



HAL
open science

Strong and weak error estimates for the solutions of elliptic partial differential equations with random coefficients

Julia Charrier

► **To cite this version:**

Julia Charrier. Strong and weak error estimates for the solutions of elliptic partial differential equations with random coefficients. [Research Report] RR-7300, 2010. inria-00490045v1

HAL Id: inria-00490045

<https://inria.hal.science/inria-00490045v1>

Submitted on 7 Jun 2010 (v1), last revised 5 Apr 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Strong and weak error estimates for the solutions of
elliptic partial differential equations with random
coefficients*

Julia Charrier

N° 7300

Jun 2010

Domaine 5

*R*apport
de recherche

Strong and weak error estimates for the solutions of elliptic partial differential equations with random coefficients

Julia Charrier *†

Domaine : STIC pour les sciences de la vie et de l'environnement
Équipes-Projets Sage

Rapport de recherche n° 7300 — Juin 2010 — 30 pages

Abstract: We consider the problem of numerically approximating the solution of an elliptic partial differential equation with random coefficients and homogeneous Dirichlet boundary conditions. We focused on the case of a lognormal coefficient, we have then to deal with the lack of uniform coercivity and uniform boundedness with respect to the randomness. This model is frequently used in hydrogeology. We approximate this coefficient by a finite dimensional noise by using a truncated Karhunen-Loève expansion. We give then estimates of the corresponding error on the solution, both a strong error estimate and a weak error estimate, that is to say an estimate of the error committed on the law of the solution. We obtain a weak rate of convergence which is twice the strong one. Besides this, we give a complete error estimate for the stochastic collocation method in this case, where neither coercivity nor boundedness are stochastically uniform. To conclude, we apply these results of strong and weak convergence to two classical cases of covariance kernel choices: the case of an exponential covariance kernel on a box and the case of an analytic covariance kernel, yielding explicit weak and strong convergence rates.

Key-words: uncertainty quantification, elliptic PDE with random coefficients, Karhunen-Loève expansion, strong error estimate, weak error estimate, stochastic collocation method

* Inria Rennes, Irisa

† Ens Cachan antenne de Bretagne

Estimations d'erreurs fortes et faibles pour des solutions d'équations aux dérivées partielles elliptiques à coefficients aléatoires

Résumé : On s'intéresse au problème de l'approximation numérique de la solution d'une équation aux dérivées partielles elliptique à coefficients aléatoires, avec conditions de Dirichlet homogènes. On se place dans le cas d'un coefficient lognormal, qui n'est ni uniformément coercif, ni uniformément borné par rapport à l'aléatoire. C'est un modèle fréquemment utilisé en hydrogéologie. On approche ce coefficient aléatoire dans un espace aléatoire de dimension finie en utilisant un développement de Karhunen-Loève tronqué. On donne alors des majorations de l'erreur qui en découle sur la solution, pour l'erreur forte mais aussi pour l'erreur faible, c'est à dire l'erreur commise sur la loi de la solution. On obtient un ordre de convergence faible qui est le double de l'ordre de convergence forte. De plus, on donne une estimation d'erreur complète pour la méthode de collocation stochastique dans ce cas où le coefficient n'est ni uniformément borné ni uniformément coercif par rapport à l'aléatoire. Pour conclure on applique ces résultats de convergence forte et faible à deux exemples de choix de covariance : le cas d'une covariance exponentielle sur un pavé et le cas d'une fonction de covariance analytique, donnant des taux de convergences explicites pour ces deux cas.

Mots-clés : quantification des incertitudes, EDP elliptique à coefficients aléatoires, développement de Karhunen-Loève, estimation d'erreur forte, estimation d'erreur faible, méthode de collocation stochastique

1 Introduction

Many engineering applications involve uncertainty on the input data, such as material properties. This uncertainty results from heterogeneity of the medium and incomplete knowledge about the medium properties and can be modeled by partial differential equations with random coefficients. This work addresses elliptic partial differential equations with random coefficients and focuses on application to hydrogeology, namely the prediction of flow in porous media, but there are many other applications, e.g., random vibrations, composite materials, seismic activity and deformations of inhomogeneous materials such as wood or biomaterials. The aim is to compute the law of the solution, but in practice we are usually interested only in some moments.

Several methods have been developed: Monte-Carlo and Monte-Carlo based methods, moment equations, perturbation methods which are adapted to the case of small uncertainty, homogenization and multiscale analysis and stochastic spectral methods [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 19, 22, 23, 24, 25, 26], which regroup stochastic galerkin methods and stochastic collocation methods.

Here we consider the model equation

$$-\operatorname{div}(a(\omega, x)\nabla_x u(\omega, x)) = f(\omega, x).$$

Both stochastic Galerkin methods and stochastic collocation methods are based on the approximation of a in a finite dimensional probability space, i.e. by using a finite number of random variables. These methods are therefore adapted to the case where the probability space has a low dimensionality, i.e. in the case where we have a good approximation a_N of a such that a_N is a function of N random variables with N small. Such approximations a_N of a can be obtained by using either a Karhunen-Loève or a polynomial chaos expansion. We compute then the solution u_N of the approximated equation resulting from replacing a by a_N . In this paper we focus on the convergence of u_N to u .

More precisely, we work here with homogeneous Dirichlet boundary conditions and that a is an homogeneous lognormal random field, which is a frequently used model for flow equation in porous media. The truncated Karhunen-Loève expansion of $\log(a)$ at order N provides then an estimate a_N of a . It is important to notice that in such a case, unlike what is frequently assumed, neither the random field a nor its approximations a_N are uniformly coercive with respect to ω . We give a strong convergence result of u_N to u , i.e. a bound for the error in $L^p(H_0^1)$ -norm, and a weak convergence result, i.e. we bound the error committed on the law of u . We obtain a bound for the weak error whose order is twice the strong order, which presents a significant interest since the number N of random variables has to be small in order to be able to compute u_N and since the law of u is what we are interested in. For simplicity we assume that f is deterministic, but all the results can be easily extended to the case where f is random, under adequate assumptions.

To begin with, we prove the existence and uniqueness of the solution u , remarking once again that in the considered case, the random field a is not uniformly coercive with respect to ω . Then we make assumptions on the eigenvalues and on the eigenfunctions of the Karhunen-Loève expansion, which enable us to prove two preliminary results: the strong convergence of a_N to a and the existence of a uniform bound in L^p -norm for a_N^{\max} and $\frac{1}{a_N^{\min}}$. We can then give a strong convergence result of u_N to u , with almost surely convergence and L^p convergence. The strong error is basically bounded by the squared root of the remainder of the series of the eigenvalues. We prove next a weak convergence result, showing that the error committed on the law of u is bounded by the remainder of the series of the eigenvalues. Besides this work gives a complete convergence analysis of the collocation method, adapting

the results of I. Babuska, F. Nobile and R. Tempone in [1], in which uniform coercivity of a with respect to ω is assumed. Finally we give examples of covariance kernel for which these results apply. In particular the exponential case on a box and the gaussian case are studied, these covariance kernels being frequently used to model the hydraulic conductivity for the flow equation in porous media.

2 Equation, existence and uniqueness of the solution

In this section, we first define the homogeneous lognormal random field a and make some regularity assumptions on its covariance kernel, then we define a linear elliptic partial differential equation with random coefficients, namely a , and finally show the existence and uniqueness of the solution of this equation. Let D be an open bounded domain in \mathbb{R}^d with C^2 boundary and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a function $k \in C^{0,1}(\mathbb{R}^+, \mathbb{R})$, and $g : \Omega \times \bar{D} \rightarrow \mathbb{R}$ a mean-free gaussian field with covariance kernel $\text{cov}[g](x, y) = k(\|x - y\|)$.

Proposition 1. *Under these assumptions, g admits a version whose trajectories belong to $C^{0,\alpha}$ a.s. for $\alpha < 1/2$.*

Proof. We denote by L the Lipschitz constant of k .

$$\begin{aligned} \mathbb{E}[|g(x) - g(y)|^2] &= \mathbb{E}[g(x)^2] - 2\mathbb{E}[g(x)g(y)] + \mathbb{E}[g(y)^2] \\ &= 2(k(0) - k(\|x - y\|)) \\ &\leq 2L\|x - y\|. \end{aligned}$$

We recall that there exists, for any positive integer p , a constant $c_p = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{2p} e^{-\frac{x^2}{2}} dx$ such that for all mean-free gaussian random variable X ,

$$\mathbb{E}[|X|^{2p}] \leq c_p \mathbb{E}[|X|^2]^p.$$

Therefore, since $g(x) - g(y)$ is a mean-free gaussian random variable, we have, for any positive integer p ,

$$\mathbb{E}[|g(x) - g(y)|^{2p}] \leq c_p (2L)^p |x - y|^p.$$

According to the Kolmogorov continuity theorem [5], there exists a version of g which is a.s. Hölder-continuous with any exponent $\beta < \frac{p-d-1}{2p}$. Since this hold for any positive integer p , letting $p \rightarrow +\infty$, it follows that there exists a version of g which is a.s. Hölder continuous with any exponent $\beta < 1/2$. □

We define the lognormal homogeneous random field $a : \Omega \times \bar{D} \rightarrow \mathbb{R}$ as $a(\omega, x) = e^{g(\omega, x)}$. By Proposition 1, the trajectories of a are a.s. continuous on the compact set \bar{D} , we can then define a.s. $a^{\min}(\omega) = \min_{x \in \bar{D}} a(\omega, x)$ and $a^{\max}(\omega) = \max_{x \in \bar{D}} a(\omega, x)$. These random variables have then the following integrability properties:

Proposition 2. $\frac{1}{a^{\min}(\omega)} \in L^p(\Omega)$ and $a^{\max}(\omega) \in L^p(\Omega)$ for any $p > 0$.

Proof. We have $a^{\min}(\omega) \geq e^{-\|g(\omega)\|_{C^0(D)}}$ and $a^{\max}(\omega) \leq e^{\|g(\omega)\|_{C^0(D)}}$. By Fernique's theorem [5], there exists $\lambda > 0$ such that $\mathbb{E}[e^{\lambda \|g(\omega)\|_{C^0(D)}^2}] < +\infty$. Thus, for any $p > 0$,

$$\mathbb{E}[e^{p \|g(\omega)\|_{C^0(D)}}] \leq \mathbb{E}[e^{\lambda \|g(\omega)\|_{C^0(D)}^2 + \frac{p^2}{4\lambda}}] < +\infty.$$

For more details, see the proof of Proposition 8. Therefore $e^{\|g(\omega)\|_{C^0(D)}} \in L^p(\Omega)$, and finally $\frac{1}{a^{\min}(\omega)} \in L^p(\Omega)$ et $a^{\max}(\omega) \in L^p(\Omega)$, for any $p > 0$. \square

Proposition 3. *Let f in $L^2(D)$, then the equation:*

$$\begin{aligned} -\operatorname{div}(a(\omega, x)\nabla u(\omega, x)) &= f(x) \text{ on } D \\ u &= 0 \text{ on } \partial D \end{aligned} \quad (1)$$

admits a unique solution u , which belongs to $L^k(\Omega, H^1(D))$, for any $k > 0$.

Remark 1. *All the following results hold in the case where the forcing term f is stochastic, under adequate assumptions.*

Proof. For almost all $\omega \in \Omega$, the equation admits a unique solution $u(\omega) \in H_0^1(D)$, the mapping $\omega \mapsto u(\omega)$ is measurable and we have, a.s. :

$$\|u(\omega, x)\|_{H_0^1(D)} \leq \frac{\|f\|_{L^2(D)}}{a^{\min}(\omega)}.$$

For every $k > 0$, by proposition 2, $\frac{1}{a^{\min}} \in L^k(\Omega)$, so for any $k > 0$,

$$\mathbb{E} \left[\|u(\omega, x)\|_{H_0^1(D)}^k \right] \leq \mathbb{E} \left[\left(\frac{1}{a^{\min}(\omega)} \right)^k \right] \|f\|_{L^2(D)} < +\infty.$$

\square

Our aim is to compute the law of the solution u , and especially its first moments, by using a stochastic collocation method. This method is based on the approximation of a by a function of a finite number of independent random variables a_N .

3 Strong convergence of a_N to a

In this section, we define the approximated random field a_N and study the strong convergence of a_N to a , i.e. the $L^p(\Omega, \mathcal{C}^0(D))$ -convergence, and the almost-sure convergence. Let $\{(\lambda_n, b_n)\}$ denote the sequence of eigenpairs associated with the compact self-adjoint operator that maps

$$f \in L^2(D) \mapsto \int_D \operatorname{cov}[g](x, \cdot) f(x) dx \in L^2(D),$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, and the eigenfunctions are orthonormal. We recall that $\mathbb{E}[g(\omega, x)] = 0$ and that $\operatorname{cov}[g](x, y) = k(\|x - y\|)$. Therefore $\sum_{n \geq 1} \lambda_n = |D|k(0)$. Then the truncated Karhunen-Loève expansion [14, 15] g_N of the stochastic gaussian process g and its exponential a_N are defined by

$$g_N(\omega, x) = \sum_{n=1}^N \sqrt{\lambda_n} b_n(x) Y_n(\omega), \quad a_N(\omega, x) = e^{\sum_{n=1}^N \sqrt{\lambda_n} b_n(x) Y_n(\omega)}.$$

where the real random variables $(Y_n)_{n \geq 1}$ are uniquely determined by $Y_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D g(\omega, x) b_n(x) dx$. They are independent gaussian random variables with mean zero and unit variance. Mercer theorem [20] gives the following convergence result:

$$\sup_{x \in D} \mathbb{E}[(g - g_N)^2](x) \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

This convergence result does not actually enable us to conclude on the convergence of the solution.

From now on, we make the following assumptions:

- The eigenfunctions b_n are continuously differentiable.
- There exists constants C and $a > 0$ such that, for any $n \in \mathbb{N}$,

$$\|b_n\|_\infty \leq C \text{ and } \|b'_n\|_\infty \leq Cn^a$$

- There exists $b > 0$ such that $\sum_{n \geq 1} \lambda_n n^b$ is convergent.

Remark 2. *Such assumptions are fulfilled in the case of an exponential covariance kernel on a rectangular domain. Although these assumptions are not fulfilled in the case of a gaussian covariance kernel, similar results as the following ones hold in this case, and more generally in the case of an analytic covariance kernel, for more details see section 6.*

Definition 1. For $\alpha \leq b$, we define

$$R_N^\alpha = \sum_{n > N} \lambda_n n^\alpha.$$

Then $R_N^\alpha \leq N^{\alpha-b}(\sum_{n > N} \lambda_n n^b)$, and therefore, $R_N^\alpha = o(N^{\alpha-b})$ when $N \rightarrow +\infty$.

Proposition 4. For any p, α, β such that $0 < \alpha \leq \min\{b, 2a\}$ and $\beta < \frac{\alpha}{2a}$, there exists a constant $A_{\alpha, \beta, p}$ such that for all N in \mathbb{N} :

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0, \beta})} \leq A_{\alpha, \beta, p} (R_N^\alpha)^{\frac{1}{2}}.$$

In particular, for all p , for any $0 < \alpha \leq \min\{b, 2a\}$, there exists a constant $A_{\alpha, p}$ such that for all N in \mathbb{N} ,

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^0)} \leq A_{\alpha, p} (R_N^\alpha)^{\frac{1}{2}}.$$

Proof. For all $x, y \in D$, and $\theta \in]0, 1[$ such that $\theta \leq \frac{b}{2a}$,

$$\begin{aligned} (b_n(x) - b_n(y))^2 &\leq (2\|b_n\|_\infty)^{2(1-\theta)} (\|b'_n\|_\infty |x - y|)^{2\theta} \\ &\leq C_\theta n^{2a\theta} |x - y|^{2\theta}, \end{aligned}$$

where $C_\theta = 2^{2(1-\theta)} C^2$.

$$\begin{aligned} \mathbb{E} \left[((g_N - g)(x) - (g_N - g)(y))^2 \right] &= \sum_{n > N} \lambda_n (b_n(x) - b_n(y))^2 \\ &\leq C_\theta \left(\sum_{n > N} \lambda_n n^{2a\theta} \right) |x - y|^{2\theta} \\ &\leq C_\theta R_N^{2a\theta} |x - y|^{2\theta}. \end{aligned}$$

Since, for all $x, y \in D$ and N in \mathbb{N} , $(g_N - g)(x) - (g_N - g)(y)$ is a mean-free gaussian random variable, as a limit in L^2 of a linear combination of independent gaussian random variables,

$$(g_N - g)(x) - (g_N - g)(y) = \lim_{p \rightarrow +\infty} \sum_{n=N+1}^p \sqrt{\lambda_n} Y_n(\omega) (b_n(x) - b_n(y)).$$

The same result has seen in the proof of Proposition 1 yields

$$\begin{aligned} \mathbb{E} \left[((g_N - g)(x) - (g_N - g)(y))^{2p} \right] &\leq c_p \mathbb{E} \left[((g_N - g)(x) - (g_N - g)(y))^2 \right]^p \\ &\leq C_\theta^p c_p (R_N^{2a\theta})^p |x - y|^{2\theta p}. \end{aligned}$$

Thus, for any $0 < \alpha \leq \min\{b, 2a\}$, for any positive integer p , there exists a constant $C_{\alpha,p}$ such that:

$$\mathbb{E} \left[((g_N - g)(x) - (g_N - g)(y))^{2p} \right] \leq C_{\alpha,p} (R_N^\alpha)^p |x - y|^{\frac{2p}{\alpha}}$$

We now are going to use the Kolmogorov continuity theorem [5]. For any $0 < \alpha \leq \min\{b, 2a\}$, let ν be such that $\nu < 1$ and

$\alpha \frac{p}{a} - 1 - 2p\nu > -d$ i.e. $\nu < \frac{ad + \alpha p - a}{2ap}$, then

$$\begin{aligned} \mathbb{E}[\|g_N - g\|_{W^{\nu,2p}}^{2p}] &= \int_{\Omega} \int_D \int_D \frac{|(g_N - g)(x) - (g_N - g)(y)|^{2p}}{|x - y|^{1+2p\nu}} \\ &\leq C_{\alpha,p} (R_N^\alpha)^p \int_D \int_D |x - y|^{\alpha \frac{p}{a} - 1 - 2p\nu} \\ &\leq C_{\alpha,p,\nu} (R_N^\alpha)^p. \end{aligned}$$

with $C_{\alpha,p,\nu} = C_{\alpha,p} \int_D \int_D |x - y|^{\alpha \frac{p}{a} - 1 - 2p\nu} < +\infty$.

Next we use that $W^{\nu,2p}$ is continuously embedded in $C^{0,\beta}$ for $\beta < \min\{1, \nu - \frac{d}{2p}\}$, let $k_{\beta,\nu}$ denote the norm of this continuous embedding.

For any α such that $0 < \alpha \leq \min\{b, 2a\}$, for any β such that $\beta < \frac{\alpha}{2a}$, there exists $p_0 > 0$ such that for all $p \geq p_0$, $\beta < \frac{\alpha p - a}{2ap}$, then we choose $\nu < 1$ such that $\beta + \frac{d}{2p}\nu < \frac{ad + \alpha p - a}{2ap}$, finally $g_N - g \in C^{0,\beta}$ for almost all ω , with:

$$\mathbb{E}[\|g_N - g\|_{C^{0,\beta}}^{2p}] \leq k_{\beta,\nu}^{2p} C_{\alpha,p,\nu} (R_N^\alpha)^p.$$

Therefore, for any $0 < \alpha \leq \min\{b, 2a\}$, for any β such that $\beta < \frac{\alpha}{2a}$, there exists $p_0 > 0$ such that for any $p \geq p_0$, there exists a constant $A_{\alpha,\beta,p}$ such that for any N in \mathbb{N} :

$$\|g_N - g\|_{L^{2p}(\Omega, C^{0,\beta})} \leq A_{\alpha,\beta,p} (R_N^\alpha)^{\frac{1}{2}}.$$

Since in a probability space, if $p \leq q$, $f \in L^q$ implies $f \in L^p$ with $\|f\|_{L^p} \leq \|f\|_{L^q}$, we can conclude that for any p and $0 < \alpha < b$, for any β such that $\beta < \frac{\alpha}{2a}$, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in \mathbb{N} :

$$\|g_N - g\|_{L^p(\Omega, C^{0,\beta})} \leq A_{\alpha,\beta,p} (R_N^\alpha)^{\frac{1}{2}}.$$

□

Proposition 5. For any $\beta < \min\{1, \frac{b}{2a}\}$, for almost all ω , $g_N \rightarrow g$ in $C^{0,\beta}(D)$ as $N \rightarrow +\infty$ and so $a_N \rightarrow a$ in $C^0(D)$ as $N \rightarrow +\infty$.

Proof. We are going to use the Borel-Cantelli lemma: let $0 < \alpha \leq \min\{b, 2a\}$ such that $\beta < \frac{\alpha}{2a}$ and p such that $p > \frac{2}{b-\alpha}$, by the previous proposition, there exists a constant $A_{\alpha,\beta,p}$ such that for all N in \mathbb{N} :

$$\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0,\beta})} \leq A_{\alpha,\beta,p} (R_N^\alpha)^{\frac{1}{2}},$$

which implies that for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}(\|g_N - g\|_{\mathcal{C}^{0,\beta}} \geq \epsilon) &\leq \frac{\|g_N - g\|_{L^p(\Omega, \mathcal{C}^{0,\beta})}^p}{\epsilon^p} \\ &\leq \frac{A_{\alpha,\beta,p}^p (R_N^\alpha)^{\frac{p}{2}}}{\epsilon^p}. \end{aligned}$$

Let $\gamma > 0$ such that $\frac{2}{p} < \gamma < b - \alpha$, then $R_N^\alpha \leq R_N^{\alpha+\gamma} N^{-\gamma} = o(N^{-\gamma})$, when $N \rightarrow +\infty$ and since $\sum_{N \geq 1} N^{-\frac{\gamma p}{2}} < +\infty$, we have $\sum_{N \geq 1} \mathbb{P}(\|g_N - g\|_{\mathcal{C}^{0,\beta}} \geq \epsilon) < +\infty$. The Borel-Cantelli lemma yields $\mathbb{P}(\limsup(\|g_N - g\|_{\mathcal{C}^{0,\beta}} \geq \epsilon)) = 0$, which show that $g_N \rightarrow g$ in $\mathcal{C}^{0,\beta}$ as $N \rightarrow +\infty$ a.s. and finally, thanks to the continuity of the exponential function, we conclude that $a_N = e^{g_N} \rightarrow a = e^g$ a.s. in \mathcal{C}^0 as $N \rightarrow +\infty$. \square

Proposition 6. *It follows that a_N^{max} converges a.s. to a^{max} and a_N^{min} converges a.s. to a^{min} as $N \rightarrow +\infty$.*

The following results, which give uniform bounds for the random variables a_N^{max} and $\frac{1}{a_N^{min}}$ in the L^p -norm will be used to conclude this section and in the next two sections.

Definition 2. *For $N \in \mathbb{N}$, $\tau \in [0, 1]^N$, we define $g_{\tau,N} : \Omega \times D \rightarrow \mathbb{R}$ and $a_{\tau,N} : \Omega \times D \rightarrow \mathbb{R}$ by: $g_{\tau,N}(\omega, x) = \sum_{n=1}^N \sqrt{\lambda_n} Y_n(\omega) \tau_n b_n(x)$, and $a_{\tau,N}(\omega, x) = e^{g_{\tau,N}(\omega, x)}$*

Proposition 7. *For any $\beta < \min\{\frac{b}{2a}, 1\}$, $p > 0$, there exists a constant $B_{\beta,p}$ such that for any N in \mathbb{N} , and τ in $[0, 1]^N$,*

$$\|g_{\tau,N}\|_{L^p(\Omega, \mathcal{C}^{0,\beta})} \leq B_{\beta,p}$$

Proof. For any θ such that $0 < \theta < \min\{1, \frac{b}{2a}\}$, for any $N \in \mathbb{N}$, $\tau \in [0, 1]^N$

$$\begin{aligned} \mathbb{E}[(g_{\tau,N}(\omega, x) - g_{\tau,N}(\omega, y))^2] &= \sum_{n=1}^N \lambda_n \tau_n^2 (b_n(x) - b_n(y))^2 \\ &\leq C_\theta \left(\sum_{n=1}^N \lambda_n n^{2a\theta} \right) |x - y|^{2\theta} \\ &\leq C_\theta R_0^{2a\theta} |x - y|^{2\theta} \end{aligned}$$

Since, for any x, y in D , N in \mathbb{N} , $(g^{\tau,N}(x) - g^{\tau,N}(y))$ is a gaussian random variable, we have:

$$\mathbb{E}[(g_{\tau,N}(\omega, x) - g_{\tau,N}(\omega, y))^{2p}] \leq c_p (R_0^{2a\theta})^p |x - y|^{2\theta p}$$

Therefore, for any $0 < \alpha \leq \min\{b, 2a\}$, for any p , there exists a constant $M_{\alpha,p}$ such that, for any τ , for any N , we have:

$$\mathbb{E}[(g_{\tau,N}(\omega, x) - g_{\tau,N}(\omega, y))^{2p}] \leq M_{\alpha,p} |x - y|^{\alpha \frac{2}{a}}$$

where $M_{\alpha,p} = c_p (R_0^{2a\theta})^p$, then we use Kolmogorov continuity theorem [5] and conclude as in the proof of proposition 4. \square

Definition 3. By the previous proposition, for any $N \in \mathbb{N}$ and $\tau \in [0, 1]^N$, the trajectories of $a_{\tau, N}$ are continuous on the compact set \bar{D} a.s, so we can define, for almost all $\omega \in \Omega$, $a_{\tau, N}^{max}(\omega) = \max_{x \in \bar{D}} a_{\tau, N}(\omega, x)$ and $a_{\tau, N}^{min}(\omega) = \min_{x \in \bar{D}} a_{\tau, N}(\omega, x)$.

We can finally bound $a_{\tau, N}^{max}$ and $\frac{1}{a_{\tau, N}^{min}}$ in L^p -norm, independently from N and τ .

Proposition 8. For any $p > 0$, $a_{\tau, N}^{max}$ and $\frac{1}{a_{\tau, N}^{min}} \in L^p(\Omega)$, and there exists a constant D_p such that for any $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$

$$\left\| \frac{1}{a_{\tau, N}^{min}} \right\|_{L^p(\Omega)} \leq D_p,$$

and

$$\|a_{\tau, N}^{max}\|_{L^p(\Omega)} \leq D_p.$$

In particular,

$$\|a_N\|_{L^p(\Omega, \mathcal{C}^0)} \leq D_p.$$

Proof. We are going to apply Fernique's theorem [5], uniformly with respect to N and τ . There exists $x_0 \in]0, 1[$ such that for all $x \in [x_0, 1[$, $\ln(\frac{1-x}{x}) \leq -2$. The previous proposition yields the existence of a constant B_2 such that for any $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$:

$$\|g_{\tau, N}\|_{L^2(\Omega, \mathcal{C}^0)} \leq B_2.$$

Thus, setting $r_0 = \frac{B_2}{\sqrt{1-x_0}}$, we have, for every $r \geq r_0$,

$$\begin{aligned} \mathbb{P}(\|g_{\tau, N}\|_{\mathcal{C}^0} \geq r) &\leq \frac{\|g_{\tau, N}\|_{L^2(\Omega, \mathcal{C}^0)}^2}{r^2} \\ &\leq \frac{B_2^2}{r^2} \\ &\leq 1 - x_0. \end{aligned}$$

We now choose λ such that $32\lambda r_0^2 \leq 1$, and we denote by $\mu_{\tau, N}$ the law of $g_{\tau, N} : \Omega \rightarrow \mathcal{C}^0(D)$. Since the $\mu_{\tau, N}$ are centred gaussian measures on the Banach space $\mathcal{C}^0(D)$, we have then, for any $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$,

$$\ln \left(\frac{1 - \mu_{\tau, N}(\bar{B}(0, r_0))}{\mu_{\tau, N}(\bar{B}(0, r_0))} \right) + 32\lambda r_0^2 \leq -1.$$

Use Fernique theorem [5], set $k = e^{16\lambda r_0^2} + \frac{e^2}{e^2 - 1}$, to obtain that for all $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$,

$$\mathbb{E}[e^{\lambda \|g_{\tau, N}\|_{\mathcal{C}^0(D)}^2}] \leq k.$$

Hence, for any $p > 0$, $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$,

$$\begin{aligned} \mathbb{E}[e^{p \|g_{\tau, N}\|_{\mathcal{C}^0(D)}^2}] &\leq e^{\frac{p^2}{4\lambda}} \mathbb{E}[e^{\lambda \|g_{\tau, N}\|_{\mathcal{C}^0(D)}^2}] \\ &\leq k e^{\frac{p^2}{4\lambda}} \end{aligned}$$

Denoting $D_p = (ke^{\frac{p^2}{4x}})^{\frac{1}{p}}$, we conclude that:

$$\begin{aligned} \left\| \frac{1}{a_{\tau,N}^{\min}} \right\|_{L^p(\Omega)} &\leq \|e^{\|g_{\tau,N}\|_{C^0(\bar{D})}}\|_{L^p(\Omega)} \\ &\leq D_p, \end{aligned}$$

and

$$\begin{aligned} \|a_{\tau,N}^{\max}\|_{L^p(\Omega)} &\leq \|e^{\|g_{\tau,N}\|_{C^0(\bar{D})}}\|_{L^p(\Omega)} \\ &\leq D_p. \end{aligned}$$

□

Proposition 9. *For any $p > 0$, and $0 < \alpha \leq \min\{b, 2a\}$, there exists a constant $E_{\alpha,p}$ such that for any $N \in \mathbb{N}$,*

$$\|a_N - a\|_{L^p(\Omega, \mathcal{C}^0)} \leq E_{\alpha,p} (R_N^\alpha)^{\frac{1}{2}}$$

Proof. Take $p > 0$, choose $q, r > 0$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then the following inequality

$$\forall x, y \in \mathbb{R} \quad |e^x - e^y| \leq |x - y|(e^x + e^y),$$

together with Hölder's inequality leads to :

$$\|e^{g_N} - e^g\|_{L^r(\Omega, \mathcal{C}^0)} \leq \|g_N - g\|_{L^p(\Omega, \mathcal{C}^0)} \|e^{g_N} + e^g\|_{L^q(\Omega, \mathcal{C}^0)},$$

which we rewrite as

$$\begin{aligned} \|a_N - a\|_{L^r(\Omega, \mathcal{C}^0)} &\leq \|g_N - g\|_{L^p(\Omega, \mathcal{C}^0)} \|a_N + a\|_{L^q(\Omega, \mathcal{C}^0)} \\ &\leq A_{\alpha,p} (R_N^\alpha)^{\frac{1}{2}} (D_q + \|a\|_{L^q(\Omega, \mathcal{C}^0)}). \end{aligned}$$

We conclude by setting $E_{\alpha,r} = A_{\alpha,p} (D_q + \|a\|_{L^q(\Omega, \mathcal{C}^0(D))})$, with $p = q = \frac{r}{2}$ for instance. □

4 Strong convergence of u_N to u

Thanks to the results of the previous section, we can now estimate the strong error committed on the solution u , resulting from the approximation of a by a_N .

Definition 4. *Since for all $N \in \mathbb{N}$ and $\tau \in [0, 1]^N$, the random variables $a_{\tau,N}^{\max}$ and $\frac{1}{a_{\tau,N}^{\min}}$ belong to $L^p(\Omega)$ for all $p > 0$, the equation*

$$\begin{aligned} -\operatorname{div}(a_{\tau,N}(\omega, x) \nabla u_N^\tau(\omega, x)) &= f(x) \quad \text{on } D \\ u_N^\tau(\omega, x) &= 0 \quad \text{on } \partial D, \end{aligned}$$

admits a unique solution $u_N^\tau \in L^p(\Omega, H_0^1(D))$ for all $p > 0$. In particular, for $\tau = 1^N$, $a_{\tau,N} = a_N$ and we denote by u_N the solution.

We have $a_N(\omega, x) = \tilde{a}_N(Y_1(\omega), \dots, Y_N(\omega), x)$, therefore by the Doob-Dynkin lemma $u_N(\omega, x) = \tilde{u}_N(Y_1(\omega), \dots, Y_N(\omega), x)$, with $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}^N, H_0^1(D))$, since the application that maps the coefficients to the solution is smooth. For convenience, \tilde{a}_N will still be denoted by a_N and \tilde{u}_N by u_N . Then, for almost all $y \in \mathbb{R}^N$, $u_N(y, \cdot)$ solves the equation:

$$\begin{aligned} -\operatorname{div}(a_N(y, x) \nabla u_N(y, x)) &= f(x) \quad \text{on } D \\ u_N(y, x) &= 0 \quad \text{on } \partial D. \end{aligned}$$

We first show the almost sure convergence of u_N to u .

Proposition 10. $u_N(\omega, x)$ converges to $u(\omega, x)$ in $H_0^1(D)$, for almost all ω .

Proof. By proposition (5), for almost all ω , a_N converges to a in $\mathcal{C}^0(D)$ i.e. uniformly. Then we use the continuity of the solution u with respect to the coefficient a of the equation, indeed we have a.s.:

$$\begin{aligned} a_N^{\min} \|u - u_N\|_{H_0^1(D)}^2 &\leq \int_D a_N |\nabla(u - u_N)|^2 \\ &= \int_D (a_N - a) \nabla u \nabla(u - u_N) \\ &\leq \|a - a_N\|_{\mathcal{C}^0(D)} \|u - u_N\|_{H_0^1(D)} \|u\|_{H_0^1(D)}. \end{aligned}$$

Therefore a.s. we have:

$$\|u - u_N\|_{H_0^1(D)} \leq \frac{1}{a_N^{\min}} \|a - a_N\|_{\mathcal{C}^0(D)} \|f\|_{L^2(D)} \frac{C_D}{a^{\min}},$$

where C_D is the constant given by Poincaré inequality. The right-hand side of this inequality converges a.s. to 0 as $N \rightarrow +\infty$ by Propositions 5 and 6. \square

Next we give a convergence result in L^p -norm.

Theorem 1. For all $p > 0$, u_N converges to u in $L^p(\Omega, H_0^1(D))$, and for any $0 < \alpha \leq \min\{b, 2a\}$, there exists a constant $F_{\alpha,p}$ such that

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq F_{\alpha,p} (R_N^\alpha)^{\frac{1}{2}}.$$

Proof. For all $p > 0$, for almost all ω ,

$$\|u - u_N\|_{H_0^1(D)} \leq \frac{1}{a_N^{\min}} \|a - a_N\|_{\mathcal{C}^0(\bar{D})} \|f\|_{L^2} \frac{C_D}{a^{\min}}.$$

Hence, by choosing $q, r, s > 0$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$ it follows from Hölder inequality that for any $0 < \alpha \leq b$

$$\begin{aligned} \|u - u_N\|_{L^p(\Omega, H_0^1(D))} &\leq \left\| \frac{1}{a_N^{\min}} \right\|_{L^q(\Omega)} \|a - a_N\|_{L^r(\Omega, \mathcal{C}^0)} \|f\|_{L^2} C_D \left\| \frac{1}{a^{\min}} \right\|_{L^s(\Omega)} \\ &\leq D_q \|f\|_{L^2} D_s E_{\alpha,r} (R_N^\alpha)^{\frac{1}{2}}. \end{aligned}$$

\square

The following proposition gives a bound for u_N^τ in $L^p(H_0^1)$ -norm independent of N and τ , which will be useful in the next section.

Proposition 11. For all $p > 0$, there exists a constant G_p such that for all $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$

$$\|u_N^\tau\|_{L^p(\Omega, H_0^1(D))} \leq G_p$$

Proof. For almost all ω , for any $p > 0$, $N \in \mathbb{N}$, and $\tau \in [0, 1]^N$

$$\begin{aligned} \|u_N^\tau\|_{H_0^1(D)} &\leq \frac{C_D}{a_{\min}^{\tau,N}} \|f\|_{L^2(D)} \\ \|u_N^\tau\|_{L^p(\Omega, H_0^1(D))} &\leq C_D \left\| \frac{1}{a_{\min}^{\tau,N}} \right\|_{L^p} \|f\|_{L^2(D)} \\ &\leq C_D D_p \|f\|_{L^2(D)} \end{aligned}$$

\square

5 Weak convergence of u_N to u

In this section we are interested in the error committed on the law of u , more precisely we show that the order of the bound for the weak convergence of u_N to u is twice the order of the bound for the strong convergence.

Proposition 12. *For any multi-index $\alpha \in \mathbb{N}^{\mathbb{N}}$ with finite support, we have the following estimate on the growth of the derivatives of u_N with respect to y :*

$$\left\| \frac{\partial^\alpha u_N}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq C_{|\alpha|} \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \|u_N\|_{H_0^1} C^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}}.$$

where $C_{|\alpha|}$ only depends of α through his length.

Proof. We recall that for all $y \in \mathbb{R}^N$, $u(y, \cdot)$ solves the following equation

$$\int_D a_N(y, x) \nabla_x u_N(y, x) \nabla_x v(x) = \int_D f(x) v(x) \quad \forall v \in H_0^1(D).$$

We compute the derivatives of a_N with respect to the y_i : for all $1 \leq i, j \leq N$:

$$\begin{aligned} a_N(y) &= e^{\sum_{n=1}^N \sqrt{\lambda_n} b_n(x) y_n} \\ \frac{\partial a_N}{\partial y_i}(y, x) &= \sqrt{\lambda_i} b_i(x) a_N(y, x) \\ \frac{\partial a_N}{\partial y_i \partial y_j}(y, x) &= \sqrt{\lambda_i} b_i(x) \sqrt{\lambda_j} b_j(x) a_N(y, x). \end{aligned}$$

In every point $y \in \mathbb{R}^N$, the derivatives of u with respect to y_i and with respect to y_i and y_j , for $1 \leq i, j \leq N$ satisfy:

$$\begin{aligned} \int_D \frac{\partial a_N}{\partial y_i}(y, x) \nabla u_N(y, x) \nabla v(x) + \int_D a_N(y, x) \nabla \frac{\partial u_N}{\partial y_i}(y, x) \nabla v(x) &= 0 \\ \int_D \left(\frac{\partial^2 a_N}{\partial y_i \partial y_j} \nabla u_N + \frac{\partial a_N}{\partial y_i} \nabla \frac{\partial u_N}{\partial y_j} + \frac{\partial a_N}{\partial y_j} \nabla \frac{\partial u_N}{\partial y_i} + a_N \nabla \frac{\partial^2 u_N}{\partial y_i \partial y_j} \right) (y, x) \nabla v(x) &= 0. \end{aligned}$$

Namely, we recall that C is a constant such that for every $i \in \mathbb{N}$, $\|b_i\|_\infty \leq C$. By choosing $v(x) = \frac{\partial u_N}{\partial y_i}(y, x)$ in the first variational formulation, we have:

$$\begin{aligned} \left\| \sqrt{a_N}(y, x) \nabla \frac{\partial u_N}{\partial y_i} \right\|_{L^2} &\leq \sqrt{\lambda_i} \|b_i\|_\infty \|\sqrt{a_N}(y, x) \nabla u_N\|_{L^2} \\ &\leq \sqrt{\lambda_i} C \sqrt{a_{max}^N(y)} \|u_N\|_{H_0^1} \\ \left\| \frac{\partial u_N}{\partial y_i} \right\|_{H_0^1} &\leq \sqrt{\lambda_i} C \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \|u_N\|_{H_0^1}. \end{aligned}$$

By choosing $v(x) = \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y, x)$ in the second variational formulation, we have:

$$\begin{aligned} \left\| \sqrt{a_N}(y, x) \nabla \frac{\partial^2 u_N}{\partial y_i \partial y_j} \right\|_{L^2} &\leq \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 \|\sqrt{a_N} \nabla u_N\|_{L^2} \\ &+ \sqrt{\lambda_i} C \|\sqrt{a_N} \nabla \frac{\partial u_N}{\partial y_j}\|_{L^2} + \sqrt{\lambda_j} C \|\sqrt{a_N} \nabla \frac{\partial u_N}{\partial y_i}\|_{L^2} \\ &\leq 3 \sqrt{a_{max}^N(y)} \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 \|u_N\|_{H_0^1}. \end{aligned}$$

Therefore:

$$\left\| \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} \leq 3 \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 \|u_N\|_{H_0^1}.$$

The result follows by a recurrence. \square

In order to estimate the weak error, that is to say the expected value of $\varphi(u_N) - \varphi(u)$, for some regular function φ , we need estimates on the growth of the derivatives of $\varphi(u_N)$ with respect to the y_i , which follow directly from the previous proposition. More precisely, let φ in $C^4(\mathbb{R}, \mathbb{R})$, we assume that its derivatives (including φ itself) are bounded by a constant C_φ .

Proposition 13. *For any multi-index $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq 4$, we have the following estimate on the growth of the derivatives of $\varphi \circ u_N$ with respect to y :*

$$\left\| \frac{\partial^\alpha \varphi \circ u_N}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq C'_{|\alpha|} C_\varphi C^{|\alpha|} \left(1 + \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \|u_N\|_{H_0^1} \right)^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}}.$$

where $C'_{|\alpha|}$ only depends of α through his length.

Proof.

$$\begin{aligned} \frac{\partial \varphi \circ u_N}{\partial y_i}(y) &= \varphi' \circ u_N(y) \frac{\partial u_N}{\partial y_i}(y) \\ \left\| \frac{\partial \varphi \circ u_N}{\partial y_i}(y) \right\|_{H_0^1} &\leq C_\varphi \left\| \frac{\partial u_N}{\partial y_i}(y) \right\|_{H_0^1} \\ &\leq C_\varphi \sqrt{\lambda_i} C \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \|u_N\|_{H_0^1}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \varphi \circ u_N}{\partial y_i \partial y_j}(y) &= \varphi' \circ u_N(y) \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) + \varphi''(u_N(y)) \frac{\partial u_N}{\partial y_i}(y) \frac{\partial u_N}{\partial y_j}(y) \\ \left\| \frac{\partial^2 \varphi \circ u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} &\leq C_\varphi \left\| \frac{\partial^2 u_N}{\partial y_i \partial y_j}(y) \right\|_{H_0^1} + C_\varphi \left\| \frac{\partial u_N}{\partial y_i}(y) \right\|_{H_0^1} \left\| \frac{\partial u_N}{\partial y_j}(y) \right\|_{H_0^1} \\ &\leq 2C_\varphi \sqrt{\lambda_i} \sqrt{\lambda_j} C^2 \left(1 + \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} \|u_N(y)\|_{H_0^1} \right)^2 \end{aligned}$$

The result follows by a recurrence and Faà di Bruno's formula. \square

We are now ready to estimate the weak error, i.e. the quantity $\mathbb{E}[\varphi(u_N) - \varphi(u)]$ in H_0^1 -norm. Before expressing and proving the estimate of the weak error, we give the basic idea of the proof. To estimate the weak error, we consider the Taylor development at order 2 of $\varphi(u_N) - \varphi(u)$ and remark that first order terms and second order terms such that $i \neq j$ are mean-free. In the case where φ is the identity, formally the second

order development is:

$$\begin{aligned}
u(\omega, x) - u_N(\omega, x) &= u(Y_1(\omega), \dots, Y_N(\omega), Y_{N+1}(\omega), \dots, x) - u(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) \\
&= \sum_{i>N} \frac{\partial u}{\partial y_i}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) Y_i(\omega) \\
&\quad + \sum_{i,j>N} \frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) Y_i(\omega) Y_j(\omega) + \dots
\end{aligned}$$

Combining the independence of the Y_i with the fact that the Y_i are mean-free yields that the following terms are mean-free:

$$\begin{aligned}
\mathbb{E} \left[\frac{\partial u}{\partial y_i}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) Y_i(\omega) \right] &= \mathbb{E} \left[\frac{\partial u}{\partial y_i}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) \right] \mathbb{E}[Y_i(\omega)] \\
&= 0.
\end{aligned}$$

Analogously, for $i \neq j$,

$$\begin{aligned}
&\mathbb{E} \left[\frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) Y_i(\omega) Y_j(\omega) \right] \\
&= \mathbb{E} \left[\frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) \right] \mathbb{E}[Y_i(\omega)] \mathbb{E}[Y_j(\omega)] \\
&= 0.
\end{aligned}$$

The proof below shows that indeed the dominant in the error on the expected value is

$$\sum_{i>N} \mathbb{E} \left[\frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, x) \right]$$

We now give the general and precise result and its proof.

Theorem 2. *There exists a constant c such that for all $N \in \mathbb{N}$, for all $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ , we have:*

$$\|\mathbb{E}_\omega[(\varphi(u_N) - \varphi(u))]\|_{H_0^1(D)} \leq c C_\varphi \sum_{i>N} \lambda_i$$

Remark 3. *We do not need φ to be bounded in this proof, but only its derivatives. More generally, the result can be easily extended to the case where the derivatives of φ are bounded by a polynomial, which enables to treat the case of any moment of u .*

Remark 4. *The weak error at order N is bounded by R_N^0 , whereas the strong error at order N is bounded by $\sqrt{R_N^\alpha}$ for any $0 < \alpha \leq \min\{2a, b\}$, that is why we said that the weak order is twice the strong order.*

Proof. Let $M > N$, and $x \in D$, the first order Taylor theorem with integral remainder gives:

$$\begin{aligned}
&\mathbb{E}_\omega[(\varphi(u_M) - \varphi(u_N))(\omega, x)] \\
&= \mathbb{E}_\omega[\varphi(u_M)(Y_1(\omega), \dots, Y_M(\omega), x) - \varphi(u_M)(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, 0, x)] \\
&= \mathbb{E}_\omega[D_y(\varphi \circ u_M)(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, 0, x) \cdot (0, \dots, 0, Y_{N+1}(\omega), \dots, Y_M(\omega))] \\
&\quad + \mathbb{E}_\omega \left[\int_0^1 (1-t) D_y^2(\varphi \circ u_M)(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) \cdot (0, \dots, 0, Y_{N+1}, \dots, Y_M)^2 dt \right]
\end{aligned}$$

Since the random variables Y_i are independent, with mean zero and unit variance, the first order term is mean-free:

$$\begin{aligned}
& \mathbb{E}_\omega [D_y(\varphi \circ u_M)(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, 0, x) \cdot (0, \dots, 0, Y_{N+1}(\omega), \dots, Y_M(\omega))] \\
= & \mathbb{E}_\omega \left[\sum_{i=N+1}^M \frac{\partial(\varphi \circ u_M)}{\partial y_i}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, 0, x) Y_i(\omega) \right] \\
= & \sum_{i=N+1}^M \mathbb{E}_\omega \left[\frac{\partial(\varphi \circ u_M)}{\partial y_i}(Y_1(\omega), \dots, Y_N(\omega), 0, \dots, 0, x) \right] \mathbb{E}_\omega [Y_i(\omega)] \\
= & 0
\end{aligned}$$

We are now going to bound the integral remainder term, to begin with we split it into two terms:

$$\begin{aligned}
& \mathbb{E}_\omega [(\varphi(u_M) - \varphi(u_N))(\omega, x)] \\
= & \sum_{N+1 \leq i, j \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i Y_j \right] dt \\
= & \sum_{N+1 \leq i \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^2 \right] dt \\
+ & \sum_{N+1 \leq i \neq j \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i Y_j \right] dt
\end{aligned}$$

First we give an estimate for the first error contribution, by using the bound of the derivatives of $\varphi \circ u_M$ given in Proposition 13:

$$\begin{aligned}
& \left\| \sum_{N+1 \leq i \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^2 \right] dt \right\|_{H_0^1(D)} \\
\leq & \sum_{N+1 \leq i \leq M} \left(\int_0^1 (1-t)^2 \left\| \mathbb{E}_\omega \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^2 \right] \right\|_{H_0^1(D)}^2 dt \right)^{1/2} \\
\leq & \sum_{N+1 \leq i \leq M} \left(\int_0^1 (1-t)^2 \mathbb{E}_\omega \left[\left\| \frac{\partial^2(\varphi \circ u_M)}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) \right\|_{H_0^1(D)}^2 Y_i^4 \right] dt \right)^{1/2} \\
\leq & C'_2 C_\varphi C^2 \sum_{N+1 \leq i \leq M} \lambda_i \left(\int_0^1 \mathbb{E}_\omega \left[\left(1 + \sqrt{\frac{a_N^{max}}{a_N^{min}}} \|u_M\|_{H_0^1} \right)^4 (Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^4 \right] dt \right)^{1/2} \\
\leq & 8C'_2 C_\varphi C^2 \sum_{N+1 \leq i \leq M} \lambda_i \left(\int_0^1 \mathbb{E}_\omega \left[\left(1 + \left(\frac{a_N^{max}}{a_N^{min}} \right)^2 \|u_M\|_{H_0^1}^4 \right) (Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^4 \right] dt \right)^{1/2}.
\end{aligned}$$

We define, for $t \in [0, 1]$, $\tau_t \in [0, 1]^M$ by $\tau_t(i) = 0$ for $i \leq N$ and $\tau_t(i) = t$ for $i > N$, then by Hölder inequality and Propositions 8 and 11 we have:

$$\begin{aligned} & \mathbb{E}_\omega \left[\left(\frac{a_M^{max}}{a_M^{min}} \right)^2 \|u_M\|_{H_0^1(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x)}^4 Y_i^4 \right] \\ & \leq \|a_{\tau_t, M}^{max}\|_{L^8(\Omega)}^2 \left\| \frac{1}{a_{\tau_t, M}^{min}} \right\|_{L^8(\Omega)}^2 \|u_M^{\tau_t}\|_{L^{16}(\Omega, H_0^1)}^4 \|Y_i\|_{L^{16}(\Omega)}^4 \\ & \leq D_8^4 G_{16}^4 m_{16}^{1/4}. \end{aligned}$$

Where m_{16} is the moment of order 16 of a gaussian with mean zero and unit variance. We obtain finally the following bound for the first term of the error contribution.

$$\begin{aligned} & \left\| \sum_{N+1 \leq i \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 u_M}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^2 \right] dt \right\|_{H_0^1(D)} \\ & \leq 8C_2' C_\varphi C^2 \sqrt{m_4 + D_8^4 G_{16}^4 m_{16}^{1/4}} \sum_{N+1 \leq i \leq M} \lambda_i = C_\varphi k_1 \sum_{N+1 \leq i \leq M} \lambda_i. \end{aligned}$$

Where $k_1 = 8C_2' C^2 \sqrt{m_4 + D_8^4 G_{16}^4 m_{16}^{1/4}}$. Next we give an estimate for the second term of the error contribution, by using once again the independence of the random variables Y_i :

$$\begin{aligned} & \sum_{N+1 \leq i < j \leq M} \int_0^1 (1-t) \mathbb{E} \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M) Y_i Y_j \right] dt \\ & = \sum_{N+1 \leq i < j \leq M} \int_0^1 (1-t) \mathbb{E} \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M) Y_i Y_j \right] dt \\ & - \sum_{N+1 \leq i < j \leq M} \int_0^1 (1-t) \mathbb{E} \left[\frac{\partial^2(\varphi \circ u_M)}{\partial y_i \partial y_j}(Y_1, \dots, tY_{i-1}, 0, tY_{i+1}, \dots, tY_M) Y_i Y_j \right] dt \\ & = \sum_{N+1 \leq i < j \leq M} \mathbb{E} \left[\int_0^1 (1-t)(1-u)t \frac{\partial^3(\varphi \circ u_M)}{\partial y_i^2 \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tuY_i, tY_M) Y_i^2 Y_j dt du \right] \\ & = \sum_{N+1 \leq i < j \leq M} \mathbb{E} \left[\int_0^1 (1-t)(1-u)t \frac{\partial^3(\varphi \circ u_M)}{\partial y_i^2 \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tuY_i, tY_M) Y_i^2 Y_j dt du \right] \\ & - \sum_{N+1 \leq i < j \leq M} \mathbb{E} \left[\int_0^1 (1-t)(1-u)t \frac{\partial^3(\varphi \circ u_M)}{\partial y_i^2 \partial y_j}(Y_1, \dots, tuY_i, \dots, tY_{j-1}, 0, tY_{j+1}, \dots, tY_M) Y_i^2 Y_j dt du \right] \\ & = \sum_{N+1 \leq i < j \leq M} \mathbb{E} \left[\int_0^1 (1-t)(1-u)(1-s)t^2 \frac{\partial^4(\varphi \circ u_M)}{\partial y_i^2 \partial y_j^2}(Y_1, \dots, tuY_i, \dots, tsY_j, \dots, tY_M) Y_i^2 Y_j^2 dt du ds \right]. \end{aligned}$$

By Proposition 13, we have then:

$$\begin{aligned}
& \left\| \mathbb{E} \left[\int_0^1 (1-t)(1-u)(1-s)t^2 \frac{\partial^4(\varphi \circ u_M)}{\partial y_i^2 \partial y_j^2}(Y_1, tuY_i, \dots, tsY_j, \dots, tY_M, x) Y_i^2 Y_j^2 dt duds \right] \right\|_{H_0^1(D)} \\
& \leq \left(\int_0^1 (1-t)^2 (1-u)^2 (1-s)^2 t^4 \mathbb{E} \left[\left\| \frac{\partial^4(\varphi \circ u_M)}{\partial y_i^2 \partial y_j^2}(Y_1, \dots, tuY_i, \dots, tsY_j, \dots, tY_M) \right\|_{H_0^1(D)}^2 Y_i^4 Y_j^4 dt duds \right] \right)^{1/2} \\
& \leq C_4' C^4 C_\varphi \lambda_i \lambda_j \\
& \quad \left(\int_0^1 \mathbb{E} \left[\left(1 + \|u_M\|_{H_0^1} \sqrt{\frac{a_M^{max}}{a_M^{min}}} \right)^8 (Y_1, \dots, tY_{N+1}, \dots, tuY_i, \dots, tsY_j, \dots, tY_M) Y_i^4 Y_j^4 dt duds \right] \right)^{1/2} \\
& \leq 2^7 C_4' C^4 C_\varphi \lambda_i \lambda_j \\
& \quad \left(m_8 + \int_0^1 \mathbb{E} \left[\left(\|u_M\|_{H_0^1} \sqrt{\frac{a_M^{max}}{a_M^{min}}} \right)^8 (Y_1, \dots, tY_{N+1}, \dots, tuY_i, \dots, tsY_j, \dots, tY_M) Y_i^4 Y_j^4 dt duds \right] \right)^{1/2}.
\end{aligned}$$

We define, for $t, s, u \in [0, 1]$, $\tau_{t,s,u} \in [0, 1]^M$ by $\tau_{t,s,u}(n) = 0$ for $n \leq N$, $\tau_{t,s,u}(n) = t$ for $n > N$ such that $n \neq i, n \neq j$, $\tau_{t,s,u}(i) = tu$, and $\tau_{t,s,u}(j) = ts$.

Then, Hölder inequality combined with Propositions 8 and 11 yields the following estimate:

$$\begin{aligned}
& \mathbb{E}_\omega \left[\left(\sqrt{\frac{a_M^{max}}{a_M^{min}}} \|u_M\|_{H_0^1} \right)^8 (Y_1, \dots, tY_{N+1}, \dots, tuY_i, \dots, tsY_j, \dots, tY_M) Y_i^4 Y_j^4 \right] \\
& \leq \|a_{\tau_{t,s,u}, M}^{max}\|_{L^{20}(\Omega)}^4 \left\| \frac{1}{a_{\tau_{t,s,u}, M}^{min}} \right\|_{L^{20}(\Omega)}^4 \|u_M^{\tau_{t,s,u}}\|_{L^{40}(\Omega, H_0^1(D))}^8 \|Y_i^4\|_{L^5(\Omega)} \|Y_j^4\|_{L^5(\Omega)} \\
& \leq D_{20}^8 \|u_M^{\tau_{t,s,u}}\|_{L^{40}(\Omega, H_0^1(D))}^8 m_{20}^{\frac{2}{5}} \\
& \leq D_{20}^8 G_{40}^8 m_{20}^{\frac{2}{5}}.
\end{aligned}$$

We have finally the following estimate for the second term of the error contribution:

$$\begin{aligned}
& \left\| \sum_{N+1 \leq i \neq j \leq M} \int_0^1 (1-t) \mathbb{E} \left[\frac{\partial^2 u_M}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i Y_j \right] dt \right\|_{H_0^1(D)} \\
& \leq 2^7 C_4' C^4 C_\varphi \sqrt{m_8 + D_{40}^{16} G_{40}^8 m_{20}^{\frac{2}{5}}} \sum_{N+1 \leq i \neq j \leq M} \lambda_i \lambda_j = C_\varphi k_2 \sum_{N+1 \leq i \neq j \leq M} \lambda_i \lambda_j.
\end{aligned}$$

Where $k_2 = 2^7 C_4' C^4 \sqrt{m_8 + D_{40}^{16} G_{40}^8 m_{20}^{\frac{2}{5}}}$.

We have finally the following estimate for the total error:

$$\begin{aligned}
& \mathbb{E}_\omega [(\varphi(u_M) - \varphi(u_N))(\omega, x)] \\
& = \sum_{N+1 \leq i \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 u_M}{\partial y_i^2}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i^2 \right] dt \\
& + \sum_{N+1 \leq i \neq j \leq M} \int_0^1 (1-t) \mathbb{E}_\omega \left[\frac{\partial^2 u_M}{\partial y_i \partial y_j}(Y_1, \dots, Y_N, tY_{N+1}, \dots, tY_M, x) Y_i Y_j \right] dt
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|\mathbb{E}_\omega[(\varphi(u_M) - \varphi(u_N))(\omega, x)]\|_{H_0^1(D)} \\
\leq & C_\varphi k_1 \sum_{N+1 \leq i \leq M} \lambda_i + C_\varphi k_2 \sum_{N+1 \leq i \neq j \leq M} \lambda_i \lambda_j \\
\leq & C_\varphi k_1 \sum_{N+1 \leq i \leq M} \lambda_i + k_2 C_\varphi \left(\sum_{N+1 \leq i \leq M} \lambda_i \right)^2 \\
\leq & C_\varphi (k_1 + k_2 |D| k(0)) \sum_{i > N} \lambda_i.
\end{aligned}$$

Indeed we recall that $\sum_{n \geq 0} \lambda_n = |D|k(0)$. We define then $c = k_1 + k_2 k(0) |D|$.

We are now ready to conclude, by giving a bound for $\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]$ in H_0^1 -norm.

Let $N \geq 1$, then for all $M > N$, we have:

$$\begin{aligned}
\|\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]\|_{H_0^1} & \leq \|\mathbb{E}_\omega[(\varphi(u) - \varphi(u_M))]\|_{H_0^1} + \|\mathbb{E}_\omega[(\varphi(u_M) - \varphi(u_N))]\|_{H_0^1} \\
& \leq \|\varphi(u) - \varphi(u_M)\|_{L^2(\Omega, H_0^1(D))} + c C_\varphi \sum_{i > N} \lambda_i \\
& \leq C_\varphi \|u - u_M\|_{L^2(\Omega, H_0^1(D))} + c C_\varphi \sum_{i > N} \lambda_i
\end{aligned}$$

Letting $M \rightarrow +\infty$, by proposition 1 we have:

$$\|\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]\|_{H_0^1} \leq c C_\varphi \sum_{i > N} \lambda_i.$$

□

Remark 5. *The independance of the Y_i is crucial in this proof.*

6 An estimate of the total error for the collocation method

In this section, we recall the stochastic collocation method and slightly generalize the proof of the convergence result given in [1] to the case considered here where the assumption of uniform coercivity with respect to ω is not valid. Since we use many preliminary results of [1], we keep their notations. As we will see later, it is unfortunately difficult to get a relevant estimate of the dependance with respect to N of the constant appearing by bounding the collocation error. With the same notations and assumptions as above, we have a regularity result for the solution u_N with respect to y . We introduce a weight $\sigma(y) = \prod_{n=1}^N \sigma_n(y_n)$, where $\sigma_n(y_n) = e^{-\alpha_n |y_n|}$, for any $\alpha \in \mathbb{R}^N$ such that $\alpha_n \geq C\sqrt{\lambda_n}$ for all $1 \leq n \leq N$, and the functional space

$$\mathcal{C}_\sigma^0(\mathbb{R}^N, H_0^1(D)) = \{v : \mathbb{R}^N \rightarrow H_0^1(D), v \text{ continuous in } y, \sup_{y \in \mathbb{R}^N} \|\sigma(y)v(y)\|_{H_0^1(D)} < +\infty\}.$$

We denote by ρ the density of $Y = (Y_1, \dots, Y_N)$, we have then $\rho(y) = \prod_{n=1}^N \rho_n(y) = \frac{1}{(2\pi)^{N/2}} e^{-\sum_{n=1}^N \frac{y_n^2}{2}}$. Under the above assumptions, the following inclusion hold true:

$$\mathcal{C}_\sigma^0(\mathbb{R}^N, H_0^1(D)) \subset L_\rho^2(\mathbb{R}^N, H_0^1(D)).$$

with continuous embedding. More precisely, for any $v \in C_\sigma^0(\mathbb{R}^N, H_0^1(D))$, we have

$$\|v\|_{L_\rho^2(\mathbb{R}^N, H_0^1(D))} \leq k_\alpha \|v\|_{C_\sigma^0(\mathbb{R}^N, H_0^1(D))}$$

where $k_\alpha = \prod_{n=1}^N \frac{1}{\sqrt{2\pi}} \int e^{-\frac{y_n^2}{2} + 2\alpha_n |y_n|} dy_n$.

Proposition 14. *The solution u_N of the equation:*

$$\begin{aligned} -\operatorname{div}_x(a_N(y, x)\nabla_x u_N(y, x)) &= f(x) \text{ on } D \\ u_N(y, x) &= 0 \text{ on } \partial D \end{aligned}$$

satisfies $u_N \in C_\sigma^0(\mathbb{R}^N, H_0^1(D))$.

Proof. We have seen previously that $y \mapsto u(y)$ is continuous from \mathbb{R}^N to $H_0^1(D)$. For all $y \in \mathbb{R}^N$, since $\frac{1}{a_N^{\min}(y)} \leq e^{C \sum_{n=1}^N \sqrt{\lambda_n} y_n}$, we have:

$$\begin{aligned} \sigma(y) \|u(y)\|_{H_0^1(D)} &\leq \sigma(y) \frac{C_D}{a_N^{\min}(y)} \|f\|_{L^2(D)} \\ &\leq C_D \|f\|_{L^2(D)} e^{\sum_{n=1}^N (C\sqrt{\lambda_n} - \alpha_n) |y_n|} \\ &\leq C_D \|f\|_{L^2(D)}. \end{aligned}$$

□

The next result concerns the analyticity of the solution u_N with respect to y , we will perform a one-dimensional analysis in each direction y_n . For this, we introduce the following notation: for any $1 \leq n \leq N$ and $y \in \mathbb{R}^N$, $y = (y_n, y_n^*)$, where $y_n^* \in \mathbb{R}^{N-1}$. We set $\rho_n^*(y) = \prod_{j \neq n} \rho_j(y)$ and $\sigma_n^*(y) = \prod_{j \neq n} \sigma_j(y)$.

Proposition 15. *For any $1 \leq n \leq N$, the solution $u(y_n, y_n^*, x)$ as a function of y_n , $u : \mathbb{R} \rightarrow C_{\sigma_n^*}^0(\mathbb{R}^{N-1}, H_0^1(D))$ admits an analytic extension $u(z, y_n^*, x)$, $z \in \mathbb{C}$, in the region of the complex plane $\Sigma(\tau) = \{z \in \mathbb{C}, \operatorname{dist}(z, \mathbb{R}) \leq \tau\}$, for τ such that $\tau < 1$, moreover, for all $z \in \Sigma(\tau)$,*

$$\|\sigma_n(\operatorname{Re} z) u_N(z)\|_{C_{\sigma_n^*}^0(\mathbb{R}^{N-1}, H_0^1(D))} \leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1 - \tau} \|f\|_{L^2(D)} e^{\tau \alpha_n} \quad (2)$$

Proof. In every point $y \in \mathbb{R}^N$, $u_N(y)$ satisfies the problem

$$\int_D a_N(y, x) \nabla u_N(y, x) \nabla v(x) dx = \int_D f(x) v(x) dx \quad \forall v \in H_0^1(D).$$

Therefore, in every point $y \in \mathbb{R}^N$, for any $k \geq 1$, the k th derivative of u_N with respect to y_n satisfies the problem

$$\begin{aligned} &\int_D a_N(y, x) \nabla \frac{\partial^k u_N}{\partial y_n^k}(y, x) \nabla v(x) dx \\ &= - \sum_{l=1}^k \binom{k}{l} \int_D \frac{\partial^l a_N}{\partial y_n^l}(y, x) \nabla \frac{\partial^{k-l} u_N}{\partial y_n^{k-l}}(y, x) \nabla v(x) dx \quad \forall v \in H_0^1(D). \end{aligned}$$

Since $|\frac{\partial^l a_N}{\partial y_n^l}(y, x)| \leq (\sqrt{\lambda_n} C)^l |a_N(y, x)|$, we obtain the recursive inequality

$$\|\sqrt{a_N(y, x)} \nabla \frac{\partial^k u_N}{\partial y_n^k}(y, x)\|_{L^2(D)} \leq \sum_{l=1}^k \binom{k}{l} (C\lambda_n)^l \|\sqrt{a_N(y, x)} \nabla \frac{\partial^{k-l} u_N}{\partial y_n^{k-l}}(y, x)\|_{L^2(D)}.$$

$$\frac{\|\sqrt{a_N(y, x)} \nabla \frac{\partial^k u_N}{\partial y_n^k}(y, x)\|_{L^2(D)}}{k!} \leq \sum_{l=1}^k \frac{(C\lambda_n)^l}{l!} \frac{\|\sqrt{a_N(y, x)} \nabla \frac{\partial^{k-l} u_N}{\partial y_n^{k-l}}(y, x)\|_{L^2(D)}}{(k-l)!}.$$

A recurrence yields:

$$\begin{aligned} \frac{\|\sqrt{a_N(y, x)} \nabla \frac{\partial^k u_N}{\partial y_n^k}(y, x)\|_{L^2(D)}}{k!} &\leq \|\sqrt{a_N(y, x)} \nabla u_N(y, x)\|_{L^2(D)} e^{\sum_{l=1}^k \frac{(C\sqrt{\lambda_n})^l}{l!}} \\ &\leq \|\sqrt{a_N(y, x)} \nabla u_N(y, x)\|_{L^2(D)} e^{e^{C\sqrt{\lambda_n}}}. \end{aligned}$$

And finally :

$$\frac{\|\frac{\partial^k u_N}{\partial y_n^k}(y, x)\|_{H_0^1(D)}}{k!} \leq \frac{C_D}{a^{\min}(y)} \|f\|_{L^2(D)} e^{e^{C\sqrt{\lambda_n}}}.$$

We now define for every $y_n \in \mathbb{R}$ the power series $u_N : \mathbb{C} \rightarrow \mathcal{C}_{\sigma_n^*}^0(\mathbb{R}^{N-1}, H_0^1(D))$ as

$$u(z, y_n^*, x) = \sum_{k=0}^{+\infty} \frac{(z - y_n)^k}{k!} \frac{\partial^k u_N}{\partial y_n^k}(y_n, y_n^*, x).$$

Since

$$\left\| \frac{|z - y_n|^k}{k!} \frac{\partial^k u_N}{\partial y_n^k}(y_n, y_n^*, x) \right\|_{\mathcal{C}_{\sigma_n^*}^0(\mathbb{R}^{N-1}, H_0^1(D))} \leq \frac{C_D}{a^{\min}(y_n, y_n^*)} \|f\|_{L^2(D)} e^{e^{C\sqrt{\lambda_n}}} |z - y_n|^k$$

the radius of convergence of this series is 1. Moreover, take $\tau \in]0, 1[$, since for all $z \in \mathbb{C}$ such that $|z - y_n| \leq \tau$, $\sigma_n(\operatorname{Re}(z)) \leq e^{\alpha_n \tau} \sigma_n(y_n)$ we have the following estimate:

$$\begin{aligned} \|\sigma_n(\operatorname{Re}(z)) u(z)\|_{\mathcal{C}_{\sigma_n^*}^0(\mathbb{R}^{N-1}, H_0^1(D))} &\leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1 - \tau} \|f\|_{L^2(D)} e^{\sum_{i=1}^N C\sqrt{\lambda_i} - \alpha_i |y_i|} e^{\alpha_n \tau} \\ &\leq \frac{C_D e^{e^{C\sqrt{\lambda_n}}}}{1 - \tau} \|f\|_{L^2(D)} e^{\alpha_n \tau}. \end{aligned}$$

Hence, by a continuation argument, the function $u_N(y)$ can be extended analytically on the whole region $\Sigma(\tau)$ and estimate (2) follows. \square

We recall here the stochastic collocation method: we seek a numerical approximation to the exact solution u_N of the equation

$$\begin{aligned} -\operatorname{div}_x(a_N(y, x) \nabla_x u_N(y, x)) &= f(x) \text{ on } D \\ u_N(y, x) &= 0 \text{ on } \partial D \end{aligned} \quad (3)$$

in a finite dimensional subspace $V_{p,h}$ based on a tensor product, $V_{p,h} = \mathbb{R}_p(\mathbb{R}^N) \otimes H_h(D)$, where the following hold:

- $H_h(D) \subset H_0^1(D)$ is a standard finite element space of dimension N_h , which contains continuous piecewise polynomials defined on regular triangulations \mathcal{T}_h that have a maximum mesh spacing parameter $h > 0$.
- $\mathbb{R}_p(\mathbb{R}^N) \subset L_\rho^2(\mathbb{R}^N)$ is the span of tensor product polynomials with degree at most $p = (p_1, \dots, p_N)$ i.e., $\mathbb{R}_p(\mathbb{R}^N) = \otimes_{n=1}^N \mathbb{R}_{p_n}(\mathbb{R})$, with

$$\mathbb{R}_{p_n}(\mathbb{R}) = \operatorname{span}(y_n^m, m = 0, \dots, p_n), \quad n = 1, \dots, N.$$

We first introduce the semidiscrete approximation $u_N^h : \mathbb{R} \rightarrow H_h(D)$, obtained by projecting (3) onto the subspace $H_h(D)$, for each $y \in \mathbb{R}$, i.e.

$$\int_D a_N \nabla u_N^h(y, x) \nabla v(x) dx = \int_D f(x) v(x) dx \quad \forall v \in H_h(D). \quad (4)$$

The next step consists in collocating (4) on the zeros of orthogonal polynomials and building the discrete solution $u_N^{h,p} \in R_p(\mathbb{R}^N) \otimes H_h(D)$ by interpolating in y the collocated solutions. For each dimension $n = 1, \dots, N$, let y_{n,k_n} , $1 \leq k_n \leq p_n + 1$, be the $p_n + 1$ roots of the Hermite polynomial q_{p_n+1} of degree $p_n + 1$, which then satisfies $\int_{\mathbb{R}} q_{p_n+1}(y) v(y) \rho(y) dy = 0$ for all $v \in R_{p_n}(\mathbb{R})$. To any vector of indexes $[k_1, \dots, k_N]$ we associate the global index

$$k = k_1 + p_1(k_2 - 1) + p_1 p_2(k_3 - 1) + \dots$$

and we denote by y_k the point $y_k = [y_{1,k_1}, y_{2,k_2}, \dots, y_{N,k_N}] \in \mathbb{R}^N$. We also introduce, for each $n = 1, 2, \dots, N$ the Lagrange basis $\{l_{n,j}\}_{j=1}^{p_n+1}$ of the space $R_{p_n}(\mathbb{R})$,

$$l_{n,j} \in R_{p_n}(\mathbb{R}), \quad l_{n,j}(y_{n,k}) = \delta_{jk}, \quad j, k = 1, \dots, p_n + 1,$$

where δ_{jk} is the Kronecker symbol, and we set $l_k(y) = \prod_{n=1}^N l_{n,k_n}(y_n)$. The points y_k are then the nodes of the Gaussian quadrature formula associated the the weight ρ . Hence, the final approximation is given by

$$u_N^{h,p}(y, x) = \sum_{k=1}^{N_p} u_N^h(y_k, x) l_k(y),$$

where $u_n^h(y_k, x)$ is the solution of problem (4) for $y = y_k$.

Equivalently, if we introduce the Lagrange interpolant operator $\mathcal{L}_p : \mathcal{C}^0(\mathbb{R}^N, H_0^1(D)) \rightarrow R_p(\mathbb{R}^N) \otimes H_0^1(D)$, such that

$$\mathcal{L}_p v(y) = \sum_{n=1}^N v(y_n) l_n(y) \quad \forall v \in \mathcal{C}^0(\mathbb{R}^N, H_0^1(D)),$$

then we have simply $u_N^{h,p} = \mathcal{L}_p u^h$.

Finally, for any continuous function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ we introduce the Gauss quadrature formula $E_\rho^p[g]$ approximating the integral $\int_{\mathbb{R}^N} g(y) \rho(y) dy$ as

$$E_\rho^p[g] = \sum_{k=1}^{N_p} \omega_k g(y_k), \quad \omega_k = \prod_{n=1}^N \omega_{k_n}, \quad \omega_{k,n} = \int_{\mathbb{R}} l_{k_n}^2(y) \rho_n(y) dy.$$

This can be used to approximate the law of u_N , i.e. we approximate the expected values $\mathbb{E}[\varphi(u_N(Y(\omega), x))]$ as $\mathbb{E}_\rho^p[\varphi(u_N^h)]$, for some function φ .

Our aim is to give a priori estimates for the total error $\varepsilon = u_N - u_N^{h,p}$ in the natural norm $L_\rho^2(\mathbb{R}^N, H_0^1(D))$. This total error naturally splits into

$$\varepsilon = (u_N - u_N^h) + (u_N^h - u_N^{h,p}).$$

The first term is a term of space discretization error and can be estimate easily, indeed, for all $y \in \mathbb{R}^N$, the function $u_N^h(y)$ is the orthogonal projection of $u_N(y)$ onto the subspace $H_h(D)$ with respect to the inner product

$(u, v) \rightarrow \int_D a_N(y, x) \nabla u(x) \nabla v(x) dx$. Therefore, for all $y \in \mathbb{R}^N$,

$$\begin{aligned} \|(u_N - u_N^h)(y)\|_{H_0^1(D)} &\leq \sqrt{\frac{1}{a_N^{\min}(y)}} \inf_{v \in H_h(D)} \left(\int_D a_N(y, x) |\nabla(u_N(y) - v)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{a_N^{\max}(y)}{a_N^{\min}(y)}} \inf_{v \in H_h(D)} \|u_N(y) - v\|_{H_0^1(D)} \\ &\leq e^{C \sum_{n=1}^N \sqrt{\lambda_n} |y_n|} \inf_{v \in H_h(D)} \|u_N(y) - v\|_{H_0^1(D)}. \end{aligned}$$

We can finally conclude for the first term, by using the standard approximation estimate for the finite element space $H_h(D)$: there exists a constant C_{fe} such that for any $v \in H^2(D)$ and $h > 0$

$$\min_{w \in H_h(D)} \|v - w\|_{H_0^1(D)} \leq C_{fe} h \|v\|_{H^2(D)},$$

Since $a_N(y)$ is smooth for any $y \in \mathbb{R}^N$, elliptic regularity yields that $u_N(y) \in H^2(D)$ for all $y \in \mathbb{R}^N$, with:

$$\begin{aligned} \|u_N(y)\|_{H^2(D)} &\leq k \frac{\|f\|_{L^2(D)}}{a_N^{\min}(y)} \left(1 + \frac{a_N^{\max}(y)}{a_N^{\min}(y)} \right) \left(1 + \frac{a_N^{\max}(y)}{a_N^{\min}(y)} + \frac{\|a'_N(y)\|_{L^\infty(D)}}{a_N^{\min}(y)} \right) \\ &\leq k \|f\|_{L^2(D)} e^{5C \sum_{n=1}^N \sqrt{\lambda_n} |y_n|} (1 + C \sum_{n=1}^N \sqrt{\lambda_n} |y_n| n^a) \end{aligned}$$

where k is a constant independent of f , N and y whose value changes. We can then bound the spacial discretization error.

$$\|u_N - u_N^h\|_{L_\rho^2(\mathbb{R}, H_0^1(D))} \leq \frac{k}{(2\pi)^{N/2}} \|f\|_{L^2(D)} C_{fe} h \sqrt{\int_{\mathbb{R}^N} e^{\sum_{n=1}^N (12C \sqrt{\lambda_n} |y_n| (1+n^a) - \frac{y_n^2}{2})} dy}$$

Proposition 16. *There exists a constant k independent of N such that for all $h > 0$*

$$\|u_N - u_N^h\|_{L_\rho^2(\mathbb{R}, H_0^1(D))} \leq k \|f\|_{L^2(D)} 2^{N/2} e^{\sum_{n=1}^N 72C^2 \lambda_n (1+n^a)^2} h.$$

Remark 6. *It is natural to obtain a bound that explodes as $N \rightarrow +\infty$, but this bound is rough, it would be interesting to have a better estimate of the constant.*

The second term $u_N^h - u_N^{h,p}$ is an interpolation error, indeed $u_N^{h,p} = \mathcal{L}_p u_N^h$. First, we recall some known results of approximation theory for functions defined on a one-dimensional domain with values in a Banach space denoted by V . Here we assume that the functions ρ and σ are defined on \mathbb{R} by $\sigma(y) = e^{-c\lambda y}$ and $\rho(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, with $\lambda, C > 0$. The proof of the following three propositions can be found in [1].

Proposition 17. *For any weight ν such that $\nu = O(\sigma)$, the operator $\mathcal{L}_\nu : \mathcal{C}_\nu^0(\mathbb{R}, V) \rightarrow L_\rho^2(\mathbb{R}, V)$ is continuous. The norm of this linear continuous operator will be denoted by C_1 .*

The following proposition, which is a consequence of a result from Uspensky [21], relates the approximation error $v - \mathcal{L}_p v$ in the L_ρ^2 -norm with the best approximation error in the weighted \mathcal{C}_σ^0 -norm.

Proposition 18. *For any weight ν such that $\nu = O(\sigma)$, for every function $v \in \mathcal{C}_\nu^0(\mathbb{R}, V)$ the interpolation error satisfies*

$$\|v - \mathcal{L}_p v\|_{L_\rho^2(\mathbb{R}, V)} \leq C_2 \inf_{w \in R_p(\mathbb{R}) \otimes V} \|v - w\|_{\mathcal{C}_\nu^0(\mathbb{R}, V)}$$

with a constant C_2 independent of p .

We now analyze the best approximation error for a function $v : \mathbb{R} \rightarrow V$ which admits an analytic extension in the complex plane, in the region $\Sigma(\tau) = \{z \in \mathbb{C}, \text{dist}(z, \mathbb{R}) \leq \tau\}$ for some $\tau > 0$. We will still denote the extension by v ; in this case, τ represents the distance between \mathbb{R} and the nearest singularity of $v(z)$ in the complex plane. The following result is a consequence of a result from Hille [11].

Proposition 19. *Let v be a function in $\mathcal{C}_\sigma^0(\mathbb{R}, V)$. We assume that v admits an analytic extension in the strip of the complex plane $\Sigma(\tau)$ for some $\tau > 0$, and that*

$$\forall r = (y + iw) \in \Sigma(\tau), \quad \sigma(y) \|v(z)\|_V \leq C_v(\tau).$$

Then, for any $\delta > 0$ there exists a constant C , independent of p , and a function $\Theta(p) = O(\sqrt{p})$ such that

$$\min_{w \in R_p(\mathbb{R}) \otimes V} \max_{y \in \mathbb{R}} \left\| \|v(y) - w(y)\|_V e^{-\frac{(\delta y)^2}{4}} \right\| \leq C \Theta(p) e^{-\tau \delta \sqrt{p}}.$$

We are now ready to prove the following proposition, which gives an estimate of the total interpolation error:

Proposition 20. *For any $\tau < 1$, there exists a constant $C^{\tau, N}$, independent of h and p such that*

$$\|u_N^h - u_N^{h,p}\|_{L_\rho^2(\mathbb{R}, H_0^1(D))} \leq C_{\tau, N} \sum_{n=1}^N \sqrt{p_n} e^{-\frac{\tau}{\sqrt{2}} \sqrt{p_n}}.$$

Remark 7. *We could precise how the constant $C_{\tau, N}$ depends on N , but it will be few explicit and not necessary relevant.*

Proof. From now on, we are back to the the multi-dimensional problem, we are going to split the total interpolation error into N partial errors linked to one-dimensional interpolation errors.

To begin with, we admit that u_N^h has the same regularity with respect to y as the exact solution u_N , i.e. more precisely that u_N^h verifies the same properties as proved for u_N in proposition 15. We focus on the first direction y_1 and define a one-dimensional interpolation operator

$\mathcal{L}_1 : \mathcal{C}_{\sigma_1}^0(\mathbb{R}, L_{\rho_1}^2(\mathbb{R}, H_0^1(D))) \rightarrow L_{\rho_1}^2(\mathbb{R}, L_{\rho_1^*}^2(\mathbb{R}, H_0^1(D)))$, by

$$\mathcal{L}_1(y_1, y_1^*, x) = \sum_{k=1}^{p_1+1} v(y_{1,k}, y_1^*, x) l_{1,k}(y_1).$$

Then, the global interpolant \mathcal{L}_p can be written as the composition of two interpolation operators $\mathcal{L}_p = \mathcal{L}_1 \circ \mathcal{L}_p^{(1)}$, where $\mathcal{L}_p^{(1)}$ is the interpolation in all the directions y_2, y_3, \dots, y_N except y_1 , i.e. $\mathcal{L}_p^{(1)} : \mathcal{C}_{\sigma_1}^0(\mathbb{R}^{N-1}, H_0^1(D)) \rightarrow L_{\rho_1^*}^2(\mathbb{R}^{N-1}, H_0^1(D))$. We have then

$$\|u_N^h - \mathcal{L}_p u_N^h\|_{L_\rho^2(\mathbb{R}^n, H_0^1(D))} \leq \|u_N^h - \mathcal{L}_1 u_N^h\|_{L_\rho^2(\mathbb{R}^n, H_0^1(D))} + \|\mathcal{L}_1(u_N^h - \mathcal{L}_p^{(1)} u_N^h)\|_{L_\rho^2(\mathbb{R}^n, H_0^1(D))}.$$

Let us bound the first term. We think of u_N^h as a function of y_1 with values in a Banach space V , $u_N^h \in L_{\rho_1}^2(\mathbb{R}, V)$, where $V = L_{\rho_1^*}^2(\mathbb{R}^{N-1}, H_0^1(D))$. The following inclusions hold true:

$$\mathcal{C}_{\sigma_1}^0(\mathbb{R}, V) \subset \mathcal{C}_{G_1}^0(\mathbb{R}, V) \subset L_{\rho_1}^2(\mathbb{R}, V)$$

with $G_1(y_1) = e^{-\frac{y_1^2}{8}}$. Since $G_1 = O(\sigma)$, proposition (18) yields the following estimate

$$\|u_N^h - \mathcal{L}_1 u_N^h\|_{L_{\rho}^2(\mathbb{R}^n, H_0^1(D))} \leq C_2 \inf_{w \in R_{p_1}(\mathbb{R}) \otimes V} \|u_N^h - w\|_{\mathcal{C}_{G_1}^0(\mathbb{R}, V)}.$$

To bound the best approximation error in $\mathcal{C}_{G_1}^0(\mathbb{R}, V)$, we employ the fact that $u_N^h \in \mathcal{C}_{\sigma_1}^0(\mathbb{R}, V)$ and the analyticity result of proposition (15), which yields the analyticity of u_N^h as a function of \mathbb{R} into V on $\sigma(\tau)$ for every $\tau < 1$ and the following bound, for all $z \in \sigma(\tau)$:

$$\sigma_1(z) \|u_N^h(z)\|_{\mathcal{C}_{\sigma_1^*}^0(\mathbb{R}^{N-1}, H_0^1(D))} \leq \frac{C_D e^{e^C \sqrt{\lambda_1}}}{1 - \tau} \|f\|_{L^2(D)} e^{\alpha_1 \tau},$$

which implies, thanks to the continuous embedding of $\mathcal{C}_{\sigma_1^*}^0(\mathbb{R}^{N-1}, H_0^1(D))$ into $L_{\rho_1^*}^2(\mathbb{R}^{N-1}, H_0^1(D))$,

$$\sigma_1(z) \|u_N^h(z)\|_{L_{\rho_1^*}^2(\mathbb{R}^{N-1}, H_0^1(D))} \leq k_{\alpha_1^*} \frac{C_D e^{e^C \sqrt{\lambda_1}}}{1 - \tau} \|f\|_{L^2(D)} e^{\alpha_1 \tau}.$$

Recalling that we have fixed $V = L_{\sigma_1^*}^2(\mathbb{R}^{N-1}, H_0^1(D))$ in this proof, we are now ready to conclude with the first term by applying proposition (19) to $u_N^h \in \mathcal{C}_{\sigma_1}^0(\mathbb{R}, V)$, for $\tau < 1$ and $\delta_1 = \frac{1}{\sqrt{2}}$, which gives the following bound for the best approximation error:

$$\inf_{w \in R_{p_1}(\mathbb{R}) \otimes V} \|u_N^h - w\|_{\mathcal{C}_{G_1}^0(\mathbb{R}, V)} \leq C\Theta(p_1) e^{-\frac{\tau}{\sqrt{2}} \sqrt{p_1}}.$$

It gives then the following bound for the first term:

$$\|u_N^h - \mathcal{L}_1 u_N^h\|_{L_{\rho}^2(\mathbb{R}^n, H_0^1(D))} \leq C_2 C\Theta(p_1) e^{-\frac{\tau}{\sqrt{2}} \delta_1 \sqrt{p_1}}.$$

Let us bound the second term. By proposition (17), we bound the second term:

$$\|\mathcal{L}_1(u_N^h - \mathcal{L}_p^{(1)} u_N^h)\|_{L_{\rho}^2(\mathbb{R}^n, H_0^1(D))} \leq C_1 \|u_N^h - \mathcal{L}_p^{(1)} u_N^h\|_{\mathcal{C}_{\sigma_1}^0(\mathbb{R}, V)}.$$

The term on the right-hand side is again an interpolation error. Thus we have to bound the interpolation error in all the other $N - 1$ directions, uniformly with respect to y_1 (in the weighted norm $\mathcal{C}_{\sigma_1}^0$). We can proceed iteratively, defining an interpolation L^2 , bounding the resulting error in the direction y_2 , and so on. \square

7 Examples

In this section, we give examples of covariance kernel $cov[g]$ and the strong and weak convergence results corresponding.

7.1 The exponential kernel case on a box

In this subsection, we consider the case where the covariance kernel $\text{cov}[g]$ is exponential, i.e. $k(x) = \sigma^2 e^{-\frac{x}{l}}$ and D is a box, the length l is called the correlation length and the norm choosed on D is $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$. In this case the eigenvalues and the eigenfunctions can be found analytically, we first show that the assumptions done in section 1 are fulfilled and then give the convergence rate for the strong and weak convergence of u_N to u . We first treat the mono-dimensional case, for convenience we assume that the domain D is $(0, 1)$. We have then analytic expressions for the eigenvalues and eigenfunctions, the proof can be found in [27]. We consider the characteristic equation

$$(l^2 w^2 - 1) \sin(w) = 2lw \cos(w)$$

and denote by $(w_n)_{n \geq 1}$ the sequence of its positive roots sorted in an increasing number, then the eigenvalues of the Karhunen-Loève development can be expressed as $\lambda_n = \frac{2l\sigma^2}{l^2 w_n^2 + 1}$ and the eigenfunctions as $b_n(x) = \alpha_n (\sin(w_n x) + l w_n \cos(w_n x))$ where $\alpha_n = \frac{1}{\sqrt{(l^2 w_n^2 + 1)/2 + l}}$. The sequence of the roots (w_n) satisfies $w_n \underset{n \rightarrow +\infty}{\sim} n\pi$, which yields the following equivalents for the eigenvalues and eigenfunctions:

$$\lambda_n \underset{n \rightarrow +\infty}{\sim} \frac{2\sigma^2}{l\pi^2 n^2}$$

$$b_n(x) \underset{n \rightarrow +\infty}{\sim} \sqrt{2} \cos(w_n x).$$

We can now show that such a covariance kernel fulfills the assumptions done in section 1: $k \in \mathcal{C}^{0,1}(\mathbb{R})$, the eigenfunctions b_n are continuously differentiable, and we have $\|b_n\|_\infty \leq 2\sqrt{2}$, $\|b'_n\|_\infty \leq 2\sqrt{2}(n + 1/2)\pi$ for every $n \geq 0$ and for any $b < 1$, $\sum_{n \geq 1} \lambda_n n^b$ is convergent. We can therefore apply the results of sections 3 and 4. The application of Theorem 1 gives the following strong convergence result:

Proposition 21. *For every $p > 0$ and $0 < \alpha < 1$, there exists a constant $C_{\alpha,p}$ such that for any $N \in \mathbb{N}$,*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq C_{\alpha,p} N^{\frac{\alpha-1}{2}}.$$

Proof. Take $0 < \alpha < 1$, we have

$$R_N^\alpha = \sum_{n > N} \lambda_n n^\alpha \leq \frac{2\sigma^2}{l\pi^2} \sum_{n > N} n^{\alpha-2} \leq \frac{2\sigma^2}{l\pi^2(1-\alpha)} N^{\alpha-1}.$$

□

On the other hand, Theorem 2 yields the following weak convergence result:

Proposition 22. *There exists a constant C such that for any $N \in \mathbb{N}$ and $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ , we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq C C_\varphi \frac{1}{N}.$$

We now treat the case where the spatial dimension is $d = 2$, the following result can be easily extended for any dimension d . We choose for the sake of simplicity $D = (0, 1)^2$, but it can be immediately generalized to the case where D is a box. We denote by $(\mu_n)_{n \geq 1}$

the sequence of the eigenvalues sorted in a decreasing order and c_n the corresponding eigenfunctions, for any $n \geq 1$, there exists (i, j) such that $\mu_n = \lambda_i \lambda_j$, because the eigenvalues in dimension 2 are the products of two eigenvalues in dimension 1, since we choose the norm $\|\cdot\|_1$. We can then define the following bijective function :

$$g : \mathbb{N}^* \rightarrow (\mathbb{N}^*)^2 \\ n \mapsto (i, j) \text{ such that } \mu_n = \lambda_i \lambda_j$$

First we notice that $(n - \frac{1}{2})\pi \leq w_n \leq (n + \frac{1}{2})\pi$, therefore, for any $n \geq 1$

$$\lambda_n \leq \frac{2l\sigma^2}{l^2 (n - \frac{1}{2})^2 \pi^2} \leq \frac{8\sigma^2}{l\pi^2} \frac{1}{n^2}$$

and

$$\lambda_n \geq \frac{2l\sigma^2}{l^2 (n + \frac{1}{2})^2 \pi^2 + 1} \geq \frac{2l\sigma^2}{1 + 4\pi^2 l^2} \frac{1}{n^2}.$$

We recall that the mono-dimensional eigenfunctions b_n are continuously differentiable and that there exists a constant C such that $\|b_n\|_\infty \leq C$ and $\|b'_n\|_\infty \leq Cn$ for all $n \geq 1$. Therefore, for any $(i, j) \in (\mathbb{N}^*)^2$, $c_{g(i,j)}(x) = b_i(x)b_j(x)$ is continuously differentiable, $\|c_{g(i,j)}\|_\infty \leq \|g_i\|_\infty \|g_j\|_\infty \leq C^2$, and $\|c'_{g(i,j)}\|_\infty \leq C^2(i+j) \leq 2C^2 ij$

$$g(i, j) = \text{Card}\{(p, q) | \lambda_p \lambda_q \geq \lambda_i \lambda_j\} \\ \geq \text{Card}\left\{(p, q) | pq \leq \frac{l^2 \pi^2}{4(1 + 4\pi^2 l^2)} ij\right\} \\ \geq C_l ij,$$

where C_l is a constant which depends only on l , because

$$\text{Card}\{(i, j) | ij \leq n\} \geq n + E\left[\frac{n}{2}\right] + E\left[\frac{n}{3}\right] + \dots + 1 \\ \geq n(1 + \log(n)).$$

We now have the required bound on the first derivatives of the eigenfunctions : for any $i, j \geq 1$

$$\|c'_{g(i,j)}\|_\infty \leq 2C^2 ij \leq \frac{2C^2}{C_l} g(i, j),$$

which means that for any $n \geq 1$, $\|c'_n\|_\infty \leq \frac{2C^2}{C_l} n$.

We now prove that the assumptions on the decay of the eigenvalues.

Take (i, j) such that $ij = n$, then if (p, q) is such that $\lambda_p \lambda_q \geq \lambda_i \lambda_j$ we have:

$$\left(\frac{8l\sigma^2}{l^2 \pi^2}\right)^2 \frac{1}{p^2 q^2} \geq \lambda_p \lambda_q \geq \lambda_i \lambda_j \geq \left(\frac{2l\sigma^2}{1 + 4\pi^2 l^2}\right)^2 \frac{1}{n^2}$$

which yields the following bound for $g(i, j)$:

$$g(i, j) = \text{Card}\{(p, q) | \lambda_p \lambda_q \geq \lambda_i \lambda_j\} \\ \leq \text{Card}\left\{(p, q) | pq \leq \frac{4(1 + 4\pi^2 l^2)}{l^2 \pi^2} n\right\} \\ \leq C'_l n \log(n),$$

where C'_l is a constant which depends only on l , because

$$\begin{aligned} \text{Card}\{(i, j) | ij \leq n\} &\leq n + \frac{n}{2} + \frac{n}{3} + \dots + 1 \\ &\leq n(1 + \log(n + 1)). \end{aligned}$$

We can now prove that the last assumption made in section 3 is fulfilled. Take $b < 1$

$$\sum_{ij=n} \lambda_i \lambda_j g(i, j)^b \leq d(n) C'_l \left(\frac{8l\sigma^2}{l^2\pi^2} \right) (n \log(n))^b \frac{1}{n^2}$$

where $d(n)$ is the number of divisors of n . For any $\varepsilon > 0$ such that $b + \varepsilon < 1$, there exists a constant $k_{l,\varepsilon}$ which depends only on l and ε such that:

$$\sum_{ij=n} \lambda_i \lambda_j g(i, j)^b \leq k_{l,\varepsilon} n^{b+\varepsilon-2}.$$

Therefore the series $\sum_{n \geq 1} \mu_n n^b$ is convergent for any $b < 1$. We can then apply Theorem 1 which gives a strong convergence result

Proposition 23. *For every $p > 0$ and $0 < \beta < 1$, there exists a constant $C_{\beta,p}$ such that for any $N \in \mathbb{N}$,*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq C_{\beta,p} N^{\frac{\beta-1}{2}}.$$

Proof. For any $\varepsilon > 0$ there exists a constant $C_{l,\varepsilon}$ such that for all $i, j \geq 1$

$$g(i, j) \leq C_{l,\varepsilon} (ij)^{1+\varepsilon}.$$

Therefore $g(i, j) > N$ implies $ij > p_{N,\varepsilon} = \left(\frac{N}{C_{l,\varepsilon}} \right)^{\frac{1}{1+\varepsilon}}$ for any $\varepsilon > 0$. Take $0 < \alpha < 1, \varepsilon > 0$, we have

$$\begin{aligned} R_N^\alpha &= \sum_{n>N} \mu_n n^\alpha \leq \sum_{n>p_{N,\varepsilon}} \sum_{ij=n} \lambda_i \lambda_j g(i, j)^\alpha \\ &\leq k_{l,\varepsilon} \sum_{n>p_{N,\varepsilon}} n^{\alpha+\varepsilon-2} \\ &\leq k_{l,\varepsilon} \frac{(p_{N,\varepsilon})^{\alpha+\varepsilon-1}}{\alpha + \varepsilon - 1} \leq \frac{k_{l,\varepsilon}}{(\alpha + \varepsilon - 1) C_{l,\varepsilon}^{\frac{\alpha+\varepsilon-1}{1+\varepsilon}}} N^{\frac{\alpha+\varepsilon-1}{1+\varepsilon}} \end{aligned}$$

□

On the other hand, Theorem 2 yields the following weak convergence result:

Proposition 24. *For any $\varepsilon > 0$, there exists a constant C_ε such that for any $N \in \mathbb{N}$ and $\varphi \in C^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ , we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq C_\varepsilon C_\varphi N^{-1+\varepsilon}.$$

Proof. Take $\varepsilon > 0$, we have

$$\begin{aligned} R_N^0 &= \sum_{n>N} \mu_n \leq \sum_{n>p_{N,\varepsilon}} \sum_{ij=n} \lambda_i \lambda_j \\ &\leq \frac{k_{l,\varepsilon}}{(\varepsilon - 1) C_{l,\varepsilon}^{\frac{\varepsilon-1}{1+\varepsilon}}} N^{\frac{\varepsilon-1}{1+\varepsilon}} \end{aligned}$$

□

7.2 The analytic covariance kernel case

We assume here that the covariance kernel $\text{cov}[g]$ is analytic on $D \times D$, which is the case of a gaussian covariance kernel $\text{cov}[g](x, y) = \sigma^2 e^{-\frac{\|x-y\|^2}{l^2|D|^2}}$ in particular. Then we have the following result from Frauenfelder, Schwab and Todor given in [6] about the eigenvalues decay and about the decay of the derivatives of the eigenfunctions:

Proposition 25. *There exists two constants $c_1, c_2 > 0$ such that for all $n \geq 1$*

$$\lambda_n \leq c_1 e^{-c_2 n^{1/d}}.$$

For any $s > 0$ there exists a constant c_s such that for any $n \geq 1$,

$$\|b_n\|_\infty \leq c_s |\lambda_n|^{-s} \text{ and } \|b'_n\|_\infty \leq c_s |\lambda_n|^{-s}.$$

The methods developed to prove the results given in the previous sections apply with some slight modifications, leading to similar results. In particular we have the following strong error and weak error results:

Proposition 26. *For any $0 < s < \frac{1}{2}$, and $p > 0$, there exists a constant $H_{s,p}$ such that for all $N \in \mathbb{N}$*

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq H_{s,p} \sqrt{\sum_{n>N} \lambda_n^{1-2s}},$$

therefore, for any $0 < s < 1/2$ and $p > 0$ there exists a constant $I_{d,s,p}$ depending on p, s, d, c_1 and c_2 such that for any $N \in \mathbb{N}$

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq I_{d,s,p} N^{\frac{d-1}{2d}} e^{-\frac{c_2(1-2s)}{2} N^{1/d}}.$$

Proposition 27. *For any $0 < s < \frac{1}{2}$, there exists a constant J_s such that for all $N \in \mathbb{N}$, for all $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ , we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq J_s C_\varphi \sum_{n>N} \lambda_n^{1-2s},$$

therefore, for any $0 < s < \frac{1}{2}$, there exists a constant $K_{d,s}$ depending on d, s, c_1 and c_2 such that for all $N \in \mathbb{N}$, for all $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq K_{d,s} C_\varphi N^{\frac{d-1}{d}} e^{-c_2(1-2s)N^{1/d}}.$$

References

- [1] Ivo Babuška, Fabio Nobile, and Raúl Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.*, 45(3):1005–1034 (electronic), 2007.
- [2] Ivo Babuška, Raúl Tempone, and Georgios E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.*, 42(2):800–825 (electronic), 2004.
- [3] Fred Espen Benth and Jon Gjerde. Convergence rates for finite element approximations of stochastic partial differential equations. *Stochastics Stochastics Rep.*, 63(3-4):313–326, 1998.

-
- [4] Marcel Bieri and Christoph Schwab. Sparse high order FEM for elliptic sPDEs. *Comput. Methods Appl. Mech. Engrg.*, 198(13-14):1149–1170, 2009.
- [5] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [6] Philipp Frauenfelder, Christoph Schwab, and Radu Alexandru Todor. Finite elements for elliptic problems with stochastic coefficients. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):205–228, 2005.
- [7] Benjamin Ganis, Hector Klie, Mary F. Wheeler, Tim Wildey, Ivan Yotov, and Dongxiao Zhang. Stochastic collocation and mixed finite elements for flow in porous media. *Comput. Methods Appl. Mech. Engrg.*, 197(43-44):3547–3559, 2008.
- [8] R. G. Ghanem and P. D. Spanos. Spectral techniques for stochastic finite elements. *Arch. Comput. Methods Engrg.*, 4(1):63–100, 1997.
- [9] Roger Ghanem and John Red-Horse. Propagation of probabilistic uncertainty in complex physical systems using a stochastic finite element approach. *Phys. D*, 133(1-4):137–144, 1999. Predictability: quantifying uncertainty in models of complex phenomena (Los Alamos, NM, 1998).
- [10] Roger G. Ghanem and Pol D. Spanos. *Stochastic finite elements: a spectral approach*. Springer-Verlag, New York, 1991.
- [11] Einar Hille. Contributions to the theory of Hermitian series. II. The representation problem. *Trans. Amer. Math. Soc.*, 47:80–94, 1940.
- [12] O. P. Le Maître, O. M. Knio, H. N. Najm, and R. G. Ghanem. Uncertainty propagation using Wiener-Haar expansions. *J. Comput. Phys.*, 197(1):28–57, 2004.
- [13] O. P. Le Maître, H. N. Najm, P. P. Pébay, R. G. Ghanem, and O. M. Knio. Multi-resolution-analysis scheme for uncertainty quantification in chemical systems. *SIAM J. Sci. Comput.*, 29(2):864–889 (electronic), 2007.
- [14] Michel Loève. *Probability theory. I*. Springer-Verlag, New York, fourth edition, 1977. Graduate Texts in Mathematics, Vol. 45.
- [15] Michel Loève. *Probability theory. II*. Springer-Verlag, New York, fourth edition, 1978. Graduate Texts in Mathematics, Vol. 46.
- [16] Hermann G. Matthies. Stochastic finite elements: computational approaches to stochastic partial differential equations. *ZAMM Z. Angew. Math. Mech.*, 88(11):849–873, 2008.
- [17] Hermann G. Matthies and Christian Bucher. Finite elements for stochastic media problems. *Comput. Methods Appl. Mech. Engrg.*, 168(1-4):3–17, 1999.
- [18] F. Nobile, R. Tempone, and C. G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2411–2442, 2008.
- [19] F. Nobile, R. Tempone, and C. G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2309–2345, 2008.

-
- [20] Frigyes Riesz and Béla Sz.-Nagy. *Functional analysis*. Dover Books on Advanced Mathematics. Dover Publications Inc., New York, 1990. Translated from the second French edition by Leo F. Boron, Reprint of the 1955 original.
- [21] J. V. Uspensky. On the convergence of quadrature formulas related to an infinite interval. *Trans. Amer. Math. Soc.*, 30(3):542–559, 1928.
- [22] Tobias von Petersdorff and Christoph Schwab. Sparse finite element methods for operator equations with stochastic data. *Appl. Math.*, 51(2):145–180, 2006.
- [23] Xiaoliang Wan and George Em Karniadakis. Solving elliptic problems with non-Gaussian spatially-dependent random coefficients. *Comput. Methods Appl. Mech. Engrg.*, 198(21-26):1985–1995, 2009.
- [24] Dongbin Xiu. Efficient collocational approach for parametric uncertainty analysis. *Commun. Comput. Phys.*, 2(2):293–309, 2007.
- [25] Dongbin Xiu and Jan S. Hesthaven. High-order collocation methods for differential equations with random inputs. *SIAM J. Sci. Comput.*, 27(3):1118–1139 (electronic), 2005.
- [26] Dongbin Xiu and George Em Karniadakis. The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM J. Sci. Comput.*, 24(2):619–644 (electronic), 2002.
- [27] Dongxiao Zhang and Zhiming Lu. An efficient, high-order perturbation approach for flow in random porous media via karhunen-loève and polynomial expansions. *J. Comput. Phys.*, 194(2):773–794, 2004.



Centre de recherche INRIA Rennes – Bretagne Atlantique
IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399