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# Choosability of the square of planar subcubic graphs with large girth 

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#### Abstract

We show that the choice number of the square of a subcubic graph with maximum average degree less than $18 / 7$ is at most 6 . As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6 . We then show that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5 .


## 1 Introduction

Let $G$ be a (simple) graph. The neighbourhood of a vertex $v$ of $G$, denoted $N_{G}(v)$, is the set of its neighbours, i.e. is the set of vertices $y$ such that $x y$ is an edge. The degree of a vertex $v$ in $G$, denoted $d_{G}(v)$, is its number of neighbours. Often, when the graph $G$ is clearly understood from the context, we omit the subscript $G$. A graph is subcubic if every vertex has degree at most 3 .

Let $p: V(G) \rightarrow \mathbb{N}$. A p-list-assignment is a list-assignment $L$ such that $|L(v)|=p(v)$ for any $v \in V(G) . G$ is $p$-choosable if it is $L$-colourable for any $p$-list-assignment. By extension, if $k$ is an integer, we say that $G$ is $k$-choosable if it is $p$-choosable when $p$ is the constant function with value $k$ (i. e. $p(v)=k$ for all $v \in V)$. The choice number of $G$, denoted $c h(G)$, is the smallest integer $k$ such that $G$ is $k$-choosable. Clearly the choice number of $G$ is at least as large as $\chi(G)$, the chromatic number of $G$.

The square of $G$ is the graph $G^{2}$ with vertex set $V(G)$ such that two vertices are linked by an edge of $G^{2}$ if and only if $x$ and $y$ are at distance at most 2 in $G$. A graph is called planar if it can be embedded in the plane. Wegner [?] proved that the square of a subcubic planar graph is 8 -colourable. He also conjectured it is 7 -colourable. Recently, this conjecture was proved by Thomassen [?].

Theorem 1 (Thomassen [?]) Let $G$ be a subcubic planar graph. Then $\chi\left(G^{2}\right) \leq 7$.
Kostochka and Woodall [?] conjectured that, for every square of a graph, the chromatic number equals the choice number.

Conjecture 2 (Kostochka and Woodall [?]) For all $G$, $\operatorname{ch}\left(G^{2}\right)=\chi\left(G^{2}\right)$.

[^0]If true, this conjecture together with Theorem ?? implies that every subcubic planar graph is 7 -choosable. Very recently, Cranston and Kim [?] showed that the square of every subcubic graph (non necessarily planar) other than the Petersen graph is 8-choosable.

The average degree of $G$, denoted $A d(G)$ is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|}=\frac{2|E(G)|}{|V(G)|}$. The maximum average degree of $G$, denoted $\operatorname{Mad}(G)$, is max $\{\operatorname{Ad}(H), H$ subgraph of $G\}$. In [?], Dvořák, Škrekovski and Tancer proved that the choice number of the square of a subcubic graph $G$ is at most 4 if $\operatorname{Mad}(G)<24 / 11$ and $G$ has no 5 -cycle, at most 5 if $\operatorname{Mad}(G)<7 / 3$ and at most 6 if $\operatorname{Mad}(G)<5 / 2$.

The girth of a graph is the smallest length of a cycle in $G$. Planar graphs with prescribed girth have bounded maximum average degree:

Proposition 3 Every planar graph with girth at least $g$ has maximum average degree less than $2+\frac{4}{g-2}$.
Hence the results of Dvořák, Škrekovski and Tancer imply that the choice number of the square of a planar graph with girth $g$ is at most 6 if $g \geq 10$, at most 5 if $g \geq 14$ and at most 4 if $g \geq 24$. The two latter results had been previously proved by Montassier and Raspaud [?].

In this paper, we improve some of these results. We first show (Theorem ??) that the choice number of the square of a subcubic graph with maximum average degree less than $18 / 7$ is at most 6 . As a corollary, we get that the choice number of the square of a subcubic planar graph with girth at least 9 is at most 6 . Note that this corollary has been proved later and independently by Cranston and Kim [?]. We then show (Theorem ??) that the choice number of the square of a subcubic planar graph with girth at least 13 is at most 5 .

## 2 The main results

The general frame of the proofs is classical. We consider a $k$-minimal graph, that is a subcubic graph such that its square is not $k$-choosable but the square of every proper subgraph is $k$-choosable. We prove that some configurations (i.e. induced subgraphs) are forbidden in such a graph and then deduce a contradiction. To do so, we will need the following definitions:

An $i$-vertex is a vertex of degree $i$. We denote by $V_{i}$ the set of $i$-vertices of $G$ and by $v_{i}$ its cardinality. Let $v$ be a vertex. An $i$-neighbour of $v$ is a neighbour of $v$ with degree $i$. The $i$-neighbourhood of $v$ is $N_{i}(v)=N(v) \cap V_{i}$ and its $i$-degree is $d_{i}(v)=\left|N_{i}(v)\right|$.

Some properties of 6 - and 5 -minimal graphs have already been proved in [?]. The easy first one is that $V_{0} \cup V_{1}=\emptyset$, so $G$ has minimum degree 2. This will allow us to use the following definitions for 6and 5 -minimal graphs.

Let $G$ be a subcubic graph with minimum degree 2. A thread of $G$ is a path whose endvertices are 3 -vertices and whose internal vertices are 2 -vertices. The kernel of $G$ is the weighted graph $K_{G}$ such that $V\left(K_{G}\right)=V_{3}(G)$ and $x y$ is an edge in $K_{G}$ with weight $l$ if and only if $x$ and $y$ are connected by a thread of length $l$ in $G$. An edge of weight $l$ is also called $l$-edge. Let $x$ be a 3 -vertex of $G$. The type of $x$ is the triple ( $l_{1}, l_{2}, l_{3}$ ) such that $l_{1} \leq l_{2} \leq l_{3}$ and the three edges (a loop being counted twice) incident to $x$ have weight $l_{1}, l_{2}$ and $l_{3}$ in $K_{G}$. We denote by $Y_{l_{1}, l_{2}, l_{3}}$ the set of 3 -vertices of type ( $l_{1}, l_{2}, l_{3}$ ) and $y_{l_{1}, l_{2}, l_{3}}$ its cardinality. Moreover, for every integer $i$, we define $Z_{i}:=\bigcup_{l_{1}+l_{2}+l_{3}=i} Y_{l_{1}, l_{2}, l_{3}}$ and $z_{i}=\left|Z_{i}\right|$. The number of vertices and edges and thus the average degree of $G$ may be easily expressed in terms of the $z_{i}$ :

$$
\begin{aligned}
|V(G)| & =\sum_{i \geq 3} \frac{i-1}{2} z_{i} \\
2|E(G)| & =\sum_{i \geq 3} i . z_{i}
\end{aligned}
$$

$$
\begin{equation*}
A d(G)=\frac{\sum_{i \geq 3} i . z_{i}}{\sum_{i \geq 3} \frac{i-1}{2} z_{i}} \tag{1}
\end{equation*}
$$

### 2.1 6-choosability

The aim of this subsection is to prove the following result.
Theorem 4 Let $G$ be a subcubic graph of maximum average degree $d<18 / 7$. Then $G^{2}$ is 6 -choosable.
Remark 5 Theorem ?? is tight. Indeed, the graph $J_{7}$ depicted in Figure ?? has average degree 18/7 and its square is the complete graph on seven vertices $K_{7}$ which is not 6 -choosable (nor 6 -colourable).


Figure 1: The graph $J_{7}$

Theorem ?? and Proposition ?? yield that the square of a subcubic planar graph with girth at least 9 is 6 -choosable.

Corollary 6 The square of a subcubic planar graph with girth at least 9 is 6 -choosable.
In order to prove Theorem ??, we need to establish some properties of 6-minimal graphs. Some of them have been proved in [?].

Lemma 7 (Dvořák, Škrekovski and Tancer [?]) Let $G$ be a 6-minimal graph. Then the following hold:
(i) all the edges of $K_{G}$ have weight at most 2;
(ii) every 3-cycle of $G$ has its vertices in $V_{3}$;
(iii) every 4-cycle of $G$ has at least three vertices in $V_{3}$;
(iv) a vertex of $Y_{2,2,2}$ is not adjacent in $K_{G}$ to a vertex of $Y_{1,2,2} \cup Y_{2,2,2}$.

We will prove in Subsection ?? some new properties.
Lemma 8 Let $G$ be a 6-minimal graph. Then the following hold:
(i) if $\left(v_{1}, v_{2}, v_{3}, v_{4}, v_{1}\right)$ is a 4-cycle with $v_{2} \in V_{2}$ then $v_{1}$ or $v_{3}$ is not in $Y_{1,2,2}$;
(ii) a vertex of $Y_{1,2,2}$ is adjacent in $K_{G}$ to at most one vertex of $Y_{1,2,2}$ by 2-edges.

Proof of Theorem ??. Let $G$ be a 6-minimal planar graph. $G$ has minimum degree 2, so its kernel $K_{G}$ is defined. Moreover by Lemma ?? (i), $Z_{i}$ is empty for $i \geq 7$ and $Z_{6}=Y_{2,2,2}$ and $Z_{5}=Y_{1,2,2}$.

Let us consider a vertex of $Z_{4}=Y_{1,1,2}$. Its neighbour in $K_{G}$ via the 2-edge is in $Z_{4} \cup Z_{5} \cup Z_{6}$ because a vertex of $Z_{3}=Y_{1,1,1}$ is incident to no edge of weight 2 . For $i=4,5,6$, let $Z_{4}^{i}$ be the set of vertices of
$Z_{4}$ which are adjacent to a vertex of $Z_{i}$ by their unique 2-edge and $z_{4}^{i}$ its cardinality. $\left(Z_{4}^{4}, Z_{4}^{5}, Z_{4}^{6}\right)$ is a partition of $Z_{4}$ so $z_{4}=z_{4}^{4}+z_{4}^{5}+z_{4}^{6}$. Hence Equation (??) becomes

$$
A d(G)=\frac{6 z_{6}+5 z_{5}+4 z_{4}^{6}+4 z_{4}^{5}+4 z_{4}^{4}+3 z_{3}}{\frac{5}{2} z_{6}+2 z_{5}+\frac{3}{2} z_{4}^{6}+\frac{3}{2} z_{4}^{5}+\frac{3}{2} z_{4}^{4}+z_{3}}
$$

By Lemma ?? (iv), the three neighbours in $K_{G}$ of a vertex of $Z_{6}$ are not in $Z_{6} \cup Z_{5}$. So they must be in $Z_{4}^{6}$. It follows that $3 z_{6}=z_{4}^{6}$. So

$$
A d(G)=\frac{5 z_{5}+6 z_{4}^{6}+4 z_{4}^{5}+4 z_{4}^{4}+3 z_{3}}{2 z_{5}+\frac{7}{3} z_{4}^{6}+\frac{3}{2} z_{4}^{5}+\frac{3}{2} z_{4}^{4}+z_{3}}
$$

By Lemma ?? (ii), a vertex of $Z_{5}$ is adjacent to at least one vertex of $Z_{4}^{5}$. Thus $z_{5} \leq z_{4}^{5}$. But $\operatorname{Ad}(G)$ is decreasing as a function of $z_{5}$ since $z_{4}^{6}, z_{4}^{5}, z_{4}^{4}$ and $z_{3}$ are non-negative. It follows that

$$
A d(G) \geq \frac{6 z_{4}^{6}+9 z_{4}^{5}+4 z_{4}^{4}+3 z_{3}}{\frac{7}{3} z_{4}^{6}+\frac{7}{2} z_{4}^{5}+\frac{3}{2} z_{4}^{4}+z_{3}} \geq \frac{18}{7} .
$$

### 2.2 5-choosability

Dvořák, Škrekovski and Tancer [?] proved that the square of a subcubic graph $G$ with maximum average degree less than $7 / 3$ is 5 -choosable. This result is tight since the graph $J_{6}$ depicted in Figure ?? has average degree $7 / 3$ and its square is the complete graph on six vertices $K_{6}$ which is not 5 -choosable (nor 5 -colourable). However, we will prove that the square of a subcubic planar graph with girth at least 13


Figure 2: The graph $J_{6}$
is 5-choosable, which improves the result of Montassier and Raspaud.
Theorem 9 The square of a subcubic planar graph with girth at least 13 is 5-choosable.
In order to prove this theorem, we need to establish some properties of 5 -minimal graphs. Some of them have been proved in [?].

Lemma 10 (Dvořák, Škrekovski and Tancer [?]) Let $G$ be a 5-minimal graph. Then the following hold:
(i) all the edges of $K_{G}$ have weight at most 3;
(ii) if $i \geq 8, Z_{i}$ is empty.

We will prove in Subsection ?? some new properties.
Lemma 11 Let $G$ be a 5-minimal graph of girth at least 13. Then in $K_{G}$ the following hold:
(i) a vertex of $Y_{2,2,3}$ and a vertex of $Y_{1,2,3} \cup Y_{2,2,3}$ are not linked by a 2-edge;
(ii) a vertex of $Y_{1,3,3}$ and a vertex of $Y_{1,2,3} \cup Y_{1,3,3}$ are not linked by a 1-edge;
(iii) $s$ vertex of $Y_{2,2,2}$ is not adjacent in $K_{G}$ to three vertices of $Y_{2,2,3}$ (by 2-edges).

Proof of Theorem ??. Let $G$ be a 5 -minimal planar graph with girth at least 13. $G$ has minimum degree 2, so its kernel $K_{G}$ is defined. Moreover, by Lemma ?? (i), $Z_{7}=Y_{2,2,3} \cup Y_{1,3,3}$, so

$$
\begin{equation*}
z_{7}=y_{2,2,3}+y_{1,3,3} \tag{2}
\end{equation*}
$$

Let us count the number $e_{2}$ of 2-edges incident to vertices of $Y_{2,2,3}$. Recall that $Z_{4}=Y_{1,1,2}$ and $Z_{3}=Y_{1,1,1}$. Since 2-edges may not link two vertices of type (2,2,3) according to Lemma ?? (i), we have $e_{2}=2 y_{2,2,3}$. Moreover, the ends of such edges which are not in $Y_{2,2,3}$ have to be in $Y_{2,2,2} \cup Y_{1,2,2} \cup Z_{4}$ by Lemmas ?? and ?? (i). Furthermore, a vertex of $Y_{2,2,2}$ is incident to at most two edges of $e_{2}$ according to Lemma ?? (iii) and a vertex of $Y_{1,2,2}$ (resp. $Z_{4}$ ) is incident to at most two (resp. one) 2-edges. Therefore $e_{2} \leq 2 y_{2,2,2}+2 y_{1,2,2}+z_{4}$. So,

$$
\begin{equation*}
2 y_{2,2,3} \leq 2 y_{2,2,2}+2 y_{1,2,2}+z_{4} \tag{3}
\end{equation*}
$$

Let us now count the number $e_{1}$ of 1-edges incident to vertices of $Y_{1,3,3}$. Since 1-edges may not link two vertices of type $(1,3,3)$ according to Lemma ?? (ii), we have $e_{1}=y_{1,3,3}$. Moreover, the ends of such edges which are not in $Y_{1,1,3}$ have to be in $Y_{1,2,2} \cup Y_{1,1,3} \cup Z_{4} \cup Z_{3}$ by Lemmas ?? and ?? (ii). Furthermore, vertices of $Y_{1,2,2}$ (resp. $Y_{1,1,3} \cup Z_{4}, Z_{3}$ ) are incident to at most one (resp. two, three) 1-edges. Thus $e_{1} \leq y_{1,2,2}+2 y_{1,1,3}+2 z_{4}+3 z_{3}$. So,

$$
\begin{equation*}
y_{1,3,3} \leq y_{1,2,2}+2 y_{1,1,3}+2 z_{4}+3 z_{3} . \tag{4}
\end{equation*}
$$

$2 \times(\boldsymbol{?} \boldsymbol{?})+(\boldsymbol{?} \boldsymbol{?})$ yields $2 y_{2,2,3}+2 y_{1,3,3} \leq 2 y_{2,2,2}+4 y_{1,2,2}+4 y_{1,1,3}+5 z_{4}+6 z_{3}$. Hence, by Equation (??), $2 z_{7} \leq 2 z_{6}+4 z_{5}+5 z_{4}+6 z_{3}$, so

$$
z_{7} \leq z_{6}+2 z_{5}+\frac{5}{2} z_{4}+3 z_{3} .
$$

Now by Equation (??) the average degree of $G$ is

$$
A d(G)=\frac{7 z_{7}+6 z_{6}+5 z_{5}+4 z_{4}+3 z_{3}}{3 z_{7}+\frac{5}{2} z_{6}+2 z_{5}+\frac{3}{2} z_{4}+z_{3}} .
$$

As a function of $z_{7}$, this is a decreasing function (on $\mathbb{R}^{+}$); so it is minimum when $z_{7}$ is maximum that is equal to $z_{6}+2 z_{5}+\frac{5}{2} z_{4}+3 z_{3}$. So,

$$
A d(G) \geq \frac{13 z_{6}+19 z_{5}+\frac{43}{2} z_{4}+24 z_{3}}{\frac{11}{2} z_{6}+8 z_{5}+9 z_{4}+10 z_{3}} \geq \frac{26}{11}
$$

This contradicts the fact that $G$ has girth 13 by Proposition ??.

Remark 12 It is very likely that using the method below, one can prove that a graph $G$ with maximum average degree less than $\frac{26}{11}$ is 5 -choosable unless it contains $J_{6}$ as an induced subgraph. However, this will require the tedious study of a large number of configurations.

## 3 Proofs of Lemmas ?? and ??

In order to prove Lemmas ?? and ??, we need the following lemma proved in [?]. Let $S$ be a set of vertices of a $k$-minimal graph $G$. The function $p_{S}: S \rightarrow \mathbb{N}$ is defined by $p_{S}(v)=k-\left|N_{G^{2}}(v) \backslash S\right|$. Then $p_{S}(v)$ represents the minimum number of available colours at a vertex $v \in S$ once we have precoloured the square of $G-S$. Hence if $(G-S)^{2}$ is $k$-choosable, $(G-S)^{2}=G^{2}-S$ and $G^{2}[S]$ is $p_{S}$-choosable, one can extend any $k$-list-colouring of $G-S$ into a $k$-list-colouring of $G$, which is a contradiction.

Lemma 13 (Dvořák, Škrekovski and Tancer [?]) Let $S$ be a set of vertices of a k-minimal graph $G$. If $(G-S)^{2}=G^{2}-S$, then $G^{2}[S]$ is not $p_{S}$-choosable.

In order to use Lemma ??, we need some results on the choosability of some graphs.

### 3.1 Some choosability tools

Definition 14 Let $x$ and $y$ be two vertices of a graph $G$. An $(x-y)$-ordering of $G$ is an ordering of the vertices such that $x$ is the minimum and $y$ the maximum. An $(x, y-z)$-ordering is an ordering of the vertices such that $x$ is minimum, $y$ is the second minimum and $z$ is maximum.

Let $\sigma=\left(v_{1}<v_{2}<\ldots<v_{n}\right)$ be an ordering of the vertices of $G$ and $p$ a function $V(G) \rightarrow$ $\mathbb{N}$. $\sigma$ is p-greedy if, for every $i,\left|N\left(v_{i}\right) \cap\left\{v_{1}, \ldots, v_{i-1}\right\}\right|<p\left(v_{i}\right)$. It is $p$-nice if, for every $i$ except $n,\left|N\left(v_{i}\right) \cap\left\{v_{1}, \ldots, v_{i-1}\right\}\right|<p\left(v_{i}\right)$ and $d\left(v_{n}\right)=p\left(v_{n}\right)$. It is $p$-good if, for every $3 \leq i \leq n, \mid N\left(v_{i}\right) \cap$ $\left\{v_{1}, \ldots, v_{i-1}\right\} \mid-\epsilon\left(v_{i}\right)<p\left(v_{i}\right)$ with $\epsilon\left(v_{i}\right)=1$ if $v_{i}$ is adjacent to both $v_{1}$ and $v_{2}$ and $\epsilon\left(v_{i}\right)=0$ otherwise. By extension, if $k$ is an integer, we say that $\sigma$ is $k$-greedy (resp. $k$-nice, $k$-good) if it is $p$-greedy (resp. $p$-nice, $p$-good) when $p$ is the constant function with value $k$ (i. e. $p\left(v_{i}\right)=k$ for every $1 \leq i \leq n$ ).

The greedy algorithm according to greedy, nice and good orderings yields the following three lemmas.
Lemma 15 If $G$ has a p-greedy ordering then $G$ is p-choosable.
Proof. Applying the greedy algorithm according to the $p$-greedy ordering gives the desired colouring.

Lemma 16 Let $x y$ be an edge of graph $G$ and $L$ be a p-list-assignment of $G$. If $L(x) \not \subset L(y)$ and $G$ has a p-nice $(x-y)$-ordering, then $G$ is $L$-colourable.

Proof. Let $a$ be a colour in $L(x) \backslash L(y)$. Assign $a$ to $x$ and proceed the greedy algorithm according to the $p$-nice $(x-y)$-ordering.. The only vertex which has not more colour in its list than previously coloured neighbours is $y$ for which $|L(y)|=d(y)$. But since $a \notin L(y)$, at most $d(y)-1$ colours of $L(y)$ are assigned to the neighbours of $y$. Hence one can colour $y$.

Lemma 17 Let $x, y$ and $z$ be three vertices of a graph $G=(V, E)$ such that $x y \notin E$. If $L(x) \cap L(y) \neq \emptyset$ and $G$ has a $p$-good $(x, y-z)$-ordering, then $G$ is $L$-colourable.

Proof. Let $a$ be a colour in $L(x) \cap L(y)$ and $\sigma=\left(v_{1}<v_{2}<\ldots<v_{n}\right)$ be a $p$-good ( $\left.x, y-z\right)$-ordering. (In particular, $v_{1}=x, v_{2}=y$ and $v_{n}=z$.) Assign $a$ to $x$ and $y$ and proceed the greedy algorithm according to $\sigma$. For every $3 \leq i \leq n$, the number of colours assigned to already coloured neighbours of $v_{i}$ is at most $\left|N\left(v_{i}\right) \cap\left\{v_{1}, \ldots, v_{i-1}\right\}\right|-\epsilon\left(v_{i}\right)$ since $v_{1}$ and $v_{2}$ are coloured the same. Hence the greedy algorithm gives an $L$-colouring.

Remark 18 Note that if $x z, y z \in E$, a $p$-nice $(x, y-z)$-ordering is also $p$-good.

Definition 19 The blocks of a graph are its maximal 2-connected components. A connected graph is said to be a Gallai tree if each of its blocks is either a complete graph or an odd cycle.

The following theorem was proved independently by Borodin [?] and Erdős, Rubin and Taylor [?]:
Theorem 20 (Borodin [?], Erdős, Rubin and Taylor [?]) Let $G$ be a connected graph and $d_{G}$ the degree function in $G$. Then $G$ is $d_{G}$-choosable if and only if $G$ is not a Gallai tree.

Lemma 21 Let $G=(V, E)$ be a graph and $p: V(G) \rightarrow \mathbb{N}$. Let $S$ be a set of vertices such that $p(v) \geq d(v)$ for all $v \in S$. If $G[S]$ is not a Gallai tree and $G-S$ is $p$-choosable then $G$ is p-choosable.

Proof. Let $L$ be a $p$-list-assignment of $G$. Since $G-S$ is $p$-choosable, its admits an $L$-colouring $c$. Let us now extend it to $S$. The list $I(v)=L(v) \backslash\{c(w), w \in N(v) \backslash S\}$ of available colours of a vertex $v \in S$ is of size at least $p^{\prime}(v)=p(v)-|N(v) \backslash S| \geq d_{G[S]}(v)$. Since $G[S]$ is not a Gallai tree, by Theorem ??, $G[S]$ is $p^{\prime}$-choosable and thus $I$-colourable. So, $G$ is $L$-colourable.

A 4-regular graph $G$ is cycle + triangles if it is the edge union of a Hamiltonian cycle $C$ and a 2-factor consisting of triangles. In other words, the graph induced by the edges of $E(G) \backslash E(C)$ is the disjoint union of 3 -cycles.

Theorem 22 (Fleischner and Stiebitz [?]) Every cycle+triangles graph is 3-choosable.

### 3.2 Proof of Lemma ??

Lemma 23 Let $q \geq 2$ and $C_{4 q}=\left(v_{1}, \ldots, v_{4 q}, v_{1}\right)$ be the $4 q$-cycle and $p$ defined by $p\left(v_{i}\right)=4$ if $i$ is odd and $p\left(v_{i}\right)=2$ otherwise. Then $C_{4 q}^{2}$ is $p$-choosable.

Proof. The set $S$ of vertices $v$ for which $p(v) \geq d_{C_{4 q}^{2}}(v)$ is the set of $v_{i}$ with odd indices. $C_{4 q}^{2}[S]$ is a $2 q$-cycle and thus is not a Gallai tree. Moreover $C_{4 q}^{2 q}-S$ is also a $2 q$-cycle and is 2 -choosable. Hence Lemma ?? gives the result.

Proposition 24 Let $P_{7}=\left(v_{1}, \ldots, v_{7}\right)$ be a path and $p$ the function defined by $p\left(v_{1}\right)=p\left(v_{2}\right)=p\left(v_{6}\right)=$ $p\left(v_{7}\right)=2, p\left(v_{3}\right)=p\left(v_{5}\right)=4$ and $p\left(v_{4}\right)=3$. Then $P_{7}^{2}$ is $p$-choosable.

Proof. Let $L$ be a $p$-list-assignment of $P_{7}^{2}$. Since ( $v_{2}<v_{4}<v_{6}<v_{7}<v_{5}<v_{3}<v_{1}$ ) is a $p$-nice ordering of $P_{7}^{2}$, by Lemma ??, we may assume that $L\left(v_{1}\right)=L\left(v_{2}\right)$, and by symmetry of $P_{7}$ and $p$ that $L\left(v_{6}\right)=L\left(v_{7}\right)$.

Since ( $v_{1}<v_{4}<v_{2}<v_{6}<v_{7}<v_{5}<v_{3}$ ) is $p$-good, by Lemma ??, we may assume that $L\left(v_{1}\right) \cap$ $L\left(v_{4}\right)=\emptyset$, and by symmetry $L\left(v_{7}\right) \cap L\left(v_{4}\right)=\emptyset$.

Now one can find $c\left(v_{1}\right) \in L\left(v_{1}\right), c\left(v_{2}\right)$ in $L\left(v_{2}\right) \backslash\left\{c\left(v_{1}\right)\right\}, c\left(v_{6}\right)$ in $L\left(v_{6}\right), c\left(v_{7}\right)$ in $L\left(v_{7}\right) \backslash\left\{c\left(v_{6}\right)\right\}$, $c\left(v_{3}\right)$ in $L\left(v_{3}\right) \backslash\left\{c\left(v_{1}\right), c\left(v_{2}\right)\right\}$, and $c\left(v_{5}\right)$ in $L\left(v_{5}\right) \backslash\left\{c\left(v_{3}\right), c\left(v_{6}\right), c\left(v_{7}\right)\right\}$. Now since $L\left(v_{1}\right) \cap L\left(v_{4}\right)=\emptyset$ and $L\left(v_{1}\right)=L\left(v_{2}\right), c\left(v_{2}\right) \notin L\left(v_{4}\right)$. Analogously, $c\left(v_{6}\right) \notin L\left(v_{4}\right)$. Hence, $L\left(v_{4}\right) \backslash\left\{c\left(v_{2}\right), c\left(v_{3}\right), c\left(v_{5}\right), c\left(v_{6}\right)\right\}=$ $L\left(v_{4}\right) \backslash\left\{c\left(v_{3}\right), c\left(v_{5}\right)\right\} \neq \emptyset$. So, one can choose $c\left(v_{4}\right)$ in this set to get an $L$-colouring $c$ of $P_{7}^{2}$.

Lemma 25 For $1 \leq i \leq 17$, let $F_{i}$ be the graphs and $p_{i}$ be the functions depicted in Figure ??.
(i) $F_{1}^{2} \cup\left\{v_{5} v_{6}\right\}$ is $p_{1}$-choosable.
(ii) $F_{2}^{2} \cup\left\{v_{1} v_{4}\right\}$ and $F_{2}^{2} \cup\left\{v_{4} v_{7}\right\}$ are $p_{2}$-choosable.
(iii) $F_{3}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{3}$-choosable.
(iv) $F_{4}^{2}$ is 6-choosable.
(v) $F_{5}^{2} \cup\left\{v_{1} v_{4}, v_{1} v_{6}\right\}$ is $p_{5}$-choosable.
(vi) $F_{6}^{2}$ is $p_{6}$-choosable.
(vii) $F_{7}^{2} \cup\left\{v_{9} v_{10}\right\}$ is $p_{7}$-choosable.
(viii) $F_{8}^{2}$ is $p_{8}$-choosable.
(ix) $F_{9}^{2} \cup\left\{v_{2} v_{9}\right\}$ and $F_{9}^{2} \cup\left\{v_{6} v_{9}\right\}$ are $p_{9}$-choosable.
(x) $F_{10}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{10}$-choosable.
(xi) $F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}\right\}, F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{9} v_{4}\right\}$ and $F_{11}^{2} \cup\left\{v_{8} v_{9}, v_{9} v_{4}\right\}$ are $p_{11}$-choosable and $F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}$ is 5 -choosable.
(xii) $F_{12}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{12}$-choosable.
(xiii) $F_{13}^{2}$ is 6-choosable.

## Proof.

(i) In $F_{1}^{2} \cup\left\{v_{5} v_{6}\right\},\left(v_{6}<v_{5}<v_{4}<v_{3}<v_{1}<v_{2}\right)$ is $p_{1}$-greedy. So, by Lemma ??, $F_{1}^{2} \cup\left\{v_{5} v_{6}\right\}$ is $p_{1}$-choosable.
(ii) In $F_{2}^{2} \cup\left\{v_{4} v_{7}\right\},\left(v_{2}<v_{4}<v_{7}<v_{6}<v_{5}<v_{3}<v_{1}\right)$ is $p_{2}$-nice and $p_{2}\left(v_{2}\right)>p_{2}\left(v_{1}\right)$. So, by Lemma ??, $F_{2}^{2} \cup\left\{v_{4} v_{7}\right\}$ is $p_{2}$-choosable.
By symmetry, one shows that $F_{2}^{2} \cup\left\{v_{1} v_{4}\right\}$ is $p_{2}$-choosable.
(iii) In $F_{3}^{2} \cup\left\{v_{4} v_{8}\right\},\left(v_{2}<v_{8}<v_{4}<v_{7}<v_{6}<v_{5}<v_{3}<v_{1}\right)$ is $p_{3}$-nice and $p_{3}\left(v_{2}\right)>p_{3}\left(v_{1}\right)$. So, by Lemma ??, $F_{3}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{3}$-choosable.
(iv) Let $L$ be a 6 -list-assignment of $F_{4}^{2}$. Every ordering with maximum $v_{1}$ and second maximum $v_{7}$ is 6nice. Thus, by Lemma ??, we may assume that $L\left(v_{j}\right)=L\left(v_{1}\right)$ for $j \in\{2,3,4,5,6,8\}$. Analogously, we may assume that $L\left(v_{j}\right)=L\left(v_{7}\right)$ for $j \in\{2,3,4,5,6,8\}$. Hence all the lists are the same, say $\{1,2,3,4,5,6\}$. Now $c\left(v_{1}\right)=c\left(v_{5}\right)=1, c\left(v_{2}\right)=2, c\left(v_{3}\right)=c\left(v_{7}\right)=3, c\left(v_{4}\right)=4, c\left(v_{6}\right)=5$ and $c\left(v_{8}\right)=6$ is an $L$-colouring of $F_{4}^{2}$.
(v) In $F_{5}^{2} \cup\left\{v_{1} v_{4}, v_{1} v_{6}\right\},\left(v_{7}<v_{6}<v_{1}<v_{4}<v_{2}<v_{3}<v_{5}\right)$ is $p_{5}$-nice and $p_{5}\left(v_{7}\right)>p_{5}\left(v_{5}\right)$. So, by Lemma ??, $F_{5}^{2} \cup\left\{v_{1} v_{5}\right\}$ is $p_{5}$-choosable.
(vi) In $F_{6}^{2},\left(v_{4}<v_{2}<v_{8}<v_{1}<v_{3}<v_{5}\right)$ is $p_{6}$-nice and $p_{6}\left(v_{4}\right)>p_{6}\left(v_{5}\right)$. So, by Lemma ??, $F_{6}^{2}$ is $p_{6}$-choosable.
(vii) Let $L$ be a $p_{7}$-list-assignment of $F_{7}^{2} \cup\left\{v_{9} v_{10}\right\} .\left(v_{2}<v_{9}<v_{10}<v_{8}<v_{6}<v_{4}<v_{7}<v_{5}<v_{3}<v_{1}\right)$, $\left(v_{2}<v_{9}<v_{10}<v_{8}<v_{6}<v_{4}<v_{5}<v_{7}<v_{1}<v_{3}\right)$ and $\left(v_{4}<v_{9}<v_{10}<v_{8}<v_{6}<v_{2}<v_{5}<\right.$ $\left.v_{7}<v_{1}<v_{3}\right)$ are $p_{7}$-nice. Thus, by Lemma ??, we may assume that $L\left(v_{2}\right) \subset L\left(v_{1}\right), L\left(v_{2}\right) \subset L\left(v_{3}\right)$ and $L\left(v_{4}\right) \subset L\left(v_{3}\right)$. It follows that $L\left(v_{1}\right) \cap L\left(v_{4}\right) \neq \emptyset$. Because $\left(v_{1}<v_{4}<v_{10}<v_{9}<v_{2}<v_{8}<\right.$ $\left.v_{6}<v_{7}<v_{5}<v_{3}\right)$ is $p_{7}$-good, by Lemma ??, $F_{7}^{2} \cup\left\{v_{9} v_{10}\right\}$ is $L$-colourable.
(viii) In $F_{8}^{2},\left(v_{6}<v_{5}<v_{7}<v_{9}<v_{8}<v_{4}<v_{3}<v_{2}<v_{1}\right)$ is $p_{8}$-greedy. So, by Lemma ??, $F_{8}^{2}$ is $p_{8}$-choosable.


Figure 3: The graphs $F_{i}$ and functions $p_{i}$ for $1 \leq i \leq 13$
(ix) Let $L$ be a $p_{9}$-list-assignment of $F_{9}^{2} \cup\left\{v_{2} v_{9}\right\}$. Then $\left(v_{2}<v_{9}<v_{6}<v_{4}<v_{8}<v_{7}<v_{5}<v_{3}<v_{1}\right)$ and ( $v_{2}<v_{9}<v_{6}<v_{4}<v_{8}<v_{7}<v_{5}<v_{1}<v_{3}$ ) are $p_{9}$-nice so by Lemma ??, we may assume that $L\left(v_{2}\right) \subset L\left(v_{3}\right) \cap L\left(v_{1}\right)$. Moreover, $\left(v_{4}<v_{2}<v_{9}<v_{6}<v_{8}<v_{7}<v_{5}<v_{1}<v_{3}\right)$ is $p_{9}$-nice so by Lemma ??, we may assume that $L\left(v_{4}\right)=L\left(v_{3}\right)$. It follows that $L\left(v_{1}\right) \cap L\left(v_{4}\right) \neq \emptyset$. Thus, by Lemma ??, since $\left(v_{1}<v_{4}<v_{2}<v_{9}<v_{8}<v_{6}<v_{7}<v_{5}<v_{3}\right)$ is $p_{9}$-good, $F_{9}^{2} \cup\left\{v_{2} v_{9}\right\}$ is L-colourable.
By symmetry, one shows that $F_{9}^{2} \cup\left\{v_{6} v_{9}\right\}$ is $p_{9}$-choosable.
(x) In $F_{10}^{2} \cup\left\{v_{4} v_{8}\right\},\left(v_{2}<v_{8}<v_{4}<v_{6}<v_{7}<v_{5}<v_{3}<v_{1}\right)$ is $p_{10}$-nice and $p_{10}\left(v_{2}\right)>p_{10}\left(v_{1}\right)$. So, by Lemma ??, $F_{10}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{10}$-colourable.
(xi) Let $F \in\left\{F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}, F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}\right\}, F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{9} v_{4}\right\}, F_{11}^{2} \cup\left\{v_{8} v_{9}, v_{9} v_{4}\right\}\right\}$ and $L$ be a 5 -list-assignment if $F=F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}$ and a $p_{11}$-list-assignment of $F$ otherwise.
Then ( $v_{1}<v_{8}<v_{9}<v_{4}<v_{2}<v_{6}<v_{7}<v_{5}<v_{3}$ ), ( $v_{7}<v_{9}<v_{8}<v_{4}<v_{6}<v_{2}<v_{1}<v_{5}<v_{3}$ ) and ( $v_{7}<v_{9}<v_{8}<v_{4}<v_{6}<v_{2}<v_{1}<v_{3}<v_{5}$ ) are $p$-nice in $F$. So by Lemma ??, we may assume that $L\left(v_{1}\right)=L\left(v_{3}\right)=L\left(v_{5}\right)=L\left(v_{7}\right)$.
If $L\left(v_{8}\right) \not \subset L\left(v_{2}\right)$, let us colour $v_{8}$ with $c_{8} \in L\left(v_{8}\right) \backslash L\left(v_{2}\right), v_{4}$ with $c_{4} \in L\left(v_{4}\right) \backslash\left\{c_{8}\right\}$, $v_{9}$ with $c_{8} \in L\left(v_{8}\right) \backslash\left\{c_{4}, c_{8}\right\}, v_{1}$ and $v_{5}$ with the same colour $c_{1} \in L\left(v_{1}\right) \backslash\left\{c_{4}, c_{8}, c_{9}\right\}$, $v_{3}$ and $v_{7}$ with the same colour $c_{3} \in L\left(v_{1}\right) \backslash\left\{c_{1}, c_{4}, c_{8}, c_{9}\right\}$, $v_{6}$ with $c_{6} \in L\left(v_{6}\right) \backslash\left\{c_{1}, c_{3}, c_{8}, c_{9}\right\}$ and finally $v_{2}$ with $c_{2} \in L\left(v_{2}\right) \backslash\left\{c_{1}, c_{3}, c_{6}, c_{8}, c_{9}\right\}=L\left(v_{2}\right) \backslash\left\{c_{1}, c_{3}, c_{6}, c_{9}\right\}$. This gives an $L$-colouring of $F$. So we may assume that $L\left(v_{8}\right) \subset L\left(v_{2}\right)$. Exchanging the role of $c_{4}$ in $c_{8}$ in the preceding argument, we may assume that $L\left(v_{4}\right) \subset L\left(v_{2}\right)$. Moreover by symmetry, we may assume that $L\left(v_{9}\right) \cup L\left(v_{4}\right) \subset L\left(v_{6}\right)$. In particular, this implies that the sets $L\left(v_{8}\right) \cap L\left(v_{9}\right), L\left(v_{8}\right) \cap L\left(v_{4}\right), L\left(v_{9}\right) \cap L\left(v_{4}\right)$ and $L\left(v_{2}\right) \cap L\left(v_{6}\right)$ are non empty.
If $F=F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{9} v_{4}\right\}$ then $v_{8} v_{9} \notin F$. Hence $\left(v_{8}<v_{9}<v_{4}<v_{2}<v_{6}<v_{7}<v_{5}<v_{3}<v_{1}\right)$ is $p_{11}$-good, so by Lemma ??, $F$ is $L$-colourable.
If $F=F_{11}^{2} \cup\left\{v_{8} v_{9}, v_{9} v_{4}\right\}$ then $v_{8} v_{4} \notin F$. Hence $\left(v_{4}<v_{8}<v_{9}<v_{1}<v_{2}<v_{6}<v_{7}<v_{3}<v_{5}\right)$ is $p_{11}$-good, so by Lemma ??, $F$ is $L$-colourable.
If $F=F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}\right\}$ then $v_{9} v_{4} \notin F$. Hence $\left(v_{4}<v_{9}<v_{8}<v_{1}<v_{2}<v_{6}<v_{7}<v_{3}<v_{5}\right)$ is

If $F=F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}$, then $\left(v_{2}<v_{6}<v_{4}<v_{8}<v_{9}<v_{7}<v_{5}<v_{3}<v_{1}\right)$ is 5 -good. So, by Lemma ??, $F$ is $L$-colourable.
(xii) In $F_{12}^{2} \cup\left\{v_{4} v_{8}\right\},\left(v_{6}<v_{8}<v_{4}<v_{2}<v_{1}<v_{3}<v_{5}<v_{7}\right)$ is $p_{12}$-nice and $p_{12}\left(v_{6}\right)>p_{12}\left(v_{7}\right)$. So by Lemma ??, $F_{12}^{2} \cup\left\{v_{4} v_{8}\right\}$ is $p_{12}$-choosable.
(xiii) Let $L$ be a 6-list-assignment of $F_{13}^{2} . \quad\left(v_{2}<v_{9}<v_{10}<v_{8}<v_{6}<v_{4}<v_{7}<v_{5}<v_{3}<v_{1}\right)$, $\left(v_{2}<v_{9}<v_{10}<v_{8}<v_{6}<v_{4}<v_{5}<v_{7}<v_{1}<v_{3}\right)$ and $\left(v_{4}<v_{9}<v_{10}<v_{8}<v_{6}<v_{2}<v_{5}<\right.$ $\left.v_{7}<v_{1}<v_{3}\right)$ are 6 -nice. Thus, by Lemma ??, we may assume that $L\left(v_{1}\right)=L\left(v_{2}\right)=L\left(v_{3}\right)=L\left(v_{4}\right)$. Because ( $v_{1}<v_{4}<v_{10}<v_{9}<v_{2}<v_{8}<v_{6}<v_{7}<v_{5}<v_{3}$ ) is 6-good, by Lemma ??, $F_{13}^{2}$ is $L$-colourable.

## Proof of Lemma ??.

To prove this lemma, we will suppose for a contradiction that it does not hold. Then we will find a set $X$ of vertices contradicting Lemma ??. Indeed Lemma ?? will show that $G^{2}[X]$ is $p_{X}$-choosable and for each set $X$ we consider, every vertex of $X$ has at most one neighbour in $G-X$, so $(G-X)^{2}=G^{2}-X$. Lemma ?? completes the proof.
(i) Suppose for a contradiction that $v_{1}$ and $v_{3}$ are in $Y_{1,2,2}$. Let $v_{5}$ (resp. $v_{6}$ ) be the neighbour of $v_{1}$ (resp. $v_{3}$ ) distinct from $v_{2}$ and $v_{4}$. By Lemma ?? (iv), $v_{4}$ is in $V_{3}$ and $v_{5} \neq v_{6}$. Set $S=$ $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$. Then $G[S]=F_{1}, p_{S} \geq p_{1}$ and $G^{2}[S] \subset F_{1}^{2} \cup\left\{v_{5} v_{6}\right\}$. So Lemma ?? contradicts Lemma ??
(ii) Suppose for a contradiction that, in $K_{G}$, a vertex $v_{4}$ of $Y_{1,2,2}$ is adjacent to two vertices of $Y_{1,2,2}$ $v_{2}$ and $v_{6}$ by 2-edges. According to Lemma ?? (iii), $v_{2} \neq v_{6}$. Let $v_{3}$ and $v_{5}$ be the 2-neighbours of $v_{4}$ common with $v_{2}$ and $v_{6}$ respectively, and $v_{1}$ (resp. $v_{7}$ ) be the 2-neighbour of $v_{2}$ (resp. $v_{6}$ ) not adjacent to $v_{4}$. Set $S=\left\{v_{1}, \ldots, v_{7}\right\}$.

We first claim that $v_{1} \neq v_{7}$. Suppose not. Then $\left(v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\right)$ is a cycle $C$. It has no chord by Lemma ?? (ii), so $C^{2}=G^{2}[S]$. Moreover, $p_{S}\left(v_{i}\right) \geq 4$ if $i$ is even and $p_{S}\left(v_{i}\right) \geq 3$ otherwise. $C^{2}$ is a cycle+triangle graph, thus, by Theorem ??, it is 3 -choosable and so $p_{S}$-choosable. This contradicts Lemma ??.

Let $w_{1}$ (resp. $w_{7}$ ) be the neighbour of $v_{1}$ (resp. $w_{7}$ ) distinct from $v_{2}$ (resp. $v_{6}$ ) and for $i \in\{2,4,6\}$, let $w_{i}$ be the neighbour of $v_{i}$ not in $\left\{v_{i-1}, v_{i+1}\right\}$. Let $W=\left\{w_{1}, w_{2}, w_{4}, w_{6}, w_{7}\right\}$.

We claim that $W \cap S \neq \emptyset$. Indeed, suppose for a contradiction that $W \cap S \neq \emptyset$. Since $G$ is simple, $w_{1} \neq v_{2}$ and $w_{7} \neq v_{6}$. Moreover by Lemma ?? (i), $w_{1}$ and $w_{7}$ are in $V_{3}$, so $w_{1} \neq v_{7}$ and $w_{7} \neq v_{1}$. Furthermore, by Lemma ?? (ii), $w_{2} \neq v_{4}$ and $w_{6} \neq v_{4}$ and by Lemma ?? (iii), $w_{1} \neq v_{4}$ and $w_{7} \neq v_{4}$. Last, we may not have $w_{1}=v_{6}$ and $w_{2}=v_{7}$ otherwise the 4 -cycle $\left(v_{1}, v_{6}, v_{7}, v_{2}, v_{1}\right)$ would contradict Lemma ?? (iii). Then, by symmetry, we only need to consider the cases $w_{2}=v_{6}, w_{2}=v_{7}$.

- Assume that $w_{2}=v_{6}$. Then $G[S]=F_{2}, p_{S} \geq p_{2}$ and $G^{2}[S] \subset F_{2}^{2} \cup\left\{v_{1} v_{4}, v_{4} v_{7}, v_{1} v_{7}\right\}$. Thus, by Lemmas ?? and ??, $F_{2}^{2} \cup\left\{v_{1} v_{7}\right\} \subset G^{2}[S]$, so $w_{1}=w_{7}=v_{8}$. Let $T=S \cup\left\{v_{8}\right\}$. If $v_{8} \neq w_{4}$, then $G[T]=F_{3}$ and $p_{T} \geq p_{3}$ and $G^{2}[T] \subset F_{3}^{2} \cup\left\{v_{4} v_{8}\right\}$. So Lemma ?? contradicts Lemma ??. If not then $G[T]=G=F_{4}$, so $G$ is 6 -choosable, by Lemma ??. This is a contradiction.
- Suppose that $w_{2}=v_{7}$. Then $G[S]=F_{5}, p_{S} \geq p_{5}$ and $G^{2}[S] \subset F_{5}^{2} \cup\left\{v_{1} v_{4}, v_{1} v_{6}\right\}$. Thus Lemma ?? contradicts Lemma??.

This proves the claim.
Note that by Lemma ?? (ii), $w_{1} \neq w_{2}$ and $w_{6} \neq w_{7}$.
Suppose $w_{1}=w_{4}=v_{8}$. Then let $R=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{8}\right\}$ and $w_{8}$ the neighbour of $v_{8}$. Then $\left(G[R], p_{R}\right)=\left(F_{6}, p_{6}\right)$ and $G^{2}[R]=F_{6}^{2}$. Thus Lemma ?? contradicts Lemma ??. Therefore, $w_{1} \neq w_{4}$ and, by symmetry, $w_{4} \neq w_{7}$.

Suppose $w_{1}=w_{7}=v_{8}$. Let $T=S \cup\left\{v_{8}\right\}$. Then $G[T]$ is the cycle $C_{8}$ and $p_{T}$ is greater or equal to the function $p$ defined in Lemma ??. So, by Lemmas ?? and ??, $G^{2}[T] \neq C_{8}^{2}$. It follows that either $w_{2}=w_{6}$ or $w_{4}=w_{8}$ with $w_{8}$ be the neighbour of $v_{8}$ not in $S$.

- Suppose $w_{2}=w_{6}=v_{9}$, and $w_{4}=w_{8}=v_{10}$. Set $W=\left\{v_{1}, \ldots, v_{10}\right\}$. If $v_{9} v_{10} \notin E(G)$ then $\left.G[W]=F_{7}, p_{W} \geq p_{7}\right)$ and $G^{2}[W] \subset F_{7}^{2} \cup\left\{v_{9} v_{10}\right\}$; so Lemma ?? contradicts Lemma ??. If not, $G=G[W]=F_{13}$, so $G^{2}$ is 6 -choosable, according to Lemma ??, a contradiction.
- Suppose $w_{2}=w_{4}=w_{6}=v_{9}$. Setting $U=\left\{v_{1}, \ldots, v_{9}\right\}$, we have $\left(G[U], p_{U}\right)=\left(F_{8}, p_{8}\right)$ and $G^{2}[U]=F_{8}^{2}$. Hence Lemma ?? contradicts Lemma ??.

By symmetry, we get a contradiction if $w_{2}=w_{6}=w_{8}, w_{2}=w_{4}=w_{8}$ or $w_{4}=w_{6}=w_{8}$.

- Suppose $w_{4}=w_{8}=v_{9}, w_{2} \neq v_{9}, w_{6} \neq v_{9}$ and $w_{2} \neq w_{6}$. Setting $U=\left\{v_{1}, \ldots, v_{9}\right\}$, we have $G[U]=F_{9}, p_{U} \geq p_{9}$ and $G^{2}[U] \subset F_{9}^{2} \cup\left\{v_{2} v_{9}\right\}$ or $G^{2}[U] \subset F_{9}^{2} \cup\left\{v_{6} v_{9}\right\}$. Hence Lemma ?? contradicts Lemma ??.
By symmetry, we get a contradiction if $w_{2}=w_{6}=v_{9}, w_{4} \neq v_{9}, w_{8} \neq v_{9}$ and $w_{4} \neq w_{8}$.

Therefore, $w_{1} \neq w_{7}$.
Suppose that $w_{2}=w_{6}=v_{8}$. Let $T=S \cup\left\{v_{8}\right\}$. Then $G[T]=F_{10}, p_{T} \geq p_{10}$, and $G^{2}[T] \subset F_{10}^{2} \cup\left\{v_{4} v_{8}\right\}$, since $w_{1}, w_{4}$ and $w_{7}$ are distinct vertices. Hence Lemma ?? contradicts Lemma ??.
Therefore, $w_{2} \neq w_{6}$.
Suppose that $w_{1}=w_{6}=v_{8}$ and $w_{2}=w_{7}=v_{9}$. Let $U=S \cup\left\{v_{8}, v_{9}\right\}$. Then $G[U]=F_{11}$ and $G^{2}[U]$ is a subgraph of $F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}$. Moreover $p_{U} \geq p_{11}$ and, if $G^{2}[U]=F_{11}^{2} \cup\left\{v_{4} v_{8}, v_{8} v_{9}, v_{9} v_{4}\right\}$, $p_{U}\left(v_{i}\right)=5$ for $1 \leq i \leq 9$. . Hence Lemma ?? contradicts Lemma ??.

Therefore, $w_{1} \neq w_{6}$ or $w_{2} \neq w_{7}$. By symmetry, $w_{2} \neq w_{7}$.
Suppose $w_{1}=w_{6}=v_{8}$. Let $T=S \cup\left\{v_{8}\right\}$ and let $w_{8}$ be the neighbour of $v_{8}$ not in $S$. Then $G[T]=F_{12}, p_{T} \geq p_{12}$ and $G^{2}[T] \subset F_{12}^{2} \cup\left\{v_{4} v_{8}\right\}$. Hence Lemma ?? contradicts Lemma ??.
Therefore, $w_{1} \neq w_{6}$.
Hence all the $w_{i}$ are distinct, so $G[S]^{2}=G^{2}[S]$. Thus Proposition ?? contradicts Lemma ??.

Remark 26 The proof of Lemma ?? in the case of planar graphs of girth at least 9 is simpler and shorter because all the configurations considered in the above proof (except the path $P_{7}$ ) have girth less than 9 . Thus Corollary ?? has a short direct proof which requires only Proposition ??.

### 3.3 Proof of Lemma??

Definition 27 For $1 \leq j \leq 4$, let $I_{j}$ and $q_{j}$ be the graphs and functions depicted in Figure ??.


Figure 4: The graphs $I_{j}$ and functions $q_{j}, 1 \leq j \leq 4$

Lemma 28 For $1 \leq j \leq 4, I_{j}^{2}$ is $q_{j}$-choosable.
Proof.

- Let $L$ be a $q_{1}$-list-assignment of $I_{1}^{2}$. The orderings ( $v_{4}<v_{3}<v_{1}<v_{2}$ ) and ( $v_{1}<v_{3}<v_{4}<v_{2}$ ) are $q_{1}$-nice. So, by Lemma ??, we may assume that $L\left(v_{1}\right) \cup L\left(v_{4}\right) \subset L\left(v_{2}\right)$. Hence $L\left(v_{1}\right) \cap L\left(v_{4}\right) \neq \emptyset$. But ( $v_{4}<v_{1}<v_{3}<v_{2}$ ) is $q_{1}$-good. Thus, by Lemma ??, $I_{1}^{2}$ is $L$-colourable.
- Let $L$ be a $q_{2}$-list-assignment of $I_{2}^{2}$.

Suppose first that $L\left(v_{3}\right) \not \subset L\left(v_{1}\right) \cup L\left(v_{6}\right)$. Then choose $c\left(v_{3}\right)$ in $L\left(v_{3}\right) \backslash\left(L\left(v_{1}\right) \cup L\left(v_{6}\right)\right)$ and $c\left(v_{4}\right) \in$ $L\left(v_{4}\right) \backslash\left\{c\left(v_{3}\right)\right\}$. Since $I_{1}^{2}$ is $q_{1}$-choosable, one can extend $c$ to $\left\{v_{5}, v_{6}, v_{7}, v_{8}\right\}$. Then one can find $c\left(v_{2}\right) \in L\left(v_{2}\right) \backslash\left\{c\left(v_{3}\right), c\left(v_{4}\right), c\left(v_{5}\right)\right\}$ and $c\left(v_{1}\right) \in L\left(v_{1}\right) \backslash\left\{c\left(v_{2}\right), c\left(v_{3}\right)\right\}=L\left(v_{1}\right) \backslash\left\{c\left(v_{2}\right)\right\}$. Hence we may assume that $L\left(v_{3}\right) \subset L\left(v_{1}\right) \cup L\left(v_{6}\right)$, so $L\left(v_{3}\right)=L\left(v_{1}\right) \cup L\left(v_{6}\right)$ and $L\left(v_{1}\right) \cap L\left(v_{6}\right)=\emptyset$.
Suppose now that $L\left(v_{4}\right) \cap L\left(v_{6}\right) \neq \emptyset$. Then colour $v_{4}$ and $v_{6}$ with the same colour $c\left(v_{4}\right)=c\left(v_{6}\right) \in$ $L\left(v_{4}\right) \cap L\left(v_{6}\right)$. Choose $c\left(v_{8}\right) \in L\left(v_{8}\right) \backslash\left\{c\left(v_{6}\right)\right\}$ and $c\left(v_{7}\right) \in L\left(v_{7}\right) \backslash\left\{c\left(v_{6}\right), c\left(v_{8}\right)\right\}$. Now since $I_{1}^{2}$ is $q_{1}$-choosable, one can extend $c$ into an $L$-colouring of $I_{2}^{2}$. So we may assume that $L\left(v_{4}\right) \cap L\left(v_{6}\right)=\emptyset$. Now ( $v_{4}<v_{1}<v_{6}<v_{8}<v_{7}<v_{5}<v_{3}<v_{2}$ ) is $q_{2}$-good so, by Lemma ??, we may assume that $L\left(v_{4}\right) \cap L\left(v_{1}\right)=\emptyset$. It follows that $L\left(v_{4}\right) \cap L\left(v_{3}\right)=\emptyset$ since $L\left(v_{3}\right)=L\left(v_{1}\right) \cup L\left(v_{6}\right)$.
The ordering ( $v_{4}<v_{8}<v_{6}<v_{7}<v_{5}<v_{3}<v_{1}<v_{2}$ ) is $q_{2}$-nice so, by Lemma ??, we may assume that $L\left(v_{4}\right) \subset L\left(v_{2}\right)$. Then one may assign $c\left(v_{4}\right) \in L\left(v_{4}\right)$ and $c\left(v_{2}\right) \in L\left(v_{4}\right) \backslash\left\{c\left(v_{4}\right)\right\}$ to the vertices $v_{4}$ and $v_{2}$. Now, because $L\left(v_{4}\right) \cap L\left(v_{3}\right)=\emptyset$, one can extend $c$ into an $L$-colouring of $I_{2}^{2}$ by colouring greedily according to the ordering ( $v_{1}<v_{8}<v_{6}<v_{7}<v_{5}<v_{3}$ ).

- Let $L$ be a $q_{3}$-list-assignment of $I_{3}^{2}$. Assign to $v_{5}$ a colour $c_{5}$ in $L\left(v_{5}\right) \backslash\left(L\left(v_{1}\right) \cup L\left(v_{9}\right)\right)$ and to $v_{6}$ a colour in $L\left(v_{6}\right) \backslash\left(L\left(v_{8}\right) \cup\left\{c_{5}\right\}\right)$. Then colour the remaining vertices greedily according to ( $v_{3}<v_{4}<v_{2}<v_{1}<v_{9}<v_{7}<v_{8}$ ) to get an $L$-colouring of $I_{3}^{2}$.
- Let $L$ be $q_{4}$-list-assignment of $I_{4}^{2}$. Pick $c\left(y_{1}\right)$ in $L\left(y_{1}\right) \backslash L\left(w_{1}\right), c\left(y_{2}\right)$ in $L\left(y_{2}\right) \backslash\left(L\left(w_{2}\right) \cup\left\{c\left(y_{1}\right)\right\}\right), c\left(y_{3}\right)$ in $L\left(y_{3}\right) \backslash\left(L\left(w_{3}\right) \cup\left\{c\left(y_{1}\right), c\left(y_{2}\right)\right\}\right)$ and $c(x)$ in $L(x) \backslash\left\{c\left(y_{1}\right), c\left(y_{2}\right), c\left(y_{3}\right)\right\}$. Since $I_{1}^{2}$ is $q_{1}$-choosable, one can extend $c$ to a colouring of $I_{4}^{2}$.


## Proof of Lemma ??.

(i) Suppose that a vertex $v_{3}$ of $Y_{2,2,3}$ and $v_{6}$ of $Y_{1,2,3} \cup Y_{2,2,3}$ are adjacent via a 2-edge in $K_{G}$. Then the subgraph of $G$ induced by $v_{3}, v_{6}$ and the 2 -vertices of their incident threads contains $I_{2}$ as an induced subgraph. (It is $I_{2}$ if $v_{6}$ is in $Y_{1,2,3}$ and has one extra vertex otherwise.) Since $G$ has girth at least 13 , then $G^{2}\left[V\left(I_{2}\right)\right]=I_{2}^{2},\left(G-V\left(I_{2}\right)\right)^{2}=G^{2}-V\left(I_{2}\right)$ and $p_{V\left[I_{2}\right]}=q_{2}$, so Lemma ?? contradicts Lemma ??.
(ii) Suppose that a vertex $v_{5}$ of $Y_{1,3,3}$ and $v_{6}$ of $Y_{1,2,3} \cup Y_{1,3,3}$ are adjacent via a 1-edge in $K_{G}$. Then the subgraph of $G$ induced by $v_{5}, v_{6}$ and the 2 -vertices of their incident threads contains $I_{3}$. So Lemma ?? contradicts Lemma ??.
(iii) Suppose that a vertex $x$ of $Y_{2,2,2}$ is adjacent to three vertices $v_{1}, v_{2}$ and $v_{3}$ of $Y_{2,2,3}$ in $K_{G}$. Then the subgraph of $G$ induced by $x, v_{1}, v_{2}, v_{3}$ and the 2 -vertices of their incident threads is $I_{4}$. So Lemma ?? contradicts Lemma ??.

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