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Clark-Ocone type formula for non-semimartingales with non-trivial quadratic variation

Formule de Clark-Ocone généralisée pour non-semimartingales à variation quadratique finie non-triviale

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Abstract

We provide a suitable framework for the concept of finite quadratic variation for processes with values in a separable Banach space B using the language of stochastic calculus via regularizations, introduced in the case $B = \mathbb{R}$ by the second author and P. Vallois. A special attention is devoted to the *window* process associated with a real finite quadratic variation process which takes naturally values in the Banach space $B = C([-τ, 0])$. An appropriated Itô formula is presented, from which we derive a generalized Clark-Ocone formula for non-semimartingales having the same quadratic variation as Brownian motion. The representation is based on solutions of an infinite dimensional PDE.

Résumé

Nous présentons un cadre adéquat pour le concept de variation quadratique finie lorsque le processus de référence est à valeurs dans un espace de Banach séparable B . Le langage utilisé est celui de l'intégrale via régularisations introduit dans le cas réel par le second auteur et P. Vallois. Une attention particulière est dédiée au cas où $B = C([-τ, 0])$ et le processus considéré est la *fenêtre* d'un processus réel à variation quadratique finie. Nous énonçons une formule d'Itô appropriée de laquelle nous déduisons une formule de Clark-Ocone relative à des non-semimartingales réelles ayant la même variation quadratique que le mouvement brownien. La représentation est basée sur des solutions d'une EDP infini-dimensionnelle.

Keywords: Calculus via regularization, Infinite dimensional analysis, Clark-Ocone formula, Itô formula, Quadratic variation, Hedging theory without semimartingales.

2010 MSC: 60H05, 60H07, 60H30, 91G80.

Version française abrégée

Dans cette Note nous développons un calcul stochastique via régularisation de type progressif (*forward*) lorsque le processus intégrateur est à valeurs dans un espace de Banach space B . Ceci est basé sur une notion sophistiquée de *variation quadratique* que nous appellerons χ -variation quadratique, où le symbole χ est référée à un sous-espace χ du dual de $B \hat{\otimes}_\pi B$. Le calcul via régularisation a été introduit lorsque $B = \mathbb{R}$ dans [12] et depuis il a été étudié par de nombreux auteurs qui ont fait avancer la théorie et ont produit plusieurs applications. Le lecteur peut consulter [13]

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pour une revue incluant une liste assez complète de références. Dans ce contexte, les auteurs introduisent une notion de covariation entre deux processus réels X et Y , notée $[X, Y]$ qui généralise le crochet droit usuel lorsque X et Y sont des semimartingales. Un vecteur de processus $\underline{X} = (X^1, \dots, X^n)$ est dit admettre tous ses crochets mutuels si $[X^i, X^j]$ existe pour tous entiers $1 \leq i, j \leq n$.

Lorsque $B = \mathbb{R}^n$, X possède une χ -quadratique variation avec $\chi = (B \hat{\otimes}_\pi B)^*$ si et seulement si X admet ses crochets mutuels. On peut voir qu'un processus à valeurs dans un espace de Banach *localement semi-sommable* X au sens de [6], admet une χ -variation quadratique avec $\chi = (B \hat{\otimes}_\pi B)^*$. Dans ce travail nous traçons une ébauche du calcul stochastique via la formule d'Itô énoncée au Théorème 5.1. Une attention spéciale est consacrée au cas où $B = C([- \tau, 0])$, pour un certain $\tau > 0$, qui est typiquement un espace de Banach non-réflexif, et à une formule de Clark-Ocone généralisée. Soit $T > 0$; tout processus réel continu $X = (X_t)_{t \in [0, T]}$ est prolongé par continuité pour $t \notin [0, T]$. Soit $0 < \tau \leq T$ et X un processus réel continu; nous appelons **fenêtre** le processus à valeurs dans $C([- \tau, 0])$ défini par

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [- \tau, 0], t \in [0, T]\}.$$

La théorie de l'intégration infini-dimensionnelle par rapport à des martingales (ou des semimartingales, [4, 10, 6]) n'est pas applicable, même lorsque l'intégrateur est la fenêtre $W(\cdot)$ associée au mouvement brownien standard W . En effet $W(\cdot)$ n'est d'aucune manière une semimartingale à valeurs dans $C([- \tau, 0])$.

Motivés par des applications liées à la couverture d'options dépendant de toute la trajectoire, nous discutons une formule de type Clark-Ocone visant à décomposer une classe significative h de v.a. dépendant de la trajectoire d'un processus X dont la variation quadratique vaut $[X]_t = t$. Ceci généralise des résultats inclus dans [14, 1, 3] visant à déterminer des formules de valorisation et de couverture d'options vanille ou asiatique dans un modèle de prix d'actif ayant la même variation quadratique que le modèle de Black-Scholes. Si le bruit dans un environnement stochastique est modélisé par la dérivée d'un mouvement brownien W , le théorème de représentation des martingales et la formule classique de Clark-Ocone sont deux outils de calcul fondamental. Le théorème 7.1 et les considérations à la fin de la section 7 montrent que dans une certaine mesure ceci reste valable lorsque la loi du processus sous-jacent n'est plus la mesure de Wiener mais le processus conserve la même variation quadratique que W . In est en fait possible de représenter des variables aléatoires $h = H(X_T(\cdot))$, où $H : C([- T, 0]) \rightarrow \mathbb{R}$, comme

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{0.1}$$

sous des conditions suffisantes raisonnables sur la fonctionnelle H , où H_0 est un nombre réel et ξ est un processus non-anticipatif qui sont donnés de façon quasi-explicite. Ces quantités sont exprimées à l'aide d'une fonctionnelle $u : [0, T] \times C([- T, 0]) \rightarrow \mathbb{R}$ de classe $C^{1,2}([0, T] \times C([- T, 0]))$, qui est solution d'une équation aux dérivées partielles; plus précisément la représentation (0.1) de h a lieu avec $H_0 = u(0, X_0(\cdot))$ et $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$, où $D^{\delta_0} u(t, \eta)$ symbolise la projection de la dérivée de Fréchet $Du(t, \eta)$ sur le sous espace vectoriel constitué par les multiples de la mesure de Dirac δ_0 ; en particulier nous avons $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\{0\})$.

Si X est un mouvement brownien standard W et $h \in \mathbb{D}^{1,2}$, l'expression (0.1) coïncide avec la formule de Clark-Ocone classique. Ces considérations montrent que la formule de Clark-Ocone classique admet une certaine robustesse relativement à la mesure de Wiener qui est la loi de $W(\cdot)$.

1. Introduction

In the whole paper $(\Omega, \mathbb{F}, \mathbb{P})$ is a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ fulfilling the usual conditions, B will be a separable Banach space and X a B -valued process. W will always denote an (\mathcal{F}_t) -real Brownian motion. Let $T > 0$ be a fixed maturity time. All the real processes $X = (X_t)_{t \in [0, T]}$ are prolonged by continuity for $t \notin [0, T]$ setting $X_t = X_0$ for $t \leq 0$ and $X_t = X_T$ for $t \geq T$.

We first recall the basic concepts of forward integral and covariation and some one-dimensional results concerning calculus via regularization, a fairly complete survey on the subject being [13]. For simplicity, all the considered integrator processes will be continuous.

Definition 1.1. Let X (respectively Y) be a continuous (resp. locally integrable) process.

The **forward integral of Y with respect to X** (resp. the **covariation of X and Y**), whenever it exists, is defined as

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \rightarrow 0^+} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \quad \text{in probability for all } t \in [0, T], \quad (1.1)$$

$$\left(\text{resp. } [X, Y]_t = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds \quad \text{in the ucp sense with respect to } t \right), \quad (1.2)$$

provided that the limiting process admits a continuous version. If $\int_0^t Y_s d^- X_s$ exists for any $0 \leq t < T$, $\int_0^T Y_s d^- X_s$ will symbolize the **improper forward integral** defined by $\lim_{t \rightarrow T} \int_0^t Y_s d^- X_s$, whenever it exists in probability.

If $[X, X]$ exists then X is said to be a **finite quadratic variation** process. $[X, X]$ will also be denoted by $[X]$ and it will be called **quadratic variation of X** . If $[X] = 0$, then X is said to be a **zero quadratic variation process**. If $X = (X^1, \dots, X^n)$ is a vector of continuous processes we say that it has all its **mutual covariations** (brackets) if $[X^i, X^j]$ exists for any $1 \leq i, j \leq n$.

When X is a (continuous) semimartingale (resp. Brownian motion) and Y is a cadlag adapted process (resp. such that $\int_0^T Y_s^2 ds < \infty$ a.s.), the integral $\int_0^t Y_s d^- X_s$ exists and coincides with classical Itô's integral $\int_0^t Y_s dX_s$, see Proposition 6 in [13]. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when Y is anticipating or X is not a semimartingale. If X, Y are (\mathcal{F}_t) -semimartingales then $[X, Y]$ coincides with the classical bracket $\langle X, Y \rangle$, see Corollary 2 in [13]. Finite quadratic variation processes will play a central role in this note: this class includes of course all (\mathcal{F}_t) -semimartingales. However that class is much richer. Typical examples of finite quadratic variation processes are (\mathcal{F}_t) -Dirichlet processes. D is called (\mathcal{F}_t) -Dirichlet process if it admits a decomposition $D = M + A$ where M is an (\mathcal{F}_t) -local martingale and A is a zero quadratic variation process. It holds in that case $[D] = [M]$. This class of processes generalizes the semimartingales since a locally bounded variation process has zero quadratic variation. A slight generalization of that notion is the notion of weak Dirichlet, which was introduced in [7]. X is called (\mathcal{F}_t) -weak Dirichlet if it admits a decomposition $X = M + A$ where M is an (\mathcal{F}_t) local martingale and A is a process such that $[A, N] = 0$ for any continuous (\mathcal{F}_t) local martingale N . An (\mathcal{F}_t) -weak Dirichlet process is not necessarily a finite quadratic variation process. On the other hand if A has finite quadratic variation then it holds $[X] = [M] + [A]$. Another interesting example is the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0, 1)$ and $K \in (0, 1]$ which has finite quadratic variation if and only if $HK \geq 1/2$, see [11]. Notice that if $K = 1$, then $B^{H,1}$ coincides with a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. If $HK = 1/2$ it holds $[B^{H,K}] = 2^{1-K}t$; if $K \neq 1$ this process is not even a Dirichlet with respect to its own filtration. Other significant examples are the so-called weak k -order Brownian motions, for fixed $k \geq 1$, constructed by [8], which are in general not Gaussian. X is a weak k -order Brownian motion if for every $0 \leq t_1 \leq \dots \leq t_k < +\infty$, $(X_{t_1}, \dots, X_{t_k})$ is distributed as $(W_{t_1}, \dots, W_{t_k})$. If $k \geq 4$ then $[X]_t = t$.

One central object of this work will be the generalization to infinite dimensional valued processes of the stochastic integral via regularization, see Definition 3.1. A stochastic calculus for Banach valued martingales was considered by [2, 15] and references therein generalizing the classical stochastic calculus of [4, 10, 6].

We introduce now a particular Banach valued process. Given $0 < \tau \leq T$ and a real continuous process X , we will call **window process** the $C([- \tau, 0])$ -valued process denoted by $X(\cdot)$ defined as

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [- \tau, 0], t \in [0, T]\}.$$

We observe that even the window Brownian motion $W(\cdot)$ is not a $C([-\tau, 0])$ -valued semimartingale, in fact $\mathcal{M}([-\tau, 0])\langle \mu, W(\cdot) \rangle_{C([-\tau, 0])}$ is even not a real semimartingale for $\mu = \delta_0 + \delta_{\tau/2}$. Motivated by the necessity of an Itô formula available also for $B = C([-\tau, 0])$ -valued processes, we introduce a quadratic variation concept which depends on a subspace χ of the dual of the tensor square of B , equipped with the projective topology, denoted by $(B\hat{\otimes}_\pi B)^*$, see Definition 4.3. We recall the fundamental identification $(B\hat{\otimes}_\pi B)^* \cong \mathcal{B}(B \times B)$, which denotes the space of \mathbb{R} -valued bounded bilinear forms on $B \times B$. An Itô formula for processes admitting a χ -quadratic variation is given in Theorem 5.1. After formulating a theory for B -valued processes with general B , in Sections 6 and 7 we fix the attention on window processes setting $B = C([-\tau, 0])$. Section 6 is devoted to the evaluation of χ -quadratic variation for window processes; in particular Proposition 6.4 gives the principal results concerning the χ -quadratic variation for windows of real finite quadratic variation processes. In Section 7 we give a representation result for random variable $h := H(X_T(\cdot))$ of the type $h = H_0 + \int_0^T \xi_s d^- X_s$ where $H : C([-T, 0]) \rightarrow \mathbb{R}$ and the integral has to be considered as a forward integral as defined in (1.1). In general when H is continuous and X is a general finite quadratic variation process such that $[X]_t = t$ this representation will be obtained by expressing h as $u(T, X_T(\cdot))$ where $u \in C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$ solves an infinite dimensional partial differential equation of type (7.5). In this case we obtain

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s ,$$

where $D^{\delta_0} u(s, \eta)$ denotes the *projection* of the Fréchet derivative $Du(t, \eta)$ on the linear space generated by Dirac measure δ_0 , i.e. $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\{0\})$.

2. Notations

Symbol $\mathcal{C}([0, T])$ denotes the linear space of continuous real processes equipped with the ucp (uniformly convergence in probability) topology, B^* will be the topological dual of the Banach space B . We introduce now some subspaces of measures that we will frequently use. Symbol $\mathcal{D}_0([-\tau, 0])$ (resp. $\mathcal{D}_{0,0}([-\tau, 0]^2)$) will denote the one dimensional Hilbert space of the multiples of Dirac measure concentrated at 0 (resp. at $(0, 0)$), i.e.

$$\mathcal{D}_0([-\tau, 0]) := \{\mu \in \mathcal{M}([-\tau, 0]); s.t. \mu(dx) = \lambda \delta_0(dx) \text{ with } \lambda \in \mathbb{R}\} \quad (2.1)$$

and

$$\mathcal{D}_{0,0}([-\tau, 0]^2) := \{\mu \in \mathcal{M}([-\tau, 0]^2); s.t. \mu(dx, dy) = \lambda \delta_0(dx) \delta_0(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_0 \hat{\otimes}_h \mathcal{D}_0 \quad (2.2)$$

where $\hat{\otimes}_h$ stands for the Hilbert tensor product.

Symbol $Diag([-\tau, 0]^2)$ (shortly $Diag$) will denote the subset of $\mathcal{M}([-\tau, 0]^2)$ defined as follows:

$$Diag([-\tau, 0]^2) := \left\{ \mu \in \mathcal{M}([-\tau, 0]^2) \text{ s.t. } \mu(dx, dy) = g(x) \delta_y(dx) dy; g \in L^\infty([-\tau, 0]) \right\} . \quad (2.3)$$

$Diag([-\tau, 0]^2)$, equipped with the norm $\|\mu\|_{Diag([-\tau, 0]^2)} = \|g\|_\infty$, is a Banach space.

3. Calculus via regularization

In this section we introduce an infinite dimensional stochastic integral via regularization. In this construction there are two main difficulties. The integrator is generally not a semimartingale or the integrand may be anticipative; B is a general non-reflexive Banach space.

Definition 3.1. Let $(X_t)_{t \in [0, T]}$ (respectively $(Y_t)_{t \in [0, T]}$) be a B -valued (respectively a B^* -valued) stochastic process. For simplicity we suppose X to be continuous and $\int_0^T \|Y_s\|_{B^*} ds < +\infty$ a.s.

For every fixed $t \in [0, T]$ we define the **definite forward integral of Y with respect to X** denoted by $\int_0^t \langle Y_s, d^- X_s \rangle_B$ as the following limit in probability:

$$\int_0^t \langle Y_s, d^- X_s \rangle_B := \lim_{\epsilon \rightarrow 0} \int_0^t \langle Y(s), \frac{X(s+\epsilon) - X(s)}{\epsilon} \rangle_B ds.$$

We say that the **forward stochastic integral of Y with respect to X** exists if the process

$$\left(\int_0^t \langle Y_s, d^- X_s \rangle_B \right)_{t \in [0, T]}$$

admits a continuous version. In the sequel indices B and B^* will often be omitted.

4. Chi-quadratic variation

Definition 4.1. A closed linear subspace χ of $(B \hat{\otimes}_\pi B)^*$, endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B \hat{\otimes}_\pi B)^*} \tag{4.1}$$

will be called a **Chi-subspace of $(B \hat{\otimes}_\pi B)^*$** .

Let χ be a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$, X be a B -valued stochastic process and $\epsilon > 0$. We denote by $[X]^\epsilon$, the application

$$[X]^\epsilon : \chi \longrightarrow \mathcal{C}([0, T]) \quad \text{defined by} \quad \phi \mapsto \left(\int_0^t \chi \langle \phi, \frac{J((X_{s+\epsilon} - X_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right)_{t \in [0, T]} \tag{4.2}$$

where $J : B \hat{\otimes}_\pi B \longrightarrow (B \hat{\otimes}_\pi B)^{**}$ denotes the canonical injection between a space and its bidual.

With application $[X]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[X]}^\epsilon$, defined, for $\omega \in \Omega$, by

$$\widetilde{[X]}^\epsilon(\omega, \cdot) : [0, T] \longrightarrow \chi^* \quad \text{given by} \quad t \mapsto \left(\phi \mapsto \int_0^t \chi \langle \phi, \frac{J((X_{s+\epsilon} - X_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right). \tag{4.3}$$

Remark 4.2.

1. We recall that $\chi \subset (B \hat{\otimes}_\pi B)^*$ implies $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$.
2. As indicated, $\chi \langle \cdot, \cdot \rangle_{\chi^*}$ denotes the duality between the space χ and its dual χ^* ; in fact, by assumption, ϕ is an element of χ and element $J((X_{s+\epsilon} - X_s) \otimes^2)$ naturally belongs to $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$.
3. With a slight abuse of notation, in the sequel application J will be omitted. The tensor product $(X_{s+\epsilon} - X_s) \otimes^2$ has to be considered as the element $J((X_{s+\epsilon} - X_s) \otimes^2)$ which belongs to χ^* .

We give now the definition of the χ -quadratic variation of a B -valued stochastic process X .

Definition 4.3. Let χ be a separable Chi-subspace of $(B \hat{\otimes}_\pi B)^*$ and X a B -valued stochastic process. We say that X **admits a χ -quadratic variation** if the following assumptions are fulfilled.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \chi \left\langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s) \otimes^2}{\epsilon_{n_k}} \right\rangle_{\chi^*} \right| ds < +\infty, \text{ a.s.}$$

H2 It exists an application denoted by $[X] : \chi \rightarrow \mathcal{C}([0, T])$, such that

$$[X]^\epsilon(\phi) \xrightarrow[\epsilon \rightarrow 0_+]{ucp} [X](\phi)$$

for all $\phi \in \mathcal{S}$, where $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

Definition 4.4. When X admits a χ -quadratic variation, the χ^* -valued measurable process $(\widetilde{[X]})_{0 \leq t \leq T}$ defined for almost all $\omega \in \Omega$ and $t \in [0, T]$ by $\phi \mapsto \widetilde{[X]}(\omega, t)(\phi) = [X](\phi)(\omega, t)$, is called χ -**quadratic variation** of X . Sometimes, with a slight abuse of notation, even $[X]$ will be called χ -quadratic variation and it will be confused with $\widetilde{[X]}$.

Remark 4.5.

1. We observe that for almost all $\omega \in \Omega$, the χ^* -valued process $\widetilde{[X]}(\omega, \cdot)$ has bounded variation on $[0, T]$.
2. The existence of $\widetilde{[X]}$ guarantees that $[X]$ admits a proper version which allows to consider it as pathwise integral.

Definition 4.6. We say that a continuous B -valued process X admits **global quadratic variation** if it admits a χ -quadratic variation with $\chi = (B \hat{\otimes}_\pi B)^*$.

When B is the finite dimensional space \mathbb{R}^n , X admits all its mutual brackets if and only if X admits a global quadratic variation. A Banach valued *locally semi summable* process X in the sense of [6], admits again a global quadratic variation, see Section 6 in [5] for more details.

We observe an interesting feature in the case $\chi = (B \hat{\otimes}_\pi B)^*$. The natural convergence topology is the weak star convergence in the space $(B \hat{\otimes}_\pi B)^{**}$ for elements $\widetilde{[X]}^\epsilon$. In fact for any $t \in [0, T]$, it exists a null subset N of Ω such and a sequence (ϵ_n) such that

$$\widetilde{[X]}^{\epsilon_n}(\omega, t) \xrightarrow[\epsilon \rightarrow 0]{} \widetilde{[X]}(\omega, t)$$

weak star for all $\omega \notin N$. We recall that $J(B \hat{\otimes}_\pi B)$ is weak star dense in $(B \hat{\otimes}_\pi B)^{**}$, so $\widetilde{[X]}$ takes values “a priori” in $(B \hat{\otimes}_\pi B)^{**}$. The weak star convergence is weaker then the strong convergence in $B \hat{\otimes}_\pi B$, i.e. the convergence with respect to the topology defined by the norm. If the Banach space $B \hat{\otimes}_\pi B$ is not reflexive, then $(B \hat{\otimes}_\pi B)^{**}$ strictly contains $B \hat{\otimes}_\pi B$. In general $B \hat{\otimes}_\pi B$ is not reflexive even if B is a Hilbert space.

5. Itô's formula

We are now able to state an Itô formula for stochastic processes with values in a general Banach space. Its proof is given in Section 8 of [5].

Theorem 5.1. Let B be a separable Banach space, χ be a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$ and X a B -valued continuous process admitting a χ -quadratic variation. Let $F : [0, T] \times B \rightarrow \mathbb{R}$ of class $C^{1,2}$ Fréchet. such that

$$D^2 F : [0, T] \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^* \text{ continuously with respect to } \chi. \quad (5.1)$$

Then the forward integral

$$\int_0^t \int_{B^*} \langle DF(s, X_s), d^- X_s \rangle_B, \quad t \in [0, T],$$

exists and the following formula holds

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \langle DF(s, X_s), d^- X_s \rangle_B + \frac{1}{2} \int_0^t \chi \langle D^2 F(s, X_s), d[\widetilde{X}]_s \rangle_{\chi^*} a.s. \quad (5.2)$$

6. Evaluation of χ -quadratic variations for window processes

From this section we fix B as the Banach space $C([- \tau, 0])$. First we give some evaluations of χ -quadratic variations of window processes $X(\cdot)$. Examples of Chi-subspaces of $(B \widehat{\otimes}_\pi B)^*$ are

- the space of measures on $[- \tau, 0]^2$ denoted by $\mathcal{M}([- \tau, 0]^2)$;
- the space $L^2([- \tau, 0]^2)$;
- the Hilbert tensor product $L^2([- \tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([- \tau, 0])$;
- the space $\mathcal{D}_{0,0}([- \tau, 0]^2)$;
- the space $Diag([- \tau, 0]^2)$;
- the space $\chi^0([- \tau, 0]^2)$, (shortly χ^0), will denote the subset of $\mathcal{M}([- \tau, 0]^2)$ defined by

$$\chi^0([- \tau, 0]^2) := (\mathcal{D}_0([- \tau, 0]) \oplus L^2([- \tau, 0])) \widehat{\otimes}_h^2 \quad (6.1)$$

which equals

$$\chi^0([- \tau, 0]^2) = L^2([- \tau, 0]^2) \oplus L^2([- \tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([- \tau, 0]) \oplus \mathcal{D}_0([- \tau, 0]) \widehat{\otimes}_h L^2([- \tau, 0]) \oplus \mathcal{D}_{0,0}([- \tau, 0]^2) . \quad (6.2)$$

Proposition 6.1. Let X be a real valued process with Hölder continuous paths of parameter $\gamma > 1/2$. Then $X(\cdot)$ admits a zero global quadratic variation.

Example 6.2. Examples of real processes with Hölder continuous paths of parameter $\gamma > 1/2$ are fractional window Brownian motion $B^H(\cdot)$ with $H > 1/2$ or a bifractional window Brownian motion $B^{H,K}(\cdot)$ with $KH > 1/2$.

Remark 6.3. Presumably the window Brownian motion $W(\cdot)$ does not admit a global (not even a $\mathcal{M}([-T, 0]^2)$ -) quadratic variation because Condition **H1** should not be verified. In fact it is possible to show that

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^T \frac{1}{\epsilon} \|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|_{C([-T, 0])}^2 ds \right] = +\infty . \quad (6.3)$$

(6.3) follows taking into account the fact that for $0 < \tau_1 < \tau_2$, there are positive constants C_1, C_2 such that

$$C_1 \leq \mathbb{E} \left[\sup_{s \in [\tau_1, \tau_2]} \frac{|W_{s+\epsilon} - W_s|^2}{\epsilon \ln(1/\epsilon)} \right] \leq C_2 .$$

For general finite quadratic variation processes we have the following.

Proposition 6.4. Let X be a real continuous process with finite quadratic variation and $0 < \tau \leq T$. The following properties hold true.

- 1) $X(\cdot)$ admits zero $L^2([- \tau, 0]^2)$ -quadratic variation.
- 2) $X(\cdot)$ admits zero $L^2([- \tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([- \tau, 0])$ -quadratic variation.

- 3) $X(\cdot)$ admits zero $\mathcal{D}_0([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0])$ -quadratic variation.
4) $X(\cdot)$ admits a $\mathcal{D}_{0,0}([-\tau, 0]^2)$ -quadratic variation given by

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \mathcal{D}_{0,0}([-\tau, 0]^2). \quad (6.4)$$

- 5) $X(\cdot)$ admits a $\chi^0([-\tau, 0]^2)$ -quadratic variation which equals

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \chi^0([-\tau, 0]^2). \quad (6.5)$$

- 6) $X(\cdot)$ admits a *Diag*-quadratic variation given by

$$\mu \mapsto [X(\cdot)]_t(\mu) = \int_0^{t \wedge \tau} g(-x)[X]_{t-x} dx \quad t \in [0, T], \quad (6.6)$$

where μ is a generic element in $\text{Diag}([-\tau, 0]^2)$ of type $\mu(dx, dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau, 0])$.

Remark 6.5. We remark that in the treated cases, the quadratic variation $[X]$ of the real finite quadratic variation process X insures the existence of (and completely determines) the χ -quadratic variation. For example if X is a real finite quadratic variation process such that $[X]_t = t$, then $X(\cdot)$ has the same χ^0 -quadratic variation as the window Brownian motion.

We conclude this section spending some words on the stability of processes admitting a quadratic variation even if this will not be the object of this paper. The class of real semimartingales with respect to a given filtration is known to be stable with respect to $C^2(\mathbb{R})$ transformations and finite quadratic variation processes are stable under $C^1(\mathbb{R})$ transformations. Also Dirichlet processes are stable with respect to $C^1(\mathbb{R})$ transformations and they admit a decomposition result. In some applications, in particular to control theory, one often needs to know the nature of process $(F(t, X_t))$ where $F \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R})$ and X is a real continuous (\mathcal{F}_t) -weak Dirichlet process with finite quadratic variation. It was shown in [9], Proposition 3.10, that $(F(t, X_t))$ is an (\mathcal{F}_t) -weak Dirichlet process. Analogous stability results are obtained in the infinite dimensional framework for the $C([-\tau, 0])$ -valued window Dirichlet and weak Dirichlet processes, see Theorems 7.33 and 7.36 in [5].

7. A generalized Clark-Ocone formula

In this section we will consider $\tau = T$ and we recall that $B = C([-T, 0])$. Let X be a real continuous stochastic process such that $X_0 = 0$ and $[X]_t = t$. Let $H : C([-T, 0]) \rightarrow \mathbb{R}$ be a Borel functional; we aim at representing the random variable

$$h = H(X_T(\cdot)). \quad (7.1)$$

The main task will consist in looking for classes of functionals H for which there is $H_0 \in \mathbb{R}$ and a predictable process ξ with respect to the canonical filtration of X such that h admits the representation

$$h = H_0 + \int_0^T \xi_s d^- X_s. \quad (7.2)$$

Moreover we look for an explicit expression for H_0 and ξ . As a consequence of Itô's formula (5.2) for path dependent functionals of the process we will observe that, in those cases, it is possible to find a function u which solves an infinite dimensional PDE and which gives at the same time the representation result (7.2). One possible representation is the following.

Theorem 7.1. Let $H : C([-T, 0]) \rightarrow \mathbb{R}$ be a Borel functional. Let $u \in C^{1,2}([0, T] \times C([-T, 0])) \cap C^0([0, T] \times C([-T, 0]))$ such that $x \mapsto D_x^{ac} u(t, \eta)$ has bounded variation, for any $t \in [0, T]$, $\eta \in C([-T, 0])$. We suppose moreover that $D^2 u(t, \eta) \in \mathcal{X}^0([-T, 0]^2)$ continuously for every $t \in [0, T]$, $\eta \in C([-T, 0])$. Suppose that u is a solution of

$$\begin{cases} \partial_t u(t, \eta) + \int_{-t, 0] D^{ac} u(t, \eta) d\eta + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.3)$$

where the integral $\int_{-t, 0] D^{ac} u(t, \eta) d\eta$ has to be understood via an integration by parts as follows:

$$\int_{-t, 0] D^{ac} u(t, \eta) d\eta = D^{ac} u(0, \eta)\eta(0) - D^{ac} u(-t, \eta)\eta(-t) - \int_{-t, 0] \eta(x) D_{dx}^{ac} u(t, \eta) .$$

Let X be a real continuous finite quadratic variation process with $[X]_t = t$ and $X_0 = 0$. Then the random variable $h := H(X_T(\cdot))$ admits the following representation

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t . \quad (7.4)$$

□

Sections 9.8 and 9.9 in [5] provide different reasonable conditions on $H : C([-T, 0]) \rightarrow \mathbb{R}$ such that there is a function u solving PDE (7.3) in general situations, i.e. fulfilling the hypotheses of Theorem 7.1. Section 9.8 considers an $L^2([-T, 0])$ -regularity on $H : C([-T, 0]) \subset L^2([-T, 0]) \rightarrow \mathbb{R}$ and Section 9.9 considers a non smooth but $L^2([-T, 0])$ -finitely based functionals H .

Remark 7.2. In relation to Theorem 7.1 we observe the following.

- Only pathwise considerations intervene and there is no need to suppose that the law of X is Wiener measure.
- Since $H(\eta) = u(T, \eta)$, we observe that H is automatically continuous by hypothesis $u \in C^0([0, T] \times C([-T, 0]))$.
- Let us suppose $X = W$.
 1. Making use of probabilistic technology, (7.4) holds in some cases even if H is not continuous and $h \notin L^1(\Omega)$; we refer to Section 9.6 in [5] for this type of results.
 2. If $\int_0^T \xi_s^2 ds < +\infty$ a.s., then the forward integral $\int_0^T \xi_t d^- W_t$ coincides with the Itô integral $\int_0^T \xi_t dW_t$.
 3. If the r.v. $h = H(W_T(\cdot))$ belongs to $\mathbb{D}^{1,2}$, by uniqueness of the martingale representation theorem and point 2., we have $H_0 = \mathbb{E}[h]$ and $\xi_t = \mathbb{E}[D_t^m h | \mathcal{F}_t]$, where D^m is the Malliavin gradient; this agrees with Clark-Ocone formula.
- If X is not a Brownian motion, in general $H_0 \neq \mathbb{E}[h]$ since $\mathbb{E}\left[\int_0^T \xi_t d^- X_t\right]$ does not generally vanish. In fact $\mathbb{E}[h]$ will specifically depend on the unknown law of X .

In chapter 9 in [5] we enlarge the discussion presented in Theorem 7.1. We can give examples where $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$ of class $C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$ such that (7.4) holds and u solves an infinite dimensional PDE of the type

$$\begin{cases} \partial_t u(t, \eta) + \int_{-t}^0 D^{ac} u(t, \eta) d\eta'' + \frac{1}{2} \langle D^2 u(t, \eta), \mathbb{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.5)$$

where $\mathbb{1}_D(x, y) := \begin{cases} 1 & \text{if } x = y, x, y \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}$ and $D^{ac}u(t, \eta)$ is the absolute continuous part of the measure $Du(t, \eta)$.

The integral “ $\int_{-t}^0 D^{ac}u(t, \eta) d\eta$ ” has to be suitably defined and term $\langle D^2u(t, \eta), \mathbb{1}_D \rangle$ indicates the evaluation of the second order derivative on the diagonal of the square $[-T, 0]^2$.

We observe that solution of (7.3) are also solutions of (7.5) since $\langle D^2u(t, \eta), \mathbb{1}_D \rangle = D^2u(t, \eta)(\{0, 0\})$.

- [1] Bender, C., Sottinen, T., Valkeila, E., 2008. Pricing by hedging and no-arbitrage beyond semimartingales. *Finance Stoch.* 12 (4), 441–468.
- [2] Brzeźniak, Z., 1995. Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.* 4 (1), 1–45.
- [3] Coviello, R., Russo, F., 2006. Modeling financial assets without semimartingales. Preprint <http://arxiv.org/abs/math.PR/0606642>.
- [4] Da Prato, G., Zabczyk, J., 1992. Stochastic equations in infinite dimensions. Vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.
- [5] Di Girolami, C., Russo, F., 2010. Infinite dimensional stochastic calculus via regularization and applications. HAL-INRIA, Preprint <http://hal.archives-ouvertes.fr/inria-00473947/fr/>.
- [6] Dinculeanu, N., 2000. Vector integration and stochastic integration in Banach spaces. *Pure and Applied Mathematics (New York)*. Wiley-Interscience, New York.
- [7] Errami, M., Russo, F., 2003. n -covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.* 104 (2), 259–299.
- [8] Föllmer, H., Wu, C.-T., Yor, M., 2000. On weak Brownian motions of arbitrary order. *Ann. Inst. H. Poincaré Probab. Statist.* 36 (4), 447–487.
- [9] Gozzi, F., Russo, F., 2006. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Processes and their Applications* 116 (11), 1563 – 1583.
- [10] Métivier, M., Pellaumail, J., 1980. *Stochastic integration*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, probability and Mathematical Statistics.
- [11] Russo, F., Tudor, C. A., 2006. On bifractional Brownian motion. *Stochastic Processes and their Applications* 116 (5), 830 – 856.
- [12] Russo, F., Vallois, P., 1991. Intégrales progressive, rétrograde et symétrique de processus non adaptés. *C. R. Acad. Sci. Paris Sér. I Math.* 312 (8), 615–618.
- [13] Russo, F., Vallois, P., 2007. Elements of stochastic calculus via regularization. In: *Séminaire de Probabilités XL*. Vol. 1899 of *Lecture Notes in Math*. Springer, Berlin, pp. 147–185.
- [14] Schoenmakers, J. G. M., Kloeden, P. E., 1999. Robust option replication for a Black-Scholes model extended with nondeterministic trends. *J. Appl. Math. Stochastic Anal.* 12 (2), 113–120.
- [15] van Neerven, J. M. A. M., Veraar, M. C., Weis, L., 2007. Stochastic integration in UMD Banach spaces. *Ann. Probab.* 35 (4), 1438–1478.