



# On the Rates of Convergence of the Wireless Multi-access Interference Distribution to the Normal Distribution

Hazer Inaltekin, Stephen V. Hanly

## ► To cite this version:

Hazer Inaltekin, Stephen V. Hanly. On the Rates of Convergence of the Wireless Multi-access Interference Distribution to the Normal Distribution. WiOpt'10: Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks, May 2010, Avignon, France. pp.569-574. inria-00503044

**HAL Id: inria-00503044**

**<https://inria.hal.science/inria-00503044>**

Submitted on 16 Jul 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the Rates of Convergence of the Wireless Multi-access Interference Distribution to the Normal Distribution

Hazer Inaltekin

Department of Electrical and Electronic Engineering  
The University of Melbourne, Melbourne, Australia  
Email: hazeri@unimelb.edu.au

Stephen V. Hanly

Department of Electrical and Computer Engineering  
National University of Singapore, Singapore  
Email: elehsv@nus.edu.sg

**Abstract**—It is of prime importance to reveal the structure of wireless multi-access interference distributions to compute many performance bounds and metrics for wireless networks such as transmission capacity, outage probability and bit-error-rate. However, at the present, there are no closed form expressions for the multi-access interference distributions in wireless networks apart from a very special case. This paper presents a principled methodology towards the resolution of this bottleneck by establishing rates of convergence of the multi-access interference distribution to a Gaussian distribution for any given *bounded* power-law decaying path-loss function  $G$ . In particular, it is shown that the interference distribution converges to the Gaussian distribution with the same mean and variance at a rate  $\frac{1}{\sqrt{\lambda}}$ , where  $\lambda > 0$  is the intensity of the homogenous planar Poisson point process generating node locations.

## I. INTRODUCTION

Wireless networks are often interference limited due to the broadcast nature of the wireless medium, which makes the interference characterization an important issue for a lot of research problems involving signal-to-interference-plus-noise-ratio calculations. Since the level of wireless multi-access interference (WMAI) at a receiver node in a wireless network depends on the overall network geometry (locations of interfering nodes as well as numerous other wireless channel dynamics), computation of the exact WMAI distributions is mathematically intractable in most practical scenarios. Except for one special case [1], i.e., the case in which the signal power decays according to the unbounded power-law decaying path-loss function  $t^{-4}$ , there are no closed form expressions available for the WMAI distributions.

Therefore, it becomes necessary to resort to numerical methods to compute the WMAI distributions by modeling WMAI as a power-law shot-noise process. Even though algorithmic perspectives based on fast Fourier transforms [2] to numerically compute power-law shot-noise distributions and densities are promising, they are of limited interest and importance in the context of wireless networking as the numerical computation cannot provide closed form expressions revealing structural dependencies between the WMAI distributions and network design parameters to assess the wireless network performance under candidate/existing wireless communication

technologies.

Other approaches in the field include various approximation techniques based on LePage series [3], Edgeworth expansion [4] or geometrical considerations [5] to estimate the WMAI distributions, and thereby obtain simple and insightful upper and lower bounds on the related performance quantities of interest in spatial wireless networks, e.g., [6] and [7]. In particular, this paper is close in spirit to [5] in that the authors of [5] also obtained upper and lower bounds on the normalized WMAI distribution with provably small gaps between the computed bounds and the actual WMAI distribution when the transmitted signal power decays according to the unbounded power-law path-loss model.<sup>1</sup>

In contrast to most earlier work in the field, one distinctive aspect of this paper is that we work with general *bounded* power-law decaying path-loss functions to calculate the WMAI distributions, which often complicates analysis significantly. The main motivation for working with bounded path-loss models is recent findings pointing out that the unrealistic singularity of the unbounded path-loss model at 0 leads to unexpected deviations on the final computed WMAI distributions, e.g., see [8] and [9].

Secondly, this paper presents a principled methodology employing advanced distribution approximation techniques [10] to handle general bounded path-loss models efficiently in WMAI distribution computations.

Last but not least, this paper establishes the rates of convergence of the WMAI distributions to the Gaussian distribution with the same mean and variance, and the bounds on the goodness of this approximation. This rate is equal to  $\frac{c(x)}{\sqrt{\lambda}}$ , where  $\lambda$  is the intensity of the homogenous planar Poisson point process generating node locations, and  $c(x) > 0$  is a function which depends on the shape of the path-loss model, and the point  $x \in \mathbb{R}$  at which we want to estimate the interference distribution function.  $c(x)$  approaches zero for large values of  $|x|$ , i.e., absolute value of  $x$ , which makes the bounds on the WMAI distributions tight for any given  $\lambda$ .

<sup>1</sup>To be more precise, they focus on the distribution of the normalized inverted signal-to-interference-ratio in [5], which is, in essence, the same as computing the WMAI distribution.

Moreover, the supremum of  $c(x)$  over  $x$  is a small constant, which allows us to obtain *uniform* rates of convergence as a function of  $\lambda$ .

#### A. Related Work

The efforts for characterizing the structure of WMAI in wireless networks by using stochastic geometry can be traced back to as early as 1978 [11]. Sousa et al. applied similar techniques, e.g., [1] and [12], in the 1990s to assess the performance of spread spectrum wireless networks as well as to find optimum transmission ranges in these networks. Subsequently, several approximation techniques appeared in the field to approximate the level of WMAI in wireless networks, and then use these results for the network performance analysis, e.g., [3], [4] and [5]. The model and the problem set-up of this work is related to all of these previous papers dealing with the WMAI characterization in wireless networks but with one very crucial distinction that this paper focuses on more realistic general *bounded* path-loss models at the physical layer.

Since interference in a wireless network is a specific instance of a shot-noise process, the results of this paper are also related to a more general body of work on shot-noise processes [13] and [14]. The paper [13] establishes many properties of the power-law shot-noise process on the line such as its moment generating functions, moments and cumulants. For a very specific bounded power-law decaying impulse response function driving the power-law shot-noise process, [13] also briefly mentions that the amplitude distribution of the power-law shot-noise process converges to the Gaussian form without any formal proof for this convergence and without establishing rates of convergence.

Our results are closest to those of [14] as they also established the uniform rates of convergence for the amplitude distribution of the shot-noise process. When compared with [14], our results are different than those of [14] in three aspects. Firstly, we give both uniform and non-uniform Berry-Esseen types of bounds on the WMAI distributions. The non-uniform bounds allow us to tightly approximate the WMAI distributions for extreme interference values even for small finite values of  $\lambda$ . Secondly, error terms appearing in our uniform Gaussian approximation formulas are seven times better than those in [14], which is a very important point since we are estimating numbers in  $[0, 1]$ . Finally, this paper presents a principled and simpler methodology to establish Gaussian approximation results, which offers an evolutionary path for many novel applications to wireless communications and networking. The proof technique used in [14] depends on obtaining general lemmas on probabilities of large deviations and on the rate of normal convergence by using characteristic functions, which is usually more complicated than the approach employed in this paper.

## II. NETWORK MODEL

We consider a planar network in which transmitters are distributed according to a homogenous planar Poisson point process with intensity  $\lambda$ , denoted by PPP( $\lambda$ ), over  $\mathbb{R}^2$ . Location

of the  $k^{\text{th}}$  transmitter is represented by  $X_k$ . The transmitter location process  $N_\lambda$  is represented by  $N_\lambda = \sum_k \delta_{X_k}$ , where  $\delta_{X_k}(A) = 1$  if  $X_k \in A \subseteq \mathbb{R}^2$ , and zero otherwise.

We consider the case in which all transmitters transmit with the same power  $P$ . For the signal power attenuation in the wireless medium, we consider a bounded monotone non-increasing path-loss function  $G : [0, \infty) \mapsto [0, \infty)$ , which asymptotically decays to zero at least as fast as  $t^{-\alpha}$  for some path-loss exponent  $\alpha > 2$ , without any fading. Therefore, the received signal power at a distance  $t$  will be given by  $PG(t)$ . The analysis in the next section can also be generalized to the case where transmission powers are random variables, and there are fading effects corrupting received signals. We also believe that the same analysis can be further extended to non-homogenous point processes by appropriately modifying Lemma 1, all of which will be subject to our future work in the field. As it is common in most of the earlier works, we will also assume an ALOHA-type medium access control (MAC) layer mediating node transmissions.

We place a test receiver node at the origin, and consider signals coming from all other transmitters, whose locations are given by  $N_\lambda$ , as interference to this test receiver node. Without loss of generality, we focus on the distribution of WMAI seen by the test receiver node placed at the origin for the rest of the paper. Due to Slivnyak's theorem [16], the interference statistics seen by any other receiver node placed at any other point in  $\mathbb{R}^2$  are the same.

The level of WMAI at the origin is given by the shot-noise process

$$I_\lambda = \sum_{k=1}^{\infty} PG(|X_k|), \quad (1)$$

where  $|X_k|$  is the distance of the  $k^{\text{th}}$  transmitter to the origin.  $I_\lambda$  is a random variable since transmitter locations  $\{X_k, k \geq 1\}$  are random variables. Therefore, different node configurations result in different levels of interference at the test receiver. In the next section, we will show that the distribution of  $I_\lambda$  can be approximated by a Gaussian distribution.

## III. WMAI DISTRIBUTION AND RATES OF CONVERGENCE TO THE GAUSSIAN DISTRIBUTION

This section presents calculations for approximating the WMAI distributions as a Gaussian distribution, and establishes the rates of convergence for this approximation as the intensity of the homogenous planar Poisson point process generating node locations increases.

There are several equivalent ways to represent a PPP( $\lambda$ ) on  $\mathbb{R}^2$ . Since the level of interference caused by a node depends on its distance to the test receiver node, the most convenient representation for our purposes in this paper is the one obtained by transforming and marking (see [15] for the details of marking and transforming of Poisson processes) a PPP(1) on  $[0, \infty)$ , which is given as

$$N_\lambda \stackrel{d}{=} \sum_k \delta \left( \sqrt{\frac{\Gamma_k}{\lambda\pi}} \cos(\Theta_k), \sqrt{\frac{\Gamma_k}{\lambda\pi}} \sin(\Theta_k) \right), \quad (2)$$

where  $X \stackrel{d}{=} Y$  means two random variables  $X$  and  $Y$  are equal in distribution,  $\Theta_k$ 's are independent and identically distributed (i.i.d) random variables with uniform distribution on  $[0, 2\pi]$ , and  $\Gamma_k = \sum_{i=1}^k E_i$ , where  $E_i$ 's are i.i.d random variables with unit exponential distribution. This representation allows us to take  $X_k \stackrel{d}{=} \left( \sqrt{\frac{\Gamma_k}{\lambda\pi}} \cos(\Theta_k), \sqrt{\frac{\Gamma_k}{\lambda\pi}} \sin(\Theta_k) \right)$ , and  $|X_k| \stackrel{d}{=} \sqrt{\frac{\Gamma_k}{\lambda\pi}}$ . By using Poisson process transformations one more time, one can further show that the distances between the origin and the points of  $N_\lambda$  form a Poisson point process on  $[0, \infty)$  with mean measure  $\mu_\lambda([0, t]) = \lambda\pi t^2$ , and the density  $p_\lambda(t) = 2\lambda\pi t \mathbf{1}_{\{t \geq 0\}}$ . This density hints us how to choose the sequence of random variables approximating  $I_\lambda$  in distribution in Lemma 1. Furthermore, by using Laplace functionals of Poisson processes (see [15] for details) and  $p_\lambda(t)$ , we have the following Laplace transform for  $I_\lambda$ :

$$\mathcal{L}_{I_\lambda}(s) = \mathbb{E}[e^{-sI_\lambda}] = \exp\left(-2\lambda\pi \int_0^\infty \left(1 - e^{-sPG(t)}\right) t dt\right).$$

We will need some auxiliary results to prove the main approximation result of the paper. The next lemma shows that the distribution of  $I_\lambda$  can be approximated as a limit distribution of a sequence of random variables  $I_n$ , i.e.,  $I_n \xrightarrow{d} I_\lambda$  as  $n \rightarrow \infty$ .

**Lemma 1:** For each  $n$ , let  $U_{n,1}, U_{n,2}, \dots, U_{n, \lceil \lambda\pi n^2 \rceil}$  be i.i.d random variables with common density  $q(t) = \frac{2t}{n^2} \mathbf{1}_{\{0 \leq t \leq n\}}$ , where  $\lceil \cdot \rceil$  is the smallest integer greater than or equal to its argument. Let  $I_n = \sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} PG(U_{n,k})$ . Then,  $I_n$  converges in distribution to  $I_\lambda$ , which is shown as  $I_n \xrightarrow{d} I_\lambda$ , as  $n \rightarrow \infty$ .

*Proof:* We will show that  $\mathcal{L}_{I_n}(s)$  converges to  $\mathcal{L}_{I_\lambda}(s)$  point-wise as  $n$  goes to infinity. Since  $U_{n,1}, U_{n,2}, \dots, U_{n, \lceil \lambda\pi n^2 \rceil}$  are independent, we have

$$\mathcal{L}_{I_n}(s) = \left(1 + \frac{2}{n^2} \int_0^n \left(e^{-sPG(t)} - 1\right) t dt\right)^{\lceil \lambda\pi n^2 \rceil}.$$

Note that  $e^{-sPG(t)} - 1 = O(t^{-\alpha})$  as  $t \rightarrow \infty$ . Since  $\alpha > 2$ ,  $\int_0^\infty (e^{-sPG(t)} - 1) t dt < \infty$ . This implies  $\lim_{n \rightarrow \infty} \mathcal{L}_{I_n}(s) = \mathcal{L}_{I_\lambda}(s)$ . ■

**Definition 1:** For any  $a > 0$ , we say a sequence of random variables  $\{Y_k\}_{k=1}^\infty$  converges to another random variable  $Y$  in the  $a^{\text{th}}$  moment if  $\lim_{k \rightarrow \infty} \mathbb{E}[|Y_k^a - Y^a|] = 0$ .

Note that, for any given  $a > 0$ , the above definition is also equivalent to the  $L_1$  convergence of the sequence  $\{Y_k^a\}_{k=1}^\infty$  to the limit random variable  $Y^a$ .

For the Gaussian convergence result, we also need the convergence of  $I_n$  to  $I_\lambda$  in the first and second moments. *With probability one convergence* of a sequence of random variables  $\{Y_k\}_{k=1}^\infty$  to another random variable  $Y$  implies the convergence in the first moment if the collection  $\{Y_k\}_{k=1}^\infty$  is *uniformly integrable*, i.e.,  $\lim_{M \rightarrow \infty} \sup_k \mathbb{E}[|Y_k| \mathbf{1}_{\{|Y_k| > M\}}] = 0$ . A sufficient condition for uniform integrability is to have  $\sup_k \mathbb{E}[\varphi(Y_k)] < \infty$  for any function  $\varphi \geq 0$  with  $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \infty$  (see [18] for details).

The next lemma, Lemma 2, establishes the uniform integrability of  $\{I_n^a\}$  for any  $a > 0$ . Therefore, by using

Skorohod's theorem [17] and Lemma 2, we can show that  $I_n$  converges to  $I_\lambda$  in the  $a^{\text{th}}$  moment for any  $a > 0$ , which is a stronger condition than the convergence in the first and second moments only.

**Lemma 2 (Uniform Integrability Lemma):** For any  $a > 0$ ,

$$\sup_n \mathbb{E}[I_n^a] < \infty.$$

*Proof:* For any  $t \geq 0$ , we have  $t^a \leq e^{at}$ . Thus,

$$\begin{aligned} \mathbb{E}[I_n^a] &\leq \mathbb{E}[e^{aI_n}] \\ &= \left(1 + \frac{2}{n^2} \int_0^n \left(e^{aPG(t)} - 1\right) t dt\right)^{\lceil \lambda\pi n^2 \rceil}. \end{aligned}$$

Since  $G$  is bounded, we have  $\|G\|_\infty = \sup_{t \in [0, \infty)} |G(t)| < \infty$ . Since  $G(t) = O(t^{-\alpha})$  as  $t \rightarrow \infty$  for some  $\alpha > 2$ , we can find positive constants  $B_1$  and  $B_2$  such that  $G(t) \leq B_1 t^{-\alpha}$  for all  $t \geq B_2$ . Thus,

$$\begin{aligned} \mathbb{E}[I_n^a] &\leq \left(1 + \frac{2}{n^2} \int_0^{B_2} \left(e^{aP\|G\|_\infty} - 1\right) t dt\right. \\ &\quad \left.+ \frac{2}{n^2} \int_{B_2}^\infty \left(e^{aPB_1 t^{-\alpha}} - 1\right) t dt\right)^{\lceil \lambda\pi n^2 \rceil}. \end{aligned} \quad (3)$$

Since  $e^{aPB_1 t^{-\alpha}} - 1 = O(t^{-\alpha})$  as  $t \rightarrow \infty$  and  $\alpha > 2$ , all the integrals in (3) are finite. Thus, there exists a constant  $C$  such that

$$\mathbb{E}[I_n^a] \leq \left(1 + \frac{C}{n^2}\right)^{\lceil \lambda\pi n^2 \rceil}$$

and

$$\sup_n \mathbb{E}[I_n^a] \leq e^{C\lambda\pi} < \infty. \quad \blacksquare$$

We are now ready to state the main theorem of the paper.

**Theorem 1:** For all  $x \in \mathbb{R}$ ,

$$\left| \Pr\left\{ \frac{I_\lambda - \mathbb{E}[I_\lambda]}{\sqrt{\text{Var}(I_\lambda)}} \leq x \right\} - \Phi(x) \right| \leq \frac{c(x)}{\sqrt{\lambda}}, \quad (4)$$

where  $\mathbb{E}[I_\lambda] = 2\lambda\pi P \int_0^\infty G(t) t dt$ ,  $\text{Var}(I_\lambda) = 2\lambda\pi P^2 \int_0^\infty G^2(t) t dt$ ,  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$ , which is the standard normal cumulative distribution function (CDF), and  $c(x) = \frac{1}{\sqrt{2\pi}} \frac{\int_0^\infty G^3(t) t dt}{\left(\int_0^\infty G^2(t) t dt\right)^{\frac{3}{2}}} \min\left(0.7975, \frac{31.935}{1+|x|^3}\right)$ .

*Proof:* We let

$$\begin{aligned} m_{n,k} &= \mathbb{E}[PG(U_{n,k})] = \frac{2P}{n^2} \int_0^n G(t) t dt, \\ m_n &= \mathbb{E}[I_n] = \lceil \lambda\pi n^2 \rceil m_{n,k}, \\ \sigma_{n,k}^2 &= \text{Var}(PG(U_{n,k})) \\ &= \frac{2P^2}{n^2} \left( \int_0^n G^2(t) t dt - \frac{2}{n^2} \left( \int_0^n G(t) t dt \right)^2 \right) \text{ and} \\ \sigma_n^2 &= \text{Var}(I_n) = \lceil \lambda\pi n^2 \rceil \sigma_{n,k}^2. \end{aligned}$$

Since  $I_n$  converges to  $I_\lambda$  in the first and second moments, we have  $\mathbb{E}[I_\lambda] = \lim_{n \rightarrow \infty} m_n = 2\lambda\pi P \int_0^\infty G(t) t dt$  and

$\text{Var}(I_\lambda) = \lim_{n \rightarrow \infty} \sigma_n^2 = 2\lambda\pi P^2 \int_0^\infty G^2(t)tdt$ .<sup>2</sup> We will focus on the following normalized random variables

$$\xi_{n,k} = \frac{PG(U_{n,k}) - m_{n,k}}{\sigma_n} \quad (5)$$

for  $n \geq 1$  and  $1 \leq k \leq \lceil \lambda\pi n^2 \rceil$ . Note that  $\mathbb{E}[\xi_{n,k}] = 0$  and  $\sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} \mathbb{E}[\xi_{n,k}^2] = 1$  for all  $k$  and  $n$ . We use the following Gaussian approximation result (see [10]) to finish the proof.

**Theorem 2:** Let  $\xi_1, \xi_2, \dots, \xi_j$  be independent random variables with zero means and  $\sum_{i=1}^j \mathbb{E}[\xi_i^2] = 1$ . Let  $\chi = \sum_{i=1}^j \mathbb{E}[|\xi_i|^3]$ . Then,  $\left| \Pr\left\{\sum_{i=1}^j \xi_i \leq x\right\} - \Phi(x) \right| \leq \chi \min\left(0.7975, \frac{31.935}{1+|x|^3}\right)$  for all  $x \in \mathbb{R}$ .

Our random variables,  $\xi_{n,k}$ 's, are already in the correct form to apply Theorem 2. We need to calculate  $\chi_n = \sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} \mathbb{E}[|\xi_{n,k}|^3]$ . To this end, we define  $t_n^*$  as  $t_n^* = \inf\{t \geq 0 : PG(t) < m_{n,k}\}$ . Then,

$$\begin{aligned} \mathbb{E}[|\xi_{n,k}|^3] &= \frac{2}{\sigma_n^3 n^2} \left( \int_0^{t_n^*} (PG(t) - m_{n,k})^3 t dt \right. \\ &\quad \left. + \int_{t_n^*}^n (m_{n,k} - PG(t))^3 t dt \right). \end{aligned}$$

First, note that

$$\begin{aligned} \int_{t_n^*}^n (m_{n,k} - PG(t))^3 t dt &\leq m_{n,k}^3 \frac{n^2}{2} \\ &= \frac{4P^3}{n^4} \left( \int_0^n G(t)tdt \right)^3 \\ &= O(n^{-4}). \end{aligned}$$

Secondly, note that  $(PG(t) - m_{n,k})^3 1_{\{0 \leq t \leq t_n^*\}} \leq P^3 G^3(t)$  and  $P^3 \int_0^\infty G^3(t)tdt < \infty$  since  $G(t) = O(t^{-\alpha})$  as  $t \rightarrow \infty$  for some  $\alpha > 2$ .  $t_n^*$  approaches infinity as  $n$  increases since  $m_{n,k}$  goes to zero with increasing  $n$ . Therefore, by using the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_0^{t_n^*} (PG(t) - m_{n,k})^3 t dt = P^3 \int_0^\infty G^3(t)tdt.$$

As a result, we calculate  $\chi_n$  as

$$\begin{aligned} \chi_n &= \lceil \lambda\pi n^2 \rceil \mathbb{E}[|\xi_{n,1}|^3] \\ &= \frac{2 \lceil \lambda\pi n^2 \rceil}{\sigma_n^3 n^2} \left( \int_0^{t_n^*} (PG(t) - m_{n,1})^3 t dt \right. \\ &\quad \left. + \int_{t_n^*}^n (m_{n,1} - PG(t))^3 t dt \right) \\ &\rightarrow \frac{1}{\sqrt{2\lambda\pi}} \frac{\int_0^\infty G^3(t)tdt}{\left( \int_0^\infty G^2(t)tdt \right)^{\frac{3}{2}}} \text{ as } n \rightarrow \infty. \end{aligned}$$

<sup>2</sup>These results can also be verified by using the Campbell's formula.

We finish the proof as follows. Let  $c'(x) = \min\left(0.7975, \frac{31.935}{1+|x|^3}\right)$ . By using Theorem 2, we have

$$\left| \Pr\left\{\sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} \xi_{n,k} \leq x\right\} - \Phi(x) \right| \leq \chi_n c'(x), \forall x \in \mathbb{R}. \quad (6)$$

By using Lemma 1 and Lemma 2, we have

$$\sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} \xi_{n,k} \xrightarrow{d} \frac{I_\lambda - \mathbb{E}[I_\lambda]}{\sqrt{\text{Var}(I_\lambda)}} \text{ as } n \rightarrow \infty. \quad (7)$$

Taking the limit of both sides in (6), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \Pr\left\{\sum_{k=1}^{\lceil \lambda\pi n^2 \rceil} \xi_{n,k} \leq x\right\} - \Phi(x) \right| &= \left| \Pr\left\{\frac{I_\lambda - \mathbb{E}[I_\lambda]}{\sqrt{\text{Var}(I_\lambda)}} \leq x\right\} - \Phi(x) \right| \\ &\leq \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} \frac{\int_0^\infty G^3(t)tdt}{\left( \int_0^\infty G^2(t)tdt \right)^{\frac{3}{2}}} \min\left(0.7975, \frac{31.935}{1+|x|^3}\right). \end{aligned} \quad (8)$$

We note that the Gaussian approximation bound we derived in Theorem 1 is a combination of two different types of Berry-Esseen bounds, one of which is a uniform bound and the other one is a non-uniform bound. The non-uniform bound is designed to be tight for large values of  $|x|$ . On the other hand, the uniform bound is tighter for moderate values of  $|x|$ . We will further analyze these points in detail in the next section. One easy corollary of Theorem 1 is the following.

**Corollary 1:**

$$\sup_{x \in \mathbb{R}} \left| \Pr\left\{\frac{I_\lambda - \mathbb{E}[I_\lambda]}{\sqrt{\text{Var}(I_\lambda)}} \leq x\right\} - \Phi(x) \right| \leq \frac{c}{\sqrt{\lambda}}, \quad (9)$$

where  $c = \frac{0.7975}{\sqrt{2\pi}} \frac{\int_0^\infty G^3(t)tdt}{\left( \int_0^\infty G^2(t)tdt \right)^{\frac{3}{2}}}$ .

We remark that the inequality (9) is given with almost seven times larger constant 2.21 (rather than  $\frac{0.7975}{\sqrt{2\pi}}$ ) in [14] (Equation 7.1).

#### IV. GAUSSIAN APPROXIMATION BOUNDS AND SIMULATION RESULTS

In this section of the paper, we present our numerically computed Gaussian approximation bounds and simulation results confirming the theoretical predictions in Section III. For the numerical study, we focused on two different path-loss models  $G_1(t) = \frac{1}{(1+t)^\alpha}$  (Fig. 1) and  $G_2(t) = \frac{1}{1+t^\alpha}$  (Fig. 2) with  $\alpha = 4$  for various values of  $\lambda$ . Similar conclusions continue to hold for other path-loss models and different values of  $\alpha$  greater than 2.

For the simulation study, we built a C-Simulator in order to perform Monte-Carlo simulations to simulate the WMAI CDFs. We simulated the WMAI distributions for the same path-loss models again:  $G_1(t) = \frac{1}{(1+t)^\alpha}$  (Fig. 3) and  $G_2(t) =$

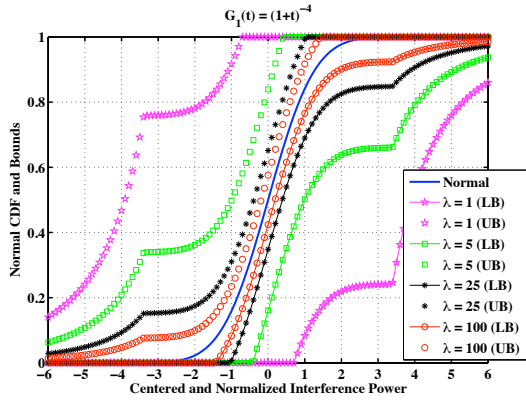


Fig. 1. Upper and lower bounds on the centered and normalized WMAI CDFs for the path-loss function  $G_1(t) = \frac{1}{(1+t)^\alpha}$  and  $\alpha = 4$ .

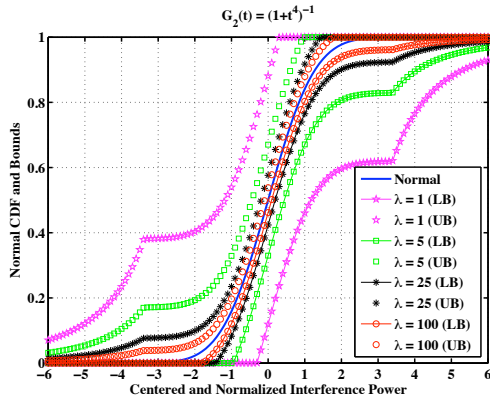


Fig. 2. Upper and lower bounds on the centered and normalized WMAI CDFs for the path-loss function  $G_2(t) = \frac{1}{1+t^\alpha}$  and  $\alpha = 4$ .

$\frac{1}{1+t^\alpha}$  (Fig. 4). In Figs. 3 and 4, we plotted the simulated WMAI distributions for two different choices of  $\alpha$ , i.e., 3 and 5, under various values of  $\lambda$ . Similar results continue to hold for other bounded path-loss models and different values of  $\alpha$  greater than 2. For each different choice of the path-loss model, path-loss exponent and node intensity, we computed the interference power level at the origin for 10000 random node configurations in order to estimate the WMAI distributions.

In Figs. 1 and 2, we observe two different regimes in our computed bounds, i.e.,  $\Phi(x) + \frac{c(x)}{\sqrt{\lambda}}$  and  $\Phi(x) - \frac{c(x)}{\sqrt{\lambda}}$ , for the the centered and normalized WMAI, i.e.,  $\frac{I_\lambda - \mathbb{E}[I_\lambda]}{\sqrt{\text{Var}(I_\lambda)}}$ , distributions. For the moderate values of  $x$ , our uniform Berry-Esseen bound gives better upper and lower bounds around the normal CDF for the interference distribution. On the other hand, for the large (greater than 3.4) absolute values of  $x$ , our non-uniform Berry-Esseen bound becomes a better estimator for the interference distribution. Our bounds can be used to bound the probability of outage in a wireless communications setting. Such bounds will be at least seven times sharper than those based on the convergence results in [14].

For any fixed value of  $\lambda$ , the gap between the upper and

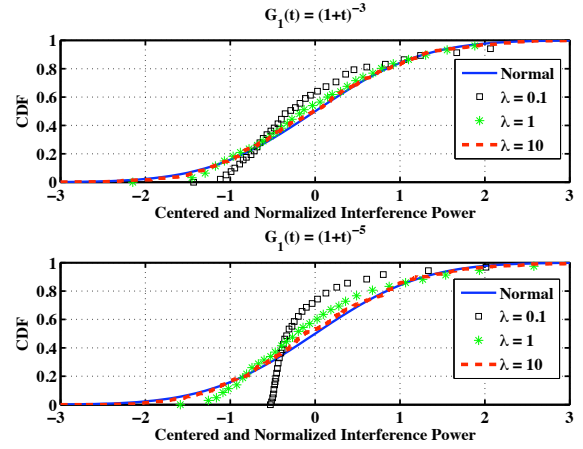


Fig. 3. Comparison of the simulated centered and normalized WMAI CDFs with the normal CDF for the path-loss function  $G_1(t) = \frac{1}{(1+t)^\alpha}$ ,  $\alpha = 3$  and 5.

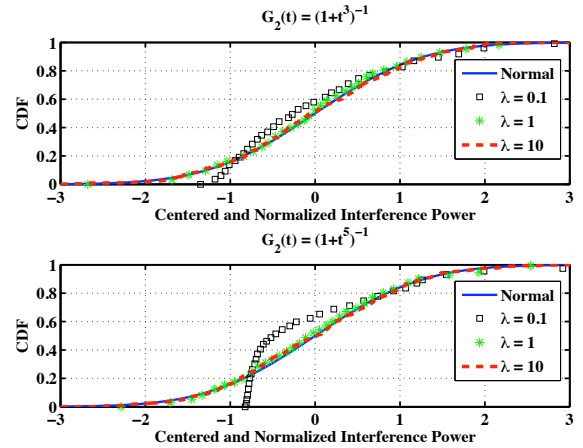


Fig. 4. Comparison of the simulated centered and normalized WMAI CDFs with the normal CDF for the path-loss function  $G_2(t) = \frac{1}{1+t^\alpha}$ ,  $\alpha = 3$  and 5.

lower bounds vanish at a rate  $O(|x|^{-3})$  as the interference power increases. When  $\lambda$  increases, the upper and lower bounds approach the normal CDF at a rate  $\frac{1}{\sqrt{\lambda}}$ , and we start to approximate the WMAI interference distribution as a Gaussian distribution increasingly more accurately. When the upper and lower bounds on the WMAI distribution are compared for different path-loss models, we see that they become tighter for  $G_2(t)$ . This is because the path-loss dependent constant, i.e.,  $\frac{\int_0^\infty G^3(t)tdt}{(\int_0^\infty G^2(t)tdt)^{\frac{3}{2}}}$ , appearing in Theorem 1 is smaller for  $G_2(t)$  than that for  $G_1(t)$  (see Table I).

For the simulation study, we focused on very small to moderate values of  $\lambda$  to illustrate the Gaussian convergence result predicted by Theorem 1, and to understand the effect of small values of  $\lambda$  on the WMAI distributions. As observed in Figs. 3 and 4, the deviations between the normal distribution and the simulated WMAI distributions are prominent for very

TABLE I  
PATH-LOSS DEPENDENT GAUSSIAN APPROXIMATION CONSTANT  

$$\left( \frac{\int_0^\infty G^3(t)tdt}{(\int_0^\infty G^2(t)tdt)^{\frac{3}{2}}} \right)$$

| Path-loss model                   | Path-loss exponent ( $\alpha$ ) |              |              |
|-----------------------------------|---------------------------------|--------------|--------------|
|                                   | $\alpha = 3$                    | $\alpha = 4$ | $\alpha = 5$ |
| $G_1(t) = \frac{1}{(1+t)^\alpha}$ | 1.564                           | 2.3838       | 3.1688       |
| $G_2(t) = \frac{1}{1+t^\alpha}$   | 1.0501                          | 1.1972       | 1.2713       |

sparse networks, i.e.,  $\lambda = 0.1$ . On the other hand, there is a good match between the normal CDF and the simulated WMAI distributions for small to moderate values of  $\lambda$ , i.e.,  $\lambda = 1$  and  $\lambda = 10$ . When  $\lambda$  is around 10 nodes per unit area, the match between the simulated distributions and the normal distribution is almost perfect for both path-loss models and path-loss exponents. Even when  $\lambda$  is around 1, it is still very good. These observations in conjunction with Theorem 1 illustrate the utility of the Gaussian approximation of the WMAI distributions for small and large values of  $\lambda$ . They also indicate the potential to further tighten the upper and lower bounds that we derive in this paper.

When the effect of small and large values of  $\alpha$  on the WMAI distributions is analyzed, it is seen that the match between the simulated WMAI distributions and the normal distribution is slightly better for small values of  $\alpha$  (e.g., for  $\lambda = 0.1$  in Fig. 3, the maximum deviation between the simulated WMAI distribution and the normal distribution is 0.14 and 0.3 for  $\alpha = 3$  and 5, respectively.). This is an expected result when we compare the path-loss model dependent constants appearing in Theorem 1 (see Table I). When we compare the effect of different path-loss models on the Gaussian approximation, we observe that the match between the simulated WMAI distributions and the normal distribution is slightly better for  $G_2(t)$  (e.g., when  $\lambda = 0.1$  and  $\alpha = 5$ , the maximum deviation between the simulated WMAI distribution and the normal distribution is 0.3 and 0.21 for Figs. 3 and 4, respectively.). This is also expected when we compare the path-loss model dependent constants appearing in Theorem 1 (see Table I).

## V. CONCLUSIONS

In this paper, we have analyzed wireless multi-access interference (WMAI) distributions for general bounded power-law decaying path-loss functions. We have shown that the WMAI distribution converges to the Gaussian distribution at a rate  $\frac{c(x)}{\sqrt{\lambda}}$  if the transmitted signal power decays to zero according to a general bounded power-law decaying path-loss function  $G$ , where  $\lambda$  is the intensity of the homogenous planar Poisson point process generating node locations, and  $c(x) > 0$  is a function which depends on  $G$  and the point  $x \in \mathbb{R}$  at which we want to estimate the interference distribution.  $c(x)$  approaches zero as  $|x|$  increases, which makes our bounds tight for any finite value of  $\lambda$ . An explicit expression for  $c(x)$  appearing in

our approximation results has also been provided.

We have performed both a numerical study and extensive simulations to illustrate the theoretical results. We have observed a very good match between the simulated (centered and normalized) WMAI distributions and the normal distribution with zero mean and variance one even for moderately small values of  $\lambda$ . Since there are no closed form expressions available for the WMAI distributions under general bounded path-loss models at the present, these results are expected to help researchers in the field significantly by simplifying the derivation of closed form expressions for various performance bounds and metrics in important wireless communications and networking research problems involving the signal-to-interference-plus-noise-ratio calculations.

## REFERENCES

- [1] E. S. Sousa and J. A. Silvester, "Optimum transmission ranges in a direct sequence spread spectrum multihop packet radio network," *IEEE J. Selected Areas Commun.*, vol. 8, no. 5, pp. 762-771, June 1990.
- [2] J. A. Gubner, "Computation of shot-noise probability distributions and densities," *SIAM Journal of Scientific Computing*, vol. 17, no. 3, pp. 750761, May 1996.
- [3] J. Ilo and D. Hatzinakos, "Analytic alpha-stable noise modeling in a Poisson field of interferers or scatterers," *IEEE Transactions on Signal Processing*, vol. 46, no. 6, pp. 16011611, June 1998.
- [4] C. C. Chan and S. V. Hanly, "Calculating the outage probability in a CDMA network with spatial Poisson traffic," *IEEE Trans. Vehicular Technology*, vol. 50, no. 1, pp. 183-204, Jan. 2001.
- [5] S. P. Weber and J. G. Andrews, "Bounds on the SIR distribution for a class of channel models in ad hoc networks," *IEEE Globecom*, San Francisco, CA, December 2006.
- [6] S. P. Weber, X. Yang, J. G. Andrews and G. de Veciana, "Transmission capacity of wireless ad hoc networks with outage constraints," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4091-4102, December 2005.
- [7] S. Weber, J. G. Andrews, X. Yang and G. de Veciana, "Transmission capacity of wireless ad hoc networks with successive interference cancellation," *IEEE Transactions on Information Theory*, vol. 53, no. 8, pp. 2799-2814, August 2007.
- [8] H. Inaltekin, M. Chiang, H. V. Poor and S. B. Wicker, "On unbounded path-loss models: effects of singularity on wireless network performance," *IEEE J. Selected Areas Commun.*, vol. 27, no. 7, pp. 1078-1092, Sept. 2009.
- [9] R. K. Ganti and M. Haenggi, "Interference and outage in clustered wireless ad hoc networks," *IEEE Trans. on Inform. Theory*, vol. 55, pp. 4067-4086, Sept. 2009.
- [10] L. H. Y. Chen and Q. M. Shao, "Stein's method for normal approximation," *IMS Lecture Notes Series, National University of Singapore*, vol. 4, pp. 1-59, February 2005.
- [11] S. Musa and W. Wasylkiwskyj, "Co-channel interference of spread spectrum systems in a multiple user environment," *IEEE Trans. Commun.*, vol. 26, no. 10, pp. 1405-1413, October 1978.
- [12] E. S. Sousa, "Performance of a spread spectrum packet radio network in a Poisson field of interferers," *IEEE Trans. Inf. Theory*, vol. 38, no. 6, pp. 1743-1754, November 1992.
- [13] S. B. Lowen and M. C. Teich, "Power-law shot noise," *IEEE Trans. on Info. Theory*, vol. IT-36, no. 6, pp. 1302-1318, Nov. 1990.
- [14] L. Heinrich and V. Schmidt, "Normal convergence of multidimensional shot noise and rates of this convergence," *Advances in Applied Probability*, vol. 17, no. 4, pp. 709-730, Dec. 1985.
- [15] J. F. C. Kingman, *Poisson Processes*, Clarendon Press, Oxford, 1993.
- [16] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE J. Selected Areas Commun.*, vol. 27, no. 7, September 2009.
- [17] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics, New York, NY, third edition, 1995.
- [18] R. Durrett, *Probability: Theory and Examples*, Duxbury Press, Belmont, CA, second edition, 1996.