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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## *On the Robustness of the Snell envelope*

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## On the Robustness of the Snell envelope

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**Abstract:** We analyze the robustness properties of the Snell envelope backward evolution equation for discrete time models. We provide a general robustness lemma, and we apply this result to a series of approximation methods, including cut-off type approximations, Euler discretization schemes, interpolation models, quantization tree models, and the Stochastic Mesh method of Broadie-Glasserman. In each situation, we provide non asymptotic convergence estimates, including  $L_p$ -mean error bounds and exponential concentration inequalities. In particular, this analysis allows us to recover existing convergence results for the quantization tree method and to improve significantly the rates of convergence obtained for the Stochastic Mesh estimator of Broadie-Glasserman. In the final part of the article, we propose a genealogical tree based algorithm based on a mean field approximation of the reference Markov process in terms of a neutral type genetic model. In contrast to Broadie-Glasserman Monte Carlo models, the computational cost of this new stochastic particle approximation is linear in the number of sampled points.

**Key-words:** Snell envelope, optimal stopping, American option pricing, genealogical trees, interacting particle model

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## Sur la robustesse de l'enveloppe de Snell

**Résumé :** On analyse les propriétés de la robustesse de l'enveloppe de Snell pour les modèles de temps discret. On fournit un lemme de robustesse générale, et on applique ce résultat à une série de méthodes d'approximation, y compris les approximations de type Cut-off, Euler discrétisation schémas, modèles d'interpolation, modèles de quantification et la méthode stochastique de Broadie-Glasserman. Dans chaque situation, on fournit des estimations de convergence non asymptotique, y compris les limites d'erreur de norme  $\mathbb{L}_p$  et les inégalités de concentration exponentielle. En particulier, cette analyse permet de récupérer les résultats de convergence existant pour la méthode de quantification et d'améliorer considérablement le taux de convergence obtenu pour l'estimateur de maillage stochastique du Broadie-Glasserman. Dans la dernière partie de l'article, on propose un algorithme d'arbre généalogique basé sur une approximation mean-field de la référence de processus de Markov en termes d'un modèle génétique de type neutre. Contrairement aux modèles Broadie-Glasserman Monte-Carlo, le coût de calcul de cette nouvelle approximation stochastique particulière est linéaire par rapport le nombre de points échantillonnés.

**Mots-clés :** enveloppe de Snell, arrêt optimal, évaluation de l'option américain, arbre génétique, modèle particulière d'interaction

## 1 Introduction

The calculation of optimal stopping time in random processes based on a given optimality criteria is one of the major problems in stochastic control and optimal stopping theory, and particularly in financial mathematics with American option pricing and hedging. In discrete time setting, these problems are related to Bermuda options and are defined in terms of given real valued stochastic process  $(Z_k)_{0 \leq k \leq n}$ , adapted to some increasing filtration  $\mathcal{F} = (\mathcal{F}_k)_{0 \leq k \leq n}$  that represents the available information at any time  $0 \leq k \leq n$ . For any  $k \in \{0, \dots, n\}$ , we let  $\mathcal{T}_k$  be the set of all stopping times  $\tau$  taking values in  $\{k, \dots, n\}$ . The Snell envelope of  $(Z_k)_{0 \leq k \leq n}$ , is the stochastic process  $(U_k)_{0 \leq k \leq n}$  defined for any  $0 \leq k < n$  by the following backward equation

$$U_k = Z_k \vee \mathbb{E}(U_{k+1} | \mathcal{F}_k) ,$$

with the terminal condition  $U_n = Z_n$ . The main property of this stochastic process is that

$$U_k = \sup_{\tau \in \mathcal{T}_k} \mathbb{E}(Z_\tau | \mathcal{F}_k) = \mathbb{E}(Z_{\tau_k^*} | \mathcal{F}_k) \quad \text{with} \quad \tau_k^* = \min \{k \leq l \leq n : U_l = Z_l\} \in \mathcal{T}_k . \quad (1.1)$$

At this level of generality, in the absence of any additional information on the sigma-fields  $\mathcal{F}_n$ , or on the terminal random variable  $Z_n$ , no numerical computation of the Snell envelope are available. To get one step further, we assume that  $(\mathcal{F}_n)_{n \geq 0}$  is the natural filtration associated with some Markov chain  $(X_n)_{n \geq 0}$  taking values in some sequence of measurable state spaces  $(E_n, \mathcal{E}_n)_{n \geq 0}$ . We let  $\eta_0 = \text{Law}(X_0)$  be the initial distribution on  $E_0$ , and we denote by  $M_n(x_{n-1}, dx_n)$  the elementary Markov transition of the chain from  $E_{n-1}$  into  $E_n$ . We also assume that  $Z_n = f_n(X_n)$ , for some collection of non negative measurable functions  $f_n$  on  $E_n$ . In this situation<sup>1</sup>, the computation of the Snell envelope amounts to solve the following backward functional equation

$$u_k = f_k \vee M_{k+1}(u_{k+1}) , \quad (1.2)$$

for any  $0 \leq k < n$ , with the terminal value  $u_n = f_n$ . In the above displayed formula,  $M_{k+1}(u_{k+1})$  stands for the measurable function on  $E_k$  defined for any  $x_k \in E_k$  by the conditional expectation formula

$$M_{k+1}(u_{k+1})(x_k) = \int_{E_{k+1}} M_{k+1}(x_k, dx_{k+1}) u_{k+1}(x_{k+1}) = \mathbb{E}(u_{k+1}(X_{k+1}) | X_k = x_k) .$$

One can check that a necessary and sufficient condition for the existence of the Snell envelope  $(u_k)_{0 \leq k \leq n}$  is that  $M_{k,l} f_l(x) < \infty$  for any  $1 \leq k \leq l \leq n$ , and any state  $x \in E_k$ . To check this claim, we simply notice that

$$f_k \leq u_k \leq f_k + M_{k+1} u_{k+1} \implies f_k \leq u_k \leq \sum_{k \leq l \leq n} M_{k,l} f_l . \quad (1.3)$$

Even if it looks innocent, the numerical solving of the recursion (1.2) often requires extensive calculations. The central problem is to compute the conditional expectations  $M_{k+1}(u_{k+1})$  on the whole state space  $E_k$ , at every time step

<sup>1</sup> Consult the last paragraph of this section for a statement of the notation used in this article.

$0 \leq k < n$ . For Markov chain models taking values in some finite state spaces (with a reasonably large cardinality), the above expectations can be easily computed by a simple backward inspection of the whole realization tree that lists all possible outcomes and every transition of the chain. In more general situations, we need to resort to some approximation strategy. Most of the numerical approximation schemes amount to replacing the pair of functions and Markov transitions  $(f_k, M_k)_{0 \leq k \leq n}$  by some approximation model  $(\widehat{f}_k, \widehat{M}_k)_{0 \leq k \leq n}$  on some possibly reduced measurable subsets  $\widehat{E}_k \subset E_k$ . We let  $\widehat{u}_k$  be the Snell envelope on  $\widehat{E}_k$  of the functions  $\widehat{f}_k$  associated with the sequence of integral operators  $\widehat{M}_k$  from  $\widehat{E}_{k-1}$  into  $\widehat{E}_k$ .

$$\widehat{u}_k = \widehat{f}_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1}) . \quad (1.4)$$

Using the elementary inequality

$$|(a \vee a') - (b \vee b')| \leq |a - b| + |a' - b'|$$

which is valid for any  $a, a', b, b' \in \mathbb{R}$ , one readily obtains, for any  $0 \leq k < n$

$$\begin{aligned} |u_k - \widehat{u}_k| &\leq |f_k - \widehat{f}_k| + |M_{k+1}u_{k+1} - \widehat{M}_{k+1}\widehat{u}_{k+1}| \\ |M_{k+1}u_{k+1} - \widehat{M}_{k+1}\widehat{u}_{k+1}| &\leq |(M_{k+1} - \widehat{M}_{k+1})u_{k+1}| + \widehat{M}_{k+1}|u_{k+1} - \widehat{u}_{k+1}|. \end{aligned}$$

Iterating the argument, one finally gets the following robustness lemma.

**Lemma 1.1** *For any  $0 \leq k < n$ , on the state space  $\widehat{E}_k$ , we have that*

$$|u_k - \widehat{u}_k| \leq \sum_{l=k}^n \widehat{M}_{k,l}|f_l - \widehat{f}_l| + \sum_{l=k}^{n-1} \widehat{M}_{k,l}|(M_{l+1} - \widehat{M}_{l+1})u_{l+1}| .$$

We quote a direct consequence of the above lemma

$$\sup_{x \in E_k} \mathbb{E} |u_k(x) - \widehat{u}_k(x)| := \|u_k - \widehat{u}_k\|_{\widehat{E}_k} \leq \sum_{l=k}^n \|f_l - \widehat{f}_l\|_{\widehat{E}_l} + \sum_{l=k+1}^n \|(M_l - \widehat{M}_l)u_l\|_{\widehat{E}_{l-1}} .$$

Lemma 1.1 provides a simple and natural way to analyze the robustness properties of the Snell equation (1.2) with respect to the pair parameters  $(f_k, M_k)$ . It also provides a simple framework to analyze in unison most of the numerical approximation models currently used in practice based on the approximation of the dynamic programming formula (1.2) by (1.4), including *cut-off techniques*, *Euler type discrete time approximations*, *quantization tree models*, *interpolation type approximations*, and *Monte Carlo importance sampling approximations*. Notice that this framework could also apply to approximations based on regression methods such as proposed in [?] and [?], however it does not directly apply to approximations based on estimation of the optimal stopping time  $\tau_k^*$  using the characterization (1.1) to deduce the Snell envelope  $U_k$ , such as the Longstaff-Schwartz algorithm (see [?]).

We emphasize that this non asymptotic robustness analysis also allows to combine in a natural way several approximation model. For instance, under appropriate tightness conditions, cut-off techniques can be used to reduce the numerical analysis of (1.2) to compact state spaces  $\widehat{E}_n$  and bounded functions  $\widehat{f}_n$ . In the same line of ideas, in designing any type of Monte Carlo approximation models, we can suppose that the transitions of the chain  $X_n$  are known based on a preliminary analysis of Euler type approximation models.

All the series of applications presented above are in discussed in Section 2, in terms robustness properties of the Snell envelope. In each situation, we provide a stochastic model that expresses the approximation scheme in terms of some pair of functions and transitions  $(\widehat{f}_n, \widehat{M}_n)_{n \geq 0}$ . We also deduce from Lemma 1.1 non asymptotic convergence theorems, including  $\mathbb{L}_p$ -mean error bounds and related exponential inequalities for the deviations of Monte Carlo type approximation models.

In the present article, three type of Monte Carlo particle models are developed:

The first one is the importance sampling type stochastic mesh method introduced by M. Broadie and P. Glasserman in their seminal paper [?] (see also [?], for some recent refinements). As any full Monte Carlo type technique, the main advantage of their approach is that it applies to high dimensional American options with a finite possibly large, number of exercise dates. In [?], the authors provide a set of conditions under which the Monte Carlo importance scheme converges as the computational effort increases. The work of the algorithm is quadratic in the number of sampled points in the stochastic mesh. In this context, in Section 3.2, we provide new non asymptotic estimates, including  $\mathbb{L}_p$ -mean error bounds and exponential concentration inequalities. To give a flavor of these results, we assume that there exists some collection of probability measures  $\eta_n$  such that  $M_n(x, \cdot) \ll \eta_n$ , for any  $n \geq 1$  and any  $x \in E_{n-1}$ . We further assume that the functions  $f_n$ , as well as the Radon Nikodym derivatives,  $R_n(x, y) = \frac{dM_n(x, \cdot)}{d\eta_n}(y)$ , are computable pointwise and it is easy to sample independent and identically distributed random variables  $(\xi_n^i)_{i \geq 1}$  with common distribution  $\eta_n$ . In this situation (1.2) can be rewritten as follows

$$u_k(x) = f_k(x) \vee \int_{E_{k+1}} \eta_{k+1}(dy) R_{k+1}(x, y) u_{k+1}(y) \quad (1.5)$$

for any  $0 \leq k < n$ , and any  $x \in E_k$ . We let  $\widehat{u}_k$  be the solution of the backward equation (1.5) defined as above *on the whole state space*  $E_k$ , by replacing the measures  $\eta_{k+1}$  by their  $N$ -empirical approximations  $\widehat{\eta}_{k+1} := \frac{1}{N} \sum_{i=1}^N \delta_{\xi_{k+1}^i}$ . Notice that the backward recursive calculation of the functions  $\widehat{u}_k$  on the stochastic mesh  $(\xi_k^i)_{1 \leq i \leq N}$  is given below

$$\widehat{u}_k(\xi_k^i) = f_k(\xi_k^i) \vee \left\{ \frac{1}{N} \sum_{j=1}^N R_{k+1}(\xi_k^i, \xi_{k+1}^j) \widehat{u}_{k+1}(\xi_{k+1}^j) \right\} \quad (1.6)$$

for any  $0 \leq k < n$ , any  $1 \leq i \leq N$ .

**Theorem 1.2** *For any  $p \geq 1$ ,  $0 \leq k \leq n$ , and any  $x \in E_k$  we have*

$$\sqrt{N} \mathbb{E} (|u_k(x) - \widehat{u}_k(x)|^p)^{\frac{1}{p}} \leq 2a(p) \sum_{k \leq l < n} \left\{ \int M'_{k,l}(x, dy) \eta_{l+1} \left[ (R_{l+1}(y, \cdot) u_{l+1})^{p'} \right] \right\}^{\frac{1}{p'}}$$

with the smallest even integer  $p'$  greater than  $p$ , and the constants  $(a(p))_{p \geq 0}$  given below

$$\forall p \geq 0 \quad a(2p)^{2p} = (2p)_p 2^{-p} \quad \text{and} \quad a(2p+1)^{2p+1} = \frac{(2p+1)_{p+1}}{\sqrt{p+1/2}} 2^{-(p+1/2)} \quad (1.7)$$

with  $(q)_p = q!/(q-p)!$ , for any  $1 \leq p \leq q$ .

The second type of Monte Carlo particle model discussed in this article is a slight variation of the Broadie-Glasserman model. The main advantage of this new strategy comes from the fact that the sampling distribution  $\eta_n$  can be chosen as the distribution of the random states  $X_n$  of the reference Markov chain, when the Radon Nikodym derivatives,  $R_n(x, y) = \frac{dM_n(x, \cdot)}{d\eta_n}(y)$  is not known explicitly. We only assume that the Markov transitions  $M_n(x, \cdot)$  are absolutely continuous with respect to some measures  $\lambda_n$  on  $E_n$ , with positive Radon Nikodym derivatives  $H_n(x, y) = \frac{dM_n(x, \cdot)}{d\lambda_n}(y)$ . Using the fact that  $\eta_n \ll \lambda_n$ , with  $\frac{d\eta_n}{d\lambda_n}(y) = \eta_{n-1}(H_n(\cdot, y))$  we notice that the backward recursion (1.2) can be rewritten as follows

$$u_k(x) = f_k(x) \vee \int_{E_{k+1}} \eta_{k+1}(dy) \frac{H_{k+1}(x, y)}{\eta_k(H_{k+1}(\cdot, y))} u_{k+1}(y) \quad (1.8)$$

for any  $0 \leq k < n$ , and any  $x \in E_k$ . Arguing as before, we let  $\hat{u}_k$  be the solution of the backward equation (1.8) defined as above *on the whole state space*  $E_k$ , by replacing the measures  $\eta_k$  by the occupation measure  $\hat{\eta}_k = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_k^i}$  associated with  $N$  independent copies  $\xi_k = (\xi_k^i)_{1 \leq i \leq N}$  of the Markov chain  $X_k$ , from the origin  $k = 0$  up to the final time horizon  $k = n$ . Hence, we recover a similar approximation to (1.6), except that the Radon Nikodym derivatives,  $R_{k+1}(\xi_k^i, \xi_{k+1}^j)$  is replaced by the approximation,

$$\hat{R}_{k+1}(\xi_k^i, \xi_{k+1}^j) = \frac{H_{k+1}(\xi_k^i, \xi_{k+1}^j)}{\frac{1}{N} \sum_{i=1}^N H_{k+1}(\xi_k^i, \xi_{k+1}^j)}.$$

The stochastic analysis of this particle model follows essentially the same line of arguments as the one of the Broadie-Glasserman model. For further details on the convergence analysis of this scheme, with the extended version of Theorem 1.2 to this class of models, we refer the reader to the second part of Section 3.2.

Several rather crude estimates can be derived from these  $\mathbb{L}_p$ -mean error bounds. For instance, let us suppose that  $R_n(x, y) \leq r_n(y)$ , for any  $x \in E_{n-1}$  and some measurable functions  $r_n \in \mathbb{L}_p(E_n, \eta_n)$ , for any  $p \geq 1$ . In this situation

$$\sqrt{N} \sup_{x \in E_k} \mathbb{E} (|u_k(x) - \hat{u}_k(x)|^p)^{\frac{1}{p}} \leq 2 a(p) \sum_{k \leq l < n} \eta_{l+1}((r_{l+1} u_{l+1})^{p'})^{1/p'}.$$

Using (1.3) and Hölder's inequalities, we prove that the r.h.s. in the above display is finite as soon as  $f_k \in \mathbb{L}_q(E_k, \eta_l M_{l,k})$ , for any  $q \geq 1$  and  $l \leq k \leq n$ . When the functions  $r_n$  and  $f_n$  are bounded, we have

$$\sqrt{N} \sup_{x \in E_k} \mathbb{E} (|u_k(x) - \hat{u}_k(x)|^p)^{\frac{1}{p}} \leq a(p) b_k(n) \quad \text{with} \quad b_k(n) \leq 2 \sum_{k \leq l < n} \|r_{l+1} u_{l+1}\|$$

for some finite constant  $b_k(n) < \infty$ , whose values do not depend on the parameter  $p$ . In this situation, we deduce the following exponential concentration inequality

$$\sup_{x \in E_k} \mathbb{P} \left( |u_k(x_k) - \hat{u}_k(x_k)| > \frac{b_k(n)}{\sqrt{N}} + \epsilon \right) \leq \exp(-N\epsilon^2/(2b_k(n)^2)). \quad (1.9)$$

This result is a direct consequence from the fact that, for any non negative random variable  $U$ ,

$$\exists b < \infty \text{ s.t. } \forall r \geq 1 \quad \mathbb{E}(U^r)^{\frac{1}{r}} \leq a(r) b \Rightarrow \mathbb{P}(U \geq b + \epsilon) \leq \exp(-\epsilon^2/(2b^2)).$$

To check this claim, we develop the exponential to check that

$$\forall t \geq 0 \quad \mathbb{E}(e^{tU}) \leq \exp\left(\frac{(bt)^2}{2} + bt\right) \Rightarrow \mathbb{P}(U \geq b + \epsilon) \leq \exp\left(-\sup_{t \geq 0} \left( \epsilon t - \frac{(bt)^2}{2} \right)\right).$$

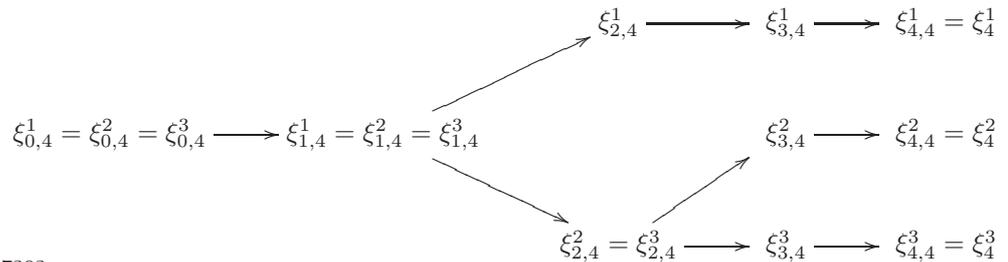
In the final part of the article, Section 3.3, we present an alternative Monte Carlo method based on the genealogical tree evolution models associated with a neutral genetic model with mutation given by the Markov transitions  $M_n$ . The main advantage of this new strategy comes from the fact that the computational effort of the algorithm is now linear in the number of sampled points. We recall that a neutral genetic is a Markov chain with a selection/mutation transition denoted by:

$$\xi_n = (\xi_n^i)_{1 \leq i \leq N} \xrightarrow{\text{selection}} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \leq i \leq N} \xrightarrow{\text{mutation}} \xi_{n+1} = (\xi_{n+1}^i)_{1 \leq i \leq N}$$

The neutral genetic selection simply consists in simulating  $N$  independent particles  $(\widehat{\xi}_n^i)_{1 \leq i \leq N}$  w.r.t. the same distribution  $\frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$ . During the mutation phase, the particles explore the state space independently (the interactions between the various particles being created by the selection steps) according to the Markov transitions  $M_{n+1}(x, dy)$ . In other terms, we have  $\widehat{\xi}_n^i \rightsquigarrow \xi_{n+1}^i$  where  $\xi_{n+1}^i$  stands for a random variable with the law  $M_{n+1}(\widehat{\xi}_n^i, \cdot)$ . This type of model is frequently used in biology, and genetic algorithms literature (see for instance [?], and references therein). An important observation concerns the genealogical tree structure of the previously defined genetic particle model. If we interpret the selection transition as a birth and death process, then arises the important notion of the ancestral line of a current individual. More precisely, when a particle  $\widehat{\xi}_{n-1}^i \rightarrow \xi_n^i$  evolves to a new location  $\xi_n^i$ , we can interpret  $\widehat{\xi}_{n-1}^i$  as the parent of  $\xi_n^i$ . Looking backwards in time and recalling that the particle  $\widehat{\xi}_{n-1}^i$  has selected a site  $\xi_{n-1}^j$  in the configuration at time  $(n-1)$ , we can interpret this site  $\xi_{n-1}^j$  as the parent of  $\widehat{\xi}_{n-1}^i$  and therefore as the ancestor denoted  $\xi_{n-1,n}^i$  at level  $(n-1)$  of  $\xi_n^i$ . Running backwards in time we may trace the whole ancestral line

$$\xi_{0,n}^i \leftarrow \xi_{1,n}^i \leftarrow \dots \leftarrow \xi_{n-1,n}^i \leftarrow \xi_{n,n}^i = \xi_n^i \quad (1.10)$$

The genealogical tree model is summarized in the following synthetic picture that corresponds to the case  $(N, n) = (3, 4)$ :



The main advantage of this path particle model comes from the fact that the occupation measure of the ancestral tree model converges in some sense to the distribution of the path of the reference Markov chain

$$\mathbb{P}_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i)} \xrightarrow{N \rightarrow \infty} \mathbb{P}_n := \text{Law}(X_0, \dots, X_n).$$

It is also well known that the Snell envelope associated with a Markov chain evolving on some finite state space is easily computed using the tree structure of the chain evolution. Therefore, replacing the reference distribution  $\mathbb{P}_n$  by its  $N$ -approximation  $\mathbb{P}_n^N$ , we define an  $N$ -approximated Markov model whose evolutions are described by the genealogical tree model defined above. Let  $\widehat{u}_k$  be the Snell envelope associated with this  $N$ -approximated Markov chain. For finite state space models, we shall prove the following result

$$\sup_{0 \leq k \leq n} \sup_{1 \leq i \leq N} \mathbb{E} (|(u_k - \widehat{u}_k)(\xi_{k,n}^i)|^p)^{1/p} \leq c_p(n)/\sqrt{N}$$

for any  $p \geq 1$ , with some collection of finite constants  $c_p(n) < \infty$  whose values only depend on the parameters  $p$  and  $n$ .

For the convenience of the reader, we end this introduction with some notation used in the present article.

We denote respectively by  $\mathcal{P}(E)$ , and  $\mathcal{B}(E)$ , the set of all probability measures on some measurable space  $(E, \mathcal{E})$ , and the Banach space of all bounded and measurable functions  $f$  equipped with the uniform norm  $\|f\|$ . Given a subset  $A \in \mathcal{E}$ , we set  $\|f\|_A := \sup_{x \in A} |f(x)|$ . We also denote by  $\text{Osc}_1(E)$  the set of functions  $f$  with oscillations  $\text{osc}(f) = \sup_{x,y} |f(x) - f(y)|$  less than 1. We let  $\mu(f) = \int \mu(dx) f(x)$ , be the Lebesgue integral of a function  $f \in \mathcal{B}(E)$ , with respect to a measure  $\mu \in \mathcal{P}(E)$ . For any  $p \geq 1$ , we also set  $\mathbb{L}_p(E, \eta)$  the set of functions  $f$  such that  $\eta(|f|^p) < \infty$ , equipped with the norm  $\|f\|_{p,\eta} = \eta(|f|^p)^{1/p}$ .

We recall that a bounded integral kernel  $M(x, dy)$  from a measurable space  $(E, \mathcal{E})$  into an auxiliary measurable space  $(E', \mathcal{E}')$  is an operator  $f \mapsto M(f)$  from  $\mathcal{B}(E')$  into  $\mathcal{B}(E)$  such that the functions

$$x \mapsto M(f)(x) := \int_{E'} M(x, dy) f(y)$$

are  $\mathcal{E}$ -measurable and bounded, for any  $f \in \mathcal{B}(E')$ . In the above displayed formulae,  $dy$  stands for an infinitesimal neighborhood of a point  $y$  in  $E'$ . Sometimes, for indicator functions  $f = 1_A$ , with  $A \in \mathcal{E}$ , we also use the notation  $M(x, A) := M(1_A)(x)$ . The kernel  $M$  also generates a dual operator  $\mu \mapsto \mu M$  from  $\mathcal{M}(E)$  into  $\mathcal{M}(E')$  defined by  $(\mu M)(f) := \mu(M(f))$ . A Markov kernel is a positive and bounded integral operator  $M$  with  $M(1) = 1$ . Given a pair of bounded integral operators  $(M_1, M_2)$ , we let  $(M_1 M_2)$  the composition operator defined by  $(M_1 M_2)(f) = M_1(M_2(f))$ . Given a sequence of bounded integral operator  $M_n$  from some state space  $E_{n-1}$  into another  $E_n$ , we set  $M_{k,l} := M_{k+1} M_{k+2} \cdots M_l$ , for any  $k \leq l$ , with the convention  $M_{k,k} = Id$ , the identity operator. For time homogenous state spaces, we denote by  $M^m = M^{m-1} M = M M^{m-1}$  the  $m$ -th composition of a given bounded integral operator  $M$ , with  $m \geq 1$ . In the context of finite state spaces, these integral operations coincides with the traditional matrix operations on multidimensional state spaces.

We also assume that the reference Markov chain  $X_n$  with initial distribution  $\eta_0 \in \mathcal{P}(E_0)$  and elementary transitions  $M_n(x_{n-1}, dx_n)$  from  $E_{n-1}$  into  $E_n$  is defined on some filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}_{\eta_0})$ , and we use the notation  $\mathbb{E}_{\mathbb{P}_{\eta_0}}$  to denote the expectations with respect to  $\mathbb{P}_{\eta_0}$ . In this notation, for all  $n \geq 1$  and for any  $f_n \in \mathcal{B}(E_n)$ , we have that

$$\mathbb{E}_{\mathbb{P}_{\eta_0}} \{f_n(X_n) | \mathcal{F}_{n-1}\} = M_n f_n(X_{n-1}) := \int_{E_n} M_n(X_{n-1}, dx_n) f_n(x_n)$$

with the  $\sigma$ -field  $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$  generated by the sequence of random variables  $X_p$ , from the origin  $p = 0$  up to the time  $p = n$ . When there are no possible confusion, we write  $\|X\|_{\mathbb{L}_p} = \mathbb{E}(|X|^p)^{1/p}$ , the  $\mathbb{L}_p$ -norm of a given real valued random variable defined on some probability space  $(\Omega, \mathbb{P})$ . We also use the conventions  $\prod_{\emptyset} = 1$ , and  $\sum_{\emptyset} = 0$ .

## 2 Some deterministic approximation models

### 2.1 Cut-off type models

We suppose that  $E_n$  are topological spaces with  $\sigma$ -fields  $\mathcal{E}_n$  that contains the Borel  $\sigma$ -field on  $E_n$ . Our next objective is to find conditions under which we can reduce the backward functional equation (1.2) to a sequence of compact sets  $\widehat{E}_n$ .

To this end, we further assume that the initial measure  $\eta_0$  and the Markov transition  $M_n$  of the chain  $X_n$  satisfy the following tightness property: For every sequence of positive numbers  $\epsilon_n \in [0, 1[$ , there exists a collection of compact subsets  $\widehat{E}_n \subset E_n$  s.t.

$$(T) \quad \eta_0(\widehat{E}_0^c) \leq \epsilon_0 \quad \text{and} \quad \forall n \geq 0 \quad \sup_{x_n \in \widehat{E}_n} M_{n+1}(x_n, \widehat{E}_{n+1}^c) \leq \epsilon_{n+1} .$$

For instance, this condition is clearly met for regular gaussian type transitions on the euclidian space, for some collection of increasing compact balls.

In this situation, a natural cut off consists in considering the Markov transitions  $\widehat{M}_k$  restricted to the compact sets  $\widehat{E}_k$

$$\forall x \in \widehat{E}_{k-1} \quad \widehat{M}_k(x, dy) := \frac{M_k(x, dy) 1_{\widehat{E}_k}}{M_k(1_{\widehat{E}_k})(x)} .$$

These transitions are well defined as soon as  $M_k(x, \widehat{E}_k) > 0$ , for any  $x \in \widehat{E}_{k-1}$ . Using the decomposition

$$\begin{aligned} [\widehat{M}_k - M_k](u_k) &= \widehat{M}_k(u_k) - M_k(1_{\widehat{E}_k} u_k) - M_k(1_{\widehat{E}_k^c} u_k) \\ &= \left(1 - \frac{1}{M_k(1_{\widehat{E}_k})}\right) M_k(u_k 1_{\widehat{E}_k}) - M_k(1_{\widehat{E}_k^c} u_k) \\ &= \frac{M_k(1_{\widehat{E}_k^c})}{M_k(1_{\widehat{E}_k})} M_k(u_k 1_{\widehat{E}_k}) - M_k(1_{\widehat{E}_k^c} u_k) . \end{aligned}$$

Then using Lemma 1.1 yields

$$\begin{aligned} \|u_k - \widehat{u}_k\|_{\widehat{E}_k} &:= \sup_{x \in \widehat{E}_k} |u_k(x) - \widehat{u}_k(x)| \\ &\leq \sum_{l=k+1}^n \left[ \left\| \frac{M_l(1_{\widehat{E}_l^c})}{M_l(1_{\widehat{E}_l})} \right\|_{\widehat{E}_{l-1}} \|M_l(u_l 1_{\widehat{E}_l})\|_{\widehat{E}_{l-1}} + \|M_l(u_l 1_{\widehat{E}_l^c})\|_{\widehat{E}_{l-1}} \right] \\ &\leq \sum_{l=k+1}^n \left[ \frac{\epsilon_l}{1 - \epsilon_l} \|M_l(u_l)\|_{\widehat{E}_{l-1}} + \|M_l(u_l^2)\|_{\widehat{E}_{l-1}}^{1/2} \epsilon_l^{1/2} \right]. \end{aligned}$$

We summarize the above discussion with the following result.

**Theorem 2.1** *We assume that the tightness condition  $(\mathcal{T})$  is met, for every sequence of positive numbers  $\epsilon_n \in [0, 1]$ , and for some collection of compact subsets  $\widehat{E}_n \subset E_n$ . In this situation, for any  $0 \leq k \leq n$ , we have that*

$$\|u_k - \widehat{u}_k\|_{\widehat{E}_k} \leq \sum_{l=k+1}^n \frac{\epsilon_l^{1/2}}{1 - \epsilon_l^{1/2}} \|M_l(u_l^2)\|_{\widehat{E}_{l-1}}^{1/2}.$$

We notice that

$$u_k \leq \sum_{l=k}^n M_{k,l}(f_l) \quad \text{and therefore} \quad \|M_k(u_k^2)\|_{\widehat{E}_{k-1}} \leq (n-k+1) \sum_{l=k}^n \|M_{k-1,l}(f_l)^2\|_{\widehat{E}_{k-1}}$$

Consequently, one can find sets  $(\widehat{E}_l)_{k < l \leq n}$  so that  $\|u_k - \widehat{u}_k\|_{\widehat{E}_k}$  is as small as one wants as soon as  $\|M_{k,l}(f_l)^2\|_{\widehat{E}_k} < \infty$ , for any  $0 \leq k < l \leq n$ .

## 2.2 Euler approximation models

In several application model areas, the discrete time Markov chain  $(X_k)_{k \geq 0}$  is often given in terms of an  $\mathbb{R}^d$ -valued and continuous time process  $(X_t)_{t \geq 0}$  given by a stochastic differential equation of the following form

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad \text{law}(X_0) = \eta_0, \quad (2.1)$$

where  $\eta_0$  is a known distribution on  $\mathbb{R}^d$ , and  $a, b$  are known functions, and  $W$  is a  $d$ -dimensional Wiener process. Except in some particular instances, the time homogeneous Markov transitions  $M_k = M$  are usually unknown, and we need to resort to an Euler approximation scheme. The discrete time approximation model with a fixed time step  $1/m$  is defined by the following recursive formulae

$$\begin{aligned} \widehat{\xi}_0(x) &= x \\ \widehat{\xi}_{\frac{i+1}{m}}(x) &= \widehat{\xi}_{\frac{i}{m}}(x) + a\left(\widehat{\xi}_{\frac{i}{m}}(x)\right) \frac{1}{m} + b\left(\widehat{\xi}_{\frac{i}{m}}(x)\right) \frac{1}{\sqrt{m}} \epsilon_i \end{aligned}$$

where the  $\epsilon_i$ 's are i.i.d. centered and  $\mathbb{R}^d$ -valued Gaussian vectors with unit covariance matrix. The chain  $(\widehat{\xi}_k)_{k \geq 0}$  is an homogeneous Markov with a transition kernel which we denote by  $\widehat{M}$ .

We further assume that the functions  $a$  and  $b$  are twice differentiable, with bounded partial derivatives of orders 1 and 2, and the matrix  $(bb^*)(x)$  is uniformly non-degenerate.

In this situation, the integral operators  $M$  and  $\widehat{M}$  admit densities, denoted by  $p$  and  $\widehat{p}$ . According to Bally and Talay [?], we have that

$$[p \vee \widehat{p}] \leq c q \quad \text{and} \quad m |\widehat{p} - p| \leq c q \quad (2.2)$$

with the Gaussian density  $q(x, x') := \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}|x-x'|^2}$ , and a pair of constants  $(c, \sigma)$  depending only on the pair of functions  $(a, b)$ . Let  $Q$ , be the Markov integral operator on  $\mathbb{R}^d$  with density  $q$ . We consider a sequence of functions  $(f_k)_{0 \leq k \leq n}$  on  $\mathbb{R}^d$ . We let  $(u_k)_{0 \leq k \leq n}$  and  $(\widehat{u}_k)_{0 \leq k \leq n}$  be the Snell envelopes on  $\mathbb{R}^d$  associated to the pair  $(M, f_k)$  and  $(\widehat{M}, f_k)$ . Using Lemma 1.1, we readily obtain the following estimate

$$|u_k - \widehat{u}_k| \leq \sum_{l=k}^{n-1} \widehat{M}^{l-k} |(M - \widehat{M})u_{l+1}| \leq \frac{c}{m} \sum_{l=k}^{n-1} \widehat{M}^{l-k} Q|u_{l+1}|.$$

Rather crude upper bounds that do not depend on the approximation kernels  $\widehat{M}$  can be derived using the first inequality in ( 2.2)

$$|u_k - \widehat{u}_k| \leq \frac{1}{m} \sum_{l=1}^{n-k} c^l Q^l |u_{l+k}|.$$

Recalling that  $u_{l+k} \leq \sum_{l+k \leq l' \leq n} M^{l'-(l+k)} f_{l'}$ , we also have that

$$\begin{aligned} |u_k - \widehat{u}_k| &\leq \frac{1}{m} \sum_{l=1}^{n-k} c^l Q^l \sum_{l+k \leq l' \leq n} c^{l'-(l+k)} Q^{l'-(l+k)} f_{l'} \\ &\leq \frac{1}{m} \sum_{l=1}^{n-k} \sum_{l+k \leq l' \leq n} c^{l'-k} Q^{l'-k} f_{l'} = \frac{1}{m} \sum_{1 \leq l \leq n-k} l c^l Q^l f_{k+l}. \end{aligned}$$

We summarize the above discussion with the following theorem.

**Theorem 2.2** *Suppose the functions  $(f_k)_{0 \leq k \leq n}$  on  $\mathbb{R}^d$  are chosen such that  $Q^l f_{k+l}(x) < \infty$ , for any  $x \in \mathbb{R}^d$ , and  $1 \leq k+l \leq n$ . Then, for any  $0 \leq l \leq n$ , we have the inequalities*

$$|u_k - \widehat{u}_k| \leq \frac{c}{m} \sum_{l=k}^{n-1} \widehat{M}^{l-k} Q|u_{l+1}| \leq \frac{1}{m} \sum_{1 \leq l \leq n-k} l c^l Q^l f_{k+l}.$$

### 2.3 Interpolation type models

Most algorithms proposed to approximate the Snell envelope provide discrete approximations  $\widehat{u}_k^i$  at some discrete (potentially random) points  $\xi_k^i$ . However, for several purposes, it can be interesting to consider approximations  $\widehat{u}_k$  of functions  $u_k$  on the whole space  $E_k$ . One motivation to do so is, for instance, to be able to define a new (low biased) estimator,  $\widehat{U}_k$ , using a Monte Carlo approximation

of (1.1), with a stopping rule  $\hat{\tau}_k$  associated with the approximate Snell envelope  $\hat{u}_k$ , by replacing  $u_k$  by  $\hat{u}_k$  in the characterization of the optimal stopping time  $\tau_k^*$  (1.1), i.e.

$$\bar{U}_k = \frac{1}{M} \sum_{i=1}^M f_{\hat{\tau}_k^i}(X_{\hat{\tau}_k^i}^i) \quad \text{with} \quad \hat{\tau}_k^i = \min \{k \leq l \leq n : \hat{u}_l(X_l^i) = f_l(X_l^i)\} . \quad (2.3)$$

where  $X^i = (X_1^i, \dots, X_n^i)$  are i.i.d. path according to the reference Markov chain dynamic.

In this section, we analyse non asymptotic errors of some specific approximation schemes providing such estimators  $\hat{u}_k$  of  $u_k$  on the whole state  $E_k$ . Let  $\widetilde{M}_{k+1} = \mathcal{I}_k \widetilde{M}_{k+1}$  be the composition of approximation Markov transition  $\widetilde{M}_{k+1}$  from a finite set  $S_k$  into the whole state space  $E_{k+1}$ , with an auxiliary interpolation type and Markov operator  $\mathcal{I}_k$  from  $E_k$  into  $S_k$ , so that

$$\forall x_k \in S_k \quad \mathcal{I}_k(x_k, ds) = \delta_{x_k}(ds)$$

and such that the integrals

$$x \in E_k \mapsto \mathcal{I}_k(\varphi_k)(x) = \int_{S_k} \mathcal{I}_k(x, ds) \varphi_k(s)$$

of any function  $\varphi_k$  on  $S_k$  are easily computed starting from any point  $x_k$  in  $E_k$ . We further assume that the finite state spaces  $S_k$  are chosen so that

$$\|f - \mathcal{I}_k f\|_{E_k} \leq \epsilon_k(f, |S_k|) \rightarrow 0 \quad \text{as} \quad |S_k| \rightarrow \infty \quad (2.4)$$

for continuous functions  $f_k$  on  $E_k$ . An example of interpolation transition  $I_k$  is provided hereafter. We let  $\widetilde{M}_k = \mathcal{I}_{k-1} \widetilde{M}_k$  be the composition operator on the state spaces  $\widehat{E}_k = E_k$ .

The approximation models  $\widetilde{M}_k$  are non necessarily deterministic. In [?], we examined the situation where

$$\forall s \in S_k \quad \widetilde{M}_k(s, dx) = \frac{1}{N_k} \sum_{1 \leq i \leq N_k} \delta_{X_k^i(s)}(dx)$$

where  $X_k^i(s)$  stands for a collection of  $N_k$  independent random variables with common law  $M_k(s, dx)$ .

**Theorem 2.3** *We suppose that the Markov transitions  $M_k$  are Feller, in the sense that  $M_k(C(E_k)) \subset C(E_{k-1})$ , where  $C(E_k)$  stands for the space of all continuous functions on the  $E_k$ . We let  $(u_k)_{0 \leq k \leq n}$ , and respectively  $(\hat{u}_k)_{0 \leq k \leq n}$  be the Snell envelope associated with the functions  $f_k = \hat{f}_k$ , and the Markov transitions  $M_k$ , and respectively  $\widetilde{M}_k = \mathcal{I}_{k-1} \widetilde{M}_k$  on the state spaces  $\widehat{E}_k = E_k$ .*

$$\|u_k - \hat{u}_k\|_{E_k} \leq \sum_{l=k}^{n-1} \left[ \epsilon_l(M_{l+1} u_{l+1}, |S_l|) + \|(M_{l+1} - \widetilde{M}_{l+1}) u_{l+1}\|_{S_l} \right] .$$

The proof of the theorem is a direct consequence of Lemma 1.1 combined with the following decomposition

$$\|u_k - \hat{u}_k\|_{E_k} \leq \sum_{l=k}^{n-1} \left[ \|(Id - \mathcal{I}_l) M_{l+1} u_{l+1}\|_{E_l} + \|\mathcal{I}_l (M_{l+1} - \widetilde{M}_{l+1}) u_{l+1}\|_{E_l} \right] . \quad (2.5)$$

We illustrate these results in the typical situation where the space  $E_k$  are the convex hull generated by the finite sets  $S_k$ . Firstly, we present the definition of the interpolation operators. We let  $\mathcal{P} = \{\mathcal{P}^1, \dots, \mathcal{P}^m\}$  be a partition of a convex and compact space  $E$  into simplexes with disjoint non empty interiors, so that  $E = \cup_{1 \leq i \leq m} \mathcal{P}_i$ . We denote by  $\delta(\mathcal{P})$  the refinement degree of the partition  $\mathcal{P}$

$$\delta(\mathcal{P}) := \sup_{1 \leq i \leq m} \sup_{x, y \in \mathcal{P}_i} \|x - y\| .$$

We let  $S = \mathcal{V}(\mathcal{P})$  be the set of vertices of these simplexes. We denote by  $\mathcal{I}$  interpolation operator defined by  $\mathcal{I}(f)(s) = f(s)$ , if  $s \in S$ , and if  $x$  belongs to some simplex  $\mathcal{P}^j$  with vertices  $\{x_1^j, \dots, x_{d_j}^j\}$

$$\mathcal{I}(f)(\sum_{1 \leq i \leq d_j} \lambda_i x_i^j) = \sum_{1 \leq i \leq d_j} \lambda_i f(x_i^j)$$

where the barycenters  $(\lambda_i)_{1 \leq i \leq d_j}$  are the unique solution of

$$x = \sum_{1 \leq i \leq d_j} \lambda_i x_i^j \quad \text{with} \quad (\lambda_i)_{1 \leq i \leq d_j} \in [0, 1]^{d_j} \quad \text{and} \quad \sum_{1 \leq i \leq d_j} \lambda_i = 1 .$$

The Markovian interpretation is that starting from  $x$ , one choses the ‘‘closest simplex’’ and then one chooses one of its vertices  $x_i$  with probability  $\lambda_i$ .

For any  $\delta > 0$ , we let  $\omega(f, \delta)$  be the  $\delta$ -modulus of continuity of a function  $f \in C(E)$

$$\omega(f, \delta) := \sup_{(x, y) \in E: \|x - y\| \leq \delta} |f(x) - f(y)| .$$

The following technical Lemma provides a simple way to check condition (2.4) for interpolation kernels.

**Lemma 2.4** *Then for any  $f, g \in C(E)$ ,*

$$\sup_{x \in E} |f(x) - \mathcal{I}g(x)| \leq \max_{x \in S} |f(x) - g(x)| + \omega(f, \delta(\mathcal{P})) + \omega(g, \delta(\mathcal{P})) . \quad (2.6)$$

*In particular, we have that*

$$\sup_{x \in E} |f(x) - \mathcal{I}f(x)| \leq \omega(f, \delta(\mathcal{P})) .$$

**Proof:**

Suppose  $x$  belongs to some simplex  $\mathcal{P}^j$  with vertices  $\{x_1^j, \dots, x_{d_j}^j\}$ , and let  $(\lambda_i)_{1 \leq i \leq d_j}$  be the barycenter parameters  $x = \sum_{1 \leq i \leq d_j} \lambda_i x_i^j$ . Since we have  $\mathcal{I}g(x_i^j) = g(x_i^j)$ , and  $\mathcal{I}g(x_i^j) = g(x_i^j)$  for any  $i \in \{1, \dots, d_j\}$ , it follows that

$$\begin{aligned} |f(x) - \mathcal{I}g(x)| &\leq \sum_{i=1}^{d_j} \lambda_i |(f(x) - f(x_i^j))| + \sum_{i=1}^{d_j} \lambda_i |f(x_i^j) - \mathcal{I}g(x_i^j)| \\ &\quad + \sum_{i=1}^{d_j} \lambda_i |\mathcal{I}g(x_i^j) - g(x)| \\ &= \sum_{i=1}^{d_j} \lambda_i |(f(x) - f(x_i^j))| + \sum_{i=1}^{d_j} \lambda_i |f(x_i^j) - g(x_i^j)| \\ &\quad + \sum_{i=1}^{d_j} \lambda_i |g(x_i^j) - g(x)| . \end{aligned}$$

This implies that

$$\sup_{x \in \mathcal{P}^j} |f(x) - \mathcal{I}g(x)| \leq \max_{x \in \mathcal{P}^j} |f(x) - g(x)| + \omega(f, \delta(\mathcal{P}^j)) + \omega(g, \delta(\mathcal{P}^j))$$

with

$$\omega(f, \delta(\mathcal{P}^j)) = \sup_{\|x-y\| \leq \delta(\mathcal{P}^j)} |f(x) - f(y)| \quad \text{and} \quad \delta(\mathcal{P}^j) := \sup_{x, y \in \mathcal{P}^j} \|x - y\|.$$

The end of the proof is now clear.  $\blacksquare$

Combining (2.5) and (2.6), we obtain the following result.

**Proposition 2.5** *We let  $\mathcal{P}_k = \{\mathcal{P}_k^1, \dots, \mathcal{P}_k^{m_k}\}$  be a partition of a convex and compact space  $E_k$  into simplexes with disjoint non empty interiors, so that  $E_k = \cup_{1 \leq i \leq m_k} \mathcal{P}_i$ . We let  $S_k = \mathcal{V}(\mathcal{P}_k)$  be the set of vertices of these simplexes. We let  $(\widehat{u}_k)_{0 \leq k \leq n}$ , be the Snell envelope associated with the functions  $\widehat{f}_k = f_k$  and the Markov transitions  $\widehat{M}_k = \mathcal{I}_{k-1} \widetilde{M}_k$  on the state spaces  $E_k = \widehat{E}_k$ .*

$$\|u_k - \widehat{u}_k\|_{E_k} \leq \sum_{l=k}^{n-1} \left[ \omega(M_{l+1} u_{l+1}, \delta(\mathcal{P}_l)) + \|(M_{l+1} - \widetilde{M}_{l+1}) u_{l+1}\|_{S_l} \right].$$

## 2.4 Quantization tree models

Quantization tree models belongs to the class of deterministic grid approximation methods. The basic idea consists in choosing finite space grids

$$\widehat{E}_k = \{x_k^1, \dots, x_k^{m_k}\} \subset E_k = \mathbb{R}^d$$

and some neighborhoods measurable partitions  $(A_k^i)_{1 \leq i \leq m_k}$  of the whole space  $E_k$  such that the random state variable  $X_k$  is suitably approximated, as  $m_k \rightarrow \infty$ , by discrete random variables of the following form

$$\widehat{X}_k := \sum_{1 \leq i \leq m_k} x_k^i 1_{A_k^i}(X_k) \simeq X_k.$$

The numerical efficiency of these quantization methods heavily depends on the choice of these grids. There exists various criteria to choose judiciously these objects, including minimal  $\mathbb{L}_p$ -quantization errors, that ensures that the corresponding Voronoi type quantized variable  $\widehat{X}_k$  minimizes the  $\mathbb{L}_p$  distance to the real state variable  $X_k$ . For further details on this subject, we refer the interested reader to the pioneering article of G. Pagès [?], and the series of articles of V. Bally, G. Pagès, and J. Printemps [?], G. Pagès and J. Printemps [?], as well as G. Pagès, H. Pham and J. Printemps [?], and references therein. The second approximation step of these quantization model consists in defining the coupled distribution of any pair of variables  $(\widehat{X}_{k-1}, \widehat{X}_k)$  by setting

$$\mathbb{P}(\widehat{X}_k = x_k^j, \widehat{X}_{k-1} = x_{k-1}^i) = \mathbb{P}(X_k \in A_k^j, X_{k-1} \in A_{k-1}^i)$$

for any  $1 \leq i \leq m_{k-1}$ , and  $1 \leq j \leq m_k$ . This allows to interpret the quantized variables  $(\widehat{X}_k)_{0 \leq k \leq n}$  as a Markov chain taking values in the states spaces  $(\widehat{E}_k)_{0 \leq k \leq n}$  with Markov transitions

$$\widehat{M}_k(x_{k-1}^i, x_k^j) := \mathbb{P}(\widehat{X}_k = x_k^j \mid \widehat{X}_{k-1} = x_{k-1}^i) = \mathbb{P}(X_k \in A_k^j \mid X_{k-1} \in A_{k-1}^i).$$

Using the decompositions

$$\begin{aligned} M_k(f)(x_{k-1}^i) &= \sum_{j=1}^{m_k} \int_{A_k^j} f(y) \mathbb{P}(X_k \in dy \mid X_{k-1} = x_{k-1}^i) \\ &= \sum_{j=1}^{m_k} \int_{A_k^j} f(y) \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i) \\ &\quad + \int [M(f)(x_{k-1}^i) - M(f)(x)] \mathbb{P}(X_{k-1} \in dx \mid X_{k-1} \in A_{k-1}^i) \end{aligned}$$

and

$$\widehat{M}_k(f)(x_{k-1}^i) = \sum_{j=1}^{m_k} \int_{A_k^j} f(x_k^j) \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i)$$

we find that

$$\begin{aligned} &[M_k - \widehat{M}_k](f)(x_{k-1}^i) \\ &= \sum_{j=1}^{m_k} \int_{A_k^j} [f(y) - f(x_k^j)] \mathbb{P}(X_k \in dy \mid X_{k-1} \in A_{k-1}^i) \\ &\quad + \int [M(f)(x_{k-1}^i) - M(f)(x)] \mathbb{P}(X_{k-1} \in dx \mid X_{k-1} \in A_{k-1}^i). \end{aligned}$$

We let  $\text{Lip}(\mathbb{R}^d)$  be the set of all Lipschitz functions  $f$  on  $\mathbb{R}^d$ , and we set

$$L(f) = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

for any  $f \in \text{Lip}(\mathbb{R}^d)$ . We further assume that  $M_k(\text{Lip}(\mathbb{R}^d)) \subset \text{Lip}(\mathbb{R}^d)$ . From previous considerations, we find that

$$\begin{aligned} |[M_k - \widehat{M}_k](f)(x_{k-1}^i)| &\leq L(f) \mathbb{E} \left[ |X_k - \widehat{X}_k|^p \mid \widehat{X}_{k-1} = x_{k-1}^i \right]^{\frac{1}{p}} \\ &\quad + L(M_k(f)) \mathbb{E} (|X_{k-1} - \widehat{X}_{k-1}|^p \mid \widehat{X}_{k-1} = x_{k-1}^i)^{\frac{1}{p}}. \end{aligned}$$

This clearly implies that

$$\begin{aligned} \widehat{M}_{k,l} |(M_{l+1} - \widehat{M}_{l+1})f|(x_k^i) &\leq L(f) \left[ \mathbb{E} (|X_{l+1} - \widehat{X}_{l+1}|^p \mid \widehat{X}_k = x_k^i) \right]^{\frac{1}{p}} \\ &\quad + L(M_{l+1}(f)) \mathbb{E} (|X_l - \widehat{X}_l|^p \mid \widehat{X}_k = x_k^i)^{\frac{1}{p}}. \end{aligned}$$

Notice that the above inequality can be refined using the fact that

$$|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$$

we observe that

$$\begin{aligned} &(f_k \text{ and } u_{k+1} \in \text{Lip}(\mathbb{R}^d)) \\ &\quad \downarrow \\ &(u_k \in \text{Lip}(\mathbb{R}^d) \text{ with } L(u_k) \leq L(f_k) \vee L(M_{k+1}(u_{k+1}))) \end{aligned}$$

Using Lemma 1.1, we readily arrive at the following theorem similar to Theorem 2 in [?].

**Theorem 2.6** *Assume that  $(f_k)_{0 \leq k \leq n} \in \text{Lip}(\mathbb{R}^d)^{n+1}$ , and  $M_k(\text{Lip}(\mathbb{R}^d)) \subset \text{Lip}(\mathbb{R}^d)$ , for any  $1 \leq k \leq n$ . In this case, we have  $(u_k)_{0 \leq k \leq n} \in \text{Lip}(\mathbb{R}^d)^{n+1}$ , and for any  $0 \leq k \leq n$ , we have the almost sure estimate*

$$\begin{aligned} |u_k - \hat{u}_k|(\hat{X}_k) &\leq L(M_{k+1}(u_{k+1})) |X_k - \hat{X}_k| \\ &\quad + \sum_{l=k+1}^{n-1} (L(u_l) + L(M_{l+1}(u_{l+1}))) \mathbb{E}(|X_l - \hat{X}_l|^p \mid \hat{X}_k)^{\frac{1}{p}} \\ &\quad + L(f_n) \left[ \mathbb{E}(|X_n - \hat{X}_n|^p \mid \hat{X}_k) \right]^{\frac{1}{p}}. \end{aligned}$$

On the other hand, we also have that

$$|u_k(\hat{X}_k) - u_k(X_k)| \leq L(u_k) |\hat{X}_k - X_k|.$$

Using the decomposition

$$\hat{u}_k(\hat{X}_k) - u_k(X_k) = [\hat{u}_k(\hat{X}_k) - u_k(\hat{X}_k)] + [u_k(\hat{X}_k) - u_k(X_k)],$$

we conclude that

$$\begin{aligned} |\hat{u}_k(\hat{X}_k) - u_k(X_k)| &\leq L(f_n) \left[ \mathbb{E}(|X_n - \hat{X}_n|^p \mid \hat{X}_k) \right]^{\frac{1}{p}} \\ &\quad + \sum_{l=k}^{n-1} (L(u_l) + L(M_{l+1}(u_{l+1}))) \mathbb{E}(|X_l - \hat{X}_l|^p \mid \hat{X}_k)^{\frac{1}{p}}. \end{aligned}$$

## 3 Monte Carlo approximation models

### 3.1 Path space models

The choice of non homogeneous state spaces  $E_n$  is not innocent. In several application areas the underlying Markov model is a path-space Markov chain

$$X_n = (X'_0, \dots, X'_n) \in E_n = (E'_0 \times \dots \times E'_n). \quad (3.1)$$

The elementary prime variables  $X'_n$  represent an elementary Markov chain with Markov transitions  $M'_k(x_{k-1}, dx'_k)$  from  $E'_{k-1}$  into  $E'_k$ . In this situation, the historical process  $X_n$  can be seen as a Markov chain with transitions given for any  $x_{k-1} = (x'_0, \dots, x'_{k-1}) \in E_{k-1}$  and  $y_k = (y'_0, \dots, y'_k) \in E_k$  by the following formula

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1}) M'_k(y'_{k-1}, dy'_k).$$

This path space framework is, for instance, well suited when dealing with path dependent options as Asian options.

Besides, this path space framework is also well suited for the analysis of Snell envelopes under different probability measures. To fix the ideas, we associate with the latter a canonical Markov chain  $(\Omega, \mathcal{F}, (X'_n)_{n \geq 0}, \mathbb{P}'_{\eta'_0})$  with initial distribution  $\eta'_0$  on  $E'_0$ , and Markov transitions  $M'_n$  from  $E'_{n-1}$  into  $E'_n$ . We use the notation  $\mathbb{E}_{\mathbb{P}'_{\eta'_0}}$  to denote the expectations with respect to  $\mathbb{P}'_{\eta'_0}$ . We further assume that there exists a sequence of measures  $(\eta_k)_{0 \leq k \leq n}$  on the state spaces  $(E'_k)_{0 \leq k \leq n}$  such that

$$\eta'_0 \sim \eta_0 \quad \text{and} \quad M'_k(x'_{k-1}, \cdot) \sim \eta_k \quad (3.2)$$

for any  $x'_{k-1} \in E'_{k-1}$ , and  $1 \leq k \leq n$ . We let  $(\Omega, \mathcal{F}, (X'_n)_{n \geq 0}, \mathbb{P}_{\eta_0})$  be the canonical space associated with a sequence of independent random variables  $X'_k$  with distribution  $\eta_k$  on the state space  $E'_k$ , with  $k \geq 1$ . Under the probability measure  $\mathbb{P}_{\eta_0}$ , the historical process  $X_n = (X'_0, \dots, X'_n)$  can be seen as a Markov chain with transitions

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1}) \eta_k(dy'_k) .$$

By construction, for any integrable function  $f'_k$  on  $E'_k$ , we have

$$\mathbb{E}_{\mathbb{P}'_{\eta'_0}}(f'_n(X'_n)) = \mathbb{E}_{\mathbb{P}_{\eta_0}}(f_n(X_n))$$

with the collection of functions  $f_k$  on  $E_k$  given for any  $x_k = (x'_0, \dots, x'_k) \in E_k$  by

$$f_k(x_k) = f'_k(x'_k) \times \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k) \quad \text{with} \quad \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k) = \frac{d\eta'_0}{d\eta_0}(x'_0) \prod_{1 \leq l \leq k} \frac{dM'_l(x'_{l-1}, \cdot)}{d\eta_l}(x'_l) . \quad (3.3)$$

**Proposition 3.1** *The Snell envelopes  $u_k$  and  $u'_k$  associated with the pairs  $(f'_k, M'_k)$  and  $(f_k, M_k)$  are given for any  $0 \leq k < n$  by the backward recursions*

$$u'_k = f'_k \vee M'_{k+1}(u'_{k+1}) \quad \text{and} \quad u_k = f_k \vee M_{k+1}(u_{k+1}) \quad \text{with} \quad (u'_n, u_n) = (f'_n, f_n) .$$

*These functions are connected by the following formulae*

$$\forall 0 \leq k \leq n \quad \forall x_k = (x'_0, \dots, x'_k) \in E_k \quad u_k(x_k) = u'_k(x'_k) \times \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}(x_k) . \quad (3.4)$$

**Proof:**

The first assertion is a simple consequence of the definition of a Snell envelope, and formula (3.4) is easily derived using the fact that

$$u'_k(x'_k) = f'_k(x'_k) \vee \left( \int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \frac{dM'_{k+1}(x'_k, \cdot)}{d\eta_{k+1}}(x'_{k+1}) u'_{k+1}(x'_{k+1}) \right) .$$

This ends the proof of the proposition.  $\blacksquare$

Under condition (3.2), the above proposition shows that the calculation of the Snell envelope associated with a given pair of functions and Markov transitions  $(f'_k, M'_k)$  reduces to that of the path space models associated with sequence of independent random variables with distributions  $\eta_n$ . More formally, the restriction  $\mathbb{P}_{\eta_0, n}$  of reference measure  $\mathbb{P}_{\eta_0}$  to the  $\sigma$ -field  $\mathcal{F}_n$  generated by the canonical random sequence  $(X'_k)_{0 \leq k \leq n}$  is given by the tensor product measure  $\mathbb{P}_{\eta_0, n} = \otimes_{k=0}^n \eta_k$ . Nevertheless, under these reference distributions the numerical solving of the backward recursion stated in the above proposition still involves integrations w.r.t. the measures  $\eta_k$ . These equations can be solved if we replace these measures by some sequence of (possibly random) measures  $\widehat{\eta}_k$  with finite support on some reduced measurable subset  $\widehat{E}'_k \subset E'_k$ , with a reasonably large and finite cardinality. We extend  $\widehat{\eta}_k$  to the whole space  $E'_k$  by setting  $\widehat{\eta}_k(E'_k - \widehat{E}'_k) = 0$ .

We let  $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$  be the distribution of a sequence of independent random variables  $\widehat{\xi}'_k$  with distribution  $\widehat{\eta}'_k$  on the state space  $\widehat{E}'_k$ , with  $k \geq 1$ . Under the probability measure  $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$ , the historical process  $X_n = (X'_0, \dots, X'_n)$  can now be seen as a Markov chain taking values in the path spaces

$$\widehat{E}_k := \left( \widehat{E}'_0 \times \dots \times \widehat{E}'_k \right)$$

with Markov transitions given for any  $x_{k-1} = (x'_0, \dots, x'_{k-1}) \in \widehat{E}_{k-1}$  and  $y_k = (y'_0, \dots, y'_k) \in \widehat{E}_k$  by the following formula

$$\widehat{M}_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1}) \widehat{\eta}'_k(dy'_k).$$

Notice that the restriction  $\widehat{\mathbb{P}}_{\widehat{\eta}'_0, n}$  of these approximated reference measure  $\widehat{\mathbb{P}}_{\widehat{\eta}'_0}$  to the  $\sigma$ -field  $\mathcal{F}_n$  generated by the canonical random sequence  $(X'_k)_{0 \leq k \leq n}$  is now given by the the tensor product measure  $\widehat{\mathbb{P}}_{\widehat{\eta}'_0, n} = \otimes_{k=0}^n \widehat{\eta}'_k$ .

We let  $\widehat{u}_k$  be the Snell envelope on the path space  $\widehat{E}_k$ , associated with the pair  $(\widehat{f}_k, \widehat{M}_k)$ , with the sequence of functions  $\widehat{f}_k = f_k$  given in (3.3). By construction, for any  $0 \leq k \leq n$ , and any path  $x_k = (x'_0, \dots, x'_k) \in \widehat{E}_k$ , we have

$$\widehat{u}_k(x_k) = \widehat{u}'_k(x'_k) \times \frac{d\widehat{\mathbb{P}}'_k}{d\mathbb{P}_k}(x_k)$$

with the collection of functions  $(\widehat{u}'_k)_{0 \leq k \leq n}$  on the state spaces  $(E'_k)_{0 \leq k \leq n}$  given by the backward recursions

$$\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left( \int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) \widehat{u}'_{k+1}(x'_{k+1}) \right) \quad (3.5)$$

with the random integral operator  $\widehat{M}'$  from  $E_k$  into  $\widehat{E}_{k+1}$  defined below

$$\widehat{M}'_{k+1}(x'_k, dx'_{k+1}) = \widehat{\eta}'_{k+1}(dx'_{k+1}) R_{k+1}(x'_k, x'_{k+1})$$

with the Radon Nikodym derivatives  $R_{k+1}(x'_k, x'_{k+1}) = \frac{dM'_{k+1}(x'_k, \cdot)}{d\eta'_{k+1}}(x'_{k+1})$ .

### 3.2 Brodie-Glasserman models

We consider the path space models associated to the changes of measures presented in Sub-section 3.1. We use the same notation as in there. We further assume that  $\widehat{\eta}'_k = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_k^i}$  is the occupation measure associated with a sequence of independent random variables  $\xi_k := (\xi_k^i)_{1 \leq i \leq N}$  with common distribution  $\eta_k$  on  $\widehat{E}'_k = E'_k$ . We further assume that  $(\xi_k)_{0 \leq k \leq n}$  are independent. This Monte Carlo type model has been introduced in 1997 by M. Brodie, and P. Glasserman (see for instance [?], and references therein). We let  $\widehat{\mathbb{E}}$  be the expectation operator associated with this additional level of randomness, and we set  $\widehat{\mathbb{E}}_{\mathbb{P}_{\eta_0}} := \widehat{\mathbb{E}} \otimes \mathbb{E}_{\mathbb{P}_{\eta_0}}$ .

In this situation, we observe that

$$(M'_{k+1} - \widehat{M}'_{k+1})(x'_k, dx'_{k+1}) = \frac{1}{N} \widehat{V}_{k+1}(dx'_{k+1}) R_{k+1}(x'_k, x'_{k+1})$$

with the random fields  $\widehat{V}_{k+1} := \sqrt{N} [\widehat{\eta}_{k+1} - \eta_{k+1}]$ . From these observations, we readily prove that the approximation operators  $\widehat{M}'_{k+1}$  are unbiased, in the sense that

$$\forall 0 \leq k \leq l \quad \forall x'_l \in E_l \quad \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \widehat{M}'_{k,l}(f)(x'_l) \mid \mathcal{F}_k \right) = M'_{k,l}(f)(x'_l) \quad (3.6)$$

for any bounded function  $f$  on  $E_{l+1}$ . Furthermore, for any even integer  $p \geq 1$ , we have

$$\sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (f)(x'_l) \right|^p \right)^{\frac{1}{p}} \leq 2 a(p) \eta_{l+1} [(R_{l+1}(x'_l, \cdot) f)^p]^{\frac{1}{p}} .$$

The above estimate is valid as soon as the r.h.s. in the above inequality is well defined.

We are now in position to state and prove the following theorem.

**Theorem 3.2** *For any integer  $p \geq 1$ , we denote by  $p'$  the smallest even integer greater than  $p$ . Then for any time horizon  $0 \leq k \leq n$ , and any  $x'_k \in E'_k$ , we have*

$$\sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( |u'_k(x'_k) - \widehat{u}'_k(x'_k)|^p \right)^{\frac{1}{p}} \leq 2a(p) \sum_{k \leq l < n} \left\{ \int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} \left[ (R_{l+1}(x'_l, \cdot) u'_{l+1})^{p'} \right] \right\}^{\frac{1}{p'}} \quad (3.7)$$

Notice that, as stated in the introduction, this result implies exponential rate of convergence in probability. Hence, this allows to improve noticeably existing convergence results stated in [?], with no rate of convergence, and in [?] with a polynomial rate of convergence in probability. **Proof:**

For any even integers  $p \geq 1$ , any  $0 \leq k \leq l$ , any measurable function  $f$  on  $E_{l+1}$ , and any  $x_k \in E'_k$ , using the generalized Minkowski inequality we find that

$$\sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \widehat{M}'_{k,l} \left[ \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (f) \right] (x'_k) \right|^p \mid \mathcal{F}_k \right)^{\frac{1}{p}} \leq 2a(p) \int \widehat{M}'_{k,l}(x'_k, dx'_l) \eta_{l+1} [(R_{l+1}(x'_l, \cdot) f)^p]^{\frac{1}{p}} .$$

By the unbiased property (3.6), we conclude that

$$\sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \widehat{M}'_{k,l} \left[ \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (f) \right] (x'_k) \right|^p \right)^{\frac{1}{p}} \leq 2a(p) \left\{ \int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} [(R_{l+1}(x'_l, \cdot) f)^p] \right\}^{1/p} .$$

For odd integers  $p = 2q + 1$ , with  $q \geq 0$ , we use the fact that

$$\mathbb{E}(Y^{2q+1})^2 \leq \mathbb{E}(Y^{2q}) \mathbb{E}(Y^{2(q+1)}) \quad \text{and} \quad \mathbb{E}(Y^{2q}) \leq \mathbb{E}(Y^{2(q+1)})^{\frac{q}{q+1}}$$

for any non negative random variable  $Y$  and

$$(2(q+1))_{q+1} = 2 (2q+1)_{q+1} \quad \text{and} \quad (2q)_q = (2q+1)_{q+1} / (2q+1)$$

so that

$$a(2q)^{2q} a(2(q+1))^{2(q+1)} \leq 2^{-(2q+1)} (2q+1)_{q+1}^2 / (q+1/2) = (a(2q+1))^{2q+1}^2$$

$$N \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \widehat{M}'_{k,l} \left[ \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (f) \right] (x'_k) \right|^{2q+1} \right)^2$$

$$\leq (2^{(2q+1)} a(2q+1)^{2q+1})^2 \int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} [(R_{l+1}(x'_l, \cdot) f)^{2(q+1)}]^{\frac{q}{q+1}}$$

using the fact that  $\mathbb{E}(Y^{\frac{q}{q+1}}) \leq \mathbb{E}(Y)^{\frac{q}{q+1}}$ , we prove that the r.h.s. term in the above display is upper bounded by

$$\left(2^{(2q+1)} a(2q+1)^{2q+1}\right)^2 \left\{ \int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} \left[ (R_{l+1}(x'_l, \cdot) f)^{2(q+1)} \right] \right\}^{2 \left(1 - \frac{1}{2(q+1)}\right)}$$

from which we conclude that

$$\begin{aligned} & \sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \widehat{M}'_{k,l} \left[ \left| M'_{l+1} - \widehat{M}'_{l+1} \right| (f) \right] (x'_k) \right|^{2q+1} \right)^{\frac{1}{2q+1}} \\ & \leq 2a(2q+1) \left\{ \int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} \left[ (R_{l+1}(x'_l, \cdot) f)^{2(q+1)} \right] \right\}^{\frac{1}{2(q+1)}} \end{aligned}$$

This ends the proof of the theorem.  $\blacksquare$

The  $\mathbb{L}_p$ -mean error estimates stated in Theorem 3.2 are expressed in terms of  $\mathbb{L}_{p'}$  norms of Snell envelope functions and Radon Nikodym derivatives. The terms in r.h.s. of (3.7) have the following interpretation:

$$\int M'_{k,l}(x'_k, dx'_l) \eta_{l+1} \left[ (R_{l+1}(x'_l, \cdot) u_{l+1})^{p'} \right] = \mathbb{E} \left( [R_{l+1}(X'_l, \xi_{l+1}^1) u_{l+1}(\xi_{l+1}^1)]^{p'} / X'_k = x'_k \right)$$

In the above display,  $\mathbb{E}(\cdot)$  stands for the expectation w.r.t. some reference probability measure under which  $X'_l$  is a Markov chain with transitions  $M'_l$ , and  $\xi_{l+1}^1$  is a independent random variable with distribution  $\eta_{l+1}$ . Loosely speaking, the above quantities can be very large when the sampling distributions  $\eta_{l+1}$  are far from the distribution of the random states  $X'_{l+1}$  of the reference Markov chain at time  $(l+1)$ . Next we provide an original strategy that allows to take  $\eta_{l+1} = \text{Law}(X'_{l+1})$  as the sampling distribution. In what follows, we let  $\widehat{\eta}_k = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_k^i}$  be the occupation measure associated with  $N$  independent copies  $\xi_k = (\xi_k^i)_{1 \leq i \leq N}$  of the Markov chain  $X'_k$ , from the origin  $k=0$  up to the final time horizon  $k=n$ . In what follows, we let  $\mathcal{F}_k$  be the sigma field generated by the random sequence  $(\xi_l)_{0 \leq l \leq k}$ .

We also assume that the Markov transitions  $M'_n(x'_{n-1}, dx'_n)$  are absolutely continuous with respect to some measures  $\lambda_n(dx'_n)$  on  $E'_n$  and we have

$$(H)_0 \quad \forall (x'_{n-1}, x'_n) \in (E'_{n-1} \times E'_n) \quad H_n(x'_{n-1}, x'_n) = \frac{dM'_n(x'_{n-1}, \cdot)}{d\lambda_n}(x'_n) > 0$$

In this situation, we have  $\eta_{k+1} \ll \lambda_{k+1}$ , with the Radon Nikodym derivative given below

$$\eta_{k+1}(dx'_{k+1}) = \eta_k M'_{k+1}(dx'_{k+1}) = \eta_k (H_{k+1}(\cdot, x'_{k+1})) \lambda_{k+1}(dx'_{k+1})$$

Also notice that the backward recursion of the Snell envelope  $u'_k$  can be rewritten as follows

$$\begin{aligned} u'_k(x'_k) &= f'_k(x'_k) \vee \left( \int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \frac{dM'_{k+1}(x'_k, \cdot)}{d\eta_{k+1}}(x'_{k+1}) u'_{k+1}(x'_{k+1}) \right) \\ &= f'_k(x'_k) \vee \left( \int_{E'_{k+1}} \eta_{k+1}(dx'_{k+1}) \frac{H_{k+1}(x'_k, x'_{k+1})}{\eta_k(H_{k+1}(\cdot, x'_{k+1}))} u'_{k+1}(x'_{k+1}) \right). \end{aligned}$$

Arguing as in (3.5), we define the approximated Snell envelope  $(\widehat{u}'_k)_{0 \leq k \leq n}$  on the state spaces  $(E'_k)_{0 \leq k \leq n}$  by setting

$$\widehat{u}'_k(x'_k) = f'_k(x'_k) \vee \left( \int_{\widehat{E}'_{k+1}} \widehat{M}'_{k+1}(x'_k, dx'_{k+1}) \widehat{u}'_{k+1}(x'_{k+1}) \right)$$

with the random integral operator  $\widehat{M}'$  from  $E_k$  into  $\widehat{E}'_{k+1}$  defined below

$$\widehat{M}'_{k+1}(x'_k, dx'_{k+1}) = \widehat{\eta}_{k+1}(dx'_{k+1}) \frac{dM'_{k+1}(x'_k, \cdot)}{d\widehat{\eta}_k M'_{k+1}}(x'_{k+1}) = \widehat{\eta}_{k+1}(dx'_{k+1}) \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\cdot, x'_{k+1}))}.$$

By construction, these random approximation operators  $\widehat{M}'_{k+1}$  satisfy the unbiased property stated in (3.6), and we have

$$(M'_{k+1} - \widehat{M}'_{k+1})(x'_k, dx'_{k+1}) = \frac{1}{N} \widehat{V}_{k+1}(dx'_{k+1}) \widehat{R}_{k+1}(x'_k, x'_{k+1})$$

with the random fields  $\widehat{V}_{k+1}$  and the  $\mathcal{F}_k$ -measurable random functions  $\widehat{R}_{k+1}$  defined below

$$\widehat{V}_{k+1} := \sqrt{N} [\widehat{\eta}_{k+1} - \widehat{\eta}_k M'_{k+1}] \quad \text{and} \quad \widehat{R}_{k+1}(x'_k, x'_{k+1}) := \frac{H_{k+1}(x'_k, x'_{k+1})}{\widehat{\eta}_k(H_{k+1}(\cdot, x'_{k+1}))}.$$

Furthermore, for any even integer  $p \geq 1$ , and any measurable function  $f$  on  $E_l$  we have

$$\sqrt{N} \widehat{\mathbb{E}}_{\mathbb{P}_{n_0}} \left( \left| \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (f)(x'_l) \right|^p \middle| \mathcal{F}_l \right)^{\frac{1}{p}} \leq 2 a(p) \widehat{\eta}_l M'_{l+1} \left[ (\widehat{R}_{l+1}(x'_l, \cdot) f)^p \right]^{\frac{1}{p}}.$$

The above estimate is valid as soon as the r.h.s. in the above inequality is well defined. For instance, assuming that

$$(H)_1 \quad \|M'_{l+1}(u_{l+1}^{2p})\| < \infty \quad \text{and} \quad \sup_{x'_l, y'_l \in E'_l} \frac{H_{l+1}(x'_l, x'_{l+1})}{H_{l+1}(y'_l, x'_{l+1})} \leq h_{l+1}(x'_{l+1}) \quad \text{with} \quad \|M'_{l+1}(h_{l+1}^{2p})\| < \infty$$

we find that

$$\sqrt{N} \mathbb{E} \left( \left| \left[ M'_{l+1} - \widehat{M}'_{l+1} \right] (u'_{l+1})(x'_l) \right|^p \middle| \mathcal{F}_l \right)^{\frac{1}{p}} \leq 2 a(p) \left( \|M'_{l+1}(h_{l+1}^{2p})\| \|M'_{l+1}((u'_{l+1})^{2p})\| \right)^{\frac{1}{2p}}.$$

Rephrasing the proof of Theorem 3.2, we prove the following result.

**Theorem 3.3** *Under the conditions  $(H)_0$  and  $(H)_1$  stated above, for any even integer  $p > 1$ , any  $0 \leq k \leq n$ , and  $x'_k \in E'_k$ , we have*

$$\sqrt{N} \mathbb{E} (|u'_k(x'_k) - \widehat{u}'_k(x'_k)|^p)^{\frac{1}{p}} \leq 2a(p) \sum_{k \leq l < n} \left( \|M'_{l+1}(h_{l+1}^{2p})\| \|M'_{l+1}((u'_{l+1})^{2p})\| \right)^{\frac{1}{2p}}. \quad (3.8)$$

### 3.3 Genealogical tree based models

#### 3.3.1 Neutral genetic models

In this section, we propose a new model whose purpose is to reduce the number of calculations in selecting  $N$  trajectories from the large exploding tree, but maintain the precision. Using the notation of Sub-section 3.1, we set

$$X_n = (X'_0, \dots, X'_n) \in E_n = (E'_0 \times \dots \times E'_n)$$

We further assume that the state spaces  $E'_n$  are finite. We denote by  $\eta_k$  the distribution of the path-valued random variable  $X_k$  on  $E_k$ , with  $0 \leq k \leq n$ .

We also set  $M'_k$  the Markov transition from  $X'_{k-1}$  to  $X'_k$ , and  $M_k$  the Markov transition from  $X_{k-1}$  to  $X_k$ . In Sub-section 3.1, we have seen that

$$M_k((x'_0, \dots, x'_{k-1}), d(y'_0, \dots, y'_k)) = \delta_{(x'_0, \dots, x'_{k-1})}(d(y'_0, \dots, y'_{k-1})) M'_k(y'_{k-1}, dy'_k).$$

In the further development, we fix the final time horizon  $n$ , and for any  $0 \leq k \leq n$ , we denote by  $\pi_k$  the  $k$ -th coordinate mapping

$$\pi_k : x_n = (x'_0, \dots, x'_n) \in E_n = (E'_0 \times \dots \times E'_n) \mapsto \pi_k(x_n) = x'_k \in E'_k.$$

In this notation, for any  $0 \leq k < n$ ,  $x'_k \in E'_k$  and any function  $f \in \mathcal{B}(E'_{k+1})$ , we have

$$\eta_n = \text{Law}(X'_0, \dots, X'_n) \quad \text{and} \quad M'_{k+1}(f)(x) := \frac{\eta_n((1_x \circ \pi_k)(f \circ \pi_{k+1}))}{\eta_n((1_x \circ \pi_k))}. \quad (3.9)$$

By construction, it is also readily checked that the flow of measure  $(\eta_k)_{0 \leq k \leq n}$  also satisfies the following equation

$$\forall 1 \leq k \leq n \quad \eta_k := \Phi_k(\eta_{k-1}) \quad (3.10)$$

with the linear mapping  $\Phi_k(\eta_{k-1}) := \eta_{k-1} M_k$ .

The genealogical tree based particle approximation associated with these recursion is defined in terms of a Markov chain  $\xi_k^{(N)} = (\xi_k^{(i,N)})_{1 \leq i \leq N_k}$  in the product state spaces  $E_k^{N_k}$ , where  $N = (N_k)_{0 \leq k \leq N}$  is a given collection of integers.

$$\mathbb{P}\left(\xi_k^{(N)} = (x_k^1, \dots, x_k^{N_k}) \mid \xi_{k-1}\right) = \prod_{1 \leq i \leq N_k} \Phi_k\left(\frac{1}{N_{k-1}} \sum_{1 \leq i \leq N_{k-1}} \delta_{\xi_{k-1}^i}\right)(x_k^i). \quad (3.11)$$

The initial particle system  $\xi_0^{(N)} = (\xi_0^{(i,N)})_{0 \leq i \leq N_0}$ , is a sequence of  $N_0$  i.i.d. random copies of  $X_0$ . We let  $\mathcal{F}_k^N$  be the sigma-field generated by the particle approximation model from the origin, up to time  $k$ .

To simplify the presentation, when there is no confusion we suppress the population size parameter  $N$ , and we write  $\xi_k$  and  $\xi_k^i$  instead of  $\xi_k^{(N)}$  and  $\xi_k^{(i,N)}$ . By construction,  $\xi_k$  is a genetic type model with a neutral selection transition and a mutation type exploration

$$\xi_k \in E_k^{N_k} \xrightarrow{\text{Selection}} \widehat{\xi}_k := (\widehat{\xi}_k^i)_{1 \leq i \leq \widehat{N}_k} \in E_k^{\widehat{N}_k} \xrightarrow{\text{Mutation}} \xi_{k+1} \in E_{k+1}^{N_{k+1}} \quad (3.12)$$

with  $\widehat{N}_k := N_{k+1}$ .

During the selection transition, we select randomly  $N_{k+1}$  path valued particles  $\widehat{\xi}_k := \left( \widehat{\xi}_k^i \right)_{1 \leq i \leq N_{k+1}}$  among the  $N_k$  path valued particles  $\xi_k = (\xi_k^i)_{1 \leq i \leq N_k}$ . Sometimes, this elementary transition is called a neutral selection transition in the literature on genetic population models. During the mutation transition  $\widehat{\xi}_k \rightsquigarrow \xi_{k+1}$ , every selected path valued individual  $\widehat{\xi}_k^i$  evolves randomly to a new path valued individual  $\xi_{k+1}^i = x$  randomly chosen with the distribution  $M_{k+1}(\widehat{\xi}_k^i, x)$ , with  $1 \leq i \leq \widehat{N}_k$ . By construction, every particle is a path-valued random variable defined by

$$\begin{aligned} \xi_k^i &:= (\xi_{0,k}^i, \xi_{1,k}^i, \dots, \xi_{k,k}^i) \\ \widehat{\xi}_k^i &:= (\widehat{\xi}_{0,k}^i, \widehat{\xi}_{1,k}^i, \dots, \widehat{\xi}_{k,k}^i) \in E_k := (E'_0 \times \dots \times E'_k). \end{aligned}$$

By definition of the transition in path space, we also have that

$$\begin{aligned} \xi_{k+1}^i &= \left( \underbrace{(\xi_{0,k+1}^i, \xi_{1,k+1}^i, \dots, \xi_{k,k+1}^i)}_{\parallel}, \xi_{k+1,k+1}^i \right) \\ &= \left( \underbrace{(\widehat{\xi}_{0,k}^i, \widehat{\xi}_{1,k}^i, \dots, \widehat{\xi}_{k,k}^i)}_{\parallel}, \xi_{k+1,k+1}^i \right) = (\widehat{\xi}_k^i, \xi_{k+1,k+1}^i) \end{aligned}$$

where  $\xi_{k+1,k+1}^i$  is a random variable with distribution  $M'_{k+1}(\widehat{\xi}_{k,k}^i, \cdot)$ . In other words, the mutation transition  $\widehat{\xi}_k^i \rightsquigarrow \xi_{k+1}^i$  simply consists in extending the selected path  $\widehat{\xi}_k^i$  with an elementary move  $\widehat{\xi}_{k,k}^i \rightsquigarrow \xi_{k+1,k+1}^i$  of the end point of the selected path.

From these observations, it is easy to check that the terminal random population model  $\xi_{k,k} = (\xi_{k,k}^i)_{1 \leq i \leq N_k}$  and  $\widehat{\xi}_{k,k} = (\widehat{\xi}_{k,k}^i)_{1 \leq i \leq N_{k+1}}$  is again defined as a genetic type Markov chain defined as above by replacing the pair  $(E_k, M_k)$  by the pair  $(E'_k, M'_k)$ , with  $1 \leq k \leq n$ . The latter coincides with the mean field particle model associated with the time evolution of the  $k$ -th time marginals  $\eta'_k$  of the measures  $\eta_k$  on  $E'_k$ . Furthermore, the above path-valued genetic model coincide with the genealogical tree evolution model associated with the terminal state random variables.

We let  $\eta_k^N$  and  $\widehat{\eta}_k^N$  be the occupation measures of the genealogical tree model after the mutation and the selection steps; that is, we have that

$$\eta_k^N := \frac{1}{N_k} \sum_{1 \leq i \leq N_k} \delta_{\xi_k^i} \quad \text{and} \quad \widehat{\eta}_k^N := \frac{1}{\widehat{N}_k} \sum_{1 \leq i \leq \widehat{N}_k} \delta_{\widehat{\xi}_k^i}.$$

In this notation, the selection transition  $\xi_k \rightsquigarrow \widehat{\xi}_k$  consists in choosing  $\widehat{N}_k$  conditionally independent and identically distributed random paths  $\widehat{\xi}_k^i$  with common distribution  $\eta_k^N$ . In other words,  $\widehat{\eta}_k^N$  is the empirical measure associated with  $\widehat{N}_k$  conditionally independent and identically distributed random paths  $\widehat{\xi}_k^i$  with common distribution  $\eta_k^N$ . Also observe  $\eta_k^N$  is the empirical measure associated with  $N_k$  conditionally independent and identically distributed random paths  $\xi_k^i$  with common distribution  $\eta_{k-1}^N M_k$ .

In practice, we can take  $N_0 = N_1 = \dots = N_n = N$  when we do not have any information on the variance of  $X_k$ . In the case when we know the approximate variance of  $X_k$ , we can take a large  $N_k$  when the variance of  $X_k$  is large. To clarify the presentation, In the further development of the article we further assume that the particle model has a fixed population size  $N_k = N$ , for any  $k \geq 0$ .

### 3.3.2 Convergence analysis

For general mean field particle interpretation models (3.11), several estimates can be derived for the above particle approximation model (see for instance [?]). For instance, for any  $n \geq 0$ ,  $r \geq 1$ , and any  $f_n \in \text{Osc}_1(E_n)$ , and any  $N \geq 1$ , we have the unbiased and the mean error estimates:

$$\mathbb{E}(\eta_n^N(f_n)) = \eta_n(f_n) = \mathbb{E}(\widehat{\eta}_n^N(f_n)) \quad \text{and} \quad \sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_n](f_n) \right|^r \right)^{\frac{1}{r}} \leq 2 a(r) \sum_{p=0}^n \beta(M_{p,n}) \quad (3.13)$$

with the Dobrushin ergodic coefficients

$$\beta(M_{p,n}) := \sup_{(x_p, y_p) \in E_p} \|M_{p,n}(x_p, \cdot) - M_{p,n}(y_p, \cdot)\|_{\text{tv}}$$

and the collection of constants  $a(p)$  defined in (1.7). Arguing as in (1.9), for time homogeneous population sizes  $N_n = N$ , for any functions  $f \in \text{Osc}_1(E_n)$ , we conclude that

$$\mathbb{P} \left( \left| [\eta_n^N - \eta_n](f) \right| \geq \frac{b(n)}{\sqrt{N}} + \epsilon \right) \leq \exp \left( -\frac{N\epsilon^2}{2b(n)^2} \right) \quad \text{with} \quad b(n) := 2 \sum_{p=0}^n \beta(M_{p,n}). \quad (3.14)$$

For the path space models (3.9), we have  $\beta(M_{p,n}) = 1$  so that the estimates (3.13) and (3.14) takes the form

$$\sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_n](f_n) \right|^r \right)^{\frac{1}{r}} \leq 2 a(r) (n+1) \quad (3.15)$$

and

$$\mathbb{P} \left( \left| [\eta_n^N - \eta_n](f) \right| \geq \frac{2(n+1)}{\sqrt{N}} + \epsilon \right) \leq \exp \left( -\frac{N\epsilon^2}{8(n+1)^2} \right).$$

In the next lemma we extend these estimates to unbounded functions and non homogeneous population size models.

**Lemma 3.4** *For any  $p \geq 1$ , we denote by  $p'$  the smallest even integer greater than  $p$ . In this notation, for any  $k \geq 0$  and any function  $f$ , we have the almost sure estimate*

$$\sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_{k-1}^N M_{k-1,n}](f) \right|^p \mid \mathcal{F}_{k-1}^N \right)^{\frac{1}{p}} \leq 2a(p) \sum_{l=k}^n \left[ \eta_{k-1}^N M_{k-1,l} (|M_{l,n}(f)|^{p'}) \right]^{\frac{1}{p'}} \quad (3.16)$$

In particular, for any  $f \in \mathbb{L}_{p'}(\eta_n)$ , we have the non asymptotic estimates

$$\sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_n](f) \right|^p \right)^{1/p} \leq 2 a(p) \|f\|_{p', \eta_n} (n+1). \quad (3.17)$$

**Proof :**

In writing  $\eta_{-1}^N M_0 = \eta_0$ , for any  $k \geq 0$  we have the decomposition

$$[\eta_n^N - \eta_{k-1}^N M_{k,n}] = \sum_{l=k}^n [\eta_l^N - (\eta_{l-1}^N M_l)] M_{l,n}$$

with the semigroup

$$M_{k,n} = M_{k+1} M_{k+2} \dots M_n$$

Using the fact that

$$\mathbb{E}(\eta_l^N(f) | \eta_{l-1}^N) = (\eta_{l-1}^N M_l)(f)$$

we prove that

$$\mathbb{E} \left( |[\eta_l^N - (\eta_{l-1}^N M_l)](f)|^p | \mathcal{F}_{l-1}^N \right)^{\frac{1}{p}} \leq \mathbb{E} \left( |[\eta_l^N - \mu_l^N](f)|^p | \mathcal{F}_{l-1}^N \right)^{\frac{1}{p}}$$

where  $\mu_l^N := \frac{1}{N} \sum_{i=1}^N \delta_{\zeta_i^l}$  stands for a independent copy of  $\eta_l^N$  given  $\eta_{l-1}^N$ . Using Kintchine's type inequalities we have

$$\begin{aligned} \sqrt{N} \mathbb{E} \left( |[\eta_l^N - \mu_l^N](f)|^p | \mathcal{F}_{l-1}^N \right)^{\frac{1}{p}} &\leq 2 a(p) \mathbb{E} \left( |f(\xi_l^1)|^{p'} | \mathcal{F}_{l-1}^N \right)^{\frac{1}{p'}} \\ &= 2 a(p) \left[ \eta_{l-1}^N M_l(|f|^{p'}) \right]^{\frac{1}{p'}}. \end{aligned}$$

Using the unbiased property of the particle scheme, we have

$$\forall k \leq l \leq n \quad \mathbb{E}(\eta_l^N(f) | \mathcal{F}_{k-1}^N) = (\eta_{k-1}^N M_{k-1,l})(f).$$

This implies that

$$\begin{aligned} \sqrt{N} \mathbb{E} \left( |[\eta_l^N - (\eta_{l-1}^N M_l)](f)|^p | \mathcal{F}_{k-1}^N \right)^{\frac{1}{p}} &\leq 2 a(p) \mathbb{E} \left( \eta_{l-1}^N M_l(|f|^{p'}) | \mathcal{F}_{k-1}^N \right)^{\frac{1}{p'}} \\ &= 2 a(p) \left[ \eta_{k-1}^N M_{k-1,l}(|f|^{p'}) \right]^{\frac{1}{p'}}. \end{aligned}$$

The end of the proof of (3.16) is now a direct application of Minkowski's inequality. The proof of (3.17) is a direct consequence of (3.16). This ends the proof of the lemma.  $\blacksquare$

**Particle approximations of the Snell envelope**

In sub-section 3.3.1 we have presented a genealogical based algorithm whose occupation measures  $\eta_n^N$  converge, as  $N \uparrow \infty$ , to the distribution  $\eta_n$  of the reference Markov chain  $(X'_0, \dots, X'_n)$  from the origin, up to the final time horizon  $n$ . Mimicking formula (3.9), we define the particle approximation of the Markov transitions  $M'_k$  as follows :

$$\widehat{M}'_{k+1}(f)(x) := \frac{\eta_n^N((1_x \circ \pi_k)(f \circ \pi_{k+1}))}{\eta_n^N((1_x \circ \pi_k))} := \frac{\sum_{1 \leq i \leq N} 1_x(\xi_{k,n}^i) f(\xi_{k+1,n}^i)}{\sum_{1 \leq i \leq N} 1_x(\xi_{k,n}^i)}$$

for every state  $x$  in the support  $\widehat{E}_{k,n}$  of the measure  $\eta_n^N \circ \pi_k^{-1}$ . Notice that  $\widehat{E}_{k,n}$  coincides with the collection of ancestors  $\xi_{k,n}^i$  at level  $k$  of the population

of individuals at the final time horizon. This random set can alternatively be defined as the set of states  $\xi_{k,k}^i$  of the particle population at time  $k$  such that  $\eta_n^N((1_{\xi_{k,k}^i} \circ \pi_k)) > 0$ ; more formally, we have

$$\widehat{E}_{k,n} := \cup_{1 \leq i \leq N} \left\{ \xi_{k,k}^i : \eta_n^N((1_{\xi_{k,k}^i} \circ \pi_k)) > 0 \right\} . \quad (3.18)$$

It is interesting to observe that the random Markov transitions  $\widehat{M}'_{k+1}$  coincides with the conditional distributions of the states  $X'_{k+1}$  given the current time states  $X'_k$  of a canonical Markov chain  $X_n := (X'_0, \dots, X'_n)$  with distribution  $\eta_n^N$  on the path space  $E_n := (E'_0 \times \dots \times E'_n)$ . Thus, the flow of  $k$ -th time marginal measures

$$\eta_{k,n}^N := \frac{1}{N} \sum_{i=1}^N \delta_{\xi_{k,n}^i}$$

are connected by the following formulae

$$\forall k \leq l \leq n \quad \eta_{k,n}^N \widehat{M}'_{k,l} = \eta_{l,n}^N$$

with the semigroup  $\widehat{M}'_{k,l}$  associated with the Markov transitions  $\widehat{M}'_{k+1}$  and given by

$$\widehat{M}'_{k,l}(f)(x) = \widehat{M}'_{k+1} \widehat{M}'_{k+1} \dots \widehat{M}'_l(f)(x) = \frac{\eta_n^N((1_x \circ \pi_k)(f \circ \pi_l))}{\eta_n^N((1_x \circ \pi_k))} \quad (3.19)$$

for every state  $x$  in  $\widehat{E}_{k,n}$ . In connection with (3.18), we also have the following formulae

$$\eta_{k,n}^N = \frac{1}{N} \sum_{i=1}^N \left( N \eta_n^N(1_{\xi_{k,k}^i} \circ \pi_k) \right) \delta_{\xi_{k,k}^i} = \sum_{i=1}^N \eta_n^N(1_{\xi_{k,k}^i} \circ \pi_k) \delta_{\xi_{k,k}^i}$$

with the proportion  $\eta_n^N(1_{\xi_{k,k}^i} \circ \pi_k)$  of individuals at the final time horizon having the common ancestor  $\xi_{k,k}^i$  at level  $k$ . It is also interesting to observe that

$$\begin{aligned} \mathbb{E}(\eta_{k,n}^N(f) | \mathcal{F}_k^N) &= \sum_{i=1}^N \mathbb{E} \left( \eta_n^N(1_{\xi_{k,k}^i} \circ \pi_k) | \mathcal{F}_k^N \right) f(\xi_{k,k}^i) \\ &= \sum_{i=1}^N \underbrace{\eta_k^N M_{k,n}(\xi_{k,k}^i \circ \pi_k)}_{=1/N} f(\xi_{k,k}^i) = \eta_k^N(f) . \end{aligned}$$

The Snell envelope associated with this particle approximation model is defined by the backward recursion:

$$\widehat{u}_k(x) = \begin{cases} f_k(x) \vee \widehat{M}'_{k+1}(u_{k+1})(x) & \forall x \in \widehat{E}_{k,n} \\ 0 & \text{otherwise} . \end{cases}$$

In terms of the ancestors at level  $k$ , this recursion takes the following form

$$\forall 1 \leq i \leq N \quad \widehat{u}_k(\xi_{k,n}^i) = f_k(\xi_{k,n}^i) \vee \widehat{M}'_{k+1}(\widehat{u}_{k+1})(\xi_{k,n}^i) .$$

For later use in the further development of this section, we quote a couple of technical lemmas. The first one we provide some  $\mathbb{L}_p$  estimates of the normalizing quantities of the Markov transitions  $\widehat{M}'_{k+1}$ . The second one allows to quantify the deviations of  $\widehat{M}'_{k+1}$  around its limiting values  $M'_{k+1}$ , as  $N \rightarrow \infty$ .

**Lemma 3.5** *For any  $p \geq 1$ , and  $0 \leq i \leq N$  we have the following uniform estimate*

$$\sup_{N \geq 1} \sup_{0 \leq l \leq k \leq n} \left\| \eta_k^N (1_{\xi_{l,k}^i} \circ \pi_l)^{-1} \right\|_{\mathbb{L}_p} < \infty. \quad (3.20)$$

**Lemma 3.6** *For any  $p \geq 1$ , and  $0 \leq i \leq N$  we have the following uniform estimate*

$$\sup_{0 \leq l \leq n} \left\| \widehat{M}'_{l+1}(f)(\xi_{l,n}^i) - M'_{l+1}(f)(\xi_{l,n}^i) \right\|_{\mathbb{L}_p} \leq c_p(n)/\sqrt{N} \quad (3.21)$$

with some collection of finite constants  $c_p(n) < \infty$  whose values only depend on the parameters  $p$  and  $n$ .

The proofs of these lemmas are rather technical, thus there are postponed to the appendix.

We are now in position to state and prove the main result of this section.

**Theorem 3.7** *For any  $p \geq 1$ , and  $0 \leq i \leq N$  we have the following uniform estimate*

$$\sup_{0 \leq k \leq n} \left\| (u_k - \widehat{u}_k)(\xi_{k,n}^i) \right\|_{\mathbb{L}_p} \leq c_p(n)/\sqrt{N} \quad (3.22)$$

with some collection of finite constants  $c_p(n) < \infty$  whose values only depend on the parameters  $p$  and  $n$ .

**Proof:**

Firstly, we use the following decomposition

$$|u_k - \widehat{u}_k| 1_{\widehat{E}_{k,n}} \leq \sum_{k \leq l \leq n-1} \widehat{M}'_{k,l} |(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})| 1_{\widehat{E}_{k,n}} \quad (3.23)$$

By construction, we have

$$\widehat{M}'_{k,l} |(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})| 1_{\widehat{E}_{k,n}} = \widehat{M}'_{k,l} |1_{\widehat{E}_{l,n}} (\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})| 1_{\widehat{E}_{k,n}}.$$

By (3.19), if we set

$$\widetilde{u}_{l+1} = |(\widehat{M}'_{l+1} - M'_{l+1})(u_{l+1})|$$

on the set  $\widehat{E}_{l,n}$ , then we have that

$$\widehat{M}'_{k,l}(\widetilde{u}_{l+1})(\xi_{k,n}^i) = \frac{\eta_n^N((1_{\xi_{k,n}^i} \circ \pi_k)(\widetilde{u}_{l+1} \circ \pi_l))}{\eta_n^N((1_{\xi_{k,n}^i} \circ \pi_k))}.$$

For any  $p \geq 1$ , we have

$$\left\| \widehat{M}'_{k,l}(\widetilde{u}_{l+1})(\xi_{k,n}^i) \right\|_{\mathbb{L}_p} \leq \left\| \eta_n^N((1_{\xi_{k,n}^i} \circ \pi_k))^{-1} \right\|_{\mathbb{L}_2}^{1/p} \times \mathbb{E} \left( \eta_n^N((1_{\xi_{k,n}^i} \circ \pi_k)(\widetilde{u}_{l+1} \circ \pi_l)^{2p}) \right)^{1/(2p)}.$$

This implies that

$$\left\| \widehat{M}'_{k,l}(\tilde{u}_{l+1})(\xi_{k,n}^i) \right\|_{\mathbb{L}_p} \leq \left\| \eta_n^N ((1_{\xi_{k,n}^i} \circ \pi_k))^{-1} \right\|_{\mathbb{L}_2}^{1/p} \times \sup_{1 \leq j \leq N} \left\| \tilde{u}_{l+1}(\xi_{l,n}^j) \right\|_{\mathbb{L}_{2p}}.$$

The proof of (3.22) is now a clear consequence of Lemma 3.5 and Lemma 3.6. This ends the proof of the theorem.  $\blacksquare$

## 4 Appendix

### 4.1 Proof of Lemma 3.6:

By construction, we have

$$\forall x \in \widehat{E}_{l,n} \quad M'_{l+1}(f)(x) = \frac{\eta_l^N M_{l,n}((1_x \circ \pi_l)(f \circ \pi_{l+1}))}{\eta_l^N M_{l,n}((1_x \circ \pi_l))}. \quad (4.1)$$

Thus, by (4.1) we have

$$\widehat{M}'_{l+1}(f)(x) - M'_{l+1}(f)(x) := \frac{\eta_n^N(g_{l,x} f_{l+1})}{\eta_n^N(g_{l,x})} - \frac{\eta_l^N M_{l,n}(g_{l,x} f_{l+1})}{\eta_l^N M_{l,n}(g_{l,x})}$$

for any  $x \in \widehat{E}_{l,n}$ , with the collection of functions

$$g_{l,x} := 1_x \circ \pi_l \quad \text{and} \quad f_{l+1} := f \circ \pi_{l+1}. \quad (4.2)$$

It is readily checked that

$$\widehat{M}'_{l+1}(f)(x) - M'_{l+1}(f)(x) = \frac{1}{\eta_n^N(\bar{g}_{l,x}^N)} \left[ \eta_n^N(\bar{f}_{l+1,x}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,x}^N) \right]$$

for any  $x \in \widehat{E}_{l,n}$ , with the pair of  $\mathcal{F}_l^N$ -measurable functions

$$\bar{f}_{l+1,x}^N := \frac{g_{l,x}}{\eta_l^N M_{l,n}(g_{l,x})} \left[ f_{l+1} - \frac{\eta_l^N M_{l,n}(g_{l,x} f_{l+1})}{\eta_l^N M_{l,n}(g_{l,x})} \right] \quad \text{and} \quad \bar{g}_{l,x}^N = \frac{g_{l,x}}{\eta_l^N M_{l,n}(g_{l,x})}.$$

It is also important to observe as  $g_{l,x}$  varies only on  $E_l'$ , then

$$\eta_l^N M_{l,n}(g_{l,x}) = \eta_l^N(g_{l,x}) \leq 1$$

In this notation, for any  $0 \leq i \leq N$  and any  $p \geq 1$ , we have

$$\begin{aligned} & \left\| \widehat{M}'_{l+1}(f)(\xi_{l,n}^i) - M'_{l+1}(f)(\xi_{l,n}^i) \right\|_{\mathbb{L}_p} \\ & \leq \left\| \eta_n^N(g_{l,\xi_{l,n}^i})^{-1} \right\|_{\mathbb{L}_{2p}} \left\| \eta_n^N(\bar{f}_{l+1,\xi_{l,n}^i}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^i}^N) \right\|_{\mathbb{L}_{2p}} \end{aligned} \quad (4.3)$$

The collection of random functions  $\bar{f}_{l+1,\xi_{l,n}^j}^N$  are well defined and we have

$$\begin{aligned} & \left( \eta_n^N(\bar{f}_{l+1,\xi_{l,n}^i}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^i}^N) \right)^\beta \\ & = \frac{1}{\eta_n^N(g_{l,\xi_{l,n}^i})} \frac{1}{N} \sum_{j=1}^N \left[ \eta_n^N(\bar{f}_{l+1,\xi_{l,n}^j}^N) - \eta_l^N M_{l,n}(\bar{f}_{l+1,\xi_{l,n}^j}^N) \right]^\beta \mathbf{1}_{\xi_{l,n}^j = \xi_{l,n}^i} \end{aligned}$$

for any  $\beta \geq 0$ . Combining the above formula for  $\beta = 2p$  and Holder's inequality, we prove that

$$\begin{aligned} & \left\| \eta_n^N(\bar{f}_{l+1, \xi_{l,n}^i}) - \eta_l^N M_{l,n}(\bar{f}_{l+1, \xi_{l,n}^i}) \right\|_{\mathbb{L}_{2p}} \\ & \leq \left\| \eta_l^N \left( g_{l, \xi_{l,n}^i} \right)^{-1} \right\|_{\mathbb{L}_q}^{1/(2p)} \times \sup_{1 \leq j \leq N} \left\| \eta_n^N(\bar{f}_{l+1, \xi_{l,l}^j}) - \eta_l^N M_{l,n}(\bar{f}_{l+1, \xi_{l,l}^j}) \right\|_{\mathbb{L}_{2pq'}} \end{aligned}$$

for any  $q, q' \geq 1$ , with  $\frac{1}{q} + \frac{1}{q'} = 1$ .

We observe that, as  $(\xi_{l,l}^j, (\xi_{l,l}^i)_{0 \leq i \leq N}, (\xi_{l,n}^i)_{0 \leq i \leq N})$  have the same distribution, for any  $1 \leq j \leq N$ , then for any function  $h$  and any  $1 \leq j, j' \leq N$  we have:

$$\mathbb{E} \left( h(\xi_{l,l}^j, (\xi_{l,l}^i)_{0 \leq i \leq N}, (\xi_{l,n}^i)_{0 \leq i \leq N}) \right) = \mathbb{E} \left( h(\xi_{l,l}^{j'}, (\xi_{l,l}^i)_{0 \leq i \leq N}, (\xi_{l,n}^i)_{0 \leq i \leq N}) \right)$$

which implies that

$$\sup_{1 \leq j \leq N} \left\| \eta_n^N(\bar{f}_{l+1, \xi_{l,l}^j}) - \eta_l^N M_{l,n}(\bar{f}_{l+1, \xi_{l,l}^j}) \right\|_{\mathbb{L}_{2pq'}} = \left\| \eta_n^N(\bar{f}_{l+1, \xi_{l,l}^1}) - \eta_l^N M_{l,n}(\bar{f}_{l+1, \xi_{l,l}^1}) \right\|_{\mathbb{L}_{2pq'}}.$$

As this equation works for any  $1 \leq j \leq N$ , in further development we take  $j = 1$  to simplify the notation.

Using Lemma 3.4, and recalling that  $\eta_l^N M_{l,n}(g_{l,x}) = \eta_l^N(g_{l,x})$ , for any  $1 \leq j \leq N$  we prove the almost sure estimate

$$\begin{aligned} & \sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_l^N M_{l,n}](\bar{f}_{l+1, \xi_{l,l}^1}) \right|^{2pq'} \middle| \mathcal{F}_l^N \right)^{\frac{1}{2pq'}} \\ & \leq 2 a(2pq')(n-l) \left[ \eta_l^N M_{l,n} \left( \left| \bar{f}_{l+1, \xi_{l,l}^1} \right|^{2pq'} \right) \right]^{\frac{1}{2pq'}} \\ & \leq 4 a(2pq')(n-l) \|f_{l+1}\| \left( \eta_l^N M_{l,n}(g_{l, \xi_{l,l}^1}) \right)^{\frac{1}{2pq'} - 1}. \end{aligned}$$

This yields that

$$\sqrt{N} \mathbb{E} \left( \left| [\eta_n^N - \eta_l^N M_{l,n}](\bar{f}_{l+1, \xi_{l,l}^1}) \right|^{2pq'} \middle| \mathcal{F}_l^N \right)^{\frac{1}{2pq'}} \leq 4 a(2pq')(n-l) \|f_{l+1}\| \eta_l^N(g_{l, \xi_{l,l}^1})^{-1}$$

and therefore

$$\begin{aligned} & \sqrt{N} \left\| \eta_n^N(\bar{f}_{l+1, \xi_{l,n}^i}) - \eta_l^N M_{l,n}(\bar{f}_{l+1, \xi_{l,n}^i}) \right\|_{\mathbb{L}_{2pq'}} \\ & \leq 4 a(2pq')(n-l) \|f_{l+1}\| \left\| \eta_l^N \left( g_{l, \xi_{l,n}^i} \right)^{-1} \right\|_{\mathbb{L}_q}^{1/(2p)} \left\| \eta_l^N(g_{l, \xi_{l,l}^1})^{-1} \right\|_{\mathbb{L}_{2pq'}} \end{aligned}$$

Finally, by (4.3), we conclude that

$$\begin{aligned} & \sqrt{N} \left\| \widehat{M}'_{l+1}(f)(\xi_{l,n}^i) - M'_{l+1}(f)(\xi_{l,n}^i) \right\|_{\mathbb{L}_p} \\ & \leq 4 a(2pq')(n-l) \|f_{l+1}\| \left\| \eta_n^N(g_{l,\xi_{l,n}^i})^{-1} \right\|_{\mathbb{L}_{2p}} \left\| \eta_l^N(g_{l,\xi_{l,n}^i})^{-1} \right\|_{\mathbb{L}_q}^{1/(2p)} \\ & \quad \times \left\| \eta_l^N(g_{l,\xi_{l,l}^1})^{-1} \right\|_{\mathbb{L}_{2pq'}}. \end{aligned}$$

We prove (3.21), by taking  $q = 1 + 2p$  and  $q' = 1 + 1/(2p)$  so that  $q = 2pq' \geq 2p$

$$\begin{aligned} & \sqrt{N} \left\| \widehat{M}'_{l+1}(f)(\xi_{l,n}^i) - M'_{l+1}(f)(\xi_{l,n}^i) \right\|_{\mathbb{L}_p} \\ & \leq 4 a(1+2p)(n-l) \|f_{l+1}\| \sup_{l \leq k \leq n} \left\| \eta_k^N(g_{l,\xi_{l,k}^1})^{-1} \right\|_{\mathbb{L}_{1+2p}}^{2+1/(2p)}. \end{aligned}$$

This end of proof is now a direct consequence of Lemma 3.5.  $\blacksquare$

## 4.2 Proof of Lemma 3.5:

We set

$$\delta_{l,n}(N) := \inf_{x \in E'_l} \eta_n^N(g_{l,x})$$

with the function  $g_{l,x}$  defined in (4.2), and we notice that

$$\mathbb{P}(\delta_{l,n}(N) = 0) \leq \sum_{x \in E'_l} \mathbb{P}(\eta_n^N(g_{l,x}) = 0).$$

On the other hand, for any  $\epsilon \in [0, 1)$  we have

$$\mathbb{P}(\eta_n^N(g_{l,x}) = 0) \leq \mathbb{P}(|\eta_n^N(g_{l,x}) - \eta_n(g_{l,x})| > \epsilon \eta_n(g_{l,x})).$$

Arguing as in (3.15), for any  $x \in E'_l$  s.t.  $\eta_n(g_{l,x}) (= \mathbb{P}(X'_l = x)) > 0$  we prove that

$$\sqrt{N} \mathbb{E} \left( |\eta_n^N(g_{l,x}) - \eta_n(g_{l,x})|^r \right)^{\frac{1}{r}} \leq 2 a(r) (n+1) \eta_n(g_{l,x})^{-1} \quad (4.4)$$

and therefore

$$\mathbb{P} \left( |\eta_n^N(g_{l,x}) - \eta_n(g_{l,x})| \geq \left( \frac{2(n+1)}{\sqrt{N}} + \epsilon \right) \eta_n(g_{l,x}) \right) \leq \exp \left( -\frac{N\epsilon^2}{8(n+1)^2} \right).$$

For any  $N \geq (2(n+1)/(1-\epsilon))^2$ , this implies that

$$\mathbb{P}(\delta_{l,n}(N) = 0) \leq \text{Card}(E'_l) \exp \left( -\frac{N\epsilon^2}{8(n+1)^2} \right).$$

If we choose,  $\epsilon = 1/2$  and  $N \geq (4(n+1))^2$ , we conclude that

$$\mathbb{P}(\delta_{l,n}(N) = 0) \leq \text{Card}(E'_l) \exp \left( -\frac{N}{32(n+1)^2} \right).$$

On the other hand, by construction we have the almost sure estimate

$$\eta_n^N(g_{l,\xi_{i,n}^i}) = \sum_{x \in E'_i} \eta_n^N(g_{l,x}) \mathbf{1}_{\xi_{i,n}^i=x} \geq \delta_{l,n}(N) \mathbf{1}_{\delta_{l,n}(N)>0} + \frac{1}{N} \mathbf{1}_{\delta_{l,n}(N)=0}$$

from which we find that

$$\eta_n^N(g_{l,\xi_{i,n}^i})^{-1} \leq \delta_{l,n}(N)^{-1} \mathbf{1}_{\delta_{l,n}(N)>0} + N \mathbf{1}_{\delta_{l,n}(N)=0}.$$

Therefore, we have

$$\begin{aligned} \left\| \eta_n^N(g_{l,\xi_{i,n}^i})^{-1} \right\|_{\mathbb{L}_p} &\leq \|\delta_{l,n}(N)^{-1} \mathbf{1}_{\delta_{l,n}(N)>0}\|_{\mathbb{L}_p} + N \|\mathbf{1}_{\delta_{l,n}(N)=0}\|_{\mathbb{L}_p} \\ &\leq \sum_{x \in E'_i} \|\eta_n^N(g_{l,x})^{-1} \mathbf{1}_{\eta_n^N(g_{l,x})>0}\|_{\mathbb{L}_p} + N \mathbb{P}(\delta_{l,n}(N) = 0)^{1/p}. \end{aligned}$$

If we set  $\bar{g}_{l,n}(x) = g_{l,x}/\eta_n(g_{l,x})$ , using the fact that

$$\frac{1}{1-u} = 1 + u + u^2 + \frac{u^3}{1-u}$$

for any  $u \neq 1$ , and  $\eta_n^N(\bar{g}_{l,x})^{-1} \mathbf{1}_{\eta_n^N(g_{l,x})>0} \leq N \eta_n(g_{l,x})$ , we find that

$$\eta_n^N(\bar{g}_{l,x})^{-1} \mathbf{1}_{\eta_n^N(g_{l,x})>0} \leq 1 + |1 - \eta_n^N(\bar{g}_{l,x})| + (1 - \eta_n^N(\bar{g}_{l,x}))^2 + N \eta_n(g_{l,x}) |1 - \eta_n^N(\bar{g}_{l,x})|^3.$$

Combining this estimate with (4.4), for any  $p \geq 1$  we prove the following upper bound

$$\begin{aligned} \|\eta_n^N(\bar{g}_{l,x})^{-1} \mathbf{1}_{\eta_n^N(g_{l,x})>0}\|_{\mathbb{L}_p} &\leq 1 + \frac{1}{\sqrt{N}} 2a(p)(n+1) + (2a(2p)(n+1))^2 \frac{1}{N} \\ &\quad + \frac{1}{\sqrt{N}} (2a(3p)(n+1))^3 \end{aligned}$$

from which we find the rather crude estimates

$$\|\eta_n^N(\bar{g}_{l,x})^{-1} \mathbf{1}_{\eta_n^N(g_{l,x})>0}\|_{\mathbb{L}_p} \leq 1 + \frac{3}{\sqrt{N}} a'(p) (n+1)^3$$

with the collection of finite constants  $a'(p) := 2a(p) + (2a(2p))^2 + (2a(3p))^3$ . Using the above exponential inequalities, we find that

$$\begin{aligned} &\left\| \eta_n^N(g_{l,\xi_{i,n}^i})^{-1} \right\|_{\mathbb{L}_p} \\ &\leq \sum_{x \in E'_i} \frac{1}{\eta_n(g_{l,x})} \left[ 1 + \frac{3}{\sqrt{N}} a'(p) (n+1)^3 \right] + N \text{Card}(E'_i)^{1/p} \exp\left(-\frac{N}{32p(n+1)^2}\right). \end{aligned}$$

This ends the proof of the lemma.  $\blacksquare$

## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Some deterministic approximation models</b>	<b>9</b>
2.1	Cut-off type models . . . . .	9
2.2	Euler approximation models . . . . .	10
2.3	Interpolation type models . . . . .	11
2.4	Quantization tree models . . . . .	14
<b>3</b>	<b>Monte Carlo approximation models</b>	<b>16</b>
3.1	Path space models . . . . .	16
3.2	Broadie-Glasserman models . . . . .	18
3.3	Genealogical tree based models . . . . .	22
3.3.1	Neutral genetic models . . . . .	22
3.3.2	Convergence analysis . . . . .	24
<b>4</b>	<b>Appendix</b>	<b>28</b>
4.1	Proof of Lemma 3.6: . . . . .	28
4.2	Proof of Lemma 3.5: . . . . .	30



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