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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Mean field Limit of Non-Smooth Systems and  
Differential Inclusions*

Nicolas Gast — Bruno Gaujal

**N° 7315**

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\_\_\_\_\_ Thème NUM \_\_\_\_\_



# Mean field Limit of Non-Smooth Systems and Differential Inclusions

Nicolas Gast, Bruno Gaujal

Thème NUM — Systèmes numériques  
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**Abstract:** In this paper, we study deterministic limits of Markov processes made of several interacting objects. While most classical results assume that the limiting dynamics has Lipschitz properties, we show that these conditions are not necessary to prove convergence to a deterministic system.

We show that under mild assumptions, the stochastic system converges to the solution of a differential inclusion and we provide a simple way to compute the limit inclusion. When this differential inclusion satisfies a one-sided Lipschitz condition (often satisfied in practice), there exists a unique solution of this differential inclusion and we show convergence in probability with explicit bounds.

This extends the applicability of mean field techniques to systems exhibiting threshold dynamics such as queuing systems with boundary conditions. This is illustrated by applying our results to push-pull queues with a large number of incoming sources and a large number of servers that are natural models of volunteer computing systems.

**Key-words:** Mean Field, Differential inclusions, Queuing systems

## Limite en Champ Moyen de systèmes non-réguliers et inclusions différentielles

**Résumé :** Dans cet article nous étudions le comportement limite de processus de Markov composés de multiples objets en interaction. Alors que la plupart des résultats classiques supposent que la dynamique limite a des propriétés de régularité de type Lipschitz, nous montrons que ces conditions ne sont pas nécessaires pour prouver la convergence vers un système déterministe. Nous montrons que sous des hypothèses assez faibles, le system stochastique converge vers la solution d'une inclusion différentielle et nous donnons un moyen simple d'obtenir cette inclusion différentielle limite. Quand elle satisfait une condition de semi-Lipschitz (souvent vérifiée en pratique), il existe une solution unique à cette inclusion différentielle et nous montrons la convergence en probabilité en donnant des bornes explicites.

Cela permet d'étendre l'applicabilité des techniques de champs moyen à des systèmes avec des dynamiques à seuil, comme c'est le cas pour des systèmes de files d'attente. Ceci est illustré par l'application de nos résultats à des files "push-pull" avec un grand nombre de sources de paquets et un grand nombre de serveurs, qui constituent des modèles naturels de systèmes de calculs à volontaires.

**Mots-clés :** Champ Moyen, Inclusions différentielles, Systèmes de files d'attente

## 1 Introduction

This paper studies the limiting behavior of a system composed by a large number of objects when the dynamics is non-smooth. Under classical smoothness assumptions, there exist general results that show that the limiting system can be described by a system of ordinary differential equations

$$\dot{y}(t) = f(y(t)). \quad (1)$$

See [4] and the references therein for examples of such convergence results. In most cases, the limiting function  $f$  is assumed to be Lipschitz. This condition limits the applicability of these results in many practical cases, in particular, for systems exhibiting thresholds dynamics or with boundary conditions.

Let us consider a simple queuing system with one buffer and many processors that can serve one packet per unit of time in average. If  $y$  denotes the number of packets in the queue, then the average decrease of  $y$  is one packet per unit of time (under a proper rescaling of time) if the queue is non-empty (*i.e.*  $y > 0$ ) and zero if the queue is empty. This leads to a deterministic limit behavior:

$$\dot{y}(t) = -1 \text{ if } y(t) > 0 \quad \text{and} \quad \dot{y}(t) = 0 \text{ if } y(t) = 0. \quad (2)$$

This dynamics is not continuous and therefore non Lipschitz which makes the classical approach inapplicable in that case.

Actually, most work using mean field for networks do not involve queues or when they do, the number of queues scale with the number of objects (as in [5]), or convergence is obtained using *ad hoc* proofs (see for example [2]).

In the case of a non-continuous right-hand side, the differential equation (1) is not well-defined since there exist no function  $y$  that is differentiable and that satisfies (2). The proper way to define solutions of (2) is to use differential inclusions (DI) instead. Equation (1) is replaced by the following equation

$$\dot{y}(t) \in F(y(t)). \quad (3)$$

where  $F$  is a set-valued mapping: if  $x \neq 0$  then  $F(x) = \{-1\}$  and  $F(0) = \{x : -1 \leq x \leq 0\}$ . Of course a differential inclusion problem may have multiple solutions.

In the following, we will provide generic convergence results that show that under few conditions on the initial system, its behavior converges to the solutions of (3) (Theorem 5). This result is generic and does not require any Lipschitz property on the function  $F$ . In particular, it shows that when (3) has a unique solution, the behavior of the system converges to it. Moreover, we also show that when  $F$  satisfies a one-sided Lipschitz condition (8), we can bound the gap with the limiting dynamics with explicit bounds (Theorem 7). The one-sided Lipschitz condition is satisfied in most systems involving queues.

The rest of the paper is organized as follows. In Section 2, we briefly describe the model and give some examples. We will briefly recall some definitions and properties of DIs in Section 2.1. Section 3 provides the main theoretical results and Section 4 give some application example.

## 2 Description of the Model and Notations

We consider of system of  $N$  objects evolving in a finite state space  $\mathcal{S} = \{1 \dots S\}$ . Time is discrete and the state of object  $n$  at time step  $k$  is denoted  $X_n^N(k)$ . The objects all evolve in a common environment, called the *context*. The context at time step  $k$  is denoted  $C^N(k) \in \mathbb{R}^d$ . The state of the global system at time  $k$  is  $(\mathcal{X}^N(k), C^N(k))$ . We denote by  $M^N(k)$  the empirical measure associated with the  $N$  objects:

$$M^N(k) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^N \delta_{X_n^N(k)}.$$

Since an object has  $S$  possible states,  $M^N(k)$  can be represented by a vector with  $S$  components, its  $i$ th component being the proportion of objects in state  $i$ :

$$M_i^N(k) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{X_n^N(k)=i}.$$

The system  $(M^N(k), C^N(k))_k$  is assumed to be a Markov chain. In particular, this is true if the evolution of the context is deterministic and if the evolution of the system is invariant by any permutation of the  $N$  objects. The state space of this Markov chain is included in  $\mathbb{R}^{S+d}$ . To simplify the notations, we call  $Y^N(k) \stackrel{\text{def}}{=} (M^N(k), C^N(k))$ .

If at time  $k$ , the system is in state  $Y^N(k) = y$ , then the expected difference between  $Y^N(k+1)$  and  $Y^N(k)$  is called the *drift* and is denoted  $f^N(y)$ :

$$f^N(y) \stackrel{\text{def}}{=} \mathbb{E}(Y^N(k+1) - Y^N(k) | Y^N(k) = y). \quad (4)$$

This only defines  $f^N$  on the state space of  $Y^N$ , which is a subset of  $\mathbb{R}^{d+S}$ . Outside the state space of  $Y^N$ ,  $f^N(y)$  is not yet defined.

We assume that as  $N$  grows, the drift vanishes with speed  $I(N)$ . This means that there exists a function  $I(N)$ , called the *intensity* of the model, a set-valued function  $F$  and a continuation of the function  $f^N$  on all  $\mathbb{R}^{d+S}$  such that

- The intensity vanishes:  $\lim_{N \rightarrow \infty} I(N) = 0$ .
- $f^N$  converges uniformly to  $F$  in the following sense: there exists  $J(N)$  with  $\lim_{N \rightarrow \infty} J(N) = 0$  and a function  $f$  such that  $f(y) \in F(y)$  satisfying

$$\sup_{y \in \mathbb{R}^{d+S}} \left\| \frac{f^N(y)}{I(N)} - f(y) \right\| \leq J(N). \quad (5)$$

where  $\|\cdot\|$  denotes the  $L^2$  norm.

Finally, we assume that the second order of the drift is bounded. This means that there exists  $b > 0$  such that:

$$\mathbb{E} \left( \left\| \frac{Y^N(k+1) - Y^N(k) - f^N(Y^N(k))}{I(N)} \right\|^2 \right) \leq b. \quad (6)$$

All these assumptions are more or less necessary to use a mean field approach.

## 2.1 Differential inclusions

In Section 3, we will see that under mild conditions, the system described by  $(M^N(\cdot), C^N(\cdot))$  converges to the solutions of a deterministic differential inclusion. In this section, we recall the main concepts on differential inclusions. For a more complete description, the reader is referred to [1]. In all that follows,  $\langle x, y \rangle$  denotes the classical inner-product on  $\mathbb{R}^{d+S}$  and  $\|x\| = \sqrt{\langle x, x \rangle}$  ( $L^2$  norm) and  $\|A\| = \sup_{x \in A} \|x\|$ .

**Definition 1.** Consider a differential inclusion problem:

$$\dot{y}(t) \in F(y(t)), \quad y(0) = y_0, \quad (7)$$

where  $F$  is a set-valued function mapping each point  $y \in \mathbb{R}^{d+S}$  to a set  $F(y) \subset \mathbb{R}^{d+S}$ . Let  $I \subset \mathbb{R}$  be an interval with  $0 \in I$ . A function  $y : I \rightarrow \mathbb{R}^{d+S}$  is a solution of the DI (7) with initial condition  $y(0) = y_0$  if there exists a function  $\varphi : I \rightarrow \mathbb{R}^{d+S}$  such that:

- (i) for all  $t \in I$ :  $y(t) = y_0 + \int_0^t \varphi(s) ds$ ;

(ii) for almost every (a.e.)  $t \in I$ :  $\varphi(t) \in F(y(t))$ .

In particular, (i) is equivalent to say that  $y$  is absolutely continuous. (i) and (ii) imply that  $y$  is differentiable at almost every  $t \in I$  with  $\dot{y}(t) \in F(y(t))$ .

**Definition 2** (Upper Semi-Continuous (USC)). *The function  $F$  is upper semi-continuous (USC) if for any  $y \in \mathbb{R}^{d+S}$ ,  $F(y)$  is a non-empty closed, convex and bounded set and if for any open set  $O$  containing  $F(y)$ , there exists a neighborhood  $V$  of  $y$  such that  $F(V) \subset O$ .*

**Definition 3** (One-Sided Lipschitz (OSL)). *A set-valued function  $F$  is one-sided Lipschitz (OSL) with constant  $L$  if for all  $y, \bar{y} \in \mathbb{R}^{d+S}$  and for all  $u \in F(y)$   $\bar{u} \in F(\bar{y})$ :*

$$\langle y - \bar{y}, u - \bar{u} \rangle \leq L \|y - \bar{y}\|^2. \quad (8)$$

These two conditions give necessary and sufficient conditions for the existence and the uniqueness of solution of the differential inclusion (7). We recall the following classical result.

**Proposition 4** (Theorems 2.2.1 and 2.2.2 of [6]).

- If  $F$  is USC, then and if there exists  $c$  such that  $\|F(x)\| \leq c(1 + \|x\|)$  then for all initial condition  $y_0$ , (7) has at least one solution on  $[0; \infty)$  with  $y(0) = y_0$ .
- If  $F$  is OSL, then for all  $T > 0$ , there exists at most one solution of (7) on  $[0; T]$ .

In our framework, if  $f^N(\cdot)$  is the drift of the system defined in Equation (4), we define an adapted set-valued function  $F$  by:

$$F(y) = \bigcap_{\epsilon > 0} \lim_{N \rightarrow \infty} \bigcup_{k \geq N} \overline{\text{conv}} \left\{ \frac{f^k(z)}{I(k)} : \|z - y\| \leq \epsilon \right\}. \quad (9)$$

where  $\overline{\text{conv}}(A)$  is the closure of the convex hull of a set  $A$ . It should be clear that Equation (9) defines a set-valued function  $F$  such that for all  $y$ :  $\lim_{N \rightarrow \infty} \inf_{u \in F(y)} \|f^N(y)/I(N) - u\| = 0$  and that the function  $F$  is USC. Moreover,  $F$  satisfies (??) if this convergence holds uniformly in  $y$ .

If we assume that the drift converges uniformly to some function  $f$ :  $\|f^N(y)/I(N) - f(y)\| \leq J(N)$ , the function  $F$  can be defined as:

$$F(y) = \bigcap_{\epsilon > 0} \overline{\text{conv}} (\{f(z) : \|z - y\| \leq \epsilon\}). \quad (10)$$

In that case, if the original function  $f$  is continuous in a point  $y$ ,  $F(y) = \{f(y)\}$  while if  $f$  is discontinuous in  $y$ ,  $F(y)$  is a set-valued function,  $F(y) = [\liminf_{z \rightarrow y} f(z) \wedge y, \limsup_{z \rightarrow y} f(z) \vee f(y)]$ .

### 3 Convergence results

This section contains the two main theoretical contributions of this paper. The first one is Theorem 5 that shows that if  $F$  is USC then the stochastic system converges to the set of solutions of the differential inclusion. In particular, this shows that if the differential inclusion has only one solution, the stochastic system converges to this solution (Corollary 6).

This theorem does not give any bound on the speed of convergence. In fact, without further conditions, the convergence may be arbitrarily slow [?]. However, when the differential satisfies an OSL condition (8), the the solution of the differential equation is unique and the speed of convergence can be lower-bounded (Theorem 7).

The rest of this section is organized as follows. We first state the two convergence results and then discuss their applicability. Finally, we give the proofs of the two theorems. The proof of the first theorem is similar to classical proofs for the existence of solutions of differential inclusions while the second one is more similar to classical proofs of the convergence of stochastic approximation algorithms like in [4, 5].

### 3.1 Two Convergence results

Recall that at time step  $k$ , the system is in state  $Y^N(k) \stackrel{\text{def}}{=} (M^N(k), C^N(k))$ . Let us denote  $\bar{Y}^N(t)$  the piecewise affine interpolation of  $Y^N(k)$  when we scale time by a factor  $I(N)$ : Let the continuous function  $\tilde{Y}^N(t)$  be the component-wise linear interpolation of  $\{Y^N(k)\}_{k \in \mathbb{N}}$ . By scaling time,  $\bar{Y}^N(t) \stackrel{\text{def}}{=} \tilde{Y}^N(t/I(N))$ .

Let us denote by  $\mathcal{S}_T(y_0)$  the set of the solutions of the DI (7) starting from  $y(0) = y_0$ , we can show that  $\bar{Y}(\cdot)$  converges (in probability) to  $\mathcal{S}_T(y_0)$ . More precisely, the following theorem holds.

**Theorem 5.** *Let  $F$  be the limit of the drift defined by Equation (??). Assume that the drift satisfies (6) and that*

- $F$  is USC and there exists  $c > 0$  s.t.  $\|F(y)\| \leq c(1 + \|y\|)$ ,
- $Y^N(0) \xrightarrow{\mathcal{P}} y_0$ .

Then for all  $T > 0$ :

$$\inf_{y \in D_T(y_0)} \sup_{0 \leq t \leq T} \|\bar{Y}^N(t) - y(t)\| \xrightarrow{\mathcal{P}} 0.$$

*Proof.* The proof is given in Section 3.2.1. □

This theorem shows that if  $N$  is large enough, the trajectory of the stochastic system  $\bar{Y}^N$  is close to a solution of the differential inclusion (7). In general a differential inclusion may have multiple solutions. Here,  $\bar{Y}^N$  may converge to any solution of the DI, depending on its random innovations, making this result rather inefficient for performance evaluation. This result is of greater interest if the DI starting from  $y_0$  has a unique solution:  $\mathcal{S}_T(y_0) = \{y\}$ . In that case, as a direct corollary of the preceding result,  $\bar{Y}^N$  converges in probability to  $y$  on all intervals  $[0; T]$ .

**Corollary 6.** *Under the conditions of Theorem 5 and if the DI (7) has a unique solution  $y$ , then for all  $T$ :*

$$\sup_{0 \leq t \leq T} \|\bar{Y}^N(t) - y(t)\| \xrightarrow{\mathcal{P}} 0.$$

In some cases, like the example of push-pull queues described in Section 4, the limiting differential inclusion clearly has a unique solution which makes the preceding corollary directly applicable. The main drawback of the previous theorem is that it does not give any insight on the speed of convergence on the stochastic system towards its limit.

This limitation can be overcome when the function  $F$  satisfies the one-sided Lipschitz condition (8). Firstly, this ensures the uniqueness of the solution. Secondly, one can get precise bounds on the gap between the stochastic system and its limit in that case.

**Theorem 7.** *Under the conditions of Theorem 5 and if  $F$  is OSL with constant  $L$ , then the DI (7) has a unique solution  $y$  and there exist constants  $A_T, B_T, C_T$  depending only on  $T, L$  and  $c$  such that for all  $\epsilon$ ,*

$$\mathcal{P} \left( \sup_{0 \leq t \leq T} \|Y^N(t) - y(t)\| \geq \|Y^N(0) - y(0)\| e^{LT} + \sqrt{I(N)}A_T + \sqrt{J(N)}B_T + \epsilon \right) \leq \frac{I(N)}{\epsilon^2} C_T.$$

The constants  $A_T, B_T$  and  $C_T$  are given by

$$\begin{aligned} A_T &\stackrel{\text{def}}{=} e^{2LT} \sqrt{\frac{I(N)}{6} c^2(1 + K_T) + \frac{K_T}{2L} (c(1 + K_T) + J(N))}; \\ B_T &\stackrel{\text{def}}{=} \frac{\sqrt{K_T} e^{2LT}}{\sqrt{L}}; \\ C_T &\stackrel{\text{def}}{=} \frac{e^{2LT}}{2L} (4c^2(1 + K_T)^2 + b) \end{aligned}$$

with  $K_T = \max\{\|y(0)\| + c, \|y_0^N\| + (c + J(N))T, \|Y_0^N\| + (c + J(N))T + \sqrt{bT}\} e^{ct}/c$  and  $c$  is defined in Theorem 5 and  $b$  in Equation (6).

*Proof.* The proof is given in Section 3.2.2.  $\square$

Note that if  $F(\cdot)$  is bounded by some  $K_F$  the terms in  $c(1 + K_T)$  can be replaced by  $K_F$ . This is in particular true if  $Y^N$  is constrained to stay in a compact space of  $\mathbb{R}^{d+S}$  or if the drift is bounded for all  $y \in \mathbb{R}^{d+S}$ .

These bounds are of a similar order than bounds that can be obtained in the case where  $f$  is Lipschitz (see [5]). The convergence speed with respect to  $N$  is in  $O(\sqrt{I(N)})$  (compared with  $O(I(N))$  is the Lipschitz case). When the unique solution  $y(\cdot)$  is piecewise Lipschitz with a finite number of pieces, this can be reduced to order  $O(I(N))$  using the results of [8].

### 3.2 Proofs of Theorem 5 and Theorem 7

The idea of the proofs is to write the evolution of the system as a stochastic approximation algorithm. The state of the system at time  $k + 1$  can be written

$$\begin{aligned} Y^N(k+1) &= Y^N(k) + I(N)f(Y^N(k)) + (f^N(Y^N(k)) - I(N)f(Y^N(k))) \\ &\quad + (Y^N(k+1) - Y^N(k) - f^N(Y^N(k))). \end{aligned} \quad (11)$$

where  $f$  is defined in (??). This equation can be seen as an Euler discretization of the DI (7) plus two error terms:

- $E^N(k) \stackrel{\text{def}}{=} f^N(Y^N(k))/I(N) - f(Y^N(k))$  which by Equation (??) is such that  $\|E^N(k)\| \leq J(N)$ ;
- $U^N(k+1) \stackrel{\text{def}}{=} (Y^N(k+1) - Y^N(k) - f^N(Y^N(k)))/I(N)$  which by Equation (4) is such that  $\mathbb{E}(U_{k+1}^N | \mathcal{F}_k^N) = 0$  where  $\mathcal{F}_k^N$  denotes the filtration associated with the process  $(Y^N(k))_k$ . Moreover, by Equation (6),  $\mathbb{E}(\|U_{k+1}^N\|^2 | \mathcal{F}_k^N) \leq b$ .

Using this notations, and the definition of  $F(y)$  by Equation (9), Equation 11 can be rewritten

$$Y^N(k+1) \in Y^N(k) + I(N)(F(Y^N(k)) + E^N(k) + U^N(k+1)). \quad (12)$$

Equation (12) is called a *stochastic approximation with constant step size* associated to the DI (7). The term *constant step size* comes from the fact that  $I(N)$  does not vary with time. Both proofs of Theorem 5 and Theorem 7 are based on the convergence of such stochastic approximation (12) as  $N$  goes to infinity.

#### 3.2.1 Proof of Theorem 5

This proof is inspired by classical proofs of differential inclusions, for example the proof of Theorem 2.2.1 of [6]. It is somewhat similar to the proof of Theorem 4.2 of [3] although there are two main differences. First, we focus on *constant step size* and are interested in the convergence over a finite time-horizon. Second, we do not need any assumption on the boundedness of the stochastic process since this is a consequence of the hypothesis in the finite time-horizon.

The idea of the proof is to show that for all sub-sequence of  $\bar{Y}^N$ , there exists a sub-sequence  $\bar{Y}^{\sigma(N)}$  (of this sub-sequence) such that  $d(\bar{Y}^{\sigma(N)}, \mathcal{D}_T(y_0)) \xrightarrow{\text{a.s.}} 0$ . In all that follows, let  $\bar{Y}^{\sigma(N)}$  be a sub-sequence of  $\bar{Y}^N$ . In order to simplify the notations and because we will take several sub-sequences of sub-sequences, we omit the  $\sigma$  in the notation and we denote all sub-sequences  $\bar{Y}^N$ . The first part of the proof, we consider the problem from a probabilistic point of view in order to make sure that the random part of the process goes almost surely to 0. Then we consider the problem from a trajectorial point of view using analytic arguments.

Developing the recurrence (12), the value of  $Y^N(k+1)$  is equal to:

$$Y^N(k+1) = Y^N(0) + \sum_{i=0}^k I(N)f(Y^N(i)) + I(N) \sum_{i=0}^k E^N(i) + I(N) \sum_{i=0}^k U^N(i+1). \quad (13)$$

We define two functions  $Z^N(t)$ , and  $V^N(t)$  to be piecewise affine functions such that for all  $t = kI(N)$ ,  $Z^N(t) = \sum_{i=0}^{k-1} I(N)f(Y^N(i))$  and  $V^N(t) = \sum_{i=0}^{k-1} I(N)U^N(i+1)$ . By Kolmogorov's inequality for martingales and since  $\mathbb{E}\left(\|U^N(k+1)\|^2 | \mathcal{F}_k^N\right) \leq b$

$$\mathcal{P}\left(\sup_{0 \leq t \leq T} \|V^N(t)\| \geq \epsilon\right) = \mathcal{P}\left(\sup_{0 \leq k \leq \lceil \frac{T}{I(N)} \rceil} \left\| I(N) \sum_{i=0}^{k-1} U^N(i+1) \right\| \geq \epsilon\right) \leq \frac{I(N)T}{\epsilon^2} b. \quad (14)$$

This shows that  $\sup_{0 \leq t \leq T} \|V^N(t)\|$  converges in probability to 0. Therefore, there exists a sub-sequence of  $V^N$  such that  $\sup_{t \leq T} \|V^N(t)\|$  converges almost surely to 0.

We now reason from a trajectorial point of view. Let us now consider a trajectory  $\omega \in \Omega$  of the system such that  $\sup_{t \leq T} \|V^N(t)\|$  converges to 0. In particular, this implies that  $\|V^N(t)\|$  is bounded for all  $N$  and  $t$ :  $\sup_{N, 0 \leq t \leq T} \|V^N(t)\| \leq d < \infty$ . Using (13) and since  $\|F(y)\| \leq c(1 + \|x\|)$ , for all  $k \leq T/I(N)$ ,  $\|Y^N(k+1)\|$  can be bounded by:

$$\begin{aligned} \|Y^N(k+1)\| &\leq \|Y^N(0)\| + \sum_{i=0}^{k-1} I(N)c(1 + \|Y^N(i)\|) + \sup_{N,t} \|V^N(t)\| \\ &\leq \|Y^N(0)\| + ckI(N) + d + \sum_{i=0}^k I(N) \|Y^N(i)\| \\ &\leq (\|Y^N(0)\| + cT + d) \exp(cT) / c, \end{aligned} \quad (15)$$

where we used the discrete Gronwall lemma and the fact that  $kI(N) \leq T$ .

Once we know that  $\sup_{N, 0 \leq t \leq T} \|Y^N(t)\|$  is bounded, the rest of the proof can be adapted from classical results on the convergence of the Euler approximation for differential inclusions, see [6] for example. There exists  $e > 0$  such that  $\sup_{N, 0 \leq t \leq T} \|Y^N(t)\| \leq e$ . Thus  $\|f(Y^N(k))\| < c(1+e) < \infty$ . This shows that the functions  $Z^N$  are Lipschitz with constant  $c(1+e)$ . Thus the sequence of functions  $(Z^N)_N$  are equicontinuous and bounded. Therefore by the Arzela-Ascoli theorem, for all sub-sequence of  $(Z^N)_N$ , there exists a sub-sequence that converges to some  $z : [0; T] \rightarrow \mathbb{R}^d$ . In the following, we will show that  $z$  is a solution of (3) which shows that  $d(Z^N, \mathcal{S}_T(y_0)) \rightarrow 0$ . As  $\|Z^N - Y^N\| = \|V^N\| \rightarrow 0$ , this implies that  $d(Y^N, \mathcal{S}_T(y_0)) \rightarrow 0$ . To prove this, we will construct a function  $\varphi$  such that:

- (i) for all  $t$ :  $z(t) = z(0) + \int_0^t \varphi(s) ds$ ;
- (ii) for almost every  $t$ :  $\varphi(t) \in F(z(t))$ .

Let  $\varphi^N(t)$  be a step function, constant on the intervals  $[kI(N), (k+1)I(N))$  and such that for  $t = kI(N)$ ,  $\varphi^N(t) = f(Y^N(k))$ . Therefore, the sequence  $\varphi^N$  is bounded in  $L_2([0; T], \mathbb{R}^d)$ . Thus, there exists a sub-sequence of  $\varphi^N$  converging (in  $L_2$ ) to a function  $\varphi$ .

Since  $L_2$  is a reflexive space, if a sequence of function  $\varphi^N$  converges to  $\varphi$ , this means that for all function  $v$ , there exists a sub-sequence of  $\varphi^N$  such that  $\langle v, \varphi^N \rangle \rightarrow \langle v, \varphi \rangle$ . Let  $\xi \in \mathbb{R}^d$  and  $t \in [0; T]$ . Let the function  $v$  be defined by  $v(s) \stackrel{\text{def}}{=} \xi$  for  $s < t$  and  $v(s) \stackrel{\text{def}}{=} 0$  for  $t \geq s$ .

Moreover, by the preceding remark and the fact that  $Z^N(t) \rightarrow z(t)$ ,

$$\begin{aligned} \langle Z^N(t), \xi \rangle &\rightarrow \langle z(t), \xi \rangle \\ \langle Z^N(t), \xi \rangle &= \langle Z^N(0), \xi \rangle + \left\langle \int_0^t \varphi^N(s) ds, \xi \right\rangle \\ &= \langle Z^N(0), \xi \rangle + \langle \varphi^N, v \rangle \\ &\rightarrow \langle z(0), \xi \rangle + \langle \varphi, v \rangle \\ &= \left\langle z(0) + \int_0^t \varphi(s) ds, \xi \right\rangle \end{aligned}$$

As this is true for all  $\xi \in \mathbb{R}^d$ ,  $z(t) = \int_0^t \varphi(s) ds$ .

As  $\varphi^N$  converges in  $L_2$  to  $\varphi$ , for almost every  $t \in [0; T]$ ,  $\varphi^N(t) \rightarrow \varphi(t)$ . Moreover, for all  $t = kI(N)$ ,  $\varphi^N(t) = f(Y^N(t)) = f(z(t) + Z^N(t) - z(t) + V^N(t))$ . Since  $V^N(t) \rightarrow 0$  and  $Z^N(t) \rightarrow z(t)$  and  $\varphi^N(t) \rightarrow \varphi(t)$ , this means that for all  $\epsilon > 0$ ,  $\varphi(t) \in F^\epsilon(z(t))$  where  $F^\epsilon(y) = \{z : \exists x \|x - y\| \leq \epsilon \wedge d(z, F(y)) \leq \epsilon\}$ . Since  $F$  is USC, this implies that  $\varphi(t) \in F(z(t))$  for almost every  $t$ . Thus  $x$  is a solution of the DI.  $\square$

### 3.2.2 Proof of Theorem 7

This proof is based on two Lemmas. Lemma 9 shows the rate of convergence of an Euler scheme without error to the solution of a differential inclusion. Lemma 10) shows that the gap between the deterministic Euler scheme and the stochastic approximation can be bounded in probability with explicit bounds. The combination of the two directly implies Theorem 7.

We define by  $y_k^N$  the Euler scheme associated with the differential inclusion (3) with step  $I(N)$  starting in  $Y^N(0)$ . That is:

$$\begin{aligned} y_k^N &= Y^N(0) \\ y_{k+1}^N &= y_k^N + I(N) (f(y_k^N) + E^N(k)) \end{aligned} \quad (16)$$

for some  $f(y_k^N) \in F(y_k^N)$ . Again,  $y^N(t)$  denotes the piecewise interpolation of  $y_k^N$  where the time is scaled by a factor  $I(N)$ . More precisely,  $y^N(t)$  is a piecewise affine function such that for  $k \in \mathbb{N}$ :  $y^N(kI(N)) = y_k^N$ . Finally, we denote by  $y(t)$  a solution of the DI (3) (which is unique if the DI is OSL by Proposition 4).

We first start by a technical lemma that bounds the growth of the functions.

**Lemma 8.** *If  $F$  is USC and OSL with constant  $L$  and if there exists  $c$  such that for all  $x$ ,  $\|F(x)\| \leq c(1 + \|x\|)$ , then: if  $y_k^N$  is the piecewise interpolation of the Euler scheme (16) with  $\|E^N(k)\| \leq J(N)$  and  $y(\cdot)$  the unique solution of the DI starting in  $y_0$ , then there exist a constant  $K_T^y, K_T^{y^N}$  and  $K_T^{Y^N}$  such that  $\sup_{0 \leq t \leq T} \|y(t)\| \leq K_T^y$ ,  $\sup_{0 \leq t \leq T} \|y^N(t)\| \leq K_T^{y^N}$  a.s. and  $\sup_{0 \leq t \leq T} \sqrt{\mathbb{E} \left( \|f(\bar{Y}^N(t))\|^2 \right)} \leq K_T^{Y^N}$  where*

$$\begin{aligned} K_T^y &\stackrel{\text{def}}{=} (\|y(0)\| + c) \frac{e^{cT}}{c} \\ K_T^{y^N} &\stackrel{\text{def}}{=} (\|y_0^N\| + (c + J(N))T) \frac{e^{cT}}{c} \\ K_T^{Y^N} &\stackrel{\text{def}}{=} (\|Y_0^N\| + (c + J(N))T + \sqrt{bT}) \frac{e^{ct}}{c}. \end{aligned}$$

*Proof.* By the definition of  $y_k^N$  of Equation (16) and since  $\|F(y)\| \leq c(1 + \|y\|)$ , we have

$$\begin{aligned} \|y_{k+1}^N\| &= \left\| y_0^N + I(N) \sum_{i=0}^k (f(y_i^N) + E^N(i)) \right\| \\ &\leq \|y_0^N\| + I(N) \sum_{i=0}^k (c(1 + \|y_i^N\|) + J(N)) \\ &= \|y_0^N\| + (k+1)I(N)(c + J(N)) + I(N)c \sum_{i=0}^k \|y_i^N\| \end{aligned}$$

Therefore, for all  $k \leq T/I(N)$ , by the discrete Gronwall's lemma,  $\|y_k^N\| \leq K_T^{y^N}$  with  $K_T^{y^N} \stackrel{\text{def}}{=} (\|y_0^N\| + (c + J(N))T) e^{cT}/c$ . Similarly,  $\|y(t)\| \leq \|y(0)\| + \int_0^t c(1 + \|y(s)\|) ds \leq (\|y(0)\| + ct) e^{cT}/c$ .

Similarly to the computation for  $y_k^N$ , and focusing on  $Y^N(k)$  leads to:

$$\|Y^N(k+1)\| \leq \|Y^N(0)\| + (k+1)I(N)(c + J(N)) + I(N) \left\| \sum_{i=0}^k U^N(i+1) \right\| + I(N)c \sum_{i=0}^k \|y^N(i)\|$$

which shows that  $\|Y^N(k)\| \leq (\|Y_0^N\| + (c + J(N))T + I(N) \left\| \sum_{i=0}^k U^N(i+1) \right\|) e^{cT}/c$ . Since  $\mathbb{E}(U^N(k+1) | \mathcal{F}_k^N) = 0$ ,  $\mathbb{E} \left( I(N) \left\| \sum_{i=0}^k U^N(i+1) \right\|^2 \right) \leq I(N) \sum_{i=0}^k \mathbb{E} \left( \|U^N(i+1)\|^2 | \mathcal{F}_k^N \right) \leq b k I(N)$ . Using the Jensen inequality on  $x \mapsto (1 + \sqrt{x})^2$ , this implies that for all  $k \leq T/I(N)$

$$\mathbb{E} \left( \|Y^N(k)\|^2 \right) \leq (\|Y_0^N\| + (c + J(N))T + \sqrt{bT})^2 \frac{e^{2cT}}{c^2}.$$

□

**Lemma 9.** *If  $F$  is USC and OSL with constant  $L$  and if there exists  $c$  such that for all  $y$ ,  $\|F(y)\| \leq c(1 + \|y\|)$ , then: if  $y^N(\cdot)$  is the piecewise interpolation of the Euler scheme (16) with  $\|E^N(k)\| \leq J(N)$  and  $y(\cdot)$  the unique solution of the DI starting in  $y_0$ , then*

$$\sup_{0 \leq t \leq T} \|y^N(t) - y(t)\| \leq e^{LT} \|y_0^N - y_0\| + \sqrt{I(N)} A_T + \sqrt{J(N)} B_T$$

where

$$A_T \stackrel{\text{def}}{=} e^{2LT} \sqrt{\frac{I(N)}{6} c^2 (1+K) + \frac{K}{2L} (c(1+K) + J(N))}$$

$$B_T \stackrel{\text{def}}{=} \frac{\sqrt{K} e^{2LT}}{\sqrt{L}}$$

and  $K_T \stackrel{\text{def}}{=} \max\{K_T^y, K_T^{y^N}\}$  with  $K_T^y$  and  $K_T^{y^N}$  defined as in Lemma 8.

If  $y_0^N$  is bounded, then for  $T$  fixed,  $A_T$  is bounded for all  $N$ . This shows that the order of convergence of  $y^N(t)$  to  $y(t)$  is of order  $O(\sqrt{I(N)})$ . In fact, the rate of convergence of the Euler's method can be improved if the solution of the differential inclusion satisfies some properties. For example, if the solution has a finite number of point at which it is not differentiable the rate of convergence is  $O(I(N))$ , see for example [8], Theorem 4.1.

*Proof.* Let fix  $N$  and let us call  $t_i = iI(N)$ . Let  $r(t) \stackrel{\text{def}}{=} \|y^N(t) - y(t)\|^2$ . Since  $y^N(\cdot)$  is piecewise affine and  $y(\cdot)$  is absolutely continuous,  $r$  is absolutely continuous and differentiable for almost every  $t$ . Therefore, for a.e.  $t$  such that  $t_i \leq t < t_{i+1}$ , we have

$$\begin{aligned} \dot{r}(t) &= 2 \langle y^N(t) - y(t), f(y^N(t_i)) + E^N(i) - f(y(t)) \rangle \\ &= 2 \langle y^N(t_i) - y(t), f(y^N(t_i)) - f(y(t)) \rangle + 2 \langle y^N(t) - y^N(t_i), f(y^N(t_i)) - f(y(t)) \rangle \\ &\quad + 2 \langle y^N(t) - y(t), E^N(i) \rangle \\ &= 2 \langle y^N(t_i) - y(t), f(y^N(t_i)) - f(y(t)) \rangle \\ &\quad + 2 \langle (t-t_i) (f(y^N(t)) + E^N(t_i)), f(y^N(t_i)) - f(y(t)) \rangle + 2 \langle y^N(t) - y(t), E^N(i) \rangle \end{aligned} \quad (17)$$

Equation (17) can be bounded by the OSL condition and is less than  $2L \|y^N(t_i) - y(t)\|^2 = 2L \|y^N(t_i) - y(t_i) + y(t_i) - y(t)\|^2 \leq 4L(r(t_i) + \|y(t_i) - y(t)\|^2)$ . Moreover, since  $y$  is a solution of the DI (7),  $\|y(t) - y(t_i)\| = \left\| \int_{t_i}^t f(y(s)) ds \right\| \leq ((t-t_i)c(1+K))^2$ . By Cauchy-Schwarz inequality, the two terms of Equation (18) are bounded by  $2(t-t_i)(c(1+K) + J(N))2K + 4KJ(N)$ .

Since  $r(t)$  is absolutely continuous,  $r(t_{i+1}) = r(t_i) + \int_{t_i}^{t_{i+1}} \dot{r}(t) dt$ . Thus

$$r(t_{i+1}) \leq r(t_i) (1 + 4LI(N)) + \frac{4L}{3} I(N)^3 c^2 (1+K)^2 + I(N)^2 (c(1+K) + J(N)) 2K + 4KJ(N) I(N).$$

Using that  $(1+4LI(N))^i \leq e^{4Lt_i}$  and that  $\sum_{j=1}^i (1+4LI(N))^j = ((1+4LI(N))^{i+1} - 1)/(4LI(N)) \leq e^{4Lt_i}/(4LI(N))$ , we get by a direct induction that:

$$r(t_i) \leq e^{4Lt_i} r(0) + \frac{e^{4Lt_i}}{4L} \left( \frac{4LI(N)^2}{3} c^2 (1+K)^2 + 2KI(N)(c(1+K) + J(N)) + 4KJ(N) \right)$$

This quantity is maximized for  $t_i = T$ . Therefore, we have

$$\sup_{0 \leq t \leq T} r(t) \leq e^{4LT} r(0) + I(N) e^{4LT} \left( \frac{I(N)}{6} c^2 (1+K) + \frac{K}{2L} (c(1+K) + J(N)) \right) + J(N) \frac{K e^{4LT}}{L}.$$

This gives the results using  $\sqrt{a+b+c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$ .  $\square$

The following lemma states the convergence in probability of  $Y^N(k)$  to the Euler scheme  $y^N(k)$  with explicit probabilistic bounds. Together with Lemma 9, it shows that  $Y^N$  converges to the solution of the solution of (7) with explicit bounds (*i.e.* Theorem 7).

**Lemma 10.** *If  $F$  is USC and OSL with constant  $L$  and if there exists  $c$  such that for all  $y$ ,  $\|F(y)\| \leq c(1 + \|y\|)$ , then if  $y^N(\cdot)$  is the affine interpolation of the Euler scheme (16) and  $Y^N(\cdot)$  the affine interpolation of the stochastic approximation (12) with*

- $\|E^N(k)\| \leq J(N)$
- $\mathbb{E}(U_{k+1}^N | \mathcal{F}_k^N) = 0$  and  $\mathbb{E}(\|U_{k+1}^N\|^2 | \mathcal{F}_k^N) \leq b$

then for all  $T$  and all  $\epsilon > 0$ ,

$$\mathcal{P} \left( \sup_{0 \leq t \leq T} |\bar{Y}^N(t) - y^N(t)| \geq \epsilon \right) \leq \frac{I(N)}{\epsilon^2} \frac{e^{2LT}}{2L} \left( 2c^2 \left( (1 + K_T^{y^N})^2 + (1 + K_T^{Y^N})^2 \right) + b \right).$$

where  $K_T^{y^N}$  and  $K_T^{Y^N}$  are defined as in Lemma 8.

In particular, this shows that  $\lim_{N \rightarrow \infty} \mathbb{P}(\sup_{0 \leq t \leq T} |\bar{Y}^N(t) - y^N(t)| \geq \epsilon)$  goes to 0 with rate  $I(N)$ .

*Proof.* Using the equality  $\|x + y + z\|^2 = \|x\|^2 + 2(\langle x, y \rangle + \langle x, z \rangle + \langle y, z \rangle) + \|y\|^2 + \|z\|^2$  and the definitions of  $Y^N(k+1)$  and  $y_{k+1}^N$  (Eq. (12) and (16)),  $\|Y^N(k+1) - y_{k+1}^N\|^2$  can be rewritten

$$\begin{aligned} \|Y^N(k+1) - y_{k+1}^N\|^2 &= \|Y^N(k) - y_k^N + I(N)(f(Y^N(k)) - f(y_k^N) + U^N(k+1))\|^2 \\ &= \|Y^N(k) - y_k^N\|^2 \\ &\quad + 2I(N) \langle Y^N(k) - y_k^N, f(Y^N(k)) - f(y_k^N) \rangle \\ &\quad + 2I(N) \langle Y^N(k) - y_k^N + f(Y^N(k)) - f(y_k^N), U^N(k+1) \rangle \\ &\quad + I(N)^2 \|f(Y^N(k)) - f(y_k^N)\|^2 + I(N)^2 \|U^N(k+1)\|^2 \end{aligned} \tag{19}$$

$$\leq (1 + 2LI(N)) \|Y^N(k) - y_k^N\|^2 \tag{20}$$

$$+ I(N)^2 \left( \|f(Y^N(k)) - f(y_k^N)\|^2 + \|U^N(k+1)\|^2 \right) \tag{21}$$

$$+ 2I(N) \langle Y^N(k) - y_k^N + f(Y^N(k)) - f(y_k^N), U^N(k+1) \rangle. \tag{22}$$

where we use the OSL to go from (19) to (20).

Let us define a real valued sequence  $W^N(k)$  by  $W^N(0) = 0$  and for all  $k \geq 0$ :

$$\begin{aligned} W^N(k+1) &= (1 + 2LI(N))W^N(k) + I(N)^2 \left( \|f(Y^N(k)) - f(y_k^N)\|^2 + \|U^N(k+1)\|^2 \right) \\ &\quad + 2I(N) \langle Y^N(k) - y_k^N + f(Y^N(k)) - f(y_k^N), U^N(k+1) \rangle \end{aligned}$$

Because of Equation (20)-(21)-(22), for all  $k \geq 0$ ,  $0 \leq \|Y^N(k+1) - y_{k+1}^N\|^2 \leq W^N(k+1)$  (almost surely). Moreover, since  $\mathbb{E}(U^N(k+1) | \mathcal{F}_k^N) = 0$ , the expectation of (22) knowing  $\mathcal{F}_k^N$  is also 0. This shows that:

$$\begin{aligned} \mathbb{E}(W^N(k+1) | \mathcal{F}_k^N) &= (1 + 2LI(N))W^N(k) + I(N)^2 \left( \|f(Y^N(k)) - f(y_k^N)\|^2 \right. \\ &\quad \left. + \mathbb{E}(\|U^N(k+1)\|^2 | \mathcal{F}_k^N) \right) \\ &\geq W^N(k). \end{aligned} \tag{23}$$

This shows that  $W^N(k)$  is a positive sub-martingale. Therefore, by Doob's inequality, for  $\epsilon > 0$ ,  $\mathcal{P}(\sup_{0 \leq i \leq k} W^N(i) \geq \epsilon^2) \leq \mathbb{E}(W^N(k))/\epsilon^2$ . Moreover, using Equation (23), the definition  $W^N(0) = 0$ , Lemma 8 and the hypothesis (6) on  $\mathbb{E}(\|U(k+1)\|^2 | \mathcal{F}_k^N)$ ,  $\mathbb{E}(W^N(k+1))$  can be bounded by:

$$\begin{aligned} \mathbb{E}(W^N(k+1)) &= \mathbb{E}(\mathbb{E}(W^N(k+1) | \mathcal{F}_k^N)) \\ &\leq (1 + 2LI(N))\mathbb{E}(W^N(k)) + I(N)^2 \left( \mathbb{E}(\|f(Y^N(k)) - f(y^N(k))\|^2) + b \right) \\ &\leq (1 + 2LI(N))^{k+1} W^N(0) \\ &\quad + I(N)^2 \sum_{i=0}^k (1 + 2LI(N))^i \left( 2c^2 \left( (1 + K_T^{y^N})^2 + (1 + K_T^{Y^N})^2 \right) + b \right) \\ &= I(N) \frac{(1 + 2LI(N))^{k+1} - 1}{2L} \left( 2c^2 \left( (1 + K_T^{y^N})^2 + (1 + K_T^{Y^N})^2 \right) + b \right). \end{aligned}$$

where  $K_T^{y^N}$  and  $K_T^{Y^N}$  are defined in Lemma 8. Note that we use Jensen's inequality on  $x \mapsto (1 + \sqrt{x})^2$  to show that  $\mathbb{E}(\|f(Y^N(k))\|^2) \leq c^2(1 + K_T^{Y^N})^2$ .

Taking the value for  $k = T/I(N)$ , using the preceding remarks and the fact that for  $a > 0$ ,  $(1 + a)^i \leq e^{ai}$ , we have

$$\begin{aligned} \mathcal{P}\left(\sup_{0 \leq t \leq T} \|\bar{Y}^N(t) - y^N(t)\| \geq \epsilon\right) &\leq \mathcal{P}\left(\sup_{0 \leq i \leq T/I(N)} W^N(i) \geq \epsilon^2\right) \\ &\leq \frac{I(N)}{\epsilon^2} \frac{e^{2LT}}{2L} \left( 2c^2 \left( (1 + K_T^{y^N})^2 + (1 + K_T^{Y^N})^2 \right) + b \right). \end{aligned}$$

□

## 4 Examples

### 4.1 A Simple Example: Best Response in Rock-Paper-Scissors

This example has no applicative purpose. Its goal is to illustrate the convergence result of Corollary 6 in a simple case. In particular, in this case the limiting system is USC and has a unique solution but does not satisfies a one-sided Lipschitz condition.

We consider a set of  $N$  individuals playing the game of rock-paper-scissors. The state of one individual can either be rock, paper or scissors. At each time step, 2 players are chosen at random. The first player can not change its state while the second can decide to play one of the three choices. The goal of the second player two is to beat player one (rock beats scissors, scissors beat paper and paper beats rock).

It should be clear that if there is a majority of rock (resp. paper or scissors), the second player should play paper (resp. scissor or rock). The corresponding limiting dynamics is drawn on Figure 1. The drift is not Lipschitz on the frontiers and the limit dynamics is non-smooth. The differential inclusion is USC but not OSL. However, for all initial conditions, the solution of the differential inclusion is unique and has at most one cusp point before converging to its unique attractor  $(1/3, 1/3, 1/3)$  at constant speed.

### 4.2 Volunteer Computing

Here, we consider a model of volunteer computing systems, such as BOINC <http://boinc.berkeley.edu/>. The system is made of a single buffer and  $N$  desktop machines, offered by their owners (volunteers), that serve the packets of this buffer. However, as soon the owner of a processor wants to use it, she preempts it and the processor becomes unavailable for the computing system.

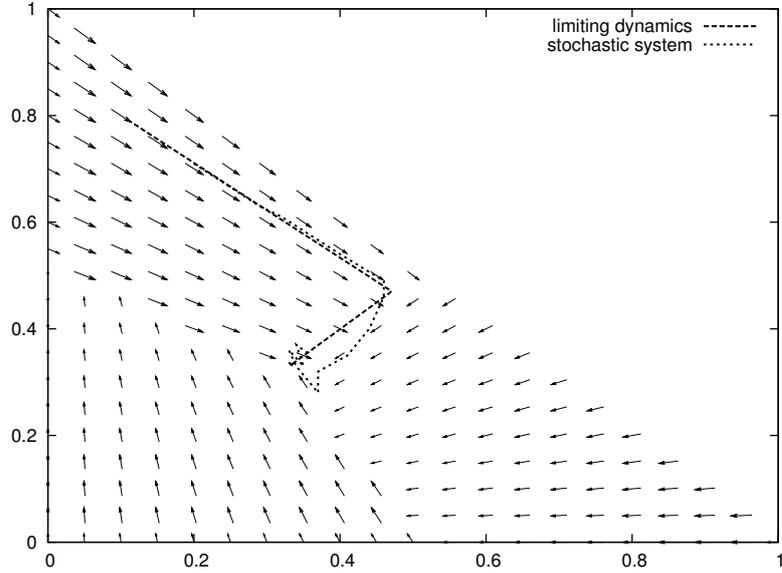


Figure 1: (Vector field for the best response dynamics. The first (second) coordinate  $M_1$  ( $M_2$ ) is the proportion of Rocks (Paper). The proportion of Scissors is  $M_3 = 1 - M_1 - M_2$ . Two trajectories starting in  $M(0) = (.8, .1, .1)$  are shown. One corresponding to a stochastic system with  $N = 10$  players and one for the deterministic limit.

As for the incoming packets, they are assumed to arrive in the buffer according to a Poisson process at rate  $\lambda$ . This kind of systems are often called push/pull modes: The distributed applications *push* jobs to a central server that stores them in a buffer and whenever a processor becomes available, it *pulls* a job from the buffer and executes it.

Such systems fit our framework. The context  $C(t)$  represents the size of the buffer while the  $N$  objects represent both the application sending jobs and the hosts executing them. The state of a host is its availability and its idleness (whether it is executing a job or not). The non-smooth part of the dynamics comes from the buffer size. When  $C(t) > 0$ , if a host asks for a job, it gets it with probability one while when  $C(t) = 0$ , a host asking for a job will get nothing. In that case, one can show that this dynamics satisfies the one-sided Lipschitz condition (OSL). Therefore, we can apply Theorem 7 to study the limiting behavior of the system when the number of hosts and applications grows.

In the simplest case, the intensity of the system is  $I(N) = 1/N$  and an application sends a job to the system with probability  $\lambda/N$  while a job is completed with probability  $\mu/N$ . To represent the communication cost, a host gets a job with probability  $\gamma$ . It becomes unavailable with probability  $p_u/N$ , available with probability  $p_a/N$  if  $C(t) > 0$  and 0 otherwise. If  $b, a, u$  denote respectively the proportion of busy, available and unavailable hosts, the limiting system is described by a DI:

$$\begin{aligned} \dot{b}(t) &= -\mu b(t) + \gamma a(t) \mathbf{1}_{C(t) > 0} \\ \dot{a}(t) &= \mu(t) b(t) + p_a u(t) - \gamma a(t) \mathbf{1}_{C(t) > 0} \\ \dot{u}(t) &= -p_a u(t) + p_u a(t) \\ \dot{C}(t) &= -\gamma a(t) \mathbf{1}_{C(t) > 0} + \lambda \mathbf{1}_{C(t) < C_{\max}}. \end{aligned}$$

The corresponding DI is obtained by replacing  $a(t) \mathbf{1}_{C(t) > 0}$  by the sets  $\{\gamma a(t)\}$  if  $C(t) > 0$  and  $[0; \gamma a(t)]$  when  $C(t) = 0$ .

At time  $t = 0$ , we consider that the size of the buffer is  $C(0) = .2$  and that all processors are available and are serving a job. The behavior of the system is represented in Figure 2. One can see that there is a point of non-differentiability in the behavior of the system when the size of the buffer reaches 0.

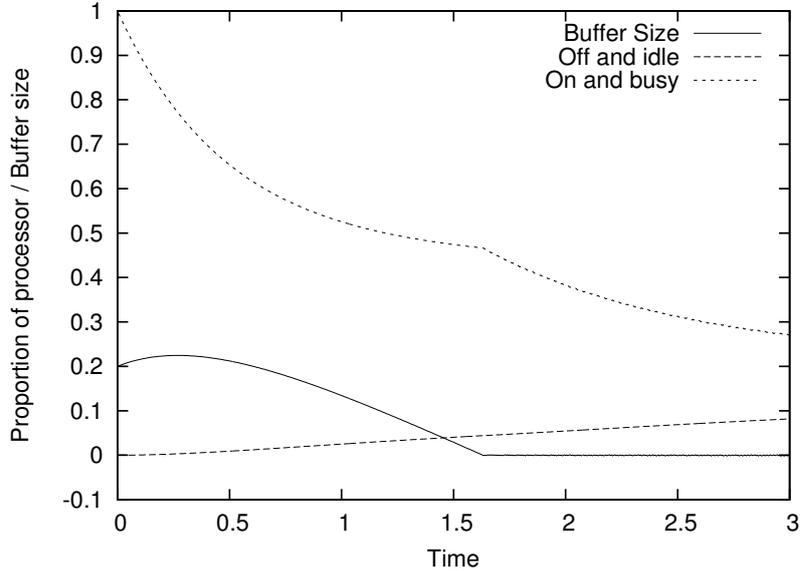


Figure 2: Limit dynamics of a volunteer computing system. A non-differentiable point occurs when the buffer becomes empty.

### 4.3 Volunteer Computing: Day and Night Scenario

We now consider a similar model as the previous one except that the processors follow a day and night behavior. We consider that some of the processors are turned off at night. Therefore, the availability is larger during the day (between 7am and 5pm) than at night.

The limiting dynamics can be represented by a differential inclusion that depends (not continuously) on time:

$$\frac{\partial y(t)}{\partial t} \in F(y(t), t). \quad (24)$$

In the previous theorems, we assumed that the function  $F$  was time-invariant. There are three ways to tackle this problem. The first one is to adapt the proofs to the time dependent case. An other idea that can be used here is the fact that for all finite interval of time, there is only a finite number of discontinuity points (7am, 5pm, ...), and to apply the convergence results on a first sub-interval  $[0; 7am]$ . Using the fact that  $Y^N(7) \rightarrow y(7)$ , the convergence holds on  $[7am, 5pm]$ , and so forth. Yet another solution is to write  $Z(t) \stackrel{\text{def}}{=} (Y(t), t)$ . Then the differential inclusion (24) can be written:

$$\frac{\partial z(t)}{\partial t} \in (F(z(t)), 1). \quad (25)$$

The fact that  $F$  is not continuous in  $t$  (and therefore in  $z(t)$ ) is not a problem in our differential inclusion setting. It should be clear that in that case there is still a unique solution to the differential inclusion 25.

On Figure 3 we can observe two kinds of non-differentiable points. The first ones are the points representing the change from day time to night time. The other ones occur when the buffer becomes empty. The small oscillations of the buffer size around 0 and of proportion of busy processors are just numerical integration artefacts (typical of numerical integration of differential inclusions). In both cases the exact solutions do not exhibit these jumps.

### 4.4 Join the Shortest Queue in Volunteer Computing

The last example is similar to the one of Section 4.2 but with two identical time-homogeneous volunteer systems. Each time a packet arrives, it is routed to the system with the smallest number

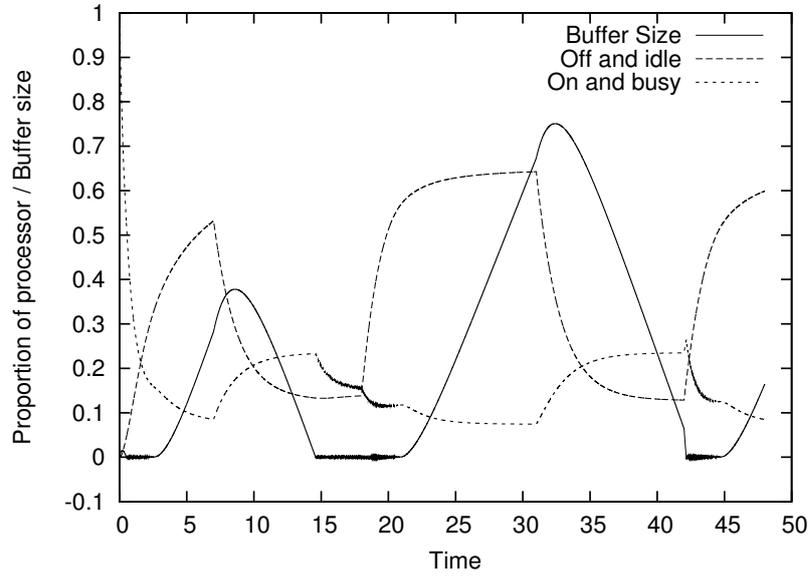


Figure 3: Limit dynamics for volunteer computing under day and night availability.

of packets. Here, the routing of packets introduces a new cause of non-smoothness: there is a threshold of the dynamics of the system when both backlogs are equal.

Figure 4 shows the behavior of the limit differential inclusion that is both USC and OSL. Therefore, the limit behavior is unique ones the initial condition is given.

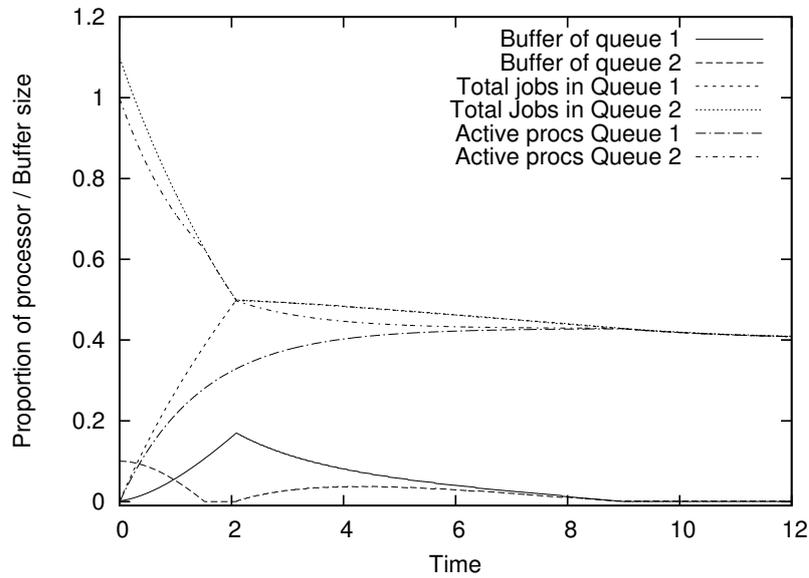


Figure 4: Join the shortest queue for volunteer computing

As expected, a new non-differential point appears when both buffers are equal.

## 5 Extension to Non-smooth Density Dependent Population Processes

In this section, we show that our results can be adapted to the case of continuous time Markov chains and in particular for the famed model of density dependent population processes of Kurtz [7]. The two theorems 5 and 7 can be transposed in this case.

Let  $D^N$  be a continuous Markov chain on  $\frac{1}{N}\mathbb{Z}^d$  ( $d \geq 1$ ) for  $N \geq 1$ .  $D^N$  is called a *density dependent population process* if there exists a set  $\mathcal{L} \subset \mathbb{Z}^d$  (with  $0 \notin \mathcal{L}$ ), such that for each  $\ell \in \mathcal{L}$  and  $x \in N^{-1}\mathbb{Z}^d$ , the rate of transition from  $x$  to  $x + \ell/N$  is  $N\beta_\ell(x) \geq 0$ , where  $\beta_\ell(\cdot)$  does not depend on  $N$ . The  $i$ th component of  $D^N(t)$ ,  $D_i^N(t)$  can be seen as the density of individuals of a population that are in state  $i$ . Each transition increases the number of individual in state  $i$  by  $\ell_i$ .

Let assume that  $\sum_{\ell \in \mathcal{L}} \sup_{x \in \mathbb{Z}^d} \|\beta_\ell(x)\| = \tau < \infty$  and let us define  $f(x) = \sum_{\ell \in \mathcal{L}} \beta_\ell(x)\ell$  (if this sum is well-defined). If  $f$  is Lipschitz, it is well-known that if  $\lim_{N \rightarrow \infty} D^N(0) = d(0)$  in probability, then there exists a differentiable function  $d(\cdot)$  such that  $\lim_{N \rightarrow \infty} \sup_{t \leq T} \|D^N(t) - d(t)\| = 0$  in probability [7]. Using uniformization of the Markov chain and results of the Section 3 we show that this convergence still hold for general drifts, replacing  $f$  by its set-valued counterpart  $F$ , defined in (9).

**Theorem 11.** *Under the foregoing assumptions, If  $\sup_{x \in \mathbb{Z}^d} \sum_{\ell \in \mathcal{L}} \beta_\ell(x) = \tau < \infty$ ,  $\sum_{\ell \in \mathcal{L}} \|\ell\|^2 \beta_\ell^2(y) < b < \infty$  and  $F$  is USC and satisfies  $\|F(x)\| \leq c(1 + \|x\|)$ . Then for all  $T > 0$ :*

$$\inf_{d \in \mathcal{S}_T(y_0)} \sup_{0 \leq t \leq T} \|D^N(t) - d(t)\| \xrightarrow{\mathcal{P}} 0.$$

where  $\mathcal{S}_T(y_0)$  is the solution set of the DI (7) starting in  $y_0$ .

Moreover, if  $F$  is OSL of constant  $L$ , then

$$\mathcal{P} \left( \sup_{0 \leq t \leq T} \|D^N(t) - d(t)\| \geq \|D^N(0) - d(0)\| e^{LT} + \frac{1}{\sqrt{N}} A_T + (1 + c(1 + K_T^y))\epsilon \right) \leq \frac{C_T + \tau}{N\epsilon^2}.$$

where  $A_T, C_T$  and  $K_T$  are defined in Theorem 7 with  $I(N) = N^{-1}$  and  $d(\cdot)$  is the unique solution of the differential inclusion (7).

*Proof.* Since  $\tau < \infty$ , the rate of transition of  $D^N(\cdot)$  is bounded by  $N\tau$ . Using uniformization of continuous time Markov chain (see [9] for example), there exist a Poisson process  $\Lambda^N$  of rate  $N\tau$  and a discrete time Markov chain  $Y^N(\cdot)$  such that  $D^N(t) = Y^N(\Lambda^N(t))$  and  $Y^N$  and  $\Lambda^N$  are independent. Moreover, for all  $x$  and  $\ell \in \mathcal{L}$ ,

$$\begin{aligned} \mathcal{P} \left( Y^N(k+1) = x + \frac{\ell}{N} | Y^N(k) = x \right) &= \frac{1}{\tau} \beta_\ell(x) \\ \mathcal{P} ( Y^N(k+1) = x | Y^N(k) = x ) &= 1 - \frac{1}{\tau} \sum_{\ell \in \mathcal{L}} \beta_\ell(x) \end{aligned}$$

For all  $y \in \mathbb{R}^d$ , the drift of  $Y^N(\cdot)$  is  $\mathbb{E} ( Y^N(k+1) - Y^N(k) | Y^N(k) = y ) = (N\tau) = N^{-1}f(y)$ . Moreover,  $\mathbb{E} \left( \|Y^N(k+1) - Y^N(k) - f(y)\|^2 | Y^N(k) = y \right) \leq (N\tau)^{-2} \sum_{\ell \in \mathcal{L}} \|\ell\|^2 \beta_\ell(y) < b$ . Therefore,  $Y^N(k)$  satisfies the conditions of Theorem 5. Moreover,  $F$  also satisfies the conditions of Theorem 5 which shows that  $\inf_{y \in \mathcal{S}_T(y_0)} \sup_{t \leq T} \|Y^N(tN) - y(t)\| = 0$ . When  $F$  satisfies the OSL condition of Theorem 7, we further get the result with explicit bounds.

As  $\Lambda^N$  is a Poisson process of rate  $N\tau$ ,  $|\Lambda^N(t) - tN\tau|^2$  is a sub-martingale and by Doob's inequality,  $\mathcal{P} (\sup_{t \leq T} |\Lambda^N(t) - N\tau t| \geq N\epsilon) \leq \mathbb{E} \left( |\Lambda^N(T) - TN\tau|^2 \right) / (N^2\epsilon^2) = \tau / (N\epsilon^2)$ . If  $y$  is a solution of the DI (7) on  $[0; T]$ , for all  $t, s \in [0, T]$ ,  $\|y(t) - y(s)\| \leq c(1 + K_T^y)|t - s|$  where  $K_T^y$  is defined in Lemma 8. This shows that if  $y$  is a solution of the differential inclusion, with probability greater than  $\tau / (N\epsilon^2)$ , we have:

$$\begin{aligned} \|D^N(t) - y(t)\| &= \|Y^N(\Lambda^N(t)) - y(t)\| \leq \|Y^N(\Lambda^N(t)) - y(\Lambda^N(t))\| + \|y(\Lambda^N(t)) - y(t)\| \\ &\leq \|Y^N(\Lambda^N(t)) - y(\Lambda^N(t))\| + c(1 + K_T^y)\epsilon. \end{aligned}$$

which concludes both parts of the Theorem.  $\square$

## References

- [1] V. Acary and B. Brogliato. *Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics*. Springer Verlag, 2008.
- [2] F. Baccelli, D. McDonald, and J. Reynier. A mean field model for multiple tcp connections through a buffer implementing red. *Performance Evaluation*, 49, 2002.
- [3] M. Benaim, J. Hofbauer, and S. Sorin. Stochastic approximation and differential inclusions. *SIAM J. Control and Optimization*, 44(1):328–348, 2005.
- [4] Michel Benaim and Jean-Yves Le Boudec. A Class Of Mean Field Interaction Models for Computer and Communication Systems. *Performance Evaluation*, 65(11-12):823–838, 2008.
- [5] N. Gast, B. Gaujal, and J.-Y. Le Boudec. Mean field for Markov Decision Processes: from Discrete to Continuous Optimization. *INRIA RR*, 7239, 2010.
- [6] M. Kunze. Non-smooth dynamical systems. *Lecture notes in mathematics*, 2000.
- [7] T.G. Kurtz. *Approximation of population processes*. Society for Industrial Mathematics, 1981.
- [8] F. Lempio. Euler’s method revisited. *Proceedings of the Steklov Institute of Mathematics*, 211(4):429–449, 1995.
- [9] W.J. Stewart. *Numerical solution of Markov chains*. CRC, 1991.



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Éditeur  
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