the incompressibility assumption – for which we can consider e.g. non-Newtonian stress laws and various forms of permeability modeling, and also for the skeleton which is modeled based on a generalized hyperelastic potential, with optional additional dissipative effects. The hyperelastic potential allows to represent a wide range of mechanical behaviors – both compressible and incompressible, finite strain conditions, arbitrary anisotropy in particular – and with some extensions could also include some plasticity effects and active components. We expect this wide range of applicability to be of utmost value in many fields, including – but not restricted to – soft tissue biomechanics which originally motivated this work.

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