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# Clark-Ocone type formula for non-semimartingales with non-trivial quadratic variation

## Formule de Clark-Ocone généralisée pour non-semimartingales à variation quadratique finie non-triviale

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### Abstract

We provide a suitable framework for the concept of finite quadratic variation for processes with values in a separable Banach space  $B$  using the language of stochastic calculus via regularizations, introduced in the case  $B = \mathbb{R}$  by the second author and P. Vallois. A special attention is devoted to the *window* process associated with a real finite quadratic variation process which takes naturally values in the Banach space  $B = C([-τ, 0])$ . An appropriated Itô formula is presented, from which we derive a generalized Clark-Ocone formula for non-semimartingales having the same quadratic variation as Brownian motion. The representation is based on solutions of an infinite dimensional PDE.

### Résumé

Nous présentons un cadre adéquat pour le concept de variation quadratique finie lorsque le processus de référence est à valeurs dans un espace de Banach séparable  $B$ . Le langage utilisé est celui de l'intégrale via régularisations introduit dans le cas réel par le second auteur et P. Vallois. Une attention particulière est dédiée au cas où  $B = C([-τ, 0])$  et le processus considéré est la *fenêtre* d'un processus réel à variation quadratique finie. Nous énonçons une formule d'Itô appropriée de laquelle nous déduisons une formule de Clark-Ocone relative à des non-semimartingales réelles ayant la même variation quadratique que le mouvement brownien. La représentation est basée sur des solutions d'une EDP infini-dimensionnelle.

*Keywords:* Calculus via regularization, Infinite dimensional analysis, Clark-Ocone formula, Itô formula, Quadratic variation, Hedging theory without semimartingales.

*2010 MSC:* 60H05, 60H07, 60H30, 91G80.

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### Version française abrégée

Dans cette Note nous développons un calcul stochastique via régularisation de type progressif (*forward*) lorsque le processus intégrateur est à valeurs dans un espace de Banach space  $B$ . Ceci est basé sur une notion sophistiquée de *variation quadratique* que nous appellerons  $\chi$ -variation quadratique, où le symbole  $\chi$  est référée à un sous-espace  $\chi$  du dual de  $B \hat{\otimes}_\pi B$ . Le calcul via régularisation a été introduit lorsque  $B = \mathbb{R}$  dans [12] et depuis il a été étudié par de nombreux auteurs qui ont fait avancer la théorie et ont produit plusieurs applications. Le lecteur peut consulter [13]

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pour une revue incluant une liste assez complète de références. Dans ce contexte, les auteurs introduisent une notion de covariation entre deux processus réels  $X$  et  $Y$ , notée  $[X, Y]$  qui généralise le crochet droit usuel lorsque  $X$  et  $Y$  sont des semimartingales. Un vecteur de processus  $\underline{X} = (X^1, \dots, X^n)$  est dit admettre tous ses crochets mutuels si  $[X^i, X^j]$  existe pour tous entiers  $1 \leq i, j \leq n$ .

Lorsque  $B = \mathbb{R}^n$ ,  $X$  possède une  $\chi$ -quadratique variation avec  $\chi = (B \hat{\otimes}_\pi B)^*$  si et seulement si  $X$  admet ses crochets mutuels. On peut voir qu'un processus à valeurs dans un espace de Banach *localement semi-sommable*  $X$  au sens de [6], admet une  $\chi$ -variation quadratique avec  $\chi = (B \hat{\otimes}_\pi B)^*$ . Dans ce travail nous traçons une ébauche du calcul stochastique via la formule d'Itô énoncée au Théorème 5.1. Une attention spéciale est consacrée au cas où  $B = C([- \tau, 0])$ , pour un certain  $\tau > 0$ , qui est typiquement un espace de Banach non-réflexif, et à une formule de Clark-Ocone généralisée. Soit  $T > 0$ ; tout processus réel continu  $X = (X_t)_{t \in [0, T]}$  est prolongé par continuité pour  $t \notin [0, T]$ . Soit  $0 < \tau \leq T$  et  $X$  un processus réel continu; nous appelons **fenêtre** le processus à valeurs dans  $C([- \tau, 0])$  défini par

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [- \tau, 0], t \in [0, T]\}.$$

La théorie de l'intégration infini-dimensionnelle par rapport à des martingales (ou des semimartingales, [4, 10, 6]) n'est pas applicable, même lorsque l'intégrateur est la fenêtre  $W(\cdot)$  associée au mouvement brownien standard  $W$ . En effet  $W(\cdot)$  n'est d'aucune manière une semimartingale à valeurs dans  $C([- \tau, 0])$ .

Motivés par des applications liées à la couverture d'options dépendant de toute la trajectoire, nous discutons une formule de type Clark-Ocone visant à décomposer une classe significative  $h$  de v.a. dépendant de la trajectoire d'un processus  $X$  dont la variation quadratique vaut  $[X]_t = t$ . Ceci généralise des résultats inclus dans [14, 1, 3] visant à déterminer des formules de valorisation et de couverture d'options vanille ou asiatique dans un modèle de prix d'actif ayant la même variation quadratique que le modèle de Black-Scholes. Si le bruit dans un environnement stochastique est modélisé par la dérivée d'un mouvement brownien  $W$ , le théorème de représentation des martingales et la formule classique de Clark-Ocone sont deux outils de calcul fondamental. Le théorème 7.1 et les considérations à la fin de la section 7 montrent que dans une certaine mesure ceci reste valable lorsque la loi du processus sous-jacent n'est plus la mesure de Wiener mais le processus conserve la même variation quadratique que  $W$ . In est en fait possible de représenter des variables aléatoires  $h = H(X_T(\cdot))$ , où  $H : C([- T, 0]) \rightarrow \mathbb{R}$ , comme

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{0.1}$$

sous des conditions suffisantes raisonnables sur la fonctionnelle  $H$ , où  $H_0$  est un nombre réel et  $\xi$  est un processus non-anticipatif qui sont donnés de façon quasi-explicite. Ces quantités sont exprimées à l'aide d'une fonctionnelle  $u : [0, T] \times C([- T, 0]) \rightarrow \mathbb{R}$  de classe  $C^{1,2}([0, T] \times C([- T, 0]))$ , qui est solution d'une équation aux dérivées partielles; plus précisément la représentation (0.1) de  $h$  a lieu avec  $H_0 = u(0, X_0(\cdot))$  et  $\xi_t = D^{\delta_0} u(t, X_t(\cdot))$ , où  $D^{\delta_0} u(t, \eta)$  symbolise la projection de la dérivée de Fréchet  $Du(t, \eta)$  sur le sous espace vectoriel constitué par les multiples de la mesure de Dirac  $\delta_0$ ; en particulier nous avons  $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\{0\})$ .

Si  $X$  est un mouvement brownien standard  $W$  et  $h \in \mathbb{D}^{1,2}$ , l'expression (0.1) coïncide avec la formule de Clark-Ocone classique. Ces considérations montrent que la formule de Clark-Ocone classique admet une certaine robustesse relativement à la mesure de Wiener qui est la loi de  $W(\cdot)$ .

## 1. Introduction

In the whole paper  $(\Omega, \mathbb{F}, \mathbb{P})$  is a fixed probability space, equipped with a given filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  fulfilling the usual conditions,  $B$  will be a separable Banach space and  $X$  a  $B$ -valued process.  $W$  will always denote an  $(\mathcal{F}_t)$ -real Brownian motion. Let  $T > 0$  be a fixed maturity time. All the real processes  $X = (X_t)_{t \in [0, T]}$  are prolonged by continuity for  $t \notin [0, T]$  setting  $X_t = X_0$  for  $t \leq 0$  and  $X_t = X_T$  for  $t \geq T$ .

We first recall the basic concepts of forward integral and covariation and some one-dimensional results concerning calculus via regularization, a fairly complete survey on the subject being [13]. For simplicity, all the considered integrator processes will be continuous.

**Definition 1.1.** Let  $X$  (respectively  $Y$ ) be a continuous (resp. locally integrable) process.

The **forward integral of  $Y$  with respect to  $X$**  (resp. the **covariation of  $X$  and  $Y$** ), whenever it exists, is defined as

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \rightarrow 0^+} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \quad \text{in probability for all } t \in [0, T], \quad (1.1)$$

$$\left( \text{resp. } [X, Y]_t = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds \quad \text{in the ucp sense with respect to } t \right), \quad (1.2)$$

provided that the limiting process admits a continuous version. If  $\int_0^t Y_s d^- X_s$  exists for any  $0 \leq t < T$ ,  $\int_0^T Y_s d^- X_s$  will symbolize the **improper forward integral** defined by  $\lim_{t \rightarrow T} \int_0^t Y_s d^- X_s$ , whenever it exists in probability.

If  $[X, X]$  exists then  $X$  is said to be a **finite quadratic variation** process.  $[X, X]$  will also be denoted by  $[X]$  and it will be called **quadratic variation of  $X$** . If  $[X] = 0$ , then  $X$  is said to be a **zero quadratic variation process**. If  $X = (X^1, \dots, X^n)$  is a vector of continuous processes we say that it has all its **mutual covariations** (brackets) if  $[X^i, X^j]$  exists for any  $1 \leq i, j \leq n$ .

When  $X$  is a (continuous) semimartingale (resp. Brownian motion) and  $Y$  is a cadlag adapted process (resp. such that  $\int_0^T Y_s^2 ds < \infty$  a.s.), the integral  $\int_0^t Y_s d^- X_s$  exists and coincides with classical Itô's integral  $\int_0^t Y_s dX_s$ , see Proposition 6 in [13]. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when  $Y$  is anticipating or  $X$  is not a semimartingale. If  $X, Y$  are  $(\mathcal{F}_t)$ -semimartingales then  $[X, Y]$  coincides with the classical bracket  $\langle X, Y \rangle$ , see Corollary 2 in [13]. Finite quadratic variation processes will play a central role in this note: this class includes of course all  $(\mathcal{F}_t)$ -semimartingales. However that class is much richer. Typical examples of finite quadratic variation processes are  $(\mathcal{F}_t)$ -Dirichlet processes.  $D$  is called  $(\mathcal{F}_t)$ -Dirichlet process if it admits a decomposition  $D = M + A$  where  $M$  is an  $(\mathcal{F}_t)$ -local martingale and  $A$  is a zero quadratic variation process. It holds in that case  $[D] = [M]$ . This class of processes generalizes the semimartingales since a locally bounded variation process has zero quadratic variation. A slight generalization of that notion is the notion of weak Dirichlet, which was introduced in [7].  $X$  is called  $(\mathcal{F}_t)$ -weak Dirichlet if it admits a decomposition  $X = M + A$  where  $M$  is an  $(\mathcal{F}_t)$  local martingale and  $A$  is a process such that  $[A, N] = 0$  for any continuous  $(\mathcal{F}_t)$  local martingale  $N$ . An  $(\mathcal{F}_t)$ -weak Dirichlet process is not necessarily a finite quadratic variation process. On the other hand if  $A$  has finite quadratic variation then it holds  $[X] = [M] + [A]$ . Another interesting example is the bifractional Brownian motion  $B^{H,K}$  with parameters  $H \in (0, 1)$  and  $K \in (0, 1]$  which has finite quadratic variation if and only if  $HK \geq 1/2$ , see [11]. Notice that if  $K = 1$ , then  $B^{H,1}$  coincides with a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$ . If  $HK = 1/2$  it holds  $[B^{H,K}] = 2^{1-K}t$ ; if  $K \neq 1$  this process is not even a Dirichlet with respect to its own filtration. Other significant examples are the so-called weak  $k$ -order Brownian motions, for fixed  $k \geq 1$ , constructed by [8], which are in general not Gaussian.  $X$  is a weak  $k$ -order Brownian motion if for every  $0 \leq t_1 \leq \dots \leq t_k < +\infty$ ,  $(X_{t_1}, \dots, X_{t_k})$  is distributed as  $(W_{t_1}, \dots, W_{t_k})$ . If  $k \geq 4$  then  $[X]_t = t$ .

One central object of this work will be the generalization to infinite dimensional valued processes of the stochastic integral via regularization, see Definition 3.1. A stochastic calculus for Banach valued martingales was considered by [2, 15] and references therein generalizing the classical stochastic calculus of [4, 10, 6].

We introduce now a particular Banach valued process. Given  $0 < \tau \leq T$  and a real continuous process  $X$ , we will call **window process** the  $C([-\tau, 0])$ -valued process denoted by  $X(\cdot)$  defined as

$$X(\cdot) = (X_t(\cdot))_{t \in [0, T]} = \{X_t(u) := X_{t+u}; u \in [-\tau, 0], t \in [0, T]\}.$$

We observe that even the window Brownian motion  $W(\cdot)$  is not a  $C([-τ, 0])$ -valued semimartingale, in fact  $\mathcal{M}([-τ, 0])\langle \mu, W(\cdot) \rangle_{C([-τ, 0])}$  is even not a real semimartingale for  $\mu = \delta_0 + \delta_{\tau/2}$ . Motivated by the necessity of an Itô formula available also for  $B = C([-τ, 0])$ -valued processes, we introduce a quadratic variation concept which depends on a subspace  $\chi$  of the dual of the tensor square of  $B$ , equipped with the projective topology, denoted by  $(B\hat{\otimes}_\pi B)^*$ , see Definition 4.3. We recall the fundamental identification  $(B\hat{\otimes}_\pi B)^* \cong \mathcal{B}(B \times B)$ , which denotes the space of  $\mathbb{R}$ -valued bounded bilinear forms on  $B \times B$ . An Itô formula for processes admitting a  $\chi$ -quadratic variation is given in Theorem 5.1. After formulating a theory for  $B$ -valued processes with general  $B$ , in Sections 6 and 7 we fix the attention on window processes setting  $B = C([-τ, 0])$ . Section 6 is devoted to the evaluation of  $\chi$ -quadratic variation for window processes; in particular Proposition 6.4 gives the principal results concerning the  $\chi$ -quadratic variation for windows of real finite quadratic variation processes. In Section 7 we give a representation result for random variable  $h := H(X_T(\cdot))$  of the type  $h = H_0 + \int_0^T \xi_s d^- X_s$  where  $H : C([-T, 0]) \rightarrow \mathbb{R}$  and the integral has to be considered as a forward integral as defined in (1.1). In general when  $H$  is continuous and  $X$  is a general finite quadratic variation process such that  $[X]_t = t$  this representation will be obtained by expressing  $h$  as  $u(T, X_T(\cdot))$  where  $u \in C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$  solves an infinite dimensional partial differential equation of type (7.5). In this case we obtain

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s ,$$

where  $D^{\delta_0} u(s, \eta)$  denotes the *projection* of the Fréchet derivative  $Du(t, \eta)$  on the linear space generated by Dirac measure  $\delta_0$ , i.e.  $D^{\delta_0} u(t, \eta) := Du(t, \eta)(\{0\})$ .

## 2. Notations

Symbol  $\mathcal{C}([0, T])$  denotes the linear space of continuous real processes equipped with the ucp (uniformly convergence in probability) topology,  $B^*$  will be the topological dual of the Banach space  $B$ . We introduce now some subspaces of measures that we will frequently use. Symbol  $\mathcal{D}_0([-τ, 0])$  ( resp.  $\mathcal{D}_{0,0}([-τ, 0]^2)$ ) will denote the one dimensional Hilbert space of the multiples of Dirac measure concentrated at 0 (resp. at  $(0, 0)$ ), i.e.

$$\mathcal{D}_0([-τ, 0]) := \{\mu \in \mathcal{M}([-τ, 0]); s.t. \mu(dx) = \lambda \delta_0(dx) \text{ with } \lambda \in \mathbb{R}\} \quad (2.1)$$

and

$$\mathcal{D}_{0,0}([-τ, 0]^2) := \{\mu \in \mathcal{M}([-τ, 0]^2); s.t. \mu(dx, dy) = \lambda \delta_0(dx) \delta_0(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_0 \hat{\otimes}_h \mathcal{D}_0 \quad (2.2)$$

where  $\hat{\otimes}_h$  stands for the Hilbert tensor product.

Symbol  $Diag([-τ, 0]^2)$  (shortly  $Diag$ ) will denote the subset of  $\mathcal{M}([-τ, 0]^2)$  defined as follows:

$$Diag([-τ, 0]^2) := \left\{ \mu \in \mathcal{M}([-τ, 0]^2) \text{ s.t. } \mu(dx, dy) = g(x) \delta_y(dx) dy; g \in L^\infty([-τ, 0]) \right\} . \quad (2.3)$$

$Diag([-τ, 0]^2)$ , equipped with the norm  $\|\mu\|_{Diag([-τ, 0]^2)} = \|g\|_\infty$ , is a Banach space.

## 3. Calculus via regularization

In this section we introduce an infinite dimensional stochastic integral via regularization. In this construction there are two main difficulties. The integrator is generally not a semimartingale or the integrand may be anticipative;  $B$  is a general non-reflexive Banach space.

**Definition 3.1.** Let  $(X_t)_{t \in [0, T]}$  (respectively  $(Y_t)_{t \in [0, T]}$ ) be a  $B$ -valued (respectively a  $B^*$ -valued) stochastic process. For simplicity we suppose  $X$  to be continuous and  $\int_0^T \|Y_s\|_{B^*} ds < +\infty$  a.s.

For every fixed  $t \in [0, T]$  we define the **definite forward integral of  $Y$  with respect to  $X$**  denoted by  $\int_0^t \langle Y_s, d^- X_s \rangle_B$  as the following limit in probability:

$$\int_0^t \langle Y_s, d^- X_s \rangle_B := \lim_{\epsilon \rightarrow 0} \int_0^t \langle Y(s), \frac{X(s+\epsilon) - X(s)}{\epsilon} \rangle_B ds.$$

We say that the **forward stochastic integral of  $Y$  with respect to  $X$**  exists if the process

$$\left( \int_0^t \langle Y_s, d^- X_s \rangle_B \right)_{t \in [0, T]}$$

admits a continuous version. In the sequel indices  $B$  and  $B^*$  will often be omitted.

#### 4. Chi-quadratic variation

**Definition 4.1.** A closed linear subspace  $\chi$  of  $(B \hat{\otimes}_\pi B)^*$ , endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B \hat{\otimes}_\pi B)^*} \tag{4.1}$$

will be called a **Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$** .

Let  $\chi$  be a Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$ ,  $X$  be a  $B$ -valued stochastic process and  $\epsilon > 0$ . We denote by  $[X]^\epsilon$ , the application

$$[X]^\epsilon : \chi \longrightarrow \mathcal{C}([0, T]) \quad \text{defined by} \quad \phi \mapsto \left( \int_0^t \chi \langle \phi, \frac{J((X_{s+\epsilon} - X_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right)_{t \in [0, T]} \tag{4.2}$$

where  $J : B \hat{\otimes}_\pi B \longrightarrow (B \hat{\otimes}_\pi B)^{**}$  denotes the canonical injection between a space and its bidual.

With application  $[X]^\epsilon$  it is possible to associate another one, denoted by  $\widetilde{[X]}^\epsilon$ , defined, for  $\omega \in \Omega$ , by

$$\widetilde{[X]}^\epsilon(\omega, \cdot) : [0, T] \longrightarrow \chi^* \quad \text{given by} \quad t \mapsto \left( \phi \mapsto \int_0^t \chi \langle \phi, \frac{J((X_{s+\epsilon} - X_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right). \tag{4.3}$$

#### Remark 4.2.

1. We recall that  $\chi \subset (B \hat{\otimes}_\pi B)^*$  implies  $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$ .
2. As indicated,  $\chi \langle \cdot, \cdot \rangle_{\chi^*}$  denotes the duality between the space  $\chi$  and its dual  $\chi^*$ ; in fact, by assumption,  $\phi$  is an element of  $\chi$  and element  $J((X_{s+\epsilon} - X_s) \otimes^2)$  naturally belongs to  $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$ .
3. With a slight abuse of notation, in the sequel application  $J$  will be omitted. The tensor product  $(X_{s+\epsilon} - X_s) \otimes^2$  has to be considered as the element  $J((X_{s+\epsilon} - X_s) \otimes^2)$  which belongs to  $\chi^*$ .

We give now the definition of the  $\chi$ -quadratic variation of a  $B$ -valued stochastic process  $X$ .

**Definition 4.3.** Let  $\chi$  be a separable Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$  and  $X$  a  $B$ -valued stochastic process. We say that  $X$  **admits a  $\chi$ -quadratic variation** if the following assumptions are fulfilled.

**H1** For every sequence  $(\epsilon_n) \downarrow 0$  there is a subsequence  $(\epsilon_{n_k})$  such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \chi \left\langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s) \otimes^2}{\epsilon_{n_k}} \right\rangle_{\chi^*} \right| ds < +\infty, \text{ a.s.}$$

**H2** It exists an application denoted by  $[X] : \chi \rightarrow \mathcal{C}([0, T])$ , such that

$$[X]^\epsilon(\phi) \xrightarrow[\epsilon \rightarrow 0_+]{ucp} [X](\phi)$$

for all  $\phi \in \mathcal{S}$ , where  $\mathcal{S} \subset \chi$  such that  $\overline{\text{Span}(\mathcal{S})} = \chi$ .

**Definition 4.4.** When  $X$  admits a  $\chi$ -quadratic variation, the  $\chi^*$ -valued measurable process  $(\widetilde{[X]})_{0 \leq t \leq T}$  defined for almost all  $\omega \in \Omega$  and  $t \in [0, T]$  by  $\phi \mapsto \widetilde{[X]}(\omega, t)(\phi) = [X](\phi)(\omega, t)$ , is called  $\chi$ -**quadratic variation** of  $X$ . Sometimes, with a slight abuse of notation, even  $[X]$  will be called  $\chi$ -quadratic variation and it will be confused with  $\widetilde{[X]}$ .

**Remark 4.5.**

1. We observe that for almost all  $\omega \in \Omega$ , the  $\chi^*$ -valued process  $\widetilde{[X]}(\omega, \cdot)$  has bounded variation on  $[0, T]$ .
2. The existence of  $\widetilde{[X]}$  guarantees that  $[X]$  admits a proper version which allows to consider it as pathwise integral.

**Definition 4.6.** We say that a continuous  $B$ -valued process  $X$  admits **global quadratic variation** if it admits a  $\chi$ -quadratic variation with  $\chi = (B \hat{\otimes}_\pi B)^*$ .

When  $B$  is the finite dimensional space  $\mathbb{R}^n$ ,  $X$  admits all its mutual brackets if and only if  $X$  admits a global quadratic variation. A Banach valued *locally semi summable* process  $X$  in the sense of [6], admits again a global quadratic variation, see Section 6 in [5] for more details.

We observe an interesting feature in the case  $\chi = (B \hat{\otimes}_\pi B)^*$ . The natural convergence topology is the weak star convergence in the space  $(B \hat{\otimes}_\pi B)^{**}$  for elements  $\widetilde{[X]}^\epsilon$ . In fact for any  $t \in [0, T]$ , it exists a null subset  $N$  of  $\Omega$  such and a sequence  $(\epsilon_n)$  such that

$$\widetilde{[X]}^{\epsilon_n}(\omega, t) \xrightarrow[\epsilon \rightarrow 0]{} \widetilde{[X]}(\omega, t)$$

weak star for all  $\omega \notin N$ . We recall that  $J(B \hat{\otimes}_\pi B)$  is weak star dense in  $(B \hat{\otimes}_\pi B)^{**}$ , so  $\widetilde{[X]}$  takes values “a priori” in  $(B \hat{\otimes}_\pi B)^{**}$ . The weak star convergence is weaker then the strong convergence in  $B \hat{\otimes}_\pi B$ , i.e. the convergence with respect to the topology defined by the norm. If the Banach space  $B \hat{\otimes}_\pi B$  is not reflexive, then  $(B \hat{\otimes}_\pi B)^{**}$  strictly contains  $B \hat{\otimes}_\pi B$ . In general  $B \hat{\otimes}_\pi B$  is not reflexive even if  $B$  is a Hilbert space.

## 5. Itô's formula

We are now able to state an Itô formula for stochastic processes with values in a general Banach space. Its proof is given in Section 8 of [5].

**Theorem 5.1.** Let  $B$  be a separable Banach space,  $\chi$  be a Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$  and  $X$  a  $B$ -valued continuous process admitting a  $\chi$ -quadratic variation. Let  $F : [0, T] \times B \rightarrow \mathbb{R}$  of class  $C^{1,2}$  Fréchet. such that

$$D^2 F : [0, T] \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^* \text{ continuously with respect to } \chi. \quad (5.1)$$

Then the forward integral

$$\int_0^t \int_{B^*} \langle DF(s, X_s), d^- X_s \rangle_B, \quad t \in [0, T],$$

exists and the following formula holds

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \langle DF(s, X_s), d^- X_s \rangle_B + \frac{1}{2} \int_0^t \chi \langle D^2 F(s, X_s), d[\widetilde{X}]_s \rangle_{\chi^*} a.s. \quad (5.2)$$

## 6. Evaluation of $\chi$ -quadratic variations for window processes

From this section we fix  $B$  as the Banach space  $C([-\tau, 0])$ . First we give some evaluations of  $\chi$ -quadratic variations of window processes  $X(\cdot)$ . Examples of Chi-subspaces of  $(B \widehat{\otimes}_\pi B)^*$  are

- the space of measures on  $[-\tau, 0]^2$  denoted by  $\mathcal{M}([-\tau, 0]^2)$ ;
- the space  $L^2([-\tau, 0]^2)$ ;
- the Hilbert tensor product  $L^2([-\tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([-\tau, 0])$ ;
- the space  $\mathcal{D}_{0,0}([-\tau, 0]^2)$ ;
- the space  $Diag([-\tau, 0]^2)$ ;
- the space  $\chi^0([-\tau, 0]^2)$ , (shortly  $\chi^0$ ), will denote the subset of  $\mathcal{M}([-\tau, 0]^2)$  defined by

$$\chi^0([-\tau, 0]^2) := (\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])) \widehat{\otimes}_h^2 \quad (6.1)$$

which equals

$$\chi^0([-\tau, 0]^2) = L^2([-\tau, 0]^2) \oplus L^2([-\tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([-\tau, 0]) \oplus \mathcal{D}_0([-\tau, 0]) \widehat{\otimes}_h L^2([-\tau, 0]) \oplus \mathcal{D}_{0,0}([-\tau, 0]^2) . \quad (6.2)$$

**Proposition 6.1.** Let  $X$  be a real valued process with Hölder continuous paths of parameter  $\gamma > 1/2$ . Then  $X(\cdot)$  admits a zero global quadratic variation.

**Example 6.2.** Examples of real processes with Hölder continuous paths of parameter  $\gamma > 1/2$  are fractional window Brownian motion  $B^H(\cdot)$  with  $H > 1/2$  or a bifractional window Brownian motion  $B^{H,K}(\cdot)$  with  $KH > 1/2$ .

**Remark 6.3.** Presumably the window Brownian motion  $W(\cdot)$  does not admit a global (not even a  $\mathcal{M}([-T, 0]^2)$ -) quadratic variation because Condition **H1** should not be verified. In fact it is possible to show that

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[ \int_0^T \frac{1}{\epsilon} \|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|_{C([-T, 0])}^2 ds \right] = +\infty . \quad (6.3)$$

(6.3) follows taking into account the fact that for  $0 < \tau_1 < \tau_2$ , there are positive constants  $C_1, C_2$  such that

$$C_1 \leq \mathbb{E} \left[ \sup_{s \in [\tau_1, \tau_2]} \frac{|W_{s+\epsilon} - W_s|^2}{\epsilon \ln(1/\epsilon)} \right] \leq C_2 .$$

For general finite quadratic variation processes we have the following.

**Proposition 6.4.** Let  $X$  be a real continuous process with finite quadratic variation and  $0 < \tau \leq T$ . The following properties hold true.

- 1)  $X(\cdot)$  admits zero  $L^2([-\tau, 0]^2)$ -quadratic variation.
- 2)  $X(\cdot)$  admits zero  $L^2([-\tau, 0]) \widehat{\otimes}_h \mathcal{D}_0([-\tau, 0])$ -quadratic variation.



- 3)  $X(\cdot)$  admits zero  $\mathcal{D}_0([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0])$ -quadratic variation.  
4)  $X(\cdot)$  admits a  $\mathcal{D}_{0,0}([-\tau, 0]^2)$ -quadratic variation given by

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \mathcal{D}_{0,0}([-\tau, 0]^2). \quad (6.4)$$

- 5)  $X(\cdot)$  admits a  $\chi^0([-\tau, 0]^2)$ -quadratic variation which equals

$$[X(\cdot)](\mu) = \mu(\{0, 0\})[X], \quad \forall \mu \in \chi^0([-\tau, 0]^2). \quad (6.5)$$

- 6)  $X(\cdot)$  admits a *Diag*-quadratic variation given by

$$\mu \mapsto [X(\cdot)]_t(\mu) = \int_0^{t \wedge \tau} g(-x)[X]_{t-x} dx \quad t \in [0, T], \quad (6.6)$$

where  $\mu$  is a generic element in  $\text{Diag}([-\tau, 0]^2)$  of type  $\mu(dx, dy) = g(x)\delta_y(dx)dy$ , with associated  $g$  in  $L^\infty([-\tau, 0])$ .

**Remark 6.5.** We remark that in the treated cases, the quadratic variation  $[X]$  of the real finite quadratic variation process  $X$  insures the existence of (and completely determines) the  $\chi$ -quadratic variation. For example if  $X$  is a real finite quadratic variation process such that  $[X]_t = t$ , then  $X(\cdot)$  has the same  $\chi^0$ -quadratic variation as the window Brownian motion.

We conclude this section spending some words on the stability of processes admitting a quadratic variation even if this will not be the object of this paper. The class of real semimartingales with respect to a given filtration is known to be stable with respect to  $C^2(\mathbb{R})$  transformations and finite quadratic variation processes are stable under  $C^1(\mathbb{R})$  transformations. Also Dirichlet processes are stable with respect to  $C^1(\mathbb{R})$  transformations and they admit a decomposition result. In some applications, in particular to control theory, one often needs to know the nature of process  $(F(t, X_t))$  where  $F \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R})$  and  $X$  is a real continuous  $(\mathcal{F}_t)$ -weak Dirichlet process with finite quadratic variation. It was shown in [9], Proposition 3.10, that  $(F(t, X_t))$  is an  $(\mathcal{F}_t)$ -weak Dirichlet process. Analogous stability results are obtained in the infinite dimensional framework for the  $C([-\tau, 0])$ -valued window Dirichlet and weak Dirichlet processes, see Theorems 7.33 and 7.36 in [5].

## 7. A generalized Clark-Ocone formula

In this section we will consider  $\tau = T$  and we recall that  $B = C([-T, 0])$ . Let  $X$  be a real continuous stochastic process such that  $X_0 = 0$  and  $[X]_t = t$ . Let  $H : C([-T, 0]) \rightarrow \mathbb{R}$  be a Borel functional; we aim at representing the random variable

$$h = H(X_T(\cdot)). \quad (7.1)$$

The main task will consist in looking for classes of functionals  $H$  for which there is  $H_0 \in \mathbb{R}$  and a predictable process  $\xi$  with respect to the canonical filtration of  $X$  such that  $h$  admits the representation

$$h = H_0 + \int_0^T \xi_s d^- X_s. \quad (7.2)$$

Moreover we look for an explicit expression for  $H_0$  and  $\xi$ . As a consequence of Itô's formula (5.2) for path dependent functionals of the process we will observe that, in those cases, it is possible to find a function  $u$  which solves an infinite dimensional PDE and which gives at the same time the representation result (7.2). One possible representation is the following.

**Theorem 7.1.** Let  $H : C([-T, 0]) \rightarrow \mathbb{R}$  be a Borel functional. Let  $u \in C^{1,2}([0, T] \times C([-T, 0])) \cap C^0([0, T] \times C([-T, 0]))$  such that  $x \mapsto D_x^{ac} u(t, \eta)$  has bounded variation, for any  $t \in [0, T]$ ,  $\eta \in C([-T, 0])$ . We suppose moreover that  $D^2 u(t, \eta) \in \mathcal{X}^0([-T, 0]^2)$  continuously for every  $t \in [0, T]$ ,  $\eta \in C([-T, 0])$ . Suppose that  $u$  is a solution of

$$\begin{cases} \partial_t u(t, \eta) + \int_{]-t, 0]} D^{ac} u(t, \eta) d\eta + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.3)$$

where the integral  $\int_{]-t, 0]} D^{ac} u(t, \eta) d\eta$  has to be understood via an integration by parts as follows:

$$\int_{]-t, 0]} D^{ac} u(t, \eta) d\eta = D^{ac} u(0, \eta)\eta(0) - D^{ac} u(-t, \eta)\eta(-t) - \int_{]-t, 0]} \eta(x) D_{dx}^{ac} u(t, \eta) .$$

Let  $X$  be a real continuous finite quadratic variation process with  $[X]_t = t$  and  $X_0 = 0$ . Then the random variable  $h := H(X_T(\cdot))$  admits the following representation

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(t, X_t(\cdot)) d^- X_t . \quad (7.4)$$

□

Sections 9.8 and 9.9 in [5] provide different reasonable conditions on  $H : C([-T, 0]) \rightarrow \mathbb{R}$  such that there is a function  $u$  solving PDE (7.3) in general situations, i.e. fulfilling the hypotheses of Theorem 7.1. Section 9.8 considers an  $L^2([-T, 0])$ -regularity on  $H : C([-T, 0]) \subset L^2([-T, 0]) \rightarrow \mathbb{R}$  and Section 9.9 considers a non smooth but  $L^2([-T, 0])$ -finitely based functionals  $H$ .

**Remark 7.2.** In relation to Theorem 7.1 we observe the following.

- Only pathwise considerations intervene and there is no need to suppose that the law of  $X$  is Wiener measure.
- Since  $H(\eta) = u(T, \eta)$ , we observe that  $H$  is automatically continuous by hypothesis  $u \in C^0([0, T] \times C([-T, 0]))$ .
- Let us suppose  $X = W$ .
  1. Making use of probabilistic technology, (7.4) holds in some cases even if  $H$  is not continuous and  $h \notin L^1(\Omega)$ ; we refer to Section 9.6 in [5] for this type of results.
  2. If  $\int_0^T \xi_s^2 ds < +\infty$  a.s., then the forward integral  $\int_0^T \xi_t d^- W_t$  coincides with the Itô integral  $\int_0^T \xi_t dW_t$ .
  3. If the r.v.  $h = H(W_T(\cdot))$  belongs to  $\mathbb{D}^{1,2}$ , by uniqueness of the martingale representation theorem and point 2., we have  $H_0 = \mathbb{E}[h]$  and  $\xi_t = \mathbb{E}[D_t^m h | \mathcal{F}_t]$ , where  $D^m$  is the Malliavin gradient; this agrees with Clark-Ocone formula.
- If  $X$  is not a Brownian motion, in general  $H_0 \neq \mathbb{E}[h]$  since  $\mathbb{E}\left[\int_0^T \xi_t d^- X_t\right]$  does not generally vanish. In fact  $\mathbb{E}[h]$  will specifically depend on the unknown law of  $X$ .

In chapter 9 in [5] we enlarge the discussion presented in Theorem 7.1. We can give examples where  $u : [0, T] \times C([-T, 0]) \rightarrow \mathbb{R}$  of class  $C^{1,2}([0, T] \times C([-T, 0]); \mathbb{R}) \cap C^0([0, T] \times C([-T, 0]); \mathbb{R})$  such that (7.4) holds and  $u$  solves an infinite dimensional PDE of the type

$$\begin{cases} \partial_t u(t, \eta) + \int_{]-t, 0]} D^{ac} u(t, \eta) d\eta'' + \frac{1}{2} \langle D^2 u(t, \eta), \mathbb{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (7.5)$$

where  $\mathbb{1}_D(x, y) := \begin{cases} 1 & \text{if } x = y, x, y \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}$  and  $D^{ac}u(t, \eta)$  is the absolute continuous part of the measure  $Du(t, \eta)$ .

The integral “ $\int_{-t}^0 D^{ac}u(t, \eta) d\eta$ ” has to be suitably defined and term  $\langle D^2u(t, \eta), \mathbb{1}_D \rangle$  indicates the evaluation of the second order derivative on the diagonal of the square  $[-T, 0]^2$ .

We observe that solution of (7.3) are also solutions of (7.5) since  $\langle D^2u(t, \eta), \mathbb{1}_D \rangle = D^2u(t, \eta)(\{0, 0\})$ .

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