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Experimental Study of the HUM Control Operator for Linear Waves

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Abstract

We consider the problem of the numerical approximation of the linear controllability of waves. All our experiments are done in a bounded domain Ω of the plane, with Dirichlet boundary conditions and internal control. We use a Galerkin approximation of the optimal control operator of the continuous model, based on the spectral theory of the Laplace operator in Ω. This allows us to obtain surprisingly good illustrations of the main theoretical results available on the controllability of waves, and to formulate some questions for the future analysis of optimal control theory of waves.

Key words: Controllability, linear waves equation, HUM method, numerical analysis, experimental mathematics
Field: Control.
Presentation: Oral.

1 HUM control operator

1.1 Controllability of linear waves

For a given $f = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem is to find a source $v(t,x) \in L^2(0, T; L^2(\Omega))$ such that the solution $u = S(v)$ of the linear wave

equation

$$\begin{cases} \Box u = \chi v & \text{in }]0, +\infty[\times \Omega \\ u_{|\partial\Omega} = 0, & t > 0 \\ (u|_{t=0}, \partial_t u|_{t=0}) = (0, 0) \end{cases} \qquad (1)$$

reaches the state $f = (u_0, u_1) = (u(T, .), \partial_t u(T, .))$ at time T, where:

- Ω is a bounded open subset of $\mathbb{R}^d$,

- the "control domain" U is a non empty open subset of Ω,

- $\chi(t, x) = \psi(t)\chi_0(x)$ where χ_0 is a real L^∞ function on $\overline{\Omega}$, such that support$(\chi_0) = \overline{U}$ and $\chi_0(x)$ is continuous and positive for $x \in U$, $\psi \in C^\infty([0, T])$ and $\psi(t) > 0$ on $]0, T[$.

The reachable set at time T is the subspace of $H = H_0^1(\Omega) \times L^2(\Omega)$:

$$\mathcal{R}_T = \{f = (u_0, u_1) \in H, \exists v, (S(v)(T, .), \partial_t S(v)(T, .)) = (u_0, u_1)\}.$$

Then we have approximate controllability if $\mathcal{R}_T$ is dense in H and exact controllability if $\mathcal{R}_T = H$.

1.2   The HUM method

The Hilbert Uniqueness Method (HUM) of J.-L. Lions [?] consists in choosing the function v with L^2-minimal norm. Then v is necessarily of the form $\chi \partial_t w$ where w is a solution of the dual control problem:

$$\begin{cases} \Box w = 0 & \text{in }]0, +\infty[\times \Omega \\ w_{|\partial\Omega} = 0, & t > 0 \\ (w|_{t=T}, \partial_t w|_{t=T}) = (w_0, w_1) = h \in H = H_0^1(\Omega) \times L^2(\Omega) \end{cases}$$

The HUM control operator is then defined by

$$\Lambda : \begin{array}{ccc} H & \to & H \\ f = (u_0, u_1) & \mapsto & h = (w_0, w_1) \end{array}$$

Let $A = A^*$ be the operator on $H = H_0^1(\Omega) \times L^2(\Omega)$ defined by

$$iA = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix}$$

Let $\lambda = \sqrt{-\Delta_D}$. Then (??) becomes $(\partial_t - iA)u = B(t)v$ with

$$B(t) = \begin{pmatrix} 0 & 0 \\ \chi(t, .)\lambda & 0 \end{pmatrix}, \quad B^*(t) = \begin{pmatrix} 0 & \lambda^{-1}\chi(t, .) \\ 0 & 0 \end{pmatrix}$$

Then we have exact controlability iff

$$\exists C > 0, M_T = \int_0^T e^{itA} B(T - t) B^*(T - t) e^{-itA^*} \, dt \geq C \, \text{Id}$$

And in that case we have

$$\Lambda = M_T^{-1}$$

1.3 Geometric control condition

We recall that the source v in (??) is multiplied by $\chi(t,x) = \psi(t)\chi_0(x)$, where $\chi_0 \in L^\infty(\overline{\Omega})$, such that support$(\chi_0) = \overline{U}$ and $\chi_0(x)$ is continuous and positive for $x \in U$, $\psi \in C^\infty([0,T])$ and $\psi(t) > 0$ on $]0,T[$. We also assume that there is no contact of infinite order between $\partial\Omega$ and the optical rays of the wave operator in the free space. Let us recall the Geometric Control Condition of C. Bardos, G. Lebeau and J. Rauch [?]:

GCC Every geodesic ray of Ω traveling with speed 1 and starting at $t = 0$ enters the open set $U = \{x \in \Omega, \chi_0(x) \neq 0\}$ in time $t < T$.

Theorem 1 *If χ and T are such that the* **GCC** *condition holds true, then M_T is an isomorphism, i.e. one has exact controlability ($\mathcal{R}_T = H$).*

2 Numerical method

2.1 Previous numerical approach

R. Glowinski *et al.* [?] first discretize the continuous wave equation, then compute the control of the discrete system. As observed by R. Glowinski *et al.* and precisely studied by E. Zuazua [?, ?], the discrete model is not uniformly exactly controllable when the mesh size goes to zero, and the interaction of waves with the numerical mesh produces spurious high frequency oscillations. In other words, the processes of numerical discretization and control do not commute for the wave equation. Thus, some multi-grid methods were developed to overcome this problem, see e.g. [?, ?].

2.2 Spectral Galerkin method

Let (ω_j^2) be the sequence of $-\Delta_D$ eigenvalues, and (e_j) the associated orthonormal basis of $L^2(\Omega)$:

$$-\Delta e_j = \omega_j^2 e_j, e_j|_{\partial\Omega} = 0$$

For a given cutoff frequency ω, we define

$$L_\omega^2 = \text{Span}\{e_j, \omega_j \leq \omega\}$$

and we denote by Π_ω the orthogonal projector on L_ω^2, which obviously acts also on $H = H_0^1 \times L^2$:

$$H_\omega = \Pi_\omega(H_0^1 \times L^2)$$

and we define the matrix $M_{T,\omega}$:

$$M_{T,\omega} = \Pi_\omega M_T \Pi_\omega, \qquad M_{T,\omega,n,m} = (M_T \phi_n | \phi_m)_H$$

where (ϕ_n) is an orthonormal basis of H_ω.
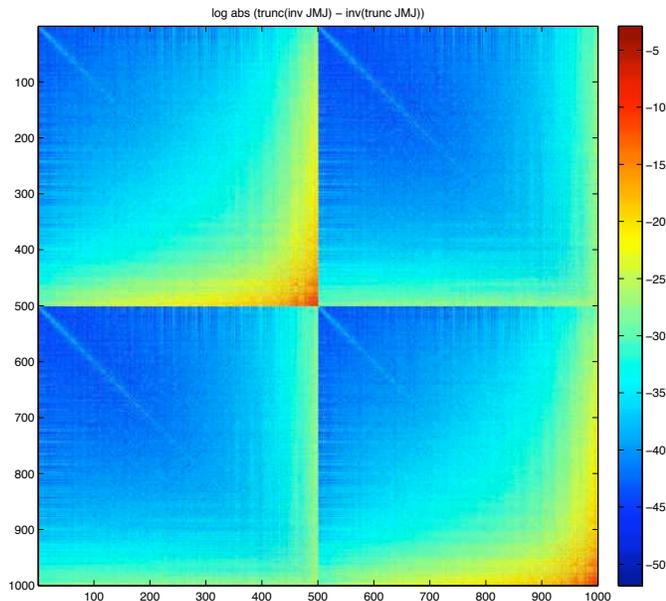Let us recall that $\Lambda = M_T^{-1}$ and $M_{T,\omega} = \Pi_\omega M_T \Pi_\omega$.

Figure 1: View of the logarithm of the coefficients of the matrix $J\left[((M_T)^{-1})_\omega - ((M_T)_\omega)^{-1}\right]J^{-1} = J\left[\Lambda_\omega - ((M_T)_\omega)^{-1}\right]J^{-1}$, for the square geometry, with smooth control. The M_T matrix is computed with 2000 eigenvalues, the cutoff frequency ω is equal to the 500th eigenvalue.

Then we can show that $M_{T,\omega}$ is invertible on H_ω and $\|M_{T,\omega}^{-1}\|$ is bounded uniformly in ω (because of exact controlability).

Lemma 2 *Assume* **GCC**. *Then there exists $c > 0$ such that for all $f \in H$:*

$$\|\Lambda(f) - M_{T,\omega}^{-1}(f_\omega)\|_H \leq c\|f - f_\omega\|_H + \|\Lambda(f_\omega) - M_{T,\omega}^{-1}(f_\omega)\|_H$$

with $f_\omega = \Pi_\omega f$ and $\lim_{\omega \to \infty} \|\Lambda(f_\omega) - M_{T,\omega}^{-1}(f_\omega)\|_H = 0$.

In other words, the processes of Galerkin approximation and inversion "almost" commute for M_T. This can be seen in Figure **??**, which represents the operator $((M_T)^{-1})_\omega - ((M_T)_\omega)^{-1}$. See [**?**] for details.

3   Experimental study of HUM operator properties

3.1   Numerical setup

Our algorithm was implemented for various 2D domains. We present three geometries: a square, a disc and a trapezoid, acting as general domain. For each geometry, we chose one "standard" control domain satisfying GCC. For the square it can be the neighbourhood of two non parallel sides, for the disc the neighbourhood of a radius, and for the trapezoid the neighbourhood of
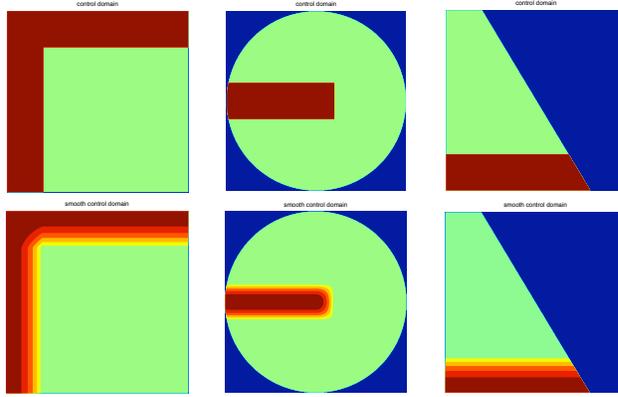
Figure 2: Geometries, control domains without (top) and with (bottom) smoothing.

the larger parallel side is suitable (see Figure **??**). As it has been shown by B. Dehman and G. Lebeau [**?**], the HUM control operator Λ has good properties when the control function $\chi(t, x)$ is smooth. So we considered two cases: non-smooth control: $\chi(t, x) = 1_{[0,T]} 1_U$ (which is the classical case, as implemented in the previous approach); smooth control: $\chi(t, x) = \psi(t)\chi_0(x)$ with $\psi(t) = \frac{4t(T-t)}{T^2} 1_{[0,T]}$ and $\chi_0(x)$ similarly smoothed.

3.2   Frequency localization

As an example of the experimental studies we have done, we show here one property of the HUM operator: the frequency localization, and the impact of smoothing on this property. We refer to G. Lebeau and M. Nodet [**?**] for an extended study of the HUM control operator.

The theoretical result states as follows. We refer to B. Dehman and G. Lebeau [**?**, **?**] for the details. Let $\psi_k(D)$, $k \in \mathbb{N}$, be the spectral localization operators associated to the Littlewood-Paley decomposition:

$$\psi_k(D)(\sum_j a_j e_j) = \sum_j \psi_k(\omega_j) a_j e_j, \quad S_k(D) = \sum_{j=0}^{k} \psi_j(D), k \geq 0$$

Theorem 3 (Dehman-Lebeau) *Assume that the geometric control condition* **GCC** *holds true, and that the control function $\chi(t, x)$ is smooth. There exists $C > 0$ such that for every $k \in \mathbb{N}$, the following inequality holds true*

$$\|\psi_k(D)\Lambda - \Lambda\psi_k(D)\|_H \leq C2^{-k}$$
$$\|S_k(D)\Lambda - \Lambda S_k(D)\|_H \leq C2^{-k}$$

Figure **??** shows the consequence of this result on the one-mode experiment, ie when the target data $f = (u_0, u_1)$ (to be reached) is equal to an eigenvector. We can see that the control (w_0, w_1) is almost equal to the same eigenvector, illustrating the above property.
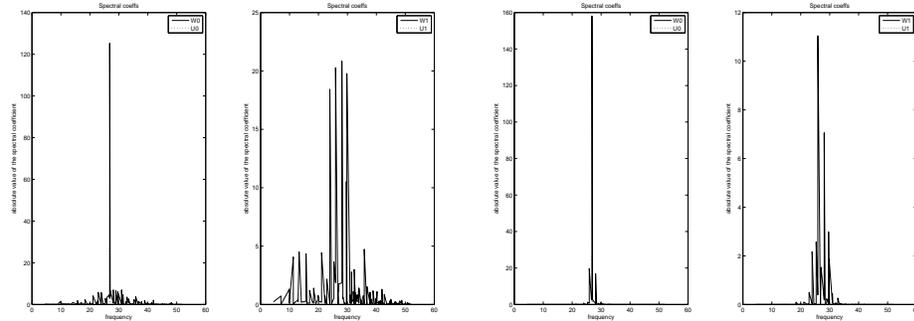
Figure 3: Frequency localization experiment in the square: localization of the Fourier frequences of (w_0, w_1) (left, right) for a given time T and a given domain U without smoothing (left) and with time- and space-smoothing (right). The x-coordinate represents the eigenvalues. The target data u_0 is equal to the 50-th eigenvector (eigenvalue of about 26.8), and $u_1 = 0$.

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