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# On the identification of hidden pointwise Hölder EXPONENTS 

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## Un court résumé en français

Mots clés: Processus, Statistique mathématique.
L'exposant de Hölder ponctuel (EHP) d'un processus stochastique $X$ permet de mesurer la régularité locale de $X$. Nombre d'auteurs se sont déjà intéressés au problème de l'estimation de cet exposant à partir de l'observation d'une trajectoire discrétisée de $X$ (voir par exemple $[1,4,5,9]$ ). Cependant, il ne semble pas toujours réaliste de supposer qu'une telle observation est directement accessible mais seulement une version corrompue de celle-ci. Est-il alors possible d'effectuer l'estimation?

A notre connaissance, cette question a été assez peu étudiée dans la littérature et les articles qui l'abordent se placent dans un cadre qui est essentiellement celui de processus Gaussiens à accoissements stationnaires [8, 11]; l'EHP a alors une structure relativement simple puisqu'il reste contant au cours du temps. L'objectif de notre exposé est d'etudier cette question dans un cadre nouveau, celui du mouvement Brownien multifractionnaire; l'EHP de ce processus a généralement une structure assez complexe parce qu'il varie d'un instant à l'autre (voir par $[3,10,2]$ ).

## A long summary in English

Let $X=\{X(t)\}_{t \in[0,1]}$ be a stochastic process with continuous and nowhere differentiable trajectories, the pointwise Hölder exponent (PHE) of $X$, is the stochastic process denoted by $\alpha_{X}=\left\{\alpha_{X}(t)\right\}_{t \in[0,1]}$ and defined as

$$
\begin{equation*}
\alpha_{X}(t):=\sup \left\{a: \limsup _{t+h \in[0,1], h \rightarrow 0} \frac{|X(t+h)-X(t)|}{|h|^{a}}=0\right\} . \tag{0.1}
\end{equation*}
$$

It measures the local smoothness of $X$ : the larger is $\alpha_{X}(t)$ the more regular is the process $X$ in a neighborhood of the point $t$. Since several years, a number of authors have been interested in the statistical problem of the estimation of $\alpha_{X}(t)$ starting from the observation of a discretized trajectory of the process $X$ (see for example $[\mathbf{1 , 4 , 5 , 5 ]}$ ). However, it does not always seem to be realistic to assume that such an observation is available but only a corrupted version of it and a natural question one can address is that whether it is still possible to estimate $\alpha_{X}(t)$. To our knowledge, only a few number
of articles in the literature deal with this problem and it has been studied only in a setting which basically remains to be that of Gaussian stationary increments processes; typically when the hidden process $X$ is a fractional Brownian motion (fBm for short) $[8,11]$; in such a setting the hidden PHE of $X$ has a rather simple structure since it can not evolve with time. Recall that the fBm of Hurst parameter $H \in(0,1)$ is usually denoted by $B_{H}=\left\{B_{H}(t)\right\}_{t \in[0,1]}$ and can be defined as the stochastic integral (harmonizable representation):

$$
\begin{equation*}
B_{H}(t):=\int_{\mathbb{R}} \frac{\left(e^{i t \xi}-1\right)}{|\xi|^{H+1 / 2}} \mathrm{~d} \hat{W}(\xi), \tag{0.2}
\end{equation*}
$$

where $d \hat{W}$ is a complex-valued Weiner measure, with adapted real and imaginary parts such that the integral in (0.2) be real-valued.

- When $H=1 / 2, \mathrm{fBm}$ reduces to Brownian motion (Bm).
- Even though fBm constitutes a quite natural extension of Bm , there is a considerable difference between these two processes when $H \neq 1 / 2$; fBm is non Markovian and its increments even display long-range dependence if $H>1 / 2$.
- $\left\{\alpha_{B_{H}}(t)\right\}_{t \in[0,1]}$ the PHE of the $\mathrm{fBm}\left\{B_{H}(t)\right\}_{t \in[0,1]}$ can not change with time (see the graph below), since it satisfies almost surely (a.s.) for all $t \in[0,1]$

$$
\begin{equation*}
\alpha_{B_{H}}(t)=H . \tag{0.3}
\end{equation*}
$$

The goal of our talk is to study the statistical problem of the estimation of a hidden PHE in a new setting where this exponent has a rather complex structure since it is allowed to evolve with time. More precisely we assume that the corresponding hidden process $X$ is a multifractional Brownian motion ( mBm ). Recall that mBm can be obtained by replacing in the harmonizable representation of fBm (see (0.2)) the constant Hurst parameter $H$ by a functional parameter $H(t)$ depending on the time variable $t$. Therefore, mBm can be represented as:

$$
\begin{equation*}
X(t):=\int_{\mathbb{R}} \frac{\left(e^{i t \xi}-1\right)}{|\xi|^{H(t)+1 / 2}} \mathrm{~d} \hat{W}(\xi) \tag{0.4}
\end{equation*}
$$

Throughout this talk we suppose that $H(\cdot)$ is a two times continuously differentiable function taking it values in $\left[H_{*}, H^{*}\right] \subset(1 / 2,1)$, where $H_{*}:=\min _{t \in[0,1]} H(t)$ and $H^{*}:=$ $\max _{t \in[0,1]} H(t)$ are assumed to be known.

- mBm reduces to fBm when the function $H(\cdot)$ is constant.
- mBm is more flexible than fBm since its local Hölder regularity can change from one point to another (see the graph below). More precisely it has been shown in [3, $10,2]$ that a.s. for all $t \in[0,1]$

$$
\begin{equation*}
\alpha_{X}(t)=H(t) . \tag{0.5}
\end{equation*}
$$



Also we assume that we observe the high frequency discrete data $Z(j /(2 n)), j=$ $0, \ldots, 2 n$; the non Gaussian process $Z=\{Z(t)\}_{t \in[0,1]}$ is the "corrupted version" of the $m B m X$, it is defined as

$$
\begin{equation*}
Z(t):=z_{0}+\int_{0}^{t} \Phi(X(s)) \mathrm{d} W(s) \tag{0.6}
\end{equation*}
$$

where:

- $z_{0}$ is a deterministic real number.
- $\{W(s)\}_{s \in[0,1]}$ is a standard Brownian motion.
- $\Phi$ is a unknown deterministic function defined on the real line which vanishes. We assume that $\Phi$ vanishes only on a Lebesgue negligible set. We also assume that the function $f$, defined for every $x \in \mathbb{R}$, as

$$
\begin{equation*}
f(x):=(\Phi(x))^{2}, \tag{0.7}
\end{equation*}
$$

is two times continuously differentiable and has a at most a slow increase at infinity as well as its two derivatives i.e. there exist two constants $c>0$ and $L>0$ such that we have for each $x \in \mathbb{R}$,

$$
\begin{equation*}
\sum_{l=0}^{2}\left|f^{(l)}(x)\right| \leq c\left(1+|x|^{L}\right) . \tag{0.8}
\end{equation*}
$$

- $\{X(s)\}_{s \in[0,1]}$ is the hidden mBm which is supposed to be independent of the Bm $\{W(s)\}_{s \in[0,1]}$.

It is worth noticing that there is a considerable loss of information when we observe $Z$ instead of $X$, in fact contrarily to $X$ the PHE of Z is at each point a.s. equal to $1 / 2$ (see the graph below).


Yet we will show that in spite of this considerable loss of information it is still possible, starting from the observation of a discretized trajectory of $Z$, to estimate the PHE of $X$ at each point.

Also it is worth noticing that models of the type (0.6) can be viewed as stochastic volatility models where $Z(t)$ is the logarithm of the price of an underlying asset at time $t$. Such models have been already considered by Gloter and Hoffmann [7] in the case where $X$ is fBm and by Rosenbaum [10] in a more general case where $X$ can nicely be expressed in terms of a Gaussian stationary increment process. Let us mention in passing that in [7], Gloter and Hoffmann were not interested in the problem of the estimation of PHE (which they suppose to be known) but in that of a parametric estimation of the hidden volatility. In [10], Rosenbaum was interested in the same problem of us and he constructed a wavelet estimator of the hidden PHE which converges at the optimal rate in a minimax sense. We preferred to adopt a rather different estimation strategy in order to be able to obtain a Central Limit Theorem for our estimator.

Let us now state our main result, to this end first we need to introduce some notations.

- We denote by $\beta \in(0,1)$ a fixed parameter and for all integer $n$ large enough, we set $N_{n}:=\left[n^{\beta}\right]$ and $m_{n}:=\left[n / N_{n}\right],[\cdot]$ being the integer part function. We also set

$$
\widehat{Y}_{i, n}:=N_{n} \sum_{k=0}^{j_{i+1}-j_{i}-1}\left(Z\left(\left(j_{i}+k+1\right) / n\right)-Z\left(\left(j_{i}+k\right) / n\right)\right)^{2}
$$

where for all $i=0, \ldots, N_{n}, j_{i}:=\left[i n / N_{n}\right]$. It is worth noticing that $\widehat{Y}_{i, n}$ provides an estimation of the following average value of the hidden process $\{Y(s)\}_{s \in[0,1]}:=$ $\{f(X(s))\}_{s \in[0,1]}:$

$$
\bar{Y}_{i, N_{n}}:=N_{n} \int_{i / N_{n}}^{(i+1) / N_{n}} Y(s) d s
$$

moreover, the smaller is the parameter $\beta$ the more precise is the estimation.

- The generalized increments of the discrete process $\left\{\widehat{Y}_{i, n}\right\}_{i}$ are defined for all $i \in$ $\left\{0, \ldots, N_{n}-p-1\right\}$ as,

$$
\Delta_{a} \widehat{Y}_{i, n}:=\sum_{k=0}^{p} a_{k} \widehat{Y}_{i+k, n}
$$

where $a=\left(a_{0}, \ldots, a_{p}\right) \in \mathbb{R}^{p+1}$ is an arbitrary finite fixed sequence having $M \geq 2$ vanishing moments i.e.:

$$
\sum_{k=0}^{p} k^{l} a_{k}=0, \text { for all } l=0, \ldots, M-1
$$

and

$$
\sum_{k=0}^{p} k^{M} a_{k} \neq 0
$$

- Let $t_{0} \in(0,1)$ be an arbitrary fixed point, We denote by $\nu_{N_{n}}\left(t_{0}\right)$ the set of indices around $t_{0}$, defined as

$$
\nu_{N_{n}}\left(t_{0}\right)=\left\{i \in\left\{0, \ldots, N_{n}-p-1\right\}:\left|i / N_{n}-t_{0}\right| \leq N_{n}^{-\gamma}\right\},
$$

where $\gamma \in(0,1)$ is a fixed parameter which allows to control the size of $\nu_{N_{n}}\left(t_{0}\right)$.

- Finally, for any $n$ big enough, we denote by $V_{n}$ the localized generalized quadratic variation defined as

$$
V_{n}=\left(\sum_{i \in \nu_{N_{n}}\left(t_{0}\right)}\left(\Delta_{a} \widehat{Y}_{i, n}\right)^{2}\right)^{1 / 2}
$$

Our main result is the following theorem:
Theorem 1 We observe $Z(i /(2 n)), i=0, \ldots, 2 n$. Let $t_{0} \in(0,1)$ be an arbitrary fixed point. We set

$$
\widehat{H}_{n, t_{0}}=\frac{1-\gamma}{2}+\frac{\log _{2}\left(V_{n} / V_{2 n}\right)}{\beta}
$$

- When the parameter $\beta$ satisfies the inequality $\beta<1 /\left(4 H^{*}+2\right)$, then

$$
\widehat{H}_{n, t_{0}} \xrightarrow[n \rightarrow+\infty]{\text { a.s. }} H\left(t_{0}\right) .
$$

- When the parameters $\beta$ and $\gamma$ satisfy the inequalities $\gamma>1 /\left(2 H_{*}+1\right)$ and $\beta<$ $1 /\left(2 H^{*}+2-\gamma\right)$, then

$$
n^{\beta(1-\gamma) / 2}\left(\widehat{H}_{n, t_{0}}-H\left(t_{0}\right)\right)
$$

converges in distribution to a centred Gaussian variable.
Recall that $H_{*}:=\min _{t \in[0,1]} H(t)$ and $H^{*}:=\max _{t \in[0,1]} H(t)$.

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