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# ON THE IDENTIFICATION OF HIDDEN POINTWISE HÖLDER EXPONENTS

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## Un court résumé en français

**Mots clés:** *Processus, Statistique mathématique.*

L'exposant de Hölder ponctuel (EHP) d'un processus stochastique  $X$  permet de mesurer la régularité locale de  $X$ . Nombre d'auteurs se sont déjà intéressés au problème de l'estimation de cet exposant à partir de l'observation d'une trajectoire discrétisée de  $X$  (voir par exemple [1, 4, 5, 9]). *Cependant, il ne semble pas toujours réaliste de supposer qu'une telle observation est directement accessible mais seulement une version corrompue de celle-ci. Est-il alors possible d'effectuer l'estimation ?*

A notre connaissance, cette question a été assez peu étudiée dans la littérature et les articles qui l'abordent se placent dans un cadre qui est essentiellement celui de processus Gaussiens à accroissements stationnaires [8, 11]; l'EHP a alors une structure relativement simple puisqu'il reste constant au cours du temps. L'objectif de notre exposé est d'étudier cette question dans un cadre nouveau, celui du mouvement Brownien multifractionnaire; l'EHP de ce processus a généralement une structure assez complexe parce qu'il varie d'un instant à l'autre (voir par [3, 10, 2]).

## A long summary in English

Let  $X = \{X(t)\}_{t \in [0,1]}$  be a stochastic process with continuous and nowhere differentiable trajectories, the pointwise Hölder exponent (PHE) of  $X$ , is the stochastic process denoted by  $\alpha_X = \{\alpha_X(t)\}_{t \in [0,1]}$  and defined as

$$\alpha_X(t) := \sup \left\{ a : \limsup_{t+h \in [0,1], h \rightarrow 0} \frac{|X(t+h) - X(t)|}{|h|^a} = 0 \right\}. \quad (0.1)$$

It measures the local smoothness of  $X$ : the larger is  $\alpha_X(t)$  the more regular is the process  $X$  in a neighborhood of the point  $t$ . Since several years, a number of authors have been interested in the statistical problem of the estimation of  $\alpha_X(t)$  starting from the observation of a discretized trajectory of the process  $X$  (see for example [1, 4, 5, 9]). *However, it does not always seem to be realistic to assume that such an observation is available but only a corrupted version of it and a natural question one can address is that whether it is still possible to estimate  $\alpha_X(t)$ .* To our knowledge, only a few number

of articles in the literature deal with this problem and it has been studied only in a setting which basically remains to be that of Gaussian stationary increments processes; typically when the hidden process  $X$  is a fractional Brownian motion (fBm for short) [8, 11]; in such a setting the hidden PHE of  $X$  has a rather simple structure since it can not evolve with time. Recall that the fBm of Hurst parameter  $H \in (0, 1)$  is usually denoted by  $B_H = \{B_H(t)\}_{t \in [0,1]}$  and can be defined as the stochastic integral (harmonizable representation):

$$B_H(t) := \int_{\mathbb{R}} \frac{(e^{it\xi} - 1)}{|\xi|^{H+1/2}} d\hat{W}(\xi), \quad (0.2)$$

where  $d\hat{W}$  is a complex-valued Weiner measure, with adapted real and imaginary parts such that the integral in (0.2) be real-valued.

- When  $H = 1/2$ , fBm reduces to Brownian motion (Bm).
- Even though fBm constitutes a quite natural extension of Bm, there is a considerable difference between these two processes when  $H \neq 1/2$ ; fBm is non Markovian and its increments even display long-range dependence if  $H > 1/2$ .
- $\{\alpha_{B_H}(t)\}_{t \in [0,1]}$  the PHE of the fBm  $\{B_H(t)\}_{t \in [0,1]}$  can not change with time (see the graph below), since it satisfies almost surely (a.s.) for all  $t \in [0, 1]$

$$\alpha_{B_H}(t) = H. \quad (0.3)$$

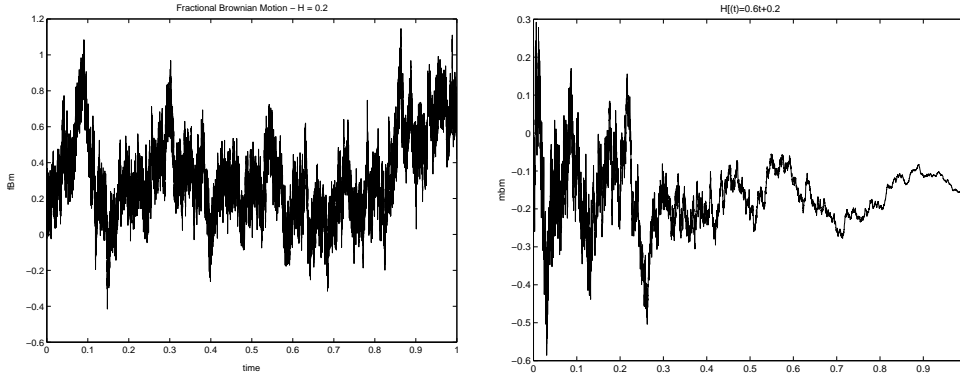
The goal of our talk is to study the statistical problem of the estimation of a hidden PHE in a new setting where this exponent has a rather complex structure since it is allowed to evolve with time. More precisely we assume that the corresponding hidden process  $X$  is a multifractional Brownian motion (mBm). Recall that mBm can be obtained by replacing in the harmonizable representation of fBm (see (0.2)) the constant Hurst parameter  $H$  by a functional parameter  $H(t)$  depending on the time variable  $t$ . Therefore, mBm can be represented as:

$$X(t) := \int_{\mathbb{R}} \frac{(e^{it\xi} - 1)}{|\xi|^{H(t)+1/2}} d\hat{W}(\xi). \quad (0.4)$$

Throughout this talk we suppose that  $H(\cdot)$  is a two times continuously differentiable function taking its values in  $[H_*, H^*] \subset (1/2, 1)$ , where  $H_* := \min_{t \in [0,1]} H(t)$  and  $H^* := \max_{t \in [0,1]} H(t)$  are assumed to be known.

- mBm reduces to fBm when the function  $H(\cdot)$  is constant.
- mBm is more flexible than fBm since its local Hölder regularity can change from one point to another (see the graph below). More precisely it has been shown in [3, 10, 2] that a.s. for all  $t \in [0, 1]$

$$\alpha_X(t) = H(t). \quad (0.5)$$



Also we assume that we observe the high frequency discrete data  $Z(j/(2n))$ ,  $j = 0, \dots, 2n$ ; the non Gaussian process  $Z = \{Z(t)\}_{t \in [0,1]}$  is the "corrupted version" of the mBm  $X$ , it is defined as

$$Z(t) := z_0 + \int_0^t \Phi(X(s)) dW(s), \quad (0.6)$$

where:

- $z_0$  is a deterministic real number.
- $\{W(s)\}_{s \in [0,1]}$  is a standard Brownian motion.
- $\Phi$  is a unknown deterministic function defined on the real line which vanishes. We assume that  $\Phi$  vanishes only on a Lebesgue negligible set. We also assume that the function  $f$ , defined for every  $x \in \mathbb{R}$ , as

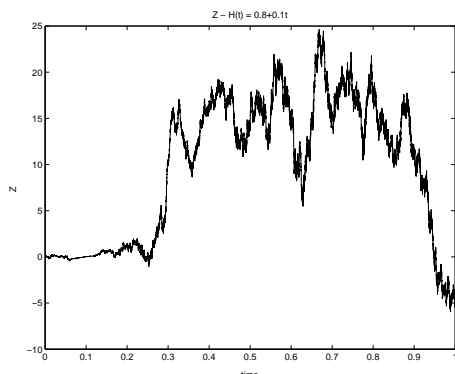
$$f(x) := (\Phi(x))^2, \quad (0.7)$$

is two times continuously differentiable and has a at most a slow increase at infinity as well as its two derivatives i.e. there exist two constants  $c > 0$  and  $L > 0$  such that we have for each  $x \in \mathbb{R}$ ,

$$\sum_{l=0}^2 |f^{(l)}(x)| \leq c(1 + |x|^L). \quad (0.8)$$

- $\{X(s)\}_{s \in [0,1]}$  is the hidden mBm which is supposed to be independent of the Bm  $\{W(s)\}_{s \in [0,1]}$ .

It is worth noticing that there is a considerable loss of information when we observe  $Z$  instead of  $X$ , in fact contrarily to  $X$  the PHE of  $Z$  is at each point a.s. equal to  $1/2$  (see the graph below).



Yet we will show that *in spite of this considerable loss of information it is still possible, starting from the observation of a discretized trajectory of  $Z$ , to estimate the PHE of  $X$  at each point.*

Also it is worth noticing that models of the type (0.6) can be viewed as stochastic volatility models where  $Z(t)$  is the logarithm of the price of an underlying asset at time  $t$ . Such models have been already considered by Gloter and Hoffmann [7] in the case where  $X$  is fBm and by Rosenbaum [10] in a more general case where  $X$  can nicely be expressed in terms of a Gaussian stationary increment process. Let us mention in passing that in [7], Gloter and Hoffmann were not interested in the problem of the estimation of PHE (which they suppose to be known) but in that of a parametric estimation of the hidden volatility. In [10], Rosenbaum was interested in the same problem of us and he constructed a wavelet estimator of the hidden PHE which converges at the optimal rate in a minimax sense. We preferred to adopt a rather different estimation strategy in order to be able to obtain a Central Limit Theorem for our estimator.

Let us now state our main result, to this end first we need to introduce some notations.

- We denote by  $\beta \in (0, 1)$  a fixed parameter and for all integer  $n$  large enough, we set  $N_n := \lfloor n^\beta \rfloor$  and  $m_n := \lfloor n/N_n \rfloor$ ,  $\lfloor \cdot \rfloor$  being the integer part function. We also set

$$\widehat{Y}_{i,n} := N_n \sum_{k=0}^{j_{i+1}-j_i-1} \left( Z((j_i+k+1)/n) - Z((j_i+k)/n) \right)^2.$$

where for all  $i = 0, \dots, N_n$ ,  $j_i := \lfloor in/N_n \rfloor$ . It is worth noticing that  $\widehat{Y}_{i,n}$  provides an estimation of the following average value of the hidden process  $\{Y(s)\}_{s \in [0,1]} := \{f(X(s))\}_{s \in [0,1]}$ :

$$\bar{Y}_{i,N_n} := N_n \int_{i/N_n}^{(i+1)/N_n} Y(s) ds;$$

moreover, the smaller is the parameter  $\beta$  the more precise is the estimation.

- The generalized increments of the discrete process  $\{\widehat{Y}_{i,n}\}_i$  are defined for all  $i \in \{0, \dots, N_n - p - 1\}$  as,

$$\Delta_a \widehat{Y}_{i,n} := \sum_{k=0}^p a_k \widehat{Y}_{i+k,n},$$

where  $a = (a_0, \dots, a_p) \in \mathbb{R}^{p+1}$  is an arbitrary finite fixed sequence having  $M \geq 2$  vanishing moments i.e.:

$$\sum_{k=0}^p k^l a_k = 0, \text{ for all } l = 0, \dots, M - 1.$$

and

$$\sum_{k=0}^p k^M a_k \neq 0.$$

- Let  $t_0 \in (0, 1)$  be an arbitrary fixed point, We denote by  $\nu_{N_n}(t_0)$  the set of indices around  $t_0$ , defined as

$$\nu_{N_n}(t_0) = \left\{ i \in \{0, \dots, N_n - p - 1\} : |i/N_n - t_0| \leq N_n^{-\gamma} \right\},$$

where  $\gamma \in (0, 1)$  is a fixed parameter which allows to control the size of  $\nu_{N_n}(t_0)$ .

- Finally, for any  $n$  big enough, we denote by  $V_n$  the localized generalized quadratic variation defined as

$$V_n = \left( \sum_{i \in \nu_{N_n}(t_0)} (\Delta_a \widehat{Y}_{i,n})^2 \right)^{1/2}.$$

Our main result is the following theorem:

**Theorem 1** *We observe  $Z(i/(2n)), i = 0, \dots, 2n$ . Let  $t_0 \in (0, 1)$  be an arbitrary fixed point. We set*

$$\widehat{H}_{n,t_0} = \frac{1 - \gamma}{2} + \frac{\log_2(V_n/V_{2n})}{\beta}.$$

- *When the parameter  $\beta$  satisfies the inequality  $\beta < 1/(4H^* + 2)$ , then*

$$\widehat{H}_{n,t_0} \xrightarrow[n \rightarrow +\infty]{a.s.} H(t_0).$$

- *When the parameters  $\beta$  and  $\gamma$  satisfy the inequalities  $\gamma > 1/(2H_* + 1)$  and  $\beta < 1/(2H^* + 2 - \gamma)$ , then*

$$n^{\beta(1-\gamma)/2} (\widehat{H}_{n,t_0} - H(t_0))$$

*converges in distribution to a centred Gaussian variable.*

*Recall that  $H_* := \min_{t \in [0,1]} H(t)$  and  $H^* := \max_{t \in [0,1]} H(t)$ .*

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