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Circular choosability ^{*}

Frédéric Havet[‡] Ross J. Kang[§] Tobias Müller[¶]
Jean-Sébastien Sereni^{||}

Abstract

We study circular choosability, a notion recently introduced by Mohar and by Zhu. First, we provide a negative answer to a question of Zhu about circular cliques. We next prove that $\text{cch}(G) = O(\text{ch}(G) + \ln |V(G)|)$ for every graph G . We investigate a generalisation of circular choosability, the circular f -choosability, where f is a function of the degrees. We also consider the circular choice number of planar graphs. Mohar asked for the value of $\tau := \sup\{\text{cch}(G) : G \text{ is planar}\}$, and we prove that $6 \leq \tau \leq 8$, thereby providing a negative answer to another question of Mohar. We also study the circular choice number of planar and outerplanar graphs with prescribed girth, and graphs with bounded density.

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1 Introduction

Let $G = (V, E)$ be a graph. If p and q are two integers, a (p, q) -colouring of G is a function c from V to $\{0, \dots, p-1\}$ such that $q \leq |c(u) - c(v)| \leq p - q$ for each edge $uv \in E$. The circular chromatic number of the graph G is

$$\chi_c(G) := \inf \{p/q : G \text{ admits a } (p, q)\text{-colouring}\} .$$

For every integer a , let $[a]_q$ be the interval $[a - q + 1, a + q - 1]$ where the computations are modulo p . Note that if $uv \in E$ then $c(u) = a$ implies that $c(v) \notin [a]_q$.

A *list assignment* L of G is a mapping that assigns to every vertex v a set of non-negative integers, called *colours*. An L -colouring of G is a mapping $c : V \rightarrow \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V$. The *list chromatic number* or *choice number* $\text{ch}(G)$ of the graph G is the smallest ℓ such that, for every list assignment L of G satisfying $|L(v)| \geq \ell$ for every $v \in V$, there is a proper L -colouring of G .

As we describe next, the concept of *circular choosability*, introduced by Mohar [13] and Zhu [26], combines the above concepts of circular colouring and list colouring, respectively, in a natural way.

A list assignment L is a t - (p, q) -list-assignment if $L(v) \subseteq \{0, \dots, p-1\}$ and $|L(v)| \geq tq$ for each vertex $v \in V$. The graph G is (p, q) - L -colourable if there exists a (p, q) - L -colouring c , i.e. c is both a (p, q) -colouring and an L -colouring. For any real number $t \geq 1$, the graph G is t - (p, q) -choosable if it is (p, q) - L -colourable for every t - (p, q) -list-assignment L . Last, G is *circularly t -choosable* if it is t - (p, q) -choosable for any p, q . The *circular list chromatic number* or *circular choice number* of G is

$$\text{cch}(G) := \inf \{t \geq 1 : G \text{ is circularly } t\text{-choosable}\} .$$

Zhu [26] proved that $\text{cch}(G) \geq \max\{\text{ch}(G) - 1, \chi_c(G)\}$ for every graph G .

Note that this definition of circular choosability differs slightly from the one introduced by Zhu [26], in which p is required to be at least $2q$. We have opted to drop this requirement, because it contradicts Lemma 7 of [26]: under the original definition [26], the circular choosability of the single edge $K_2 = (\{v_1, v_2\}, \{v_1v_2\})$ would be 1. Indeed, suppose that $p \geq 2q$ and $L(v_1), L(v_2) \subseteq \{0, \dots, p-1\}$ were given with $|L(v_1)|, |L(v_2)| \geq (1+\varepsilon)q$. Then there would exist $c_1 \in L(v_1)$ and $c_2 \in L(v_2)$ such that $q \leq |c_1 - c_2| \leq p - q$, implying that K_2 would have circular choosability one. On the other hand, Lemma 7 of [26] implies that $\text{cch}(K_2) \geq \chi_c(K_2) = 2$. If $\text{cch} > 2$, then the condition $p \geq 2q$ is redundant, so this subtle change in definition is only a minor correction.

We also note that there is an alternative, continuous definition for circular choosability, given by Mohar [13], and shown by Zhu [26] to be equivalent to the definition given here. In this paper, we prefer to work with *list* (p, q) -colouring as defined above.

The paper is organised as follows. In Section 2, we introduce a useful tool (Lemma 1) and provide a negative answer to a question of Zhu about circular cliques (Proposition 6). Next, in Section 3, we consider the question, given a graph G , of whether $\text{cch}(G)$ is bounded within a constant factor of $\text{ch}(G)$; we use a probabilistic thinning argument to bound the circular choice number of a graph by a constant multiple of $(\text{ch}(G) + \ln n)$, where n is the number of vertices (Theorem 9). We also adapt this thinning argument and obtain a better result in the specific case of complete multipartite graphs (Proposition 10). In Section 4, we investigate a generalisation of circular choosability, the circular f -choosability, where f is a function of the degrees. We attempt to establish an analogue for the result of, independently, Borodin [4] and Erdős, Rubin and Taylor [6] (Theorem 12). The results of Section 5 are motivated by a question of Mohar [13], who asked for the value of

$$\tau := \sup\{\text{cch}(G) : G \text{ is planar}\}.$$

We prove that $6 \leq \tau \leq 8$ (Theorem 20 and Proposition 22). Then, along the lines of the odd-girth conjecture, we give bounds (Proposition 24, Corollary 27 and Corollary 29) for the parameter

$$\tau(k) := \sup\{\text{cch}(G) : G \text{ is planar and has girth } \geq k\},$$

and, more generally, the circular choice number of graphs with bounded density and prescribed girth (Theorems 26 and 28). We conclude Section 5 by determining

$$\tau_o(k) := \sup\{\text{cch}(G) : G \text{ is outerplanar and has girth } \geq k\},$$

for every $k \geq 3$ (Theorem 40).

The four main sections are relatively self-contained and may, with a few exceptions, be read independently of each other.

2 Warming up

We first present two basic tools which are crucial to several of our results.

Given a graph G , integers $p \geq q$, $t \geq 1$ and a t - (p, q) -list assignment L of G , suppose that $(v_1, \dots, v_{|V(G)|})$ is an ordering of the vertices of G such

that v_1, \dots, v_m are (p, q) - L -precoloured. A colour $a \in L(v_j)$ is *extendable* if there exists some (p, q) - L -colouring c of the subgraph induced by $\{v_1, \dots, v_j\}$ that respects the precolouring, and such that $c(v_j) = a$. For every $j \in \{1, \dots, |V(G)|\}$, we set $I_j := \{i : i < j \text{ and } v_i v_j \in E(G)\}$ and we let X_j be the set of extendable colours of v_j . Note that if $1 \leq j \leq m$ then v_j has a unique extendable colour—the one precolouring it—so $|X_j| = 1$.

Lemma 1 *Let $(v_1, \dots, v_{|V(G)|})$ be an ordering of the vertices of G such that v_1, \dots, v_m are (p, q) - L -precoloured and each non-precoloured vertex has at most one neighbour later in the ordering. If $|X_i| \geq 1$ for each $i \in I_j$, then $|X_j| \geq |L(v_j)| - \sum_{i \in I_j: |X_i| < 2q} (2q - |X_i|)$.*

Proof. To determine which colours of v_j are extendable, it suffices to consider all possible colourings of its lower indexed neighbours with colours chosen from their respective lists of extendable colours. Since every non-precoloured vertex has at most one neighbour later in the ordering, a colour of v_j is not extendable for v_j if and only if it is in the intersection $\bigcap_{a \in X_i} [a]_q$ for some $i \in I_j$. These intersection sets are maximised when the X_i are intervals, giving $|\bigcap_{a \in X_i} [a]_q| = 2q - |X_i|$, if $|X_i| < 2q$, and $|\bigcap_{a \in X_i} [a]_q| = 0$, otherwise. \square

Notes. This lemma is closely related to results of Raspaud and Zhu [17]. Lemma 1 can be considered a more explicit version of their Lemma 2.5 in which the leaves of the tree are required to have colour lists of size one.

The second basic tool, due to Raspaud and Zhu, can be considered as a converse of Lemma 1. Whereas Lemma 1 is used to find valid list (p, q) -colourings, the following lemma is used to find list assignments which do not admit valid (p, q) -colourings.

Lemma 2 (Raspaud and Zhu [17, Lemma 2.4]) *Given a tree T , suppose $\ell : V(T) \rightarrow \mathbb{N}$ is a mapping such that $\sum_{x \in V(T)} \ell(x) < 2(|V(T)| - 1)q + 1$. Then there exists a list assignment L of T such that $L(V) \subseteq \{0, \dots, p - 1\}$ and $|L(x)| = \ell(x)$ for each $x \in V(T)$, and T is not L - (p, q) -colourable.*

Note. This lemma is independent of the choice of p . If we choose p large enough, it is guaranteed that there is a “large gap in the circle”, i.e. for each k and large enough p , there is an interval $I \subseteq \{0, \dots, p - 1\}$ of length k and a list assignment L as in Lemma 2 such that $L(v) \cap I = \emptyset$ for every $v \in V(T)$.

As a first application of one of these basic tools, we show the following lemma which is of use to us later on. Given a graph G , a *thread* of G is a

path whose internal vertices have degree 2, with possibly the two endvertices being the same vertex. The *length* of a thread is the number of edges along the path.

Lemma 3 *Fix a positive integer n . Let L be a $(2 + \frac{2}{n})$ - (p, q) -list-assignment of the graph G . Suppose that $v_0v_1v_2 \cdots v_nv_{n+1}$ is a thread of G of length $n + 1$. Then any (p, q) - L -precolouring of $G \setminus \{v_1, \dots, v_n\}$ can be extended to the entire graph.*

Proof. Let $t = 2 + \frac{2}{n}$. Since v_0 is coloured, v_1 has, by Lemma 1, at least $tq - 2q + 1 = \frac{2q}{n} + 1$ extendable colours. Inductively, we can thus show that v_i has at least $i\frac{2q}{n} + 1$ extendable colours, for $i < n$. Now, v_{n+1} is coloured and v_{n-1} has at least $(n - 1)\frac{2q}{n} + 1$ extendable colours; thus, by Lemma 1, v_n has at least $tq - (2q - 1) - \left(2q - (n - 1)\frac{2q}{n} - 1\right) = 2$ extendable colours, so that the required colouring indeed exists. \square

The following is a direct consequence of Lemma 3 and the fact that the circular chromatic number of C_{2k+1} is $2 + \frac{1}{k}$.

Corollary 4 (Zhu [26]) *For every odd integer $n \geq 3$, the cycle C_n has circular choice number $2 + \frac{2}{n-1}$.*

The situation for even cycles seems more complex. Clearly, $\text{cch}(C_{2n}) \geq \text{cch}(K_2) = 2$ for any $n \geq 2$. In a previous version of this work [9], it was conjectured that 2 is the circular choice number for all even cycles and we showed that $\text{cch}(C_4) = 2$. Shortly afterwards, the conjecture was verified by Norine [15] using the Combinatorial Nullstellensatz [1].

Theorem 5 (Norine [15]) $\text{cch}(C_{2n}) = 2$ for any $n \geq 2$.

2.1 Circular cliques

For any two positive integers $a \geq 2b$, the graph $K_{a/b}$, called the *circular clique*, has vertex set $\{0, \dots, a - 1\}$ and ij is an edge if and only if $b \leq |i - j| \leq a - b$. Observe that, for any $k \geq 1$, $K_{(2k+1)/k} \simeq C_{2k+1}$. Zhu [26] proved that $\text{cch}(C_{2k+1}) = 2 + \frac{1}{k}$ for every $k \geq 1$ and asked whether the circular list chromatic number of $K_{a/b}$ is $\frac{a}{b}$. This is not the case, since the circular cliques contain large complete bipartite subgraphs as we see next.

Proposition 6 *For every positive integer N , there exist two positive integers a and b such that $a \geq 2b$ and the difference between $\text{cch}(K_{a/b})$ and $\frac{a}{b}$ is more than N .*

Proof. Let $m \geq \binom{2(N+3)-1}{N+3}$. As is well-known [6], $\text{ch}(K_{m,m}) \geq N + 4$. Set $a := 2b + 2m$ for some integer $b > 2m$. The graph $K_{a/b}$ contains $K_{m,m}$ as a subgraph: take the vertices $\{0, 1, \dots, m-1\} \cup \{b+m-1, b+m, \dots, b+2m\}$. Hence, $\text{cch}(K_{a/b}) \geq \text{ch}(K_{a/b}) - 1 \geq \text{ch}(K_{m,m}) - 1 \geq N + 3$. However, $\frac{a}{b} = \frac{2b+2m}{b} < 3$ since $b > 2m$. Therefore, $\text{cch}(K_{a/b}) - \frac{a}{b} > N$. \square

3 Bounds in terms of choosability (and number of vertices)

The *degeneracy* $\delta^*(G)$ of a graph G is the maximum over all subgraphs H of G of the minimum degree of H . A graph G is *k-degenerate* if $\delta^*(G) \leq k$. It is well-known that $\text{ch}(G) \leq \delta^*(G) + 1$. The following is an analogue of this bound for circular choosability.

Lemma 7 (Zhu [26]) $\text{cch}(G) \leq 2\delta^*(G)$.

This result was also proved by Zhu [26] to be asymptotically tight by (essentially) showing that the complete bipartite graph K_{k,m^k} , which has degeneracy k , satisfies $\text{cch}(K_{k,m^k}) \geq (2 - \frac{2k}{m})k$. This led him to pose the following problem.

Problem 8 *Is there a constant α such that, for every graph G , $\text{cch}(G) \leq \alpha \text{ch}(G)$?*

Note that if such an α exists then it is at least 2, as $\text{ch}(K_{k,m^k}) \leq \delta^*(K_{k,m^k}) + 1 = k + 1$. In a previous version of this work [9], we showed that the answer to Problem 8 is yes if we restrict ourselves to 2-choosable graphs. In this case, we showed that the corresponding constant α is at most $5/2$. Since then, this bound has been improved by Norine, Wong and Zhu [16] to $16/7$.

In this section, we establish that $\text{cch}(G) = O(\text{ch}(G) + \ln(|V(G)|))$ for every graph G . We then give a better upper bound in the case of complete multipartite graphs.

3.1 General upper bound

Problem 8 asks whether $\text{cch}(G) = O(\text{ch}(G))$. We are not able to settle the question here, but Theorem 9 below does show that $\text{cch}(G) = O(\text{ch}(G) + \ln(|V(G)|))$. We make no attempt to optimise constants.

Theorem 9 For every graph G with n vertices,

$$\text{cch}(G) \leq 36 \cdot (\text{ch}(G) + \ln n) + 3.$$

Proof. Fix p, q and set $t := 36 \cdot (\text{ch}(G) + \ln n) + 3$. Suppose that lists $L(v) \subseteq \mathbb{Z}_p$ of size at least $\lfloor tq \rfloor + 1$ are given. If $q = 1$, then we can certainly (p, q) - L -colour G , as $t > \text{ch}(G)$. So we may assume that $q \geq 2$.

Let us partition $\left\{0, \dots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$ into groups $g_i := \{3i, 3i+1, 3i+2\}$ of three consecutive numbers, where the last group may contain less than three numbers. Out of each group of three, except the very last one, we pick one element at random, but in such a way that we never pick two consecutive numbers. To be more precise, for $i = 0$ we simply pick one of $0, 1, 2$ uniformly at random. Once a choice has been made for g_{i-1} , we pick one of $3i, 3i+1, 3i+2$ uniformly at random provided we did not choose $3(i-1)+2$ from g_{i-1} . Otherwise, we choose one of $3i+1, 3i+2$ at random each with probability $\frac{1}{2}$. The set of selected indices is $\mathcal{K} := \{k : k \text{ was chosen}\}$. With each index $k \in \left\{0, \dots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$, we associate an interval $I_k = \{k(q-1), \dots, (k+1)(q-1) - 1\}$ of \mathbb{Z}_p . Notice that the I_k are disjoint intervals of length $q-1$. A crucial observation for the sequel is that if k and l are two distinct elements of \mathcal{K} , then $|a-b|_p \geq q$ for every $a \in I_k$ and every $b \in I_l$. See Figure 1.

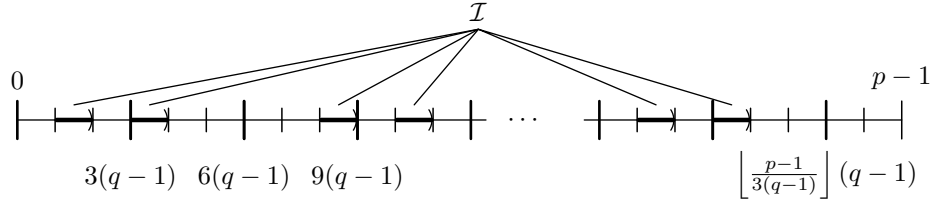


Figure 1: An illustration of the “thinning” procedure for Theorem 9.

Let us set $\mathcal{I} := \bigcup_{k \in \mathcal{K}} I_k$. For each $v \in V$, we let $S(v) := \{k \in \mathcal{K} : I_k \cap L(v) \neq \emptyset\}$. The idea for the rest of the proof is to show that t was chosen in such a way that $\mathbb{P}(|S(v)| < \text{ch}(G)) < \frac{1}{n}$ for all v . Then it follows that

$$\mathbb{P}(|S(v)| < \text{ch}(G) \text{ for some } v \in V) < n \cdot \frac{1}{n} = 1.$$

In other words, there must exist a choice of non-adjacent intervals, one from each group of three, for which $|S(v)| \geq \text{ch}(G)$ for all $v \in V$. By the definition of $\text{ch}(G)$, there exists a colouring c of G with $c(v) \in S(v)$. Let us define a new colouring f by choosing $f(v) \in I_k \cap L(v)$ if $c(v) = k$. This can be done

for each v , by the definition of $S(v)$. Now f is a (p, q) - L -colouring, because if $vw \in E(G)$ then $c(v)$ and $c(w)$ are distinct elements of \mathcal{K} . Consequently, $f(v)$ and $f(w)$ have been chosen from non-adjacent intervals $I_{c(v)}$ and $I_{c(w)}$, and hence $|f(v) - f(w)|_p \geq q$.

It remains to show that t is chosen such that $\mathbb{P}(|S(v)| < \text{ch}(G)) < \frac{1}{n}$ holds. We first assert that

$$\mathbb{P}(|S(v)| < \text{ch}(G)) \leq \mathbb{P}\left(\text{Bi}\left(s, \frac{1}{6}\right) \leq \text{ch}(G)\right), \quad (1)$$

where $s := \lceil \frac{t}{3} \rceil - 1$ and $\text{Bi}(n, p)$ denotes a binomial random variable with parameters n and p . In order to prove the assertion, first note that we can “thin” the lists $L(v)$ to get sublists $L'(v) \subseteq L(v)$ with

$$|L'(v)| \geq \left\lceil \frac{|L(v)|}{3(q-1)} \right\rceil - 1 > \frac{t}{3} - 1,$$

and a distance of at least $3(q-1)$ between elements of $L'(v)$. Indeed, we can construct $L'(v)$ by taking the first, the $(3(q-1)+1)^{\text{th}}$, the $(6(q-1)+1)^{\text{th}}$, and so on up to and including the $((M-1)(q-1)+1)^{\text{th}}$ element of $L(v)$, where $M := \lceil \frac{tq}{3(q-1)} \rceil$, and we discard the $(M(q-1)+1)^{\text{th}}$ element, to avoid possible wrap-around effects. Let $L'(v) := \{a_1, \dots, a_l\}$ with $a_i \leq a_{i+1}$. For $J \subseteq \{1, \dots, i-1\}$, let $A(i, J)$ be the event that $a_j \in \mathcal{I}$ for $j \in J$ and $a_j \notin \mathcal{I}$ for all $j \in \{1, \dots, i-1\} \setminus J$. We assert that for every $J \subseteq \{1, \dots, i-1\}$

$$\mathbb{P}(a_i \in \mathcal{I} | A(i, J)) \geq \frac{1}{6}. \quad (2)$$

To see this, note that a_1, \dots, a_{i-1} each give information about the choice made for some group g_l . Also observe that if $a_i \in I_{3k+1}$ or $a_i \in I_{3k+2}$ for some k , then the probability that a_i is covered by \mathcal{I} given that $A(i, J)$ holds is at least $\frac{1}{3}$, because regardless of which element of g_{k-1} was selected, the probability that $3k+1$ (respectively $3k+2$) is selected is at least $\frac{1}{3}$. Now, supposing that $a_i \in I_{3k}$ for some k , it follows that $a_{i-1} \notin I_{3(k-1)+1} \cup I_{3(k-1)+2}$. Therefore, the probability that a_i is covered given that $A(i, J)$ holds is at least the minimum of two probabilities: the probability that $3k$ is chosen given that $3(k-1)$ was chosen from g_{k-1} ; and the probability that $3k$ is chosen given that $3(k-1)$ was not chosen from g_{k-1} . This minimum is $\frac{1}{6}$, which proves (2) and hence also (1).

We use the following incarnation of the Chernoff Bound (see, e.g. [11, Theorem 2.1, p. 26]).

$$\forall r \geq 0, \quad \mathbb{P}(\text{Bi}(k, p) \leq kp - r) \leq \exp\left[-2\frac{r^2}{kp}\right].$$

Setting $r := \frac{s}{6} - \text{ch}(G) \geq 0$, it follows that

$$\mathbb{P}(|S(v)| < \text{ch}(G)) \leq \mathbb{P}\left(\text{Bi}\left(s, \frac{1}{6}\right) \leq \frac{s}{6} - r\right) \leq \exp\left[-3\frac{r^2}{s}\right].$$

This yields the conclusion provided that

$$3\left(\text{ch}(G) - \frac{s}{6}\right)^2 > s \ln n.$$

This certainly holds if $s \geq 12 \cdot (\text{ch}(G) + \ln n)$, which is the case as $t = 36 \cdot (\text{ch}(G) + \ln n) + 3$. \square

Alon [2] has shown that $\text{ch}(G) = \Omega(\ln d)$, where d is the average degree of G . Theorem 9 thus shows that the existence of such a constant α for Problem 8 can only be disproved by considering sparse graphs: for each $\varepsilon > 0$ there is a choice of $\alpha = \alpha(\varepsilon)$ that works for all graphs with average degree at least n^ε .

We also note that it is straightforward to give an upper bound for cch in terms of ch that is exponential in ch : the mentioned result of Alon also shows that $\text{ch}(G) = \Omega(\ln(\delta^*(G)))$. Indeed, as there is some subgraph of G with minimum degree $\delta^*(G)$, this subgraph certainly has average degree at least $\delta^*(G)$. On the other hand, $\text{cch}(G) \leq 2\delta^*(G)$ so that

$$\text{cch}(G) \leq e^{\beta \text{ch}(G)},$$

for some $\beta > 0$.

3.2 Complete multipartite graphs

Complete bipartite graphs are a natural class to consider, being the canonical examples of graphs with low chromatic number, yet high choosability. The following adaptation of our argument sharpens the general upper bound given by Theorem 9 in the special case where G is $K_{r^*m} = \underbrace{K_{m, \dots, m}}_r$, the

balanced complete r -partite graph with each part of size m . Recently, Gazit and Krivelevich [8] showed that $\text{ch}(K_{r^*m}) = (1 + o(1))\frac{\ln m}{\ln(1 + \frac{1}{r-1})}$, so that the bound on $\text{cch}(K_{r^*m})$ given by Proposition 10 below is indeed sharper than the general upper bound of Theorem 9, which gives a bound of $36 \cdot (\text{ch}(K_{r^*m}) + \ln(rm)) + 3$. Gazit and Krivelevich also considered complete multipartite graphs with not all parts of equal size, but with not too much difference in size between the smallest and the largest parts. Our result

and proof can be adapted to also cover this additional class of complete multipartite graphs, but we have chosen to omit this here. We note that Alon and Zaks [3] prove a very similar result to Proposition 10 below for T -choosability of complete bipartite graphs (and in fact give a similar proof).

Proposition 10 *For every positive integers k and m ,*

$$\text{cch}(K_{r**m}) \leq \frac{3(\ln m + \ln r)}{\ln\left(1 + \frac{1}{6r-1}\right)} + 1.$$

Proof. We proceed as in the proof of Theorem 9 and select a subset $\mathcal{K} \subseteq \left\{0, \dots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$ of indices, no two consecutive, at random. We now partition the indices that have been selected into r sets $\mathcal{K}_1, \dots, \mathcal{K}_r$, by assigning each index $k \in \mathcal{K}$ uniformly at random to one of $\mathcal{K}_1, \dots, \mathcal{K}_r$, independently from all other elements in \mathcal{K} . Let $\mathcal{I}_1 := \bigcup_{k \in \mathcal{K}_1} I_k, \dots, \mathcal{I}_r = \bigcup_{k \in \mathcal{K}_r} I_k$.

Let $V_1 \uplus \dots \uplus V_r$ be the partition of the vertex-set. We attempt to colour K_{r**m} using only the colours from \mathcal{I}_i for the vertices of V_i , for each index i . This can be done provided that, for each $i \in \{1, 2, \dots, r\}$ and each $v \in V_i$,

$$L'(v) := L(v) \cap \mathcal{I}_i \neq \emptyset. \quad (3)$$

To see this, notice that if (3) holds then we can define a colouring by picking an arbitrary $f(v) \in L'(v)$ for each $v \in V$. We obtain a proper (p, q) - L -colouring. Indeed, if $vw \in E(K_{r**m})$ then $f(v)$ and $f(w)$ are chosen from distinct—and thus also non-adjacent—intervals, as $(v, w) \in V_i \times V_j$ for some $i \neq j$ and $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset$. Therefore, $|f(v) - f(w)|_p \geq q$.

So it suffices to show that $\mathbb{P}(L'(v) = \emptyset) < \frac{1}{rm}$. We take $a_1 \leq \dots \leq a_l$ in $L(v)$ with a distance of at least $3(q-1)$ between them and $l > \frac{t}{3} - 1$. The probability that $a_j \in \mathcal{I}_i$ given that $a_1, \dots, a_{j-1} \notin \mathcal{I}_i$ is at least $\frac{1}{6r}$ (by an argument analogous to the argument used in the proof of Theorem 9), so

$$\mathbb{P}(L'(v) = \emptyset) < \left(1 - \frac{1}{6r}\right)^{\frac{t}{3}-1}.$$

Now, if $t \geq \frac{3(\ln m + \ln r)}{\ln\left(1 + \frac{1}{6r-1}\right)} + 1$ then $\mathbb{P}(L'(v) = \emptyset) < \frac{1}{rm}$, which concludes the proof. \square

4 Circular f -choosability

A graph G is *degree-choosable* if, for every list-assignment L such that every vertex has a list of size at least its degree, there exists a proper L -colouring of G . A theorem independently proved by Borodin [4] and Erdős, Rubin and Taylor [6]—see also Jensen and Toft’s book [21]—states that a connected graph is degree-choosable, unless it is a *Gallai tree*, i.e. unless each of its blocks is complete or an odd cycle. In this section, we study the analogous problem for circular choosability. The main result is interesting in itself but can also be used as a tool to extend list circular colouring from a graph to larger graphs.

Let $G = (V, E)$ be a graph and $f : V \rightarrow \mathbb{R}^+$. We say that an f - (p, q) -*list-assignment* L is a list assignment such that $L(v) \subseteq \{0, \dots, p-1\}$ and $|L(v)| \geq \max\{1, f(v)q\}$ for every $v \in V$. The graph G is f - (p, q) -*choosable* if it is (p, q) - L -colourable for every f - (p, q) -list-assignment L . Last, G is *circularly f -choosable* if it is f - (p, q) -choosable for any p, q . The condition $|L(v)| \geq \max\{1, f(v)q\}$ may seem unnatural but is just here to allow some vertices v to have $f(v) = 0$.

Every graph is circularly $2d$ -choosable. This is best possible as $\text{cch}(K_2) = 2$; however, K_2 is essentially the unique graph attaining the bound. We next consider $(2d-1)$ -choosability.

Proposition 11 *A connected graph with more than two vertices is circularly $(2d-1)$ -choosable.*

Proof. Let G be a connected graph with more than two vertices and v a vertex with degree at least 2. Consider an ordering $(v_1, v_2, \dots, v_n = v)$ of G such that v_i has at least one neighbour in $\{v_{i+1}, \dots, v_n\}$ for every $i < n$. Note that $v_{n-1}v$ is an edge. Let us greedily colour the vertices v_i for $i \in \{1, 2, \dots, n-2\}$; this is possible since, at each step, v_i has at least $(2d(v_i) - 1)q - (d(v_i) - 1)(2q - 1) = q + d(v_i) - 1$ colours available. We now extend this colouring also to v_{n-1}, v_n by treating this greedy colouring as a precolouring and applying Lemma 1. Observe that the number of extendable colours for v_{n-1} is at least

$$(2d(v_{n-1}) - 1)q - (2q - 1)(d(v_{n-1}) - 1) = q + d(v_{n-1}) - 1.$$

Therefore, again by Lemma 1, the number of extendable colours for v_n is at least

$$\begin{aligned} & (2d(v_n) - 1)q - (2q - 1)(d(v_n) - 1) - (2q - q - d(v_{n-1}) + 1) \\ & = d(v_n) + d(v_{n-1}) - 2 \geq 1. \end{aligned}$$

Hence one can extend the colouring to both v_{n-1} and v . \square

Proposition 11 is best possible since the circular choice number of K_3 is 3.

4.1 Circular $(2d - 2)$ -choosability

The main result of this section is as follows.

Theorem 12 *Let G be a connected graph on more than two vertices.*

- (i) *G is circularly $(2d - 2)$ -choosable if*
 - (a) *it is 2-connected and not an odd cycle; or*
 - (b) *it is not 2-connected and one of its blocks is neither an edge nor an odd cycle of length at least 5.*
- (ii) *G is not circularly $(2d - 2)$ -choosable if*
 - (a) *it is a tree; or*
 - (b) *it is an odd cycle.*

Given a tree, $\sum_{v \in T} (2d(v) - 2) = 4|E(T)| - 2|V(T)| = 2|V(T)| - 4$. Thus the fact that a tree is not circularly $(2d - 2)$ -choosable is implied by Lemma 2.

We now give a lemma which allows us to extend the property of a connected graph H being $(2d - 2)$ -choosable to graphs that contain H as an induced subgraph.

Lemma 13 *Let G be a connected graph. If G has an induced subgraph H that is circularly $(2d_H - 2)$ -choosable and contains at least one edge, then G is also circularly $(2d_G - 2)$ -choosable.*

Proof. It suffices to prove the result for a connected induced subgraph H . We proceed by induction on $|V(G)| - |V(H)|$, the result holding trivially if $|V(G)| - |V(H)| = 0$. Assume now that $|V(G)| > |V(H)|$. Let x be a vertex which is at maximal distance from H in G . Then, $G - x$ is a connected subgraph of G containing H as an induced subgraph.

Let L be a $(2d_G - 2)$ - (p, q) -list assignment of G , and let $\alpha \in L(x)$. Removing the elements of $[\alpha]_q$ from the list of each neighbour of x in G , we obtain a $(2d_{G-x} - 2)$ - (p, q) -list assignment of $G - x$. By the induction hypothesis, $G - x$ admits a circular (p, q) - L -colouring c , which can be extended to G by setting $c(x) := \alpha$. \square

We now show circular $(2d - 2)$ -choosability for some small important graphs. The graph composed of two vertices connected with three independent paths of length i , j and k is $\theta_{i,j,k}$. As we observe later, these graphs are important for 2-connected graphs. We also consider the *flag*, the graph with vertex set $\{t, u, v, w\}$ and edge set $\{tu, tv, uv, vw\}$.

Proposition 14 *The following hold.*

- (i) *For every i, j and k , the graph $\theta_{i,j,k}$ is circularly $(2d - 2)$ -choosable.*
- (ii) *The flag is circularly $(2d - 2)$ -choosable.*

Proof of part (i) of Proposition 14. Let $xu_1 \dots u_{i-1}y$, $xv_1 \dots v_{j-1}y$ and $xw_1 \dots w_{k-1}y$ be the three paths forming $\theta_{i,j,k}$. Since there are no multiple edges, we may assume that $i, j \geq 2$.

Let L be a $(2d - 2)$ - (p, q) -list assignment of $\theta_{i,j,k}$. Every vertex has a list of size $2q$ except x and y which have lists of size $4q$.

Suppose that $L(u_1) \cap L(v_1) \neq \emptyset$, and colour u_1 and v_1 with the same colour. Then, greedily extend this colouring according to the ordering $(u_1, v_1, u_2, \dots, u_{i-1}, v_2, \dots, v_{j-1}, y, w_{k-1}, \dots, w_1, x)$.

Assume now that $L(u_1) \cap L(v_1) = \emptyset$. Assign to x a colour $\alpha \in L(x)$. Let a and b be the number of extendable colours of u_1 and v_1 , respectively. Observe that $a + b \geq 4q - (2q - 1) = 2q + 1$. By Lemma 1, we can extend the colouring to $\theta_{i,j,k} - y$ such that u_{i-1}, v_{j-1} and w_{k-1} have at least a, b and 1 extendable colours, respectively. Then, y has at least $4q - 6q + a + b + 1 \geq 2$ extendable colours. \square

Before the proof of the next part, we note the following consequence of Lemma 1, whose straightforward proof is left to the reader.

Lemma 15 *If L is a (p, q) -list assignment of K_2 with $p \geq 2q$ such that $|L(x)| \geq q$ and $|L(y)| \geq q + 1$, then K_2 is (p, q) - L -colourable.*

Proof of part (ii) of Proposition 14. Suppose on the contrary that there exists a $(2d - 2)$ - (p, q) -list assignment L such that the flag is not (p, q) - L -colourable. We may assume that $L(w) = \{0\}$, $|L(v)| = 4q$ and $|L(u)| = |L(t)| = 2q$.

Assertion 1 $0 \notin L(t) \cup L(u)$.

Suppose otherwise. By symmetry, we may assume that $0 \in L(t)$. Then, there exists $k_u \in L(u) \setminus [0]_q$ and $k_v \in L(v) \setminus ([0]_q \cup [k_u]_q)$. Hence, $c(t) = 0, c(u) = k_u, c(v) = k_v, c(w) = 0$ is a (p, q) - L -colouring of the flag, a contradiction. This proves Assertion 1.

Assertion 2 $(L(t) \cup L(u)) \cap [0]_q \neq \emptyset$.

Suppose on the contrary that $(L(t) \cup L(u)) \cap [0]_q = \emptyset$. Let k_t, k_u , and k_v be the smallest integers in $L(t), L(u)$, and $L(v) \setminus [0]_q$, respectively. By symmetry, we may assume that $k_t \leq k_u$. If $k_t \leq k_v$, then $|L(v) \setminus ([0]_q \cup [k_t]_q)| \geq q+1$ and $|L(u) \setminus [k_t]_q| \geq q$. So, by Lemma 15, the colouring $c(t) = k_t, c(w) = 0$ can be extended to a (p, q) - L -colouring of the flag, a contradiction. If $k_v < k_t$, then $|L(t) \setminus [k_v]_q| \geq q+1$ and $|L(u) \setminus [k_v]_q| \geq q+1$. So, by Lemma 15, the colouring $c(v) = k_v, c(w) = 0$ may be extended to a (p, q) - L -colouring of the flag, a contradiction. This proves Assertion 2.

By symmetry, we assume that $(L(t) \cup L(u)) \cap [0, q-1] \neq \emptyset$. Let k_t be the minimum integer in this set. Without loss of generality, we assume that $k_t \in L(t)$.

Assertion 3 $|L(u) \cap [k_t]_q| \geq q+1$.

Since $|L(v) \setminus ([0]_q \cup [k_t]_q)| \geq q+2$, it follows that $|L(u) \setminus [k_t]_q| < q$; otherwise, by Lemma 15, the colouring $c(t) = k_t, c(w) = 0$ can be extended to a (p, q) - L -colouring of the flag, a contradiction. Thus, $|L(u) \cap [k_t]_q| \geq q+1$. This proves Assertion 3.

Hence, there exist l_u and $k_u = l_u + q$ both in $L(u) \cap [k_t]_q$. It follows from the minimality of k_t that $l_u \in [k_t - q + 1, 0] \cap L(u)$.

Analogously to Assertion 3, we deduce that $|L(t) \cap [l_u]_q| \geq q+1$. Hence, there exists $l_t \in L(t) \cap [l_u - q + 1, l_u]$. Set $J := [l_t]_q \cup [k_u]_q \cup [0]_q \subseteq [l_t - q + 1, k_u + q - 1]$. Since $l_t \geq l_u - q + 1$ and $k_u - l_u = q$, we deduce that $|J| < 4q$. It follows that there exists $l_v \in L(v) \setminus J$. Then $c(t) = l_t, c(u) = k_u, c(v) = l_v, c(w) = 0$ is a (p, q) - L -colouring of the flag, a contradiction. \square

Corollary 16 *A 2-connected graph is circularly $(2d-2)$ -choosable unless it is an odd cycle.*

Proof. Let G be a 2-connected graph. If G is an even cycle, then the result holds by Theorem 5. If G contains a complete graph K_n for some $n \geq 4$, then, by Lemmas 7 and 13, G is circularly $(2d-2)$ -choosable since $\text{cch}(K_n) = n \leq 2n-4$. So we assume that G is neither a clique nor a cycle. By Lemma 13 and Proposition 14(i), it suffices to show that G contains an induced $\theta_{i,j,k}$ for some i, j, k .

Let $C = v_1 v_2 \dots v_n v_1$ be a shortest cycle of G . As G is not a cycle, there exists a vertex v_s of C with a neighbour v outside C . If v has at most two neighbours in C , then let P be a shortest path from v to C in $G - v_s$ (such

a path exists since G is 2-connected). The subgraph induced by the vertices of P and v_s is $\theta_{i,j,k}$ for some i, j, k . If v has at least three neighbours in C , then let v_i, v_j and v_k be three neighbours of v in C with $i < j < k$. Note that C is not K_3 because G does not contain K_4 . Therefore, the subgraph of G induced by v_i, v_{i+1}, \dots, v_k and v is $\theta_{i,j,k}$ for some i, j, k , which concludes the proof. \square

Corollary 17 *A graph containing the flag as a subgraph is circularly $(2d - 2)$ -choosable.*

Proof. If a graph contains the flag, then it contains as an induced subgraph either the flag, K_4 or L_4 , the graph obtained from K_4 by deleting an edge, as an induced subgraph. By Lemma 13, it suffices to prove that each of these three graphs are circularly $(2d - 2)$ -choosable. Since the circular choice number of K_4 is 4, it is circularly $(2d - 2)$ -choosable. Note that $L_4 = \theta_{1,2,2}$. Thus, both L_4 and the flag are circularly $(2d - 2)$ -choosable by Proposition 14. \square

We are now ready to prove the main theorem.

Proof of Theorem 12. Part (ii) of the theorem follows from Lemma 2 and the fact that $\text{cch}(C_{2k+1}) = 2 + \frac{1}{k} > 2$. Part (i)(a) is Corollary 16. For part (i)(b), since a block is also an induced subgraph of G we can apply Lemma 13 and one of Theorem 5 or Corollary 17. \square

For a few graphs, we could not determine whether they are circularly $(2d - 2)$ -choosable. We offer the following conjecture.

Conjecture 18 *A connected graph is circularly $(2d - 2)$ -choosable unless it is a tree or an odd cycle.*

This conjecture would be true if every graph consisting of an odd cycle of length at least 5 and an edge with one endvertex in the cycle is $(2d - 2)$ -choosable.

Despite not completely characterising the circularly $(2d - 2)$ -choosable graphs, we can characterise those with a universal vertex. A vertex v of a graph G is said to be *universal* if it is adjacent to every other vertex of G .

Corollary 19 *A graph with a universal vertex is $(2d - 2)$ -choosable unless it is K_3 or a star.*

5 Planar graphs and graphs of bounded density

5.1 Planar graphs

Mohar [13] asked for the value of $\tau := \sup\{\text{cch}(G) : G \text{ is planar}\}$. We first show that every planar graph is circularly 8-choosable, and so $\tau \leq 8$. Afterwards, we exhibit for each $n \geq 2$ a planar graph whose circular choice number is at least $6 - \frac{1}{n}$, implying $\tau \geq 6$.

The proof of the following theorem is inspired by the celebrated proof for 5-choosability of planar graphs by Thomassen [20]. His result can be considered a subcase (in particular, the case $q = 1$) of Proposition 21.

Theorem 20 *Every planar graph is circularly 8-choosable.*

We actually establish the following stronger result.

Proposition 21 *Let G be a near triangulation, i.e. a simple plane graph consisting of a cycle C and vertices and edges inside C such that each bounded face is bounded by a triangle. Fix two integers $p \geq q$, and let L be a (p, q) -list-assignment such that $\forall v \in V, L(v) \subseteq \{0, \dots, p-1\}$ with $|L(v)| \geq 4q - 1$ if $v \in C$ and $|L(v)| \geq 8q - 3$ otherwise. Then any (p, q) - L -precolouring of two adjacent vertices of C can be extended to a (p, q) - L -colouring of G .*

Proof. The proof is by induction on the number of vertices n . The result holds if G is a triangle since there are at least $4q - 1 - 2(2q - 1) = 1$ choices to colour the last vertex. Assume now that the result is true for every near triangulation with at most $n - 1 \geq 3$ vertices, and let G be a near triangulation with n vertices. We let $u_1 u_2 \dots u_k$ be the outer cycle of G , and u_1 and u_2 be the two precoloured vertices.

Case 1: G has a chord $u_i u_j$ with $i < j$. We use the induction hypothesis on the near triangulation whose outer cycle is $u_1 u_2 \dots u_i u_j u_{j+1}, \dots, u_k, u_1$. Next we use the induction hypothesis on the near triangulation whose outer cycle is $u_i u_{i+1} \dots u_j u_i$, the two precoloured vertices being u_i and u_j .

Case 2: G has no chord. Let v_1, \dots, v_d be the neighbours of u_k that do not belong to C . Without loss of generality, we can assume that $u_{k-1} v_1 v_2 \dots v_d u_1$ is a path. Let a and b be two colours of $L(u_k) \setminus [c(u_1)]_q$ such that $[a]_q \cap [b]_q = \emptyset$. Such colours exist since $|L(u_k)| \geq 4q - 1 \geq 2q + 2q - 1$. We consider the graph G' obtained from G by removing the vertex u_k . It is a near triangulation with outer cycle $u_1 u_2 \dots u_{k-1} v_1 v_2 \dots v_d u_1$. We define the list-assignment L' of G' by $L'(v) := L(v)$ if $v \notin \{v_1, v_2, \dots, v_d\}$ and $L'(v) := L(v) \setminus ([a]_q \cup [b]_q)$

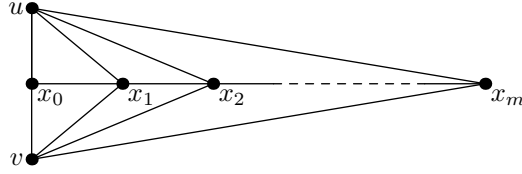


Figure 2: The graph H_m .

otherwise. Note that $|L'(v_i)| \geq 8q - 3 - 2(2q - 1) = 4q - 1$ for each i . Thus we can apply the induction hypothesis to G' and L' . Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \notin [a]_q$ and with b otherwise. \square

Proposition 22 *For each $n \geq 2$, there exists a planar graph G_n with $\text{cch}(G_n) \geq 6 - \frac{1}{n}$.*

Proof. Let $t := 6 - \frac{1}{n}$ for some $n \geq 2$. Set $q := 3n$ and let p be an integer much larger than $tq = 18n - 3$. All the computations and intervals are modulo p . We consider the planar graph H_m of Figure 2, with $m = 2q - 1$. We construct the graph G_n by taking $(tq)^2$ copies of H_m indexed by $L(u) \times L(v)$, identifying the vertices u of each copy, and identifying the vertices v of each copy. Observe that G_n is planar.

We first set $L(u) := [r, r + tq - 1]$ and $L(v) := [s, s + tq - 1]$ with r and s such that $[r - q + 1, r + tq + q - 1] \cap [s - q + 1, s + tq + q - 1] = \emptyset$. Now, for each $(a, b) \in L(u) \times L(v)$, we assign lists to the vertices of the copy $H_{a,b}$ of H_m in such a way that if u is coloured a and v is coloured b , then the subgraph $H_{a,b}$ cannot be (p, q) - L -coloured. Each $x_i \in H_{a,b}$ is adjacent to both u and v , so we may include the intervals $[a]_q$ and $[b]_q$ to $L(x_i)$, so that it remains to assign $tq - 2(2q - 1) = 6n - 1 = m$ colours using Lemma 2. Note that

$$\sum_{x \in H_{a,b} \setminus \{u,v\}} m = m^2 < m(m+1) + 1 = 2(|V(H_{a,b} \setminus \{u,v\})| - 1)q + 1.$$

Thus, there is a list assignment L' of $H_{a,b} \setminus \{u,v\}$ such that $L(x_i) = m$ for each i and $H_{a,b} \setminus \{u,v\}$ is not L' - (p, q) -colourable. By our remark after Lemma 2, we may select L' so that its lists are far away from both of the intervals $[a]_q$ and $[b]_q$. Let $L(x_i) = [a]_q \cup [b]_q \cup L'(x_i)$ and note that $|L(x_i)| = tq$ for each i .

If there is (p, q) - L -colouring of G_n , then we may assume by symmetry that u is coloured a and v is coloured b . Then for each i , the vertex $x_i \in H_{a,b}$

may not be assigned a colour from $[a]_q \cup [b]_q$. It follows that there is a (p, q) - L' -colouring of the subgraph $H_{a,b} \setminus \{u, v\}$, a contradiction. \square

5.2 Planar graphs of prescribed girth

A natural question is to ask what happens when we restrict ourselves to planar graphs with a given girth. The *girth* of a graph G is the length of a smallest cycle of G . We study the circular choice number of planar graphs with girth at least k and, for each $k \geq 3$, we define

$$\tau(k) := \sup\{\text{cch}(G) : G \text{ is planar and has girth } \geq k\}.$$

In the previous section, we have seen that $6 \leq \tau(3) = \tau \leq 8$.

The corresponding question for circular chromatic number has attracted a great deal of attention from the graph theoretic community [14, 7, 12, 25, 5]. This is motivated by a notable conjecture of Jaeger [10] on nowhere-zero flows, which when restricted to planar graphs states the following.

Conjecture 23 *Planar graphs of girth k have circular chromatic number at most $2 + \lceil \frac{k}{4} \rceil^{-1}$.*

This conjecture is somehow sharp as DeVos (cf. [5]) showed that there exist planar graphs of girth at least k and circular chromatic number greater than $2 + \lceil \frac{k+1}{4} \rceil^{-1}$. It was shown by Borodin *et al.* [5] that every planar graph with girth k has circular chromatic number at most $2 + \lceil \frac{3k+2}{20} \rceil^{-1}$.

In this section, we establish lower and upper bounds on $\tau(k)$. First, we show that Conjecture 23 cannot be resolved affirmatively by bounding the circular choice number. In particular, we generalise Proposition 22 to show that $\tau(k) \geq 2 + \frac{4}{k-2}$.

Proposition 24 *For any $k \geq 3$ and $\varepsilon > 0$, there exists a planar graph $G_{k,\varepsilon}$ of girth k with $\text{cch}(G_{k,\varepsilon}) \geq 2 + \frac{4}{k-2} - \varepsilon$.*

Proof. We may assume that $k \geq 4$ due to Proposition 22. We set $t := 2 + \frac{4}{k-2} - \frac{1}{n}$ where n is an integer greater than $\max(\frac{k-2}{4}, \varepsilon^{-1})$. Let p and q be two integers with $q = (k-2)n$, and p much larger than $tq = 2kn - (k-2)$. Note that $tq - 2q = 4n + 2 - k > 0$. All the computations and intervals are modulo p .

We consider two cases according to the parity of k . If $k = 2l + 1$, we construct the planar graph H'_m from H_m of Figure 2 by subdividing each of

the edges ux_i and vx_i exactly $(l - 1)$ times; see Figure 3. Let u_i^1, \dots, u_i^{l-1} be the vertices on the induced path of length l between u and x_i , and let v_i^1, \dots, v_i^{l-1} be the vertices on the path between v and x_i . If $k = 2l + 2$ then we further subdivide the edge $x_i x_{i+1}$ once. The arguments and computations for the two cases are similar, so from now on we only consider the case $k = 2l + 1 \geq 5$. The value of m depends on the parity of k , but for k odd, we set $m := \left\lfloor \frac{2q-1}{k^2-4k+2} \right\rfloor$.

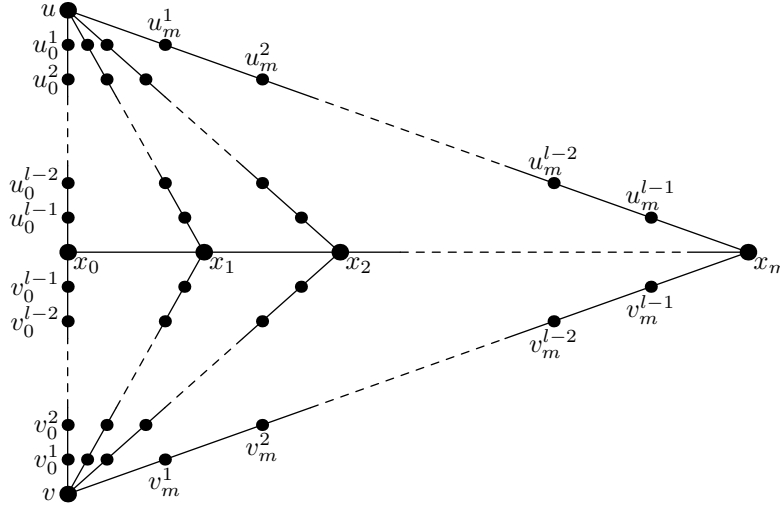


Figure 3: The graph H'_m for the case where $k = 2l + 1$.

Similarly to Proposition 22, the graph $G_{k,\varepsilon}$ is constructed from $(tq)^2$ copies of H'_m indexed by $L(u) \times L(v)$ by identifying the vertices u of each copy, and identifying the vertices v of each copy. Such a graph is planar and has girth exactly k . Again, we set $L(u) := [r, r + tq - 1]$ and $L(v) := [s, s + tq - 1]$ with r and s chosen so that the lists $L(u)$ and $L(v)$ have circular distance at least $2q$. For each $(a, b) \in L(u) \times L(v)$, we assign lists to the vertices of the copy $H_{a,b}$ of H'_m in such a way that if u is coloured a and v is coloured b , then the subgraph $H_{a,b}$ cannot be (p, q) - L -coloured.

Each $u_i^1 \in H_{a,b}$ is adjacent to u and each $v_i^1 \in H_{a,b}$ is adjacent to v , so we may include the interval $[a]_q$ to $L(u_i^1)$ and the interval $[b]_q$ to $L(v_i^1)$, so that it remains to assign $tq - (2q - 1) = 4n + 3 - k$ colours to each of these vertices. Set $\ell(u_i^1) = \ell(v_i^1) = 4n + 3 - k$ and set $\ell(x) = tq = 2kn - (k - 2)$

for every other vertex x in $H_{a,b} \setminus \{u, v\}$. Note that

$$\begin{aligned} \sum_{x \in H_{a,b} \setminus \{u,v\}} \ell(x) &= m(2(4n + 3 - k) + (k - 4)(2kn - (k - 2))) \\ &= 2(k - 2)^2 mn - (k^2 - 4k + 2)m \end{aligned}$$

and

$$2(|V(H_{a,b} \setminus \{u, v\})| - 1)q + 1 = 2(k - 2)^2 mn - 2q + 1.$$

By the choice of m , $\sum_{x \in H_{a,b} \setminus \{u,v\}} \ell(x) < 2(|V(H_{a,b} \setminus \{u, v\})| - 1)q + 1$. Since $H_{a,b} \setminus \{u, v\}$ is a tree, it follows from Lemma 2 that there is a list assignment L' of $H_{a,b} \setminus \{u, v\}$ such that $L(x) = \ell(x)$ for each $x \in H_{a,b} \setminus \{u, v\}$ and $H_{a,b} \setminus \{u, v\}$ is not L' - (p, q) -colourable. By our remark after Lemma 2, we may select L' so that its lists are far away from both of the intervals $[a]_q$ and $[b]_q$. Let $L(u_i^1) = [a]_q \cup L'(u_i^1)$ and $L(v_i^1) = [b]_q \cup L'(v_i^1)$ for every i and let $L(x) = L'(x)$ for every other vertex x in $H_{a,b} \setminus \{u, v\}$. Note that $|L(x)| = tq$ for each $x \in H_{a,b} \setminus \{u, v\}$.

If there is (p, q) - L -colouring of $G_{k,\varepsilon}$, then we may assume by symmetry that u is coloured a and v is coloured b . Then for each i , the vertex $u_i^1 \in H_{a,b}$ may not be assigned a colour from $[a]_q$ nor may the vertex $v_i^1 \in H_{a,b}$ be assigned a colour from $[b]_q$. It follows that there is a (p, q) - L' -colouring of the subgraph $H_{a,b} \setminus \{u, v\}$, a contradiction. \square

For the upper bounds, we rely on bounds for a slightly more general class of graphs—that is, for the class of graphs with bounded density (and girth). The *density* or *maximum average degree* $\text{Mad}(G)$ of a graph G is defined as the maximum, over all subgraphs H of G , of the average degree of H . Observe that $\delta(G) \leq \text{Mad}(G)$ so that by Lemma 7:

Remark 25 $\text{cch}(G) \leq 2\lceil \text{Mad}(G) \rceil$.

By Euler's formula, any planar graph with girth at least k has maximum average degree strictly less than $\text{Mad}_k = 2 + \frac{4}{k-2}$; thus, if we upper bound the circular choosability of graphs with bounded density (and prescribed girth), then we also upper bound the circular choosability of planar graphs of prescribed girth. The following minor variation of Remark 25 gives a better bound when Mad_k is integer and coincides with that given by plugging Mad_k into Remark 25 otherwise. If $\text{Mad}(G) < k + 1$, then $\delta^*(G) \leq k$ and thus $\tau(k) \leq 2\lceil \text{Mad}_k - 1 \rceil$. This bound is however is far from the correct

behaviour: the lower bound of Proposition 24 approaches 2 as k approaches infinity, while this last upper bound is 4 for $k \geq 6$.

In the next section, we outline how an adaptation of the upper bound by Borodin *et al.* [5] gives a much better bound on the circular choosability in terms of a graph's density and girth. This result is obtained using a discharging procedure.

Theorem 26 *If G has girth at least $3t + 1$ and $\text{Mad}(G) \leq 2 + \frac{6}{5t+1}$, then $\text{cch}(G) \leq 2 + \frac{2}{t}$.*

We postpone the proof of Theorem 26 until the next section and we first demonstrate how this theorem gives us a bound on $\tau(k)$.

Corollary 27 $\tau(k) \leq 2 + 2 \lceil \frac{3k-8}{10} \rceil^{-1}$.

Proof. Let G be a planar graph with girth at least k . By Euler's formula, G has maximum average degree strictly less than $\text{Mad}_k = 2 + \frac{4}{k-2}$. Some straightforward manipulations show that $\text{Mad}_k \leq 2 + \frac{6}{5t+1}$ is equivalent to $k \geq \frac{10t+8}{3}$, so that the theorem applies since $\frac{10t+8}{3} \geq 3t + 1$. We conclude that $t \leq \frac{3k-8}{10}$ implies that $\text{cch}(G) \leq 2 + \frac{2}{t}$. Therefore, $\tau(k) \leq 2 + 2 \lceil \frac{3k-8}{10} \rceil^{-1}$, as required. \square

In previous work on the odd-girth conjecture for the circular chromatic number, it was only relevant to consider graphs with large girth, since $\chi_c(G) \leq \chi(G)$ and Grötzsch's Theorem states that every triangle-free planar graph has chromatic number at most 3. In the analogous study for circular choosability, the corresponding values are higher as we have already seen; for example, $\tau(6) \geq 3$. Therefore, we find it an interesting problem to improve the upper bounds on $\tau(k)$ for the smaller values of girth k .

Observe that $\text{Mad}_4 = 4$ and $\text{Mad}_6 = 3$; thus, Remark 25 implies that $\tau(4) \leq 6$ and $\tau(6) \leq 4$. Corollary 27 gives $\tau(10) \leq 3$. By using more specific arguments, we can improve upon these bounds for $\tau(k)$ for $k \leq 9$. Again, our results rely on considering graphs of bounded density and prescribed girth and the following theorem is proved in the next section.

Theorem 28 *Let k , s and n be positive integers and set $r = \lceil \frac{s-2}{2} \rceil$. The following hold.*

- (i) *if $k \geq 2$ and $\text{Mad}(G) < k + 1 + \frac{k+1}{k+1+s}$ with $s \in \{1, 2\}$ then $\text{cch}(G) \leq 2k + \frac{4}{s+2}$.*

- (ii) If $s \leq 2k + 2$, G has girth at least 4 and $\text{Mad}(G) < k + 1 + \frac{k+1-r}{k+1+s'-r}$, where $s' = \min\{k + 2, s\}$, then $\text{cch}(G) \leq 2k + \frac{4}{s+2}$.
- (iii) If $2 \leq s \leq k + 2$, G has girth at least 6 and $\text{Mad}(G) < k + 1 + \frac{k+1-r}{k+s-r+\frac{1}{k+3-s}}$, then $\text{cch}(G) \leq 2k + \frac{4}{s+2}$.

We again postpone the proof until the next section and first give the following corollary:

Corollary 29 $\tau(5) \leq 4 + \frac{1}{2}$, $\tau(8) \leq 3 + \frac{1}{3}$, and $\tau(9) \leq 3$.

Proof. By Euler's formula, every planar graph with girth at least k has maximum average degree less than $\text{Mad}_k = 2 + \frac{4}{k-2}$. $\text{Mad}_5 = 3 + \frac{1}{3}$ and, setting $k = 2$ and $s = 6$, part (ii) of Theorem 28 gives $\tau(5) \leq 4 + \frac{1}{2}$. $\text{Mad}_8 = 2 + \frac{2}{3}$ and, setting $k = 1$ and $s = 1$, part (i) of Theorem 28 gives $\tau(8) \leq 3 + \frac{1}{3}$. $\text{Mad}_9 = 2 + \frac{4}{7}$ and, setting $k = 1$ and $s = 2$, part (iii) of Theorem 28 gives $\tau(9) \leq 3$. \square

Table 1 contains a summary of our bounds on $\tau(k)$. It is clear that there is a lot of room for improvement.

k (girth)	3	4	5	6	7	8	9	$k \geq 10$
$\tau(k)$ upper	8	6	$4 + \frac{1}{2}$	4	4	$3 + \frac{1}{3}$	3	$2 + \frac{2}{\lfloor (3k-8)/10 \rfloor}$
$\tau(k)$ lower	6	4	$3 + \frac{1}{3}$	3	$2 + \frac{4}{5}$	$2 + \frac{2}{3}$	$2 + \frac{4}{7}$	$2 + \frac{4}{k-2}$

Table 1: Bounds for planar graphs of prescribed girth k .

5.3 Graphs of bounded density (and prescribed girth)

In this section, we prove several upper bounds in terms of a graph's density and girth. We use a standard general approach several times: consider a minimal counterexample, discount certain reducible configurations, and then perform a discharging procedure. We define a graph G to be x -critical if $\text{cch}(G) > x$ and $\text{cch}(H) \leq x$ for every proper subgraph H of G .

As it is a direct adaptation of the method used by Borodin *et al.* [5], we give only an outline of the proof of Theorem 26 by stating the minor modifications and analogous intermediary lemmas required. The routine details are left to the reader. We say that two vertices are *weakly adjacent* if they are endpoints of the same thread.

Proof (outline) of Theorem 26. Let G be a $(2 + \frac{2}{t})$ -critical graph. The following are analogues of Lemma 4.1, Corollary 4.2, Lemma 4.3 and Lemma 4.4 in the paper of Borodin *et al.* [5].

Lemma 30 *Every thread in G has length at most t .*

Note. This lemma is a direct corollary of Lemma 3.

Corollary 31 *No three vertices of G with degree at least 3 are pairwise weakly adjacent and no two threads have the same set of endpoints.*

Let Y be the set of vertices of G with degree at least 3. For $v \in Y$, a *weak Y -neighbour* (resp. *weak 2-neighbour*) of v is a vertex in Y (resp. of degree 2) that is weakly adjacent to v ; let $N_Y(v)$ be the set of weak Y -neighbours of v . For each pair of weakly adjacent vertices u, v , define l_{vu} to be the length of the (unique) thread between v and u . For $v \in Y$, let $f(v) = -(t + 1) + \sum_{u \in N_Y(v)} (t + 1 - l_{vu})$.

The following lemmas are proved using Lemma 1.

Lemma 32 $f(v) \geq 1$ for all $v \in Y$.

Proof. Suppose on the contrary that $f(v) \leq 0$ for some vertex v . Colour the graph minus v and the internal vertices of its incident thread. Each Y -neighbour u of v has one incident colour, and applying Lemma 1 for each vertex after another along the thread between u and v , we obtain that the neighbour w_u of v on this thread has at least $\frac{2q}{t}(l_{vu} - 1)$ extendable colours. Hence applying Lemma 1 for v , we obtain that v has at least $1 - \frac{2q}{t}f(v)$ extendable colours. Hence as $f(v) \leq 0$ then G is circularly $(2 + \frac{2}{t})$ -choosable, a contradiction. \square

Lemma 33 $\sum_{u \in N_Y(v)} f(u) \geq t + 2$ for all $v \in Y$.

Proof. Suppose on the contrary that $\sum_{u \in N_Y(v)} f(u) \leq t + 1$ for some vertex v . Note that because the girth is at least $3t + 1$, there is no thread between neighbours of v . Let U be the set of Y -neighbours u of v such that $f(u) - t - 1 + l_{vu} \leq 0$.

Colour the graph G minus the vertices of $S \cup \{v\}$ and the internal vertices of its adjacent thread. Then every neighbour of $v \notin U$ has $1 \geq -\frac{2q}{t}(f(u) + l_{uv} - t)$ extendable colours. Let $u \in U$. Applying Lemma 1 along the threads incident to u which are not the thread between u and v similarly to the proof

of Lemma 32, we obtain that u has at least $-\frac{2q}{t}(f(u) + l_{uv} - t)$ extendable colours. Thus applying Lemma 1 along the threads incident to v , we obtain that v has at least $1 + \frac{2q}{t} \left(t + 1 - \sum_{u \in N_Y(v)} f(u) \right) \geq 1$ extendable colours. So G is circularly $(2 + \frac{2}{t})$ -choosable, a contradiction. \square

Given the above, we execute a discharging procedure as follows.

Discharging rules. Given G whose vertices have their degree as their initial charge, we define an adjusted charge $d^*(u)$ for each vertex u in G by the following operations:

- (R1). Every $v \in Y$ gives each to its weak 2-neighbours the amount $\frac{3}{5(t+1)}$.
(R2). Every $v \in Y$ gives each of its weak Y -neighbour the amount $\frac{3f(v) + (t+2)(d(v)-3)}{5(t+1)d(v)}$.

The reader can verify the following analogues of Lemmas 4.5 and 4.6 in the paper of Borodin *et al.* [5].

Lemma 34 *Under discharging, every $v \in Y$ receives from its weak Y -neighbours at least $\frac{t+2}{5(t+1)}$.*

Lemma 35 *After discharging, $d^*(v) \geq 2 + \frac{4d(v)-2}{5(t+1)}$ for all vertices v in G .*

Now, it follows that

$$\begin{aligned} 2|E(G)| &= \sum_{v \in V(G)} d^*(v) \geq \sum_{v \in V(G)} \left(2 + \frac{4d(v) - 2}{5(t+1)} \right) \\ &= \left(1 - \frac{1}{5(t+1)} \right) 2|V(G)| + \frac{4}{5(t+1)} 2|E(G)| \end{aligned}$$

so that $\frac{5t+4}{5t+5}|V(G)| \leq \frac{5t+1}{5t+5}|E(G)|$. Thus, the average degree of G is at least $\frac{2(5t+4)}{5t+1} = 2 + \frac{6}{5t+1}$, as required. \square

We now proceed with the proof of Theorem 28. The following is a lemma used to prove Lemma 7. It is also an important ingredient in the proof of Theorem 28.

Lemma 36 *Let k be a positive integer and $\alpha \geq 0$. Let G be a $(2k + \alpha)$ -critical graph. Then G has minimum degree at least $k + 1$.*

Proof. As $\text{cch}(G) > 2k + \alpha$, there exist $\varepsilon > 0$, two integers p, q and a $(2k + \alpha + \varepsilon)$ -list-assignment L such that G cannot be (p, q) - L -coloured. Observe that every proper subgraph H of G can be (p, q) - L -coloured since, by the definition of G , $\text{cch}(H) \leq 2k + \alpha$. Let $t = 2k + \alpha + \varepsilon$.

Suppose that v is a vertex of degree at most k . By minimality of G , one can (p, q) - L -colour $G - v$. To apply Lemma 1, we can consider the colouring of $G - v$ as a precolouring and, hence, the number of extendable colours for v is at least $tq - k(2q - 1) = \alpha q + \varepsilon q + k \geq 1$. This yields a (p, q) - L -colouring of G , a contradiction. \square

For part (i) of Theorem 28, we require the following lemma.

Lemma 37 *Let k be a positive integer and $\alpha \geq 0$. Let G be a $(2k + \alpha)$ -critical graph. Then,*

- (i) *if $k \geq 2$, then the neighbours of a vertex of degree $k + 1$ having degree $k + 1$ are pairwise non-adjacent;*
- (ii) *if $\alpha \geq \frac{2}{r+2}$, then a vertex of degree $k + 1$ is adjacent to at most r vertices of degree $k + 1$; and*
- (iii) *for $s \in \{1, 2\}$, if $\alpha \geq \frac{4}{s+2}$, then a vertex of degree $k + 2$ is adjacent to at most s vertices of degree $k + 1$.*

Proof. As $\text{cch}(G) > 2k + \alpha$, there exist $\varepsilon > 0$, two integers p, q and a $(2k + \alpha + \varepsilon)$ -list-assignment L such that G cannot be (p, q) - L -coloured. Let $t = 2k + \alpha + \varepsilon$.

(i) Let v be a vertex of degree $k + 1$ in G . Let H be the graph induced by v and its neighbours of degree $k + 1$ in G . By minimality, there exists a (p, q) - L -colouring c of $G - H$. We consider this as a precolouring and now consider the number of extendable colours for a vertex u of H . There are at least $(2k + \alpha + \varepsilon)q - (k + 1 - d_H(u))(2q - 1) \geq (2d_H(u) - 2)q$ colours available for u , that is not creating conflict with colours of already coloured neighbours of u . For a contradiction, let us assume that there is an edge between two of the neighbours of v , i.e. H is not a star. If H is not K_3 , then, by Corollary 19, it admits a (p, q) -colouring from the lists of extendable colours, thus completing a (p, q) - L -colouring of G .

Let us suppose now that H is the complete graph on the vertex set $\{v, y, z\}$. Since $k \geq 2$, there is a vertex u not in H adjacent to v . Observe that, if we consider the subgraph H' induced by $\{u, v, y, z\}$, then our goal is to (p, q) - L' -colour it, with $L'(u) = \{c(u)\}$, and $L'(a) = L(a) \setminus$

$\bigcup_{w \in N(a) \setminus V(H')} [c(w)]_q$ for $a \in \{v, y, z\}$. Note that, for $a \in \{v, y, z\}$, the list $L'(a)$ has size at least $4q$ if au is an edge, and at least $2q$ otherwise—in particular, $|L'(v)| \geq 4q$. Hence, without loss of generality, we can assume that $L'(y)$ has size $2q$, and if uy is an edge, then these $2q$ colours are not in $[c(u)]_q$. The same holds for z . Therefore, the fact that c can be extended to G directly follows from part (ii) of Proposition 14.

(ii) Let v a vertex of degree $k + 1$ in G . Suppose that v has $r + 1$ neighbours v_1, \dots, v_{r+1} of degree $k + 1$. By (i), the vertices v_1, \dots, v_{i+1} are pairwise non-adjacent; this observation is important for the valid application of Lemma 1. Consider any (p, q) - L -colouring of $G - \{v_1, \dots, v_{r+1}, v\}$ and an ordering which ends in (v_1, \dots, v_{r+1}, v) . By Lemma 1, the number of extendable colours for v_i , for any $i \in \{1, 2, \dots, r+1\}$, is at least $tq - k(2q - 1) = \alpha q + k + \varepsilon q$. By Lemma 1, the number of extendable colours for v is at least $tq - (k - r)(2q - 1) - (r + 1)(2q - (\alpha q + k + \varepsilon q)) = 2(\alpha - 1)q + 2(k + \varepsilon q) + r(\alpha q - 1 + k + \varepsilon q) = (\alpha(r + 2) - 2)q + 2(k + \varepsilon q) + r(k + \varepsilon q - 1) \geq \left(\frac{2}{r+2}(r + 2) - 2\right)q + 1 \geq 1$, since $\alpha \geq \frac{2}{r+2}$ and $k \geq 1$, a contradiction to the t -criticality.

(iii) Suppose that v is a vertex of degree $k + 2$ with $s + 1$ neighbours v_1, \dots, v_{s+1} of degree $k + 1$. Consider any (p, q) - L -colouring of $G - \{v_1, \dots, v_{s+1}, v\}$ and an ordering which ends in (v_1, \dots, v_{s+1}, v) . By (ii), the vertices v_1, \dots, v_{s+1} are pairwise non-adjacent, since $\alpha \geq 1$. Hence, using Lemma 1, the number of extendable colours for v_i , for any $i \in \{1, 2, \dots, s + 1\}$, is at least $tq - k(2q - 1) = \alpha q + k + \varepsilon q$. Applying the lemma again, the number of extendable colours for v is at least $tq - (k + 1 - s)(2q - 1) - (s + 1)(2q - (\alpha q + k + \varepsilon q)) = 2(\alpha - 2)q + 2(k + \varepsilon q) + 1 + s(\alpha q - 1 + k + \varepsilon q) = (\alpha(s + 2) - 4)q + 2(k + \varepsilon q) + s(k + \varepsilon q - 1) + 1 \geq 1$, since $\alpha \geq \frac{4}{s+2}$ and $k \geq 1$, a contradiction to t -criticality. \square

Proof of part (i) of Theorem 28. Suppose that G is $\left(2k + \frac{4}{s+2}\right)$ -critical. Then, by Lemmas 36 and 37, G has minimum degree at least $k + 1$, no two vertices of degree $k + 1$ are adjacent, and any vertex of degree $k + 2$ is adjacent to at most s vertices of degree $k + 1$. We perform a discharging procedure as follows.

Discharging rules. Given G whose vertices have their degree as their initial charge, we define an adjusted charge $d^*(u)$ for each vertex u in G by the following operation:

(R1). Every vertex of degree at least $k + 2$ gives the amount $\eta = \frac{1}{k+1+s}$ to

each of its neighbours of degree $k + 1$.

We want to show now that the new charge of every vertex v is at least $k + 1 + (k + 1)\eta$. If v has degree $k + 1$, then its new charge is at least $k + 1 + (k + 1)\eta$ since all of its neighbours gave it charge. If v has degree $k + 2$, then its new charge is at least $k + 2 - s\eta = k + 1 + (k + 1)\eta$ since it has at most s neighbours of degree $k + 1$ to give charge to. If v has degree at least $k + 3$, then its new charge is at least $k + 3 - (k + 3)\eta = k + 1 + (k + 2s - 1)\eta \geq k + 1 + (k + 1)\eta$ since $s \geq 1$.

It follows now that the average degree, and hence the maximum average degree, is at least $k + 1 + (k + 1)\eta$. This shows that every graph with circular choice number more than $2k + \frac{4}{s+2}$ has maximum average degree at least $k + 1 + (k + 1)\eta$, as required. \square

To prove parts (ii) and (iii) of Theorem 28, we require the following lemma. We define a *hibernian* to be a vertex of degree $k + 2$ with s neighbours of degree $k + 1$. A *barbarian* is a vertex of degree $k + 2$ with exactly $s - 1$ neighbours of degree $k + 1$.

Lemma 38 *Let k be a positive integer, $s \in \{0, 1, \dots, k + 2\}$ and $\alpha \geq 0$. Let G a $(2k + \alpha)$ -critical graph of girth g .*

- (i) *If $g \geq 4$ and $\alpha \geq \frac{4}{s+2}$, then a vertex of degree $k + 2$ is adjacent to at most s vertices of degree $k + 1$.*
- (ii) *If $g \geq 5$ and $\alpha \geq \frac{3}{s+1}$, then two hibernians cannot be adjacent.*
- (iii) *If $g \geq 6$ and $\alpha \geq \frac{4}{3s+1}$, then a barbarian is adjacent to at most one hibernian.*

Proof. Take ε, p, q, L and t as in the proof of Lemma 37.

(i) The proof is similar to the proof of Lemma 37(iii), so we omit it here. Note that the condition $g \geq 4$ ensures that the neighbours of a vertex are pairwise non-adjacent.

(ii) Suppose that v and v' are adjacent hibernians. Let $V_v = \{v_1, \dots, v_s\}$ be the set of s neighbours of v of degree $k + 1$. Let $V_{v'} = \{v'_1, \dots, v'_s\}$ be the set of s neighbours of v' of degree $k + 1$. Since the girth of G is at least 5, the sets V_v and $V_{v'}$ are disjoint. Furthermore, $V_v \cup V_{v'}$ induces an independent set in G . This observation makes our subsequent applications of Lemma 1 valid.

Consider any (p, q) - L -colouring of $G \setminus (V_v \cup V_{v'} \cup \{v, v'\})$. We wish to show that such a colouring can be extended to the entire graph G , giving a contradiction to t -criticality. Let us use an ordering of the vertices of G which ends $(v_1, \dots, v_s, v'_1, \dots, v'_s, v, v')$; it is clear that this ordering satisfies the properties for Lemma 1.

First, the number of extendable colours for v_i or v'_i , for any i , is at least

$$x_0 := tq - k(2q - 1) = \alpha q + k + \varepsilon q.$$

We have $x_0 \geq 1$, since $k \geq 1$ and $\alpha, \varepsilon \geq 0$. Now, since v is adjacent to v_1, \dots, v_s and $k + 1 - s$ precoloured vertices, the number of extendable colours for v is at least

$$\begin{aligned} x &:= tq - s(2q - x_0) - (k + 1 - s)(2q - 1) \\ &= ((s + 1)\alpha - 2)q + k + \varepsilon q + s(k + \varepsilon q - 1) + 1. \end{aligned}$$

We have $x \geq 1$, since $\alpha(s + 1) \geq 2$ and $k \geq 1$. Applying Lemma 1, since v' is adjacent to v'_1, \dots, v'_s , $k + 1 - s$ precoloured vertices and v , it follows that the number of extendable colours for v' is at least

$$\begin{aligned} x' &:= tq - s(2q - x_0) - (k + 1 - s)(2q - 1) - (2q - x) \\ &= ((s + 1)\alpha - 3)2q + 2\{k + \varepsilon q + s(k + \varepsilon q - 1) + 1\}. \end{aligned}$$

Since $\alpha \geq \frac{3}{s+1}$ and $k \geq 1$ then $x' \geq 0$. But this means that G is (p, q) - L -colourable, a contradiction.

(iii) Suppose that a barbarian w is adjacent to two hibernians v and v' . Since the girth is at least 6, the neighbourhoods of w , v and v' are pairwise disjoint and their union is an independent set. Consider any (p, q) - L -colouring of $G \setminus \{v_1, \dots, v_s, v'_1, \dots, v'_s, w_1, \dots, w_{s-1}, v, v', w\}$ and extend the colouring using the ordering above via Lemma 1. As in (ii) every neighbour of v , v' or w has at least $x_0 = \alpha q + k + \varepsilon q$ extendable colours, and v and v' have at least

$$x = ((s + 1)\alpha - 2)q + k + \varepsilon q + s(k + \varepsilon q - 1) + 1$$

extendable colours.

Now the number of extendable colours at w is at least

$$tq - (s - 1)(2q - x_0) - (k + 1 - s)(2q - 1) - 2(2q - x) \geq ((3s + 1)\alpha - 4)q + 1.$$

So G is (p, q) - L -colourable since $\alpha \geq \frac{4}{3s+1}$, a contradiction. \square

The proof of part (ii) of Theorem 28 follows from Lemma 38(i) and is similar to the proof of Theorem 28(i), so we omit it here.

Proof of part (iii) of Theorem 28. Suppose that G is $\left(2k + \frac{4}{s+2}\right)$ -critical and has girth at least 6. Then, by Lemmas 36, 37(ii) and 38(i), G has minimum degree at least $k+1$, any vertex of degree $k+1$ is adjacent to at most r vertices of degree $k+1$, and any vertex of degree $k+2$ is adjacent to at most s vertices of degree $k+1$. Furthermore, $\frac{4}{s+2} \geq \max\left\{\frac{3}{s+1}, \frac{4}{3s+1}\right\}$ because $s \geq 2$, and thus, using parts (ii), (iii) of Lemma 38, no two hibernians are adjacent and every barbarian is adjacent to at most one hibernian. We perform a discharging procedure as follows.

Discharging rules. Given G whose vertices have their degree as their initial charge, we define an adjusted charge $d^*(u)$ for each vertex u in G by the following operations:

- (R1). Every vertex of degree $k+1$ receives charge $\eta = \frac{1}{k+s-r+\frac{1}{k+3-s}}$ from every neighbour of degree at least $k+2$.
- (R2). Every hibernian receives charge $\eta_1 = \frac{\eta}{k+3-s} = \frac{1}{(k+s-r)(k+3-s)+1}$ from every non-hibernian neighbour of degree at least $k+2$.

We want to show now that the new charge of every vertex is at least $k+1 + (k+1-r)\eta$. If v has degree $k+1$, then its new charge is at least $k+1 + (k+1-r)\eta$ since it has at least $k+1-r$ neighbours that gave it charge η . If v is a hibernian, then its new charge is at least $k+2-s\eta + (k+2-s)\eta_1 = k+1 + \left(\frac{1}{\eta} - s + \frac{k+2-s}{k+3-s}\right)\eta = k+1 + (k+1-r)\eta$ since it has at most s neighbours of degree $k+1$ to give charge to and it receives η_1 charge from all of the other adjacent vertices. If v is a barbarian, then its new charge is at least $k+2 - (s-1)\eta - \eta_1 = k+1 + \left(\frac{1}{\eta} - s + 1 - \frac{1}{k+3-s}\right)\eta = k+1 + (k+1-r)\eta$ since it has $s-1$ neighbours of degree $k+1$ and at most one hibernian neighbour. If v is a non-hibernian, non-barbarian vertex of degree $k+2$, then its new charge is at least $k+2 - (s-2)\eta - (k+2 - (s-2))\eta_1 = k+1 + \left(\frac{1}{\eta} - s + 2 - \frac{k+4-s}{k+3-s}\right)\eta = k+1 + (k+1-r)\eta$ since it has at most $s-2$ neighbours of degree $k+1$ (and the rest could be hibernians). If v is a vertex of degree at least $k+3$, then its new charge is at least $k+3 - (k+3)\eta \geq k+1 + (k+1-r)\eta$; this holds if and only if $k+s-r + \frac{1}{k+3-s} \geq k+2 - \frac{r}{2}$, that is $s - \frac{r}{2} - 2 + \frac{1}{k+3-s} \geq 0$. This is true when $s \geq 2$.

It follows now that the average degree, and hence the maximum average degree, is at least $k+1 + (k+1-r)\eta$. This shows that every graph with

circular choice number more than $2k + \frac{4}{s+2}$ and girth at least 6 has maximum average degree at least $k + 1 + (k + 1 - r)\eta$, as required. \square

We note that the following result, proven in a previous version of this work [9], follows from Lemma 36 and parts (i), (ii) of Lemma 37.

Proposition 39 *Let k be a positive integer and $r \in \{0, \dots, k\}$. If $\text{Mad}(G) < k + 1 + \frac{k+1-r}{2k+3-r}$, then $\text{cch}(G) \leq 2k + \frac{2}{r+2}$.*

We do not apply this result elsewhere, but by listing this result we wish to highlight that the relationship between the circular choice number and the maximum average degree (and girth) deserves further attention. By improving the upper bounds of this type, we could gain a better understanding of the circular choice number for planar graphs (and, more generally, graphs on surfaces) of prescribed girth.

5.4 Outerplanar graphs of prescribed girth

A graph is *outerplanar* if it can be drawn in the plane such that the outer face is incident to every vertex of the graph. For any $k \geq 3$, we define

$$\tau_o(k) := \sup\{\text{cch}(G) : G \text{ is outerplanar and has girth } \geq k\}.$$

We show the following theorem.

Theorem 40 $\tau_o(k) = 2 + \frac{2}{k-2}$ for all integers $k \geq 3$.

We start by exhibiting a class of examples that show that $\tau_o(k)$ is at least the expression given in the theorem.

Proposition 41 *Fix $k \geq 3$. For any $\varepsilon > 0$, there exists an outerplanar graph $O_{k,\varepsilon}$ of girth k whose circular choice number is at least $2 + \frac{2}{k-2} - \varepsilon$.*

Proof outline. We define n to be an integer greater than $\max(\frac{k-2}{2}, \varepsilon^{-1})$. Let $t := 2 + \frac{2}{k-2} - \frac{1}{n}$, set $q := 2(k-2)n$, and choose p much larger than $tq = 4kn - 4n - 2k + 4$. Note that $tq - 2q = 4n - 2k + 4 > 0$. We consider the graph P_m of Figure 4, for some m divisible by $k-2$ large enough. We construct the graph $O_{k,\varepsilon}$ by taking tq copies of P_m and identifying the vertices u of each copy. To each $a \in L(u)$ we associate a copy P_a of P_m and a corresponding list assignment such that, if u is coloured a , then the subgraph P_a cannot be (p, q) - L -coloured. As the methods are similar to

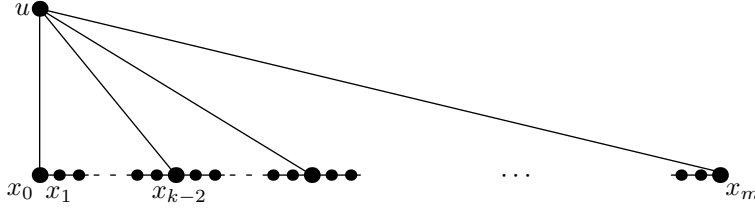


Figure 4: The graph P_m .

those of Propositions 22 and 24, we omit the remaining details of the proof. \square

Next we show that $\tau_o(k)$ is at most the expression of Theorem 40. We note that this upper bound was already proved [22] for even k . We give a more succinct presentation of the proof and also cover the case for odd k . The following is a consequence of Lemma 3.

Lemma 42 *Fix $k \geq 3$. Let L be a $\left(2 + \frac{2}{k-2}\right)$ - (p, q) -list-assignment of the cycle C_k . Every precolouring of two adjacent vertices can be extended to a (p, q) - L -colouring of the entire cycle.*

Proposition 43 *Let $k \geq 3$. Every outerplanar graph of girth at least k has circular choice number at most $2 + \frac{2}{k-2}$.*

Proof. Suppose on the contrary that there exists an outerplanar graph G of girth at least k that is $\left(2 + \frac{2}{k-2}\right)$ -critical. Let L be a $\left(2 + \frac{2}{k-2} + \varepsilon\right)$ - (p, q) -list-assignment of G such that G is not (p, q) - L -colourable. The graph G cannot have any leaves (vertices of degree one). We now note that every outerplanar graph of girth at least k with no leaves can be inductively built up from a collection of cycles of length at least k as follows. We start with a cycle of length at least k . At each step of the construction, we identify an edge of the outerplanar graph constructed so far with an edge of a new cycle of length at least k . By following this inductive sequence of long cycles (in the reverse order), we can repeatedly apply Lemma 42 to produce a (p, q) - L -colouring of G , a contradiction. \square

6 Concluding remarks

We showed that the difference between the circular choice number and circular chromatic number of a circular clique is unbounded. We provided some

evidence in support of a positive answer to the question of Problem 8, by showing that $\text{cch}(G) = O(\text{ch}(G) + \ln |V(G)|)$. We showed, perhaps counter-intuitively, that the value of $\tau := \sup\{\text{cch}(G) : G \text{ is planar}\}$ lies between 6 and 8, rather than between 4 and 5, as Mohar had suggested [13].

Much further work remains. Problem 8 remains open as does the following fundamental question posed by Zhu.

Problem 44 (Zhu [26]) *Is the circular choice number of every graph a rational?*

Norine [15] recently established that the circular choice number is not always attained, thereby answering a question of Zhu [26].

The upper bounds we obtained for graphs of bounded density and prescribed girth also imply upper bounds for graphs on surfaces with prescribed girth (for example, toroidal graphs with prescribed girth); however, it is unclear to us how our lower bound examples can be extended or generalised to guarantee higher circular choice numbers on higher surfaces.

Recently, using the Combinatorial Nullstellensatz, Norine, Wong and Zhu [16] studied bipartite graphs of bounded density and showed the following.

Theorem 45 (Norine *et al.* [16]) *If G is a connected bipartite graph that is not a tree, then $\text{cch}(G) \leq \text{Mad}(G)$.*

In particular, all bipartite planar graphs have circular choice number less than 4; furthermore, all bipartite planar graphs with girth at least k have circular choice number less than $2 + 2/(\lceil k/2 \rceil - 1)$. Since the examples $G_{k,n}$ of Proposition 24 are bipartite for even k , the above bounds are sharp in a sense. On the other hand, if we sought to improve upon Proposition 24, then we must consider more complicated, non-bipartite constructions.

As mentioned before circular choosability is closely related to the notion of T -choosability, introduced by Tesman [19] and further studied by, for instance, Alon and Zaks [3] and Waters [23, 24]. Given a set T of forbidden differences, a T -proper colouring of G is a colouring that satisfies $|c(v) - c(u)| \notin T$ whenever $uv \in E(G)$. The T -choosability $T\text{-ch}(G)$ of G is the least number k such that any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$ admits a T -proper colouring c with $c(v) \in L(v)$ for all v . The special case when $T = T_r = \{0, \dots, r\}$ has received a lot of attention. By considering the proofs it should be clear that many of the bounds on $\text{cch}(G)$ given in this paper extend to $T_r\text{-ch}(G)$ when multiplied by $r + 1$. However, as might be expected, the ‘‘circularity constraint’’ is an essential difference

between T_r -choosability and circular choosability. For instance, Sitters [18] showed that $T_r\text{-ch}(C_{2n}) = \left\lfloor \frac{4n-2}{4n-1}(2r+2) \right\rfloor + 1$, whereas Norine [15] proved that $\text{ch}(C_{2n}) = 2$. Since writing this paper it has come to our attention that the T_r -analogue of Problem 8 has been investigated by Waters [24], who has shown that $T_r\text{-ch}(G) \leq 2(r+1)$ for all 2-choosable graphs. Moreover, Waters' thesis [23] contains a number of results and proofs that are strikingly similar to results in our paper and in [26]. In particular, he gave T_r -analogues of Theorem 20 and Theorem 22 that are essentially the same as ours.

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