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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Information Propagation Speed in Bidirectional Vehicular Delay Tolerant Networks

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Abstract: This research report analyses information propagation speed in bidirectional vehicular delay tolerant networks such as roads or highways. The analysis provided in this report shows that phase transition occurs concerning information propagation speed, with respect to vehicle density in each direction of the highway. Under a certain threshold, information propagates on average at vehicle speed, while above this threshold, information propagates faster. This report gives the exact expression of this threshold and of the average propagation speed in this context, around the threshold. Simulations results, obtained on several platforms, are presented and confirm the analytical results.

Key-words: Ad hoc, DTN, vehicular network, information, propagation speed

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Vitesse de Propagation de l'Information dans les Réseaux DTN Véhiculaires Bidirectionnels

Résumé : Ce rapport étudie la vitesse de propagation de l'information dans les réseaux DTN véhiculaires bidirectionnels, tels des automobiles sur une autoroute. L'analyse présentée dans ce rapport montre qu'une transition de phase se produit concernant la vitesse de propagation de l'information, par rapport à la densité des véhicules dans chaque direction. Sous une certaine borne, l'information se propage à la vitesse des véhicules, alors qu'au dessus de cette borne, l'information se propage plus vite. Ce rapport donne l'expression exacte de cette borne et de la vitesse moyenne de propagation de l'information dans ce contexte, autour de cette borne. Des résultats de simulations, obtenus sur différentes plateformes, sont présentés et confirment ces résultats analytiques.

Mots-clés : Ad hoc, DTN, réseau véhiculaire, information, vitesse de propagation

1 Introduction

The limits of the performance of multi-hop packet radio networks have been studied for more than a decade, yielding fundamental results such as those of Gupta and Kumar [12] on the capacity of fixed ad hoc networks. These studies assume that either end-to-end paths are available or packets are dropped on the spot. Following seminal works such as [11] evaluating the potential of mobility to increase capacity, recent research studies focussed on the limits of the performance beyond the end-to-end hypothesis, *i.e.*, when end-to-end paths may not exist and communication routes may only be available through time and mobility. In this context nodes may carry packets for a while until advancing further towards the destination is possible. Such networks are generally referred as Intermittently Connected Networks (ICNs) or Delay Tolerant Networks (DTNs). Interest in DTN modeling and analysis has risen as novel network protocols and architectures are being elaborated to accommodate various forms of new, intermittently connected networks, which include vehicular ad hoc networks (VANETs), power-saving sensor networks, or even Interplanetary Internet [7].

In this paper, we study the information propagation speed in the typical case of bidirectional vehicular DTNs, such as roads or highways (e.g., about 75% of the total statute miles in the USA [18]). Our objective consists in determining the maximum speed at which a packet (or beacon) of information can propagate in such a bidirectional vehicular network. Our analysis shows that a phase transition occurs concerning information propagation speed, with respect to the vehicle density. We prove that under a certain threshold, information propagates on average at vehicle speed, while above this threshold, information propagates much faster. We provide the exact expressions of the threshold and of the average propagation speed near the threshold.

With applications such as safety, ad hoc vehicular networks are receiving increasing attention (see recent surveys [6, 17]). Delay tolerant architectures have thus been considered in this context in recent studies, and various analytical models have been proposed. In [19], the authors study vehicle traces and conclude that vehicles are very close to being exponentially distributed on highways. The authors of [5] also base themselves on traces gathered in DieselNET (the experimental vehicular network deployed by UMass) to elaborate and evaluate a novel DTN routing algorithm. In [10], the authors provide a model for critical message dissemination in vehicular networks and derive results on the average delay in delivery of messages with respect to vehicle density. The authors of [21] propose an alternative model for vehicular DTNs and derived results on node connectivity, under the hypothesis that vehicles are exponentially distributed. The study is based on queuing theory techniques and characterizes the relationship between node connectivity and several parameters including speed distribution and traffic flow. In [20], the authors model vehicles on a highway, and study message propagation among vehicles in the same direction, taking into account speed differences between vehicles, while in [16] authors study message dissemination among vehicles in opposing directions and conclude that using both directions increases dissemination significantly.

Several studies focus on characterizing the packet propagation delay in DTNs: [8] which models DTNs as Erdős-Rényi random graphs to derive results concerning packet propagation delay, [22] which uses fluid limit techniques to derive relationships between buffer space, packet duplication and dissemination delay.

Other studies focus on information propagation speed in DTNs. In [15] the authors show that when a two-dimensional network is not percolated, the latency scales linearly with the Euclidean distance between the sender and the receiver, while in [13], the authors obtained analytical estimates of the constant bounds on the speed at which information can propagate in two-dimensional DTNs. Studies such as [1, 2, 3] are the closest related work, also focusing on information propagation speed in one-dimensional DTNs. These studies introduce a model based on space discretization to derive upper and lower bounds in the highway model under the assumption that the radio propagation speed is finite. Their bounds, although not converging, clearly indicates the existence of a phase transition phenomenon for the information propagation speed. Comparatively, we introduce a model based on Poisson point process on continuous space, that allows both infinite and finite radio propagation speed, and derive more fine-grained results above and below the threshold (some of the work described in the following was presented in [4]). Using our model, we prove and explicitly characterize the phase transition.

In this context, our contributions are as follows: (1) we develop a new vehicle-to-vehicle model for information propagation in bidirectional vehicular DTNs in Section 2; (2) we show the existence of a threshold (with respect to the vehicle density), above which the information speed increases dramatically over the vehicle speed, and below which the information propagation speed is on average equal to the vehicle speed, and (3) we give the exact expression of this threshold, in Section 3; (4) in Section 4, we prove that, under the threshold, even though the average propagation speed equals the vehicle speed, DTN routing using cars moving on both directions provides a gain in the propagation distance, and this gain follows a sub-linear power law with respect to the elapsed time, in the referential of the moving cars; (5) we characterize information propagation speed as increasing quasi-exponentially with the vehicle density when the latter becomes large above the threshold, in Section 5; (6) we cover both infinite radio propagation speed cases, then finite radio propagation speed cases in Section 6, and (7) we validate the provided analysis with simulations in multiple environments (The One and Maple), in Section 7.

2 Model and Results

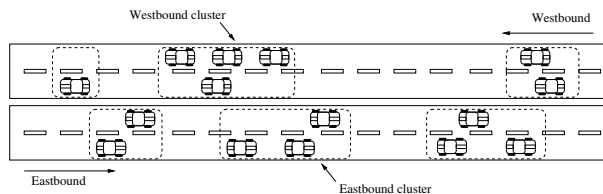


Figure 1: Model of a bidirectional vehicular network on a highway.

In the following, we consider a bidirectional vehicular network, such as a road or a highway, where vehicles move in two opposite directions (say east and west, respectively) at speed v , as depicted in Figure 1. Let us consider eastbound vehicle density as Poisson with intensity λ_e , while westbound vehicle density is

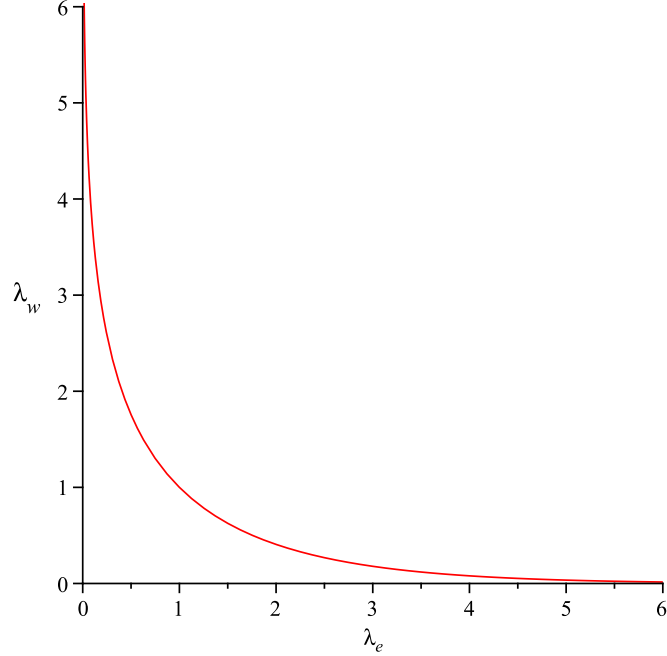


Figure 2: Information propagation threshold with respect to (λ_e, λ_w) . Below the curve, the average information propagation speed is limited to the vehicle speed (*i.e.*, the propagation speed is 0 in the referential of the eastbound cars), while above the curve, information propagates faster on average.

Poisson with intensity λ_w . We note that the Poisson distribution is indeed a reasonable approximation of vehicles moving on non-congested highways [19]. Furthermore, we consider that the radio propagation speed (including store and forward processing time) is infinite, and that the radio range of each transmission in each direction is of length R .

The main result presented in this paper is that, concerning the information propagation speed in such an environment, a phase transition occurs when $\lambda_e R$ and $\lambda_w R$ coincide on the curve $y = xe^{-x}$, *i.e.*,

$$\lambda_e R e^{-\lambda_e R} = \lambda_w R e^{-\lambda_w R}. \quad (1)$$

Figure 2 shows the corresponding threshold curve for $R = 1$ (in the following, we will always consider the case $R = 1$, without loss of generality). We show that below this threshold, the average information propagation speed is limited to the vehicle speed, while above the curve, information propagates faster on average.

We focus on the propagation of information in the eastbound lane. Our aim is the evaluation of the maximum speed at which a packet (or beacon) of information can propagate. An information beacon propagates in the following manner, illustrated in Figure 3: it moves toward the east jumping from car to car until it stops because the next car is beyond radio range. The propagation is instantaneous, since we assume that radio routing speed is infinite. The beacon

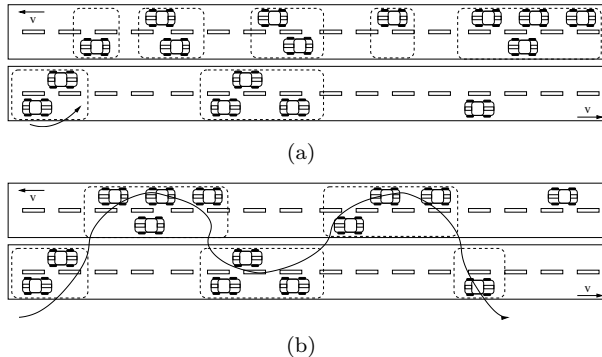


Figure 3: Eastbound information propagation: the beacon waits on the last eastbound car (a), until the gap is bridged by westbound cars so that the beacon can move again (b).

waits on the last eastbound car until the gap is filled by westbound cars, so that the beacon can move again to the next eastbound car.

We denote \mathbf{T}_i the duration the beacon waits when blocked for the i th time and \mathbf{D}_i the distance traveled by the beacon just after. The random variables \mathbf{T}_i and \mathbf{D}_i are dependent but, due to the Poisson nature of vehicle traffic, the tuples in the sequence $(\mathbf{T}_i, \mathbf{D}_i)$ are *i.i.d.*. From now on, we denote (\mathbf{T}, \mathbf{D}) the independent random variable.

We denote $L(t)$ the distance traveled by the beacon during a time t on the eastbound lane. We consider the distance traveled with respect to the referential of the eastbound cars. We also define the average information propagation speed v_p as:

$$v_p = \lim_{t \rightarrow \infty} \frac{\mathbf{E}(L(t))}{t}. \quad (2)$$

By virtue of the renewal processes, we have

$$v_p = \frac{\mathbf{E}(\mathbf{D})}{\mathbf{E}(\mathbf{T})}. \quad (3)$$

For the remainder of the paper, for $x > 0$, we denote x^* the conjugate of x with respect to the function xe^{-x} : x^* is the alternate solution of the equation $x^*e^{-x^*} = xe^{-x}$. Notice that $x^{**} = x$ and $1^* = 1$.

We prove the following theorems, concerning the information propagation phase transition threshold (Theorem 1), the distribution of the waiting time spent by information packets in the phase depicted in Figure 3a (Theorem 2), and the total propagation distance achieved (under the phase transition threshold) because of multi-hop bridging using nodes in both traffic directions, as depicted in Figure 3b (Theorem 3).

Theorem 1. For all (λ_e, λ_w) , the information propagation speed v_p with respect to the referential of the eastbound cars is $v_p < \infty$, and,

$$\lambda_e < \lambda_w^* \Rightarrow v_p = 0, \quad (4)$$

$$\lambda_e > \lambda_w^* \Rightarrow v_p > 0. \quad (5)$$

Theorem 2. When $t \rightarrow \infty$,

$$P(\mathbf{T} > t) = A(\lambda_e, \lambda_w)(2vt)^{-\frac{\lambda_e}{\lambda_w^*}}(1 + o(1)) , \quad (6)$$

for some $A(\lambda_e, \lambda_w)$, function of (λ_e, λ_w) .

Notice that Theorem 1 is in fact a corollary of Theorem 2.

Theorem 3. When $\lambda_w^* > \lambda_e$ (case $v_p = 0$), when $t \rightarrow \infty$,

$$\mathbf{E}(L(t)) = B(\lambda_e, \lambda_w)(2vt)^{\frac{\lambda_e}{\lambda_w^*}} + O(t^{2\frac{\lambda_e}{\lambda_w^*}-1}) . \quad (7)$$

for some $B(\lambda_e, \lambda_w)$, function of (λ_e, λ_w) .

3 Phase Transition: Proof of Theorem 1

3.1 Proof Outline

We call cluster a maximal sequence of cars such that two consecutive cars are within radio range. A westbound (respectively, eastbound) cluster is a cluster made exclusively of westbound (respectively, eastbound) cars. A full cluster is made of westbound and eastbound cars.

We define the length of the cluster as the distance between the first and last cars augmented by a radio range. We denote L_w a westbound cluster length. We start by proving in Section 3.2 that the Laplace transform of L_w : $f_w(\theta) = \mathbf{E}(e^{-\theta L_w})$ equals:

$$f_w(\theta) = \frac{(\lambda_w + \theta)e^{-\lambda_w - \theta}}{\theta + \lambda_w e^{-\lambda_w - \theta}} , \quad (8)$$

and the exponential tail of the distribution of L_w is given by

$$P(L_w > x) = \Theta(e^{-\lambda_w^* x}) . \quad (9)$$

To evaluate how information will propagate according to Figure 3, we compute the distribution of the gap length G_e between the cluster of eastbound cars on which the beacon is blocked and the next cluster of eastbound cars. We show in Section 3.4 that $P(G_e > x) = O(e^{-\lambda_e x})$.

Now, let $\mathbf{T}(x)$ be the time needed to meet a westbound cluster long enough to fill a gap of length x (i.e., a westbound cluster of length larger than x). We show in Section 3.3 that:

$$\mathbf{E}(\mathbf{T}(x)) = \Theta\left(\frac{1}{vP(L_w > x)}\right) = \Theta(e^{\lambda_w^* x}) . \quad (10)$$

Therefore, the average time \mathbf{T} to get a bridge over all possible gaps is

$$\begin{aligned} \mathbf{E}(\mathbf{T}) &= \int_1^\infty \mathbf{E}(\mathbf{T}(x))e^{-x\lambda_e} dx \\ &= \frac{1}{2v} \int_1^\infty \Theta(\exp((\lambda_w^* - \lambda_e)x)) dx . \end{aligned} \quad (11)$$

As a result, the threshold with respect to (λ_w, λ_e) where $\mathbf{E}(\mathbf{T})$ diverges is clearly when we have:

$$\lambda_w^* = \lambda_e, \quad (12)$$

or, in other words, since $\lambda_w^* e^{-\lambda_w^*} = \lambda_w e^{-\lambda_w}$, when we have:

$$\lambda_w e^{-\lambda_w} = \lambda_e e^{-\lambda_e}. \quad (13)$$

3.2 Cluster Length Distribution

Lemma 1. *The Laplace transform of the westbound cluster length $f_w(\theta) = \mathbf{E}(e^{-\theta L_w})$ satisfies:*

$$f_w(\theta) = \frac{(\lambda_w + \theta)e^{-\lambda_w - \theta}}{\theta + \lambda_w e^{-\lambda_w - \theta}}. \quad (14)$$

Proof. The length of the cluster is counted from the first car. The random variable L_w satisfies:

- $L_w = 1$, with probability $e^{-\lambda_w}$, when the first car has no car behind within the radio range;
- $L_w = g_w + L_w$, with probability $1 - e^{-\lambda_w}$, where g_w is the distance to the next car and $g_w < 1$.

Translating this in terms of Laplace transforms yields:

$$f_w(\theta) = e^{-\lambda_w - \theta} + f_w(\theta) \int_0^1 \lambda_w e^{-(\lambda_w + \theta)x} dx \quad (15)$$

$$= e^{-\lambda_w - \theta} + f_w(\theta) \frac{\lambda_w}{\lambda_w + \theta} (1 - e^{-\lambda_w - \theta}). \quad (16)$$

□

In passing, we get $\mathbf{E}(L_w) = -f_w'(0) = \frac{e^{\lambda_w} - 1}{\lambda_w}$.

Lemma 2. *We have the asymptotic formula:*

$$P(L_w > x) = \frac{(\lambda_w - \lambda_w^*)e^{\lambda_w^* - \lambda_w}}{(1 - \lambda_w^*)\lambda_w^*} e^{-\lambda_w^* x} (1 + o(1)) \quad (17)$$

Proof. The asymptotics on $P(L_w > x)$ are given by inverse Laplace transform:

$$\begin{aligned} P(L_w > x) &= -\frac{1}{2i\pi} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} \frac{f_w(\theta)}{\theta} e^{\theta x} d\theta \\ &= -\frac{1}{2i\pi} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} \frac{(\lambda_w + \theta)e^{-\lambda_w}}{(\theta e^\theta + \lambda_w e^{-\lambda_w})\theta} e^{\theta x} d\theta, \end{aligned}$$

for some $\varepsilon > 0$ small enough. For $\Re(\theta) < 0$ the denominator $(\theta e^\theta + \lambda_w e^{-\lambda_w})$ has two simple roots at $\theta = -\lambda_w^*$ and $\theta = -\lambda_w$ and is absolutely integrable elsewhere. The root $-\lambda_w$ does not lead to a singularity since it is canceled by the numerator $\lambda_w + \theta$. The residues theorem neutralizes the pole at $-\lambda_w$, therefore for some $\varepsilon > 0$:

$$P(L_w > x) = \frac{(\lambda_w - \lambda_w^*)e^{\lambda_w^* - \lambda_w}}{(1 - \lambda_w^*)\lambda_w^*} e^{-\lambda_w^* x} + O(e^{-(\lambda_w^* + \varepsilon)x}).$$

□

3.3 Road Length to Bridge a Gap

Now, let us assume that we want to fill a gap of length x . We want to know the average length of westbound road until the first cluster that has a length greater than $x - 1$. Figure 4 depicts a gap of length x , and the length of westbound road until a cluster is encountered which can bridge the gap. Let $f_w(\theta, x)$ be the Laplace transform of the cluster length, under the condition that it is smaller than x : $f_w(\theta, x) = \mathbf{E}(1_{(L_w < x)} e^{-\theta L_w})$.

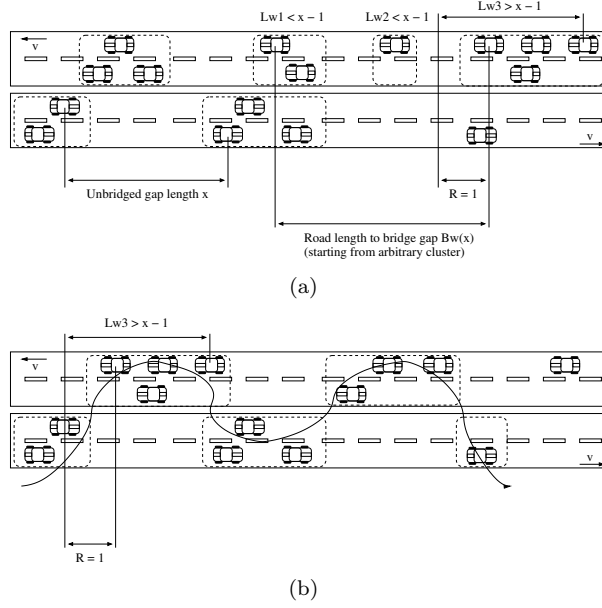


Figure 4: Illustration of the road length $B_w(x)$ until a gap x is bridged: (a) smaller clusters cannot bridge the gap, (b) until a westbound cluster of length at least $x - 1$ is encountered.

Lemma 3. *The Laplace transform of the road length $B_w(x)$ to bridge a gap of length x , starting from the beginning of an arbitrary cluster, is:*

$$\beta_w(\theta, x) = \mathbf{E}(e^{-\theta B_w(x)}) = \frac{P(L_w > x - 1)}{1 - \frac{\lambda_w}{\lambda_w + \theta} f_w(\theta, x - 1)}, \quad (18)$$

and,

$$\mathbf{E}(B_w(x)) = (1 + O(e^{-\varepsilon x})) \frac{e^{\lambda_w}}{\lambda_w} \frac{(1 - \lambda_w^*) \lambda_w^*}{(\lambda_w - \lambda_w^*) e^{\lambda_w^* - \lambda_w}} e^{(x-1) \lambda_w^*}. \quad (19)$$

Proof. Before a cluster of length greater than x appears there is a succession of clusters L_1, L_2, \dots each of length smaller than x (see Figure 4b). Several cases are possible:

- the first cluster to come is a cluster of length greater than x , with probability $P(L_w > x - 1)$; the road length to this cluster is equal to 0.

- the first cluster greater than x is the second cluster; in this case, the road length Laplace transform is equal to $f_w(\theta, x) \frac{\lambda_w}{\lambda_w + \theta}$, *i.e.*, the Laplace transform of a cluster multiplied the Laplace transform of the exponentially distributed inter-cluster distance, namely $\frac{\lambda_w}{\lambda_w + \theta}$ (as Laplace transform multiplication represents the addition of the corresponding independent random variables).
- or, in general, the first cluster greater than x is the k th cluster; in this case, the road length Laplace transform is equal to $\left(f_w(\theta, x) \frac{\lambda_w}{\lambda_w + \theta}\right)^k$

Therefore, the Laplace transform of the road length to the cluster of length greater than x is equal to the sum of the Laplace transforms of the previous cases, *i.e.*, $P(L_w > x - 1) \sum_{k=0}^{\infty} \left(f_w(\theta, x) \frac{\lambda_w}{\lambda_w + \theta}\right)^k$.

Thus, the average is (combining with Lemma 2):

$$\begin{aligned}
\mathbf{E}(B_w(x)) &= -\frac{\partial}{\partial \theta} \beta_w(0, x) \\
&= -\left(\frac{\partial}{\partial \theta} f_w(0, x - 1) - \frac{1}{\lambda_w} f_w(0, x - 1) \right) \\
&\quad \times \frac{1}{P(L_w > x - 1)} \\
&= \left(\frac{e^{\lambda_w}}{\lambda_w} + O(e^{-(x-1)\lambda_w^*}) \right) \frac{1}{P(L_w > x - 1)} \\
&= (1 + O(e^{-\varepsilon x})) \frac{e^{\lambda_w} (1 - \lambda_w^*) \lambda_w^*}{\lambda_w (\lambda_w - \lambda_w^*) e^{\lambda_w^* - \lambda_w}} e^{(x-1)\lambda_w^*}
\end{aligned}$$

□

3.4 Gap Distribution

Let us call G_e an eastbound gap which is not bridged (see Figure 5). As illustrated in Figure 6, G_e can be decomposed into a westbound cluster length L_w^* *without* eastbound cars, plus a random exponentially distributed distance I_e to the next eastbound car.

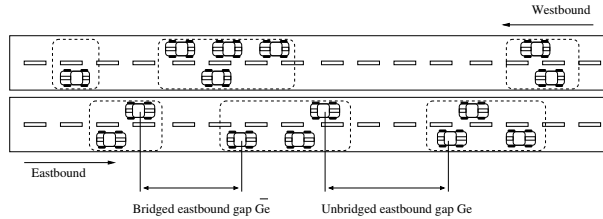


Figure 5: Illustration of a bridged gap \bar{G}_e , and an unbridged gap G_e .

Lemma 4. *The distribution of G_e satisfies*

$$\mathbf{E}(e^{-\theta G_e}) = \frac{f_w(\theta + \lambda_e)}{f_w(\lambda_e)} \frac{\lambda_e}{\lambda_e + \theta}, \quad (20)$$

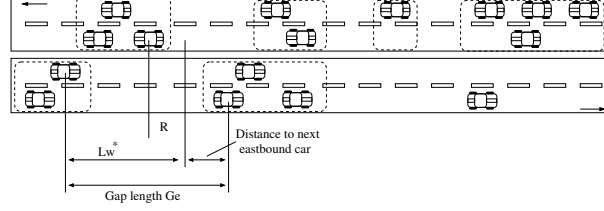


Figure 6: Unbridged gap G_e model; L_w^* corresponds to a westbound cluster length without eastbound cars.

which is defined for all $\Re(\theta) > -\lambda_e$, and

$$\mathbf{E}(G_e) = -\frac{f'_w(\lambda_e)}{f_w(\lambda_e)} + \frac{1}{\lambda_e}. \quad (21)$$

Proof. Let $p_w(x)$ be the probability density of a westbound cluster length L_w . The probability that a westbound cluster has no eastbound cars is:

$$\int_0^\infty p_w(x)e^{-\lambda_e x} dx = f_w(\lambda_e). \quad (22)$$

The Laplace transform of the westbound cluster length without eastbound cars, defined for all $\Re(\theta) > \lambda_w^* + \lambda_e$, equals:

$$\mathbf{E}(e^{-\theta L_w^*}) = \frac{\int_0^\infty p_w(x)e^{-\lambda_e x} e^{-\theta x} dx}{Pr(\text{no eastbound})} = \frac{f_w(\theta + \lambda_e)}{f_w(\lambda_e)}. \quad (23)$$

and the Laplace transform of $G_e = L_w^* + I_e$ follows, as well as its average estimate. \square

Lemma 5. *The probability density $p_e(x)$ of G_e is:*

$$p_e(x) = \frac{\lambda_e}{f_w(\lambda_e)} e^{-\lambda_e x} (1 + O(e^{-\varepsilon x})). \quad (24)$$

Proof. The proof comes from a straightforward singularity analysis on the inverse Laplace transform. \square

3.5 Distribution of Waiting Time \mathbf{T}

Lemma 6. *We have $2v\mathbf{T} = L_w^* + I_w + B_w - 1$ (where I_w is a random exponentially distributed distance to the next westbound car), and, therefore,*

$$2v\mathbf{E}(\mathbf{T}) = \mathbf{E}(L_w^*) - 1 + \frac{1}{\lambda_w} + \int_1^\infty \mathbf{E}(B_w(x))p_e(x)dx. \quad (25)$$

Proof. The total distance to bridge a gap, as depicted in Figure 7, equals the distance to the beginning of the first westbound cluster ($L_w^* + I_w$) plus the road length to bridge a gap starting from an arbitrary cluster (B_w) minus 1,

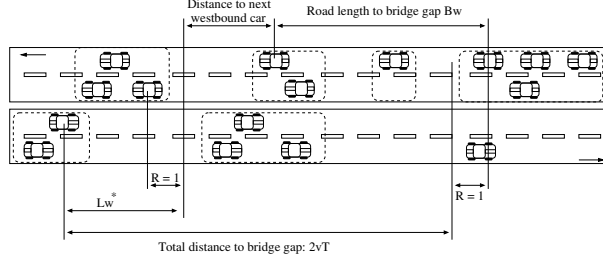


Figure 7: Waiting time \mathbf{T} : the total distance to bridge a gap is $L_w^* + I_w + B_w - 1$.

since communication can start at exactly one radio range. Since the distance is covered by cars moving in opposite directions, we have $2vT = L_w^* + I_w + B_w - 1$. We complete the proof by taking the expectations, and averaging on all possible gap lengths x . \square

Corollary 1. *The quantity $\mathbf{E}(\mathbf{T})$ converges when $\lambda_e > \lambda_w^*$ and diverges when $\lambda_e < \lambda_w^*$.*

Proof. The proof comes from the leading terms of $\mathbf{E}(B_w(x))$ and $p_e(x)$. \square

3.6 Distance \mathbf{D} Traveled after Waiting Time \mathbf{T}

We denote C_e the distance traveled beyond the first gap. As depicted in Figure 8, we have $\mathbf{D} = G_e + C_e$.

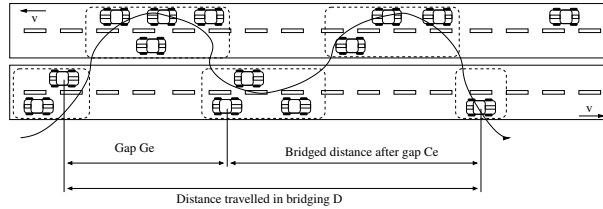


Figure 8: Total distance \mathbf{D} traveled when a bridge is created $\mathbf{D} = G_e + C_e$.

Lemma 7. *The Laplace transform $\mathbf{E}(e^{-\theta C_e})$ is defined for all $\Re(\theta) > -(\lambda_e + \lambda_w)^*$.*

Proof. The random variable C_e is smaller in probability than a full cluster. \square

Lemma 8. *The average value of C_e satisfies:*

$$\mathbf{E}(C_e) = \frac{1}{\lambda_e} \frac{1 - f_w(\lambda_e)}{f_w(\lambda_e)} + \frac{f'_w(\lambda_e)}{f_w(\lambda_e)}. \quad (26)$$

Proof. From (22), the probability that an eastbound car is not bridged to the next eastbound car equals $f_w(\lambda_e)$. The unconditional gap length is $\frac{1}{\lambda_e}$. We define \bar{G}_e as an eastbound gap, under the condition that the gap is bridged (see Figure 5). Therefore, the average gap length \bar{G}_e satisfies the following:

$$f_w(\lambda_e)\mathbf{E}(G_e) + (1 - f_w(\lambda_e))\mathbf{E}(\bar{G}_e) = \frac{1}{\lambda_e}, \quad (27)$$

which gives $\mathbf{E}(\bar{G}_e) = \frac{1}{\lambda_e} + \frac{f'_w(\lambda_e)}{1-f_w(\lambda_e)}$.

The distance C_e traveled in bridging (beyond the first gap, and extended to the next cluster, which is eventually bridged) satisfies:

$$\mathbf{E}(C_e) = (1 - f_w(\lambda_e)) (\mathbf{E}(\bar{G}_e) + \mathbf{E}(C_e)) \quad (28)$$

$$= \frac{1 - f_w(\lambda_e)}{\lambda_e} + \frac{f'_w(\lambda_e)}{f_w(\lambda_e)}. \quad (29)$$

□

Corollary 2. *The total distance D_e traveled including the first gap satisfies $\mathbf{E}(D_e) = \mathbf{E}(G_e) + \mathbf{E}(C_e) = \frac{1}{\lambda_e f_w(\lambda_e)}$, which remains finite for all vehicle densities.*

Since $\mathbf{E}(D_e)$ is finite (Corollary 2) and $\mathbf{E}(\mathbf{T})$ converges when $\lambda_e > \lambda_w^*$, and diverges when $\lambda_e < \lambda_w^*$ (Corollary 1), we obtain the proof of Theorem 1.

4 Power Laws: Proof of Theorems 2 and 3

4.1 Waiting Time Distribution

In this section, we are interested in finding an evaluation of the waiting time distribution $P(\mathbf{T} > y)$, when $y \rightarrow \infty$, in case $\lambda_e < \lambda_w^*$, *i.e.*, when the information propagation speed is 0 on average.

Lemma 9. *When y tends to infinity,*

$$P(B_w > y) = A(\lambda_e, \lambda_w) y^{-\frac{\lambda_e}{\lambda_w^*}} (1 + o(1)),$$

with

$$A(\lambda_e, \lambda_w) = \frac{\lambda_e e^{-\lambda_e}}{\lambda_w^* f_w(\lambda_e)} \Gamma\left(\frac{\lambda_e}{\lambda_w^*}\right) \beta^{-\frac{\lambda_e}{\lambda_w^*}},$$

where $\Gamma(\cdot)$ is the Euler “Gamma” function, $\beta = \frac{\lambda_w(\lambda_w - \lambda_w^*) e^{\lambda_w^* - 2\lambda_w}}{1 - \lambda_w^*}$.

Proof. See appendix. □

Lemma 10. *When $t \rightarrow \infty$:*

$$P(\mathbf{T} > t) = A(\lambda_e, \lambda_w) (t2v)^{-\frac{\lambda_e}{\lambda_w^*}} (1 + o(1)). \quad (30)$$

Proof. We have the relation $\frac{\mathbf{T}}{2v} = G_{ew} + B_w$, with $G_{ew} = L_w^* + I_w - 1$. We know that G_{ew} , in analogy with G_e , has an exponential tail, *i.e.*, $\mathbf{E}(e^{-\theta G_{ew}}) < \infty$ for all $\theta > -\lambda_w$. In other words $P(G_{ew} > y) = O(\exp(-\theta y))$. The other particularity is that G_{ew} and B_w are dependent. First we have the inequality for all y

$$P(G_{ew} + B_w > y) \geq P(B_w > y),$$

therefore, we have $P(\mathbf{T} > t) \geq A(\lambda_e, \lambda_w) (t2v)^{-\frac{\lambda_e}{\lambda_w^*}} (1 + o(1))$. Second, we have the other inequality for all (y, z) :

$$P(G_{ew} + B_w > y) \leq P(G_{ew} > z) + P(B_w > y - z).$$

Thus, by selecting $z = O(\log t)$ such that $P(G_{ew} > z) = o(t^{-\frac{\lambda_e}{\lambda_w^*}})$, we get $P(\mathbf{T} > t) \leq A(\lambda_e, \lambda_w) (t - O(\log t)2v)^{-\frac{\lambda_e}{\lambda_w^*}} + o(t^{-\frac{\lambda_e}{\lambda_w^*}})$ □

Therefore, we conclude the proof of Theorem 2.

4.2 Traveled Distance Distribution

Now, we focus on the renewal process made of the various waiting time intervals \mathbf{T} , experienced by the information beacon. Considering the sequence of waiting phases, $\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_n, \dots$: the beacon moves at time \mathbf{T}_1 , then at time $\mathbf{T}_1 + \mathbf{T}_2$, $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$, etc. We denote $n(t)$ the number of phases achieved before time t :

$$\sum_{i=1}^{i=n(t)} \mathbf{T}_i \leq t < \sum_{i=1}^{i=n(t)+1} \mathbf{T}_i .$$

Since the \mathbf{T}_i are independent, this is a renewal process, and we have the identity:

$$P(n(t) \leq n) = P(\mathbf{T}_1 + \dots + \mathbf{T}_n \geq t) .$$

We can get a precise estimate of the average number of renewals $\mathbf{E}(n(t))$ during time t .

Lemma 11. *There exists $b > 0$ such that, when $t \rightarrow \infty$, the following estimate is valid:*

$$\mathbf{E}(n(t)) = \frac{\sin^2(\pi \frac{\lambda_e}{\lambda_w^*})}{b\pi^2} \Gamma(\frac{\lambda_e}{\lambda_w^*}) t^{\frac{\lambda_e}{\lambda_w^*}} + O(t^{2\frac{\lambda_e}{\lambda_w^*} - 1}) . \quad (31)$$

Proof. See appendix. □

In parallel to the sequence of waiting intervals $\{\mathbf{T}_i\}_{i \geq 1}$, we have the sequence $\{\mathbf{D}_i\}_{i \geq 1}$ the distances traveled by the beacon after every waiting interval \mathbf{T}_i . Now we denote $L(t) = \sum_{i=1}^{i=n(t)} \mathbf{D}_i$ which is the total distance traveled by the beacon until time t .

Lemma 12.

$$\mathbf{E}(L(t)) = \mathbf{E}(n(t))\mathbf{E}(\mathbf{D}) .$$

Proof. We have the identity:

$$\mathbf{E}(L(t)) = \sum_{n>0} \mathbf{E}(1_{n(t) \geq n} D_n), \quad (32)$$

where $1_{n(t) \geq n}$ is the indicative function of the event $n(t) \geq n$. Since, from the definition of $n(t)$, we have the equivalence: $n(t) \geq n \Leftrightarrow \mathbf{T}_1 + \dots + \mathbf{T}_{n-1} < t$, then $1_{n \leq n(t)}$ and D_n are independent random variables and, therefore, $\mathbf{E}(1_{n(t) \geq n} D_n) = P(n(t) \geq n)\mathbf{E}(\mathbf{D})$. □

Quantity $\mathbf{E}(\mathbf{D})$ has a closed expression ($\mathbf{E}(\mathbf{D}) = \frac{1}{\lambda_e f_w(\lambda_e)}$, from Corollary 2. Substituting and using Lemma 11, the power law for $\mathbf{E}(L(t))$ in Theorem 3 is shown.

5 Asymptotic Estimates

5.1 Near the Threshold

First, we investigate the case where (λ_e, λ_w) is close to the threshold boundary.

Corollary 3. When $\lambda_e \rightarrow (\lambda_w^*)^+$:

$$v_p \sim 2v \frac{(\lambda_w - \lambda_w^*)\lambda_w}{\lambda_e^2(1 - \lambda_w^*)\lambda_w^*} (\lambda_e - \lambda_w^*) e^{\lambda_w^* + \lambda_e - 2\lambda_w} . \quad (33)$$

Proof. Using Lemma 6, we have:

$$\begin{aligned} 2vE(\mathbf{T}) &= -\frac{f'_w(\lambda_e)}{f_w(\lambda_e)} + \int_1^\infty \mathbf{E}(B_w(x))p_e(x)dx \\ &\sim \int_1^\infty \frac{e^{\lambda_w}}{\lambda_w} \frac{(1 - \lambda_w^*)\lambda_w^*}{\lambda_w - \lambda_w^*} e^{\lambda_w - \lambda_w^*} e^{(x-1)\lambda_w^*} \\ &\quad \times \frac{\lambda_e}{f_w(\lambda_e)} e^{-\lambda_e x} dx . \end{aligned}$$

Integrating, we obtain $v_p = \frac{E(\mathbf{D})}{E(\mathbf{T})}$. \square

5.2 Large Densities

Corollary 4. When the vehicle densities become large, i.e., $\lambda_e, \lambda_w \rightarrow \infty$:

$$v_p \sim 2v \frac{e^{\lambda_e + \lambda_w}}{1 + \frac{\lambda_w}{\lambda_e} + \frac{\lambda_e}{\lambda_w}} . \quad (34)$$

Proof. According to Lemma 4, we have:

$$E(L_w^*) = 1 + \frac{\lambda_w}{\lambda_e(\lambda_w + \lambda_e)} , \quad (35)$$

and the expected gap length tends to 1. The average road length to bridge such a gap tends to $\frac{1}{\lambda_w}$. From Lemma 6:

$$2vE(\mathbf{T}) \sim E(L_w^*) - 1 + \frac{1}{\lambda_w} . \quad (36)$$

From corollary 2, the average distance traveled in bridging is:

$$E(\mathbf{D}) = \frac{1}{\lambda_e f_w(\lambda_e)} \sim \frac{e^{\lambda_e + \lambda_w}}{\lambda_e + \lambda_w} , \quad (37)$$

and we obtain $v_p = \frac{E(\mathbf{D})}{E(\mathbf{T})}$. \square

Note that the information propagation speed grows quasi-exponentially with respect to the total vehicle density.

6 Finite radio propagation speed

If the radio propagation speed (including store and forward timings) is finite and constant, equal to v_r , then the average information propagation speed becomes:

$$s = \frac{\mathbf{E}(T_w)v + \mathbf{E}(D_e)}{\mathbf{E}(T_w) + \frac{1}{v_r}\mathbf{E}(D_e)} . \quad (38)$$

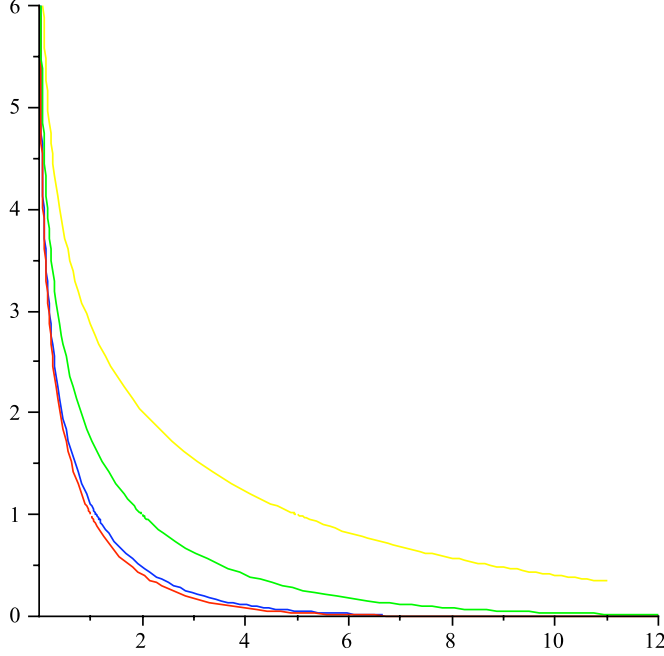


Figure 9: Threshold zones for (λ_e, λ_w) for various $\gamma = \frac{v}{v_r}$, $\gamma = 0$ red, $\gamma = 0.1$ blue, $\gamma = 0.5$ green, $\gamma = 0.8$ yellow.

But the speed v_r impacts the random variables G_e and T_w . The main impact is that, to fill an eastbound gap of length x , one needs a westbound cluster of length at least $\frac{x(1+\gamma)}{1-\gamma}$ with $\gamma = \frac{v}{v_r}$, otherwise the message will be off the gap when arriving at the end of the cluster. Thus, $\mathbf{E}(T_w(x)) = O(e^{\frac{(1+\gamma)\theta_w}{1-\gamma}x})$.

The gap length is also modified, since we must consider $\frac{1-\gamma}{1+\gamma}L_w^*$:

$$\mathbf{E}(e^{-\theta G_e}) = \frac{f_w(\frac{1-\gamma}{1+\gamma}(\theta + \lambda_e))}{f_w(\frac{1-\gamma}{1+\gamma}\lambda_e)} \frac{\lambda_e}{\lambda_e + \theta}. \quad (39)$$

But, this does not change the exponential term $e^{-\lambda_e x}$ in the asymptotic expression of $p_e(x)$. Therefore, the threshold condition becomes:

$$\lambda_w e^{-\lambda_w} = \frac{1-\gamma}{1+\gamma} \lambda_e e^{-\frac{1-\gamma}{1+\gamma}\lambda_e}. \quad (40)$$

This corresponds to a dilatation by a factor $\frac{1+\gamma}{1-\gamma}$ of the horizontal axis in the diagram of Figure 2. Notice how the diagram then loses its symmetry with respect to λ_e versus λ_w , which can be observed in Figure 9.

Notice also that, when v_r decreases to v , the threshold limit tends to infinity.

Similarly, in correspondance to Theorem 3, we have $\mathbf{E}(L(t)) = \Omega(t^{\frac{1-\gamma}{1+\gamma}} \frac{\lambda_e}{\lambda_w^*})$, due to the dilatation by the factor $\frac{1+\gamma}{1-\gamma}$.

7 Simulations

In this section, we present simulation results obtained with Maple on one hand, and the Opportunistic Network Environment (ONE [14]) simulator on the other hand.

7.1 Maple Simulations

We first compare the theoretical analysis with measurements performed using Maple. In this case, the simulations follow precisely the bidirectional highway model described in Section 2: we generate Poisson traffic of eastbound and westbound traffic on two opposite lanes moving at constant speed, which is set to $v = 1m/s$. The radio propagation range is $R = 1m$, and radio transmissions are instantaneous; the length of the highway is sufficiently large to provide a large number of bridging operations (of order at least 10^3) for all considered traffic densities.

We measure the information propagation speed which is achieved using optimal DTN routing, by selecting a source and destination pairs at large distances, taking the ratio of the propagation distance over the corresponding delay, and averaging over multiple iterations of randomly generated traffic. We vary the total traffic density, and we plot the resulting information propagation speed. Figures 10 and 11 show the evolution of the information propagation speed near the threshold versus the total vehicle density, when $\lambda_e = \lambda_w$, in linear and semilogarithmic plots, respectively. We can observe the threshold at $\lambda_e + \lambda_w = 2$ in Figure 10, which confirms the analysis presented previously in Section 3, and corresponds to $\lambda_e = \lambda_w = 1$ in Figure 2. In semilogarithmic scale (Figure 11), we observe that the simulation measurements quickly approach a straight line, and therefore are close to the theoretically predicted exponential growth above the phase transition threshold, in Section 5. In Figure 12, we present a 3-dimensional plot of the eastbound information propagation speed v_p by varying the vehicle densities in both eastbound (λ_e) and westbound (λ_w) traffic.

Finally, we perform detailed measurements of the waiting time \mathbf{T} that each packet of information spends in the buffer of an eastbound car until it encounters a westbound cluster which allows it to propagate faster to the next eastbound car. For the measurements, we set the vehicle densities to $\lambda_e = \lambda_w = 0.9$; thus the conjugate $\lambda_w^* = 1.107\dots$. In Figure 13, (*i.e.*, below the phase transition threshold), we plot the distribution of the waiting time \mathbf{T} , and we compare it to the predicted power law in Theorem 2: $P(\mathbf{T} > t) = t^{-\frac{\lambda_e}{\lambda_w^*}}$.

7.2 ONE Simulations

In this section, we depart from the exact Poisson model simulations in Maple, and we present simulation results obtained with the Opportunistic Network Environment (ONE [14]) simulator. Vehicles are distributed uniformly on the length of both lanes of a road, and move at a constant unit speed. The total number of vehicles varies from 1000 to 5000. Similarly to the previous section, we measure the fastest possible information propagation speed which is achieved using epidemic broadcast, assuming that radio transmissions are instantaneous and that there are no buffering or congestion delays, with a radio range $R = 10m$. Again, we vary the vehicle densities λ_e and λ_w , which are

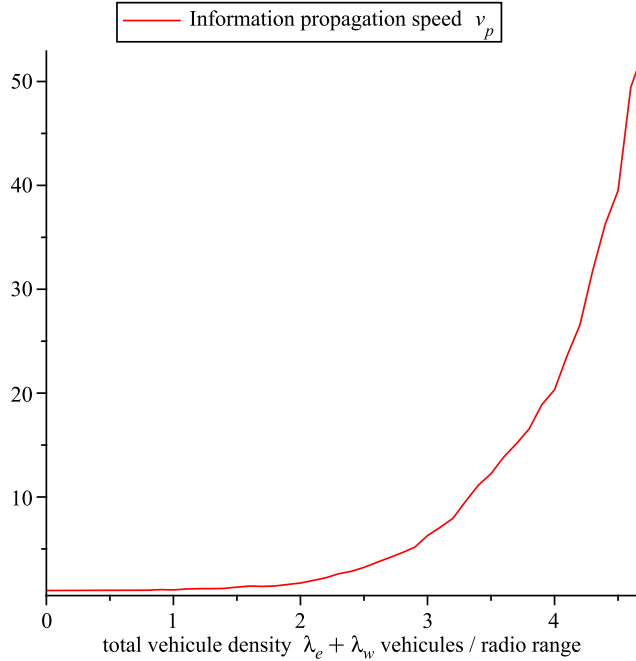


Figure 10: Maple simulations. Information propagation speed v_p for $\lambda_e = \lambda_w$, versus the total vehicle density $\lambda_e + \lambda_w$.

given in vehicles per radio range, and we perform several simulation iterations of randomly generated traffic. As shown in Figure 14, similarly to the Maple simulation results and to the analysis, we observe the threshold phenomenon at $\lambda_e = \lambda_w = 1$: the information propagation speed remains almost constant below the threshold but increases dramatically beyond it. We also observe an exponential growth above the threshold in Figure 15.

We remark that measurements below the phase transition threshold yield an average information propagation speed which is slightly larger than the vehicle speed. This is due to the finite duration of the simulations and in the computations of the expectations. This phenomenon can also be explained from the theoretical analysis in Section 4: even below the threshold, DTN routing using cars moving on both directions still provides a gain in the propagation distance, which follows a sub-linear power law with respect to the elapsed time (in the referential of the moving cars). Figure 16 confirms the predicted power law $t^{\lambda_e/\lambda_w^*}$ for $\mathbf{E}(L(t))$, the average distance traveled by information with respect to time (in Theorem 3), which is shown in Figure 16 for $\lambda_e = 1$ and $\lambda_w = 0.9$, therefore, a growth of order $t^{0.903}$.

8 Concluding Remarks

In this paper, we provided a detailed analysis for information propagation in bidirectional vehicular DTNs. We proved the existence of a threshold, concerning vehicle density, above which information speed increases dramatically over

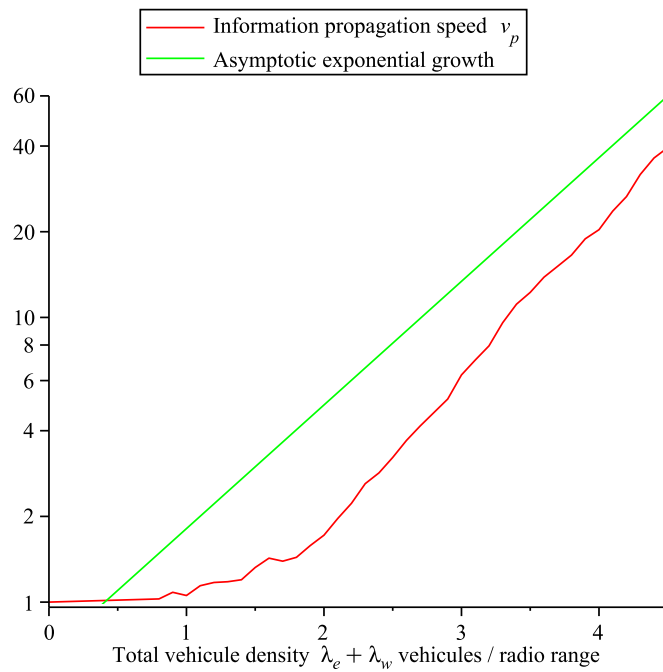


Figure 11: Maple simulations. Information propagation speed v_p for $\lambda_e = \lambda_w$, versus the total vehicle density $\lambda_e + \lambda_w$, in semi-log scale, compared to the theoretically predicted asymptotic exponential growth.

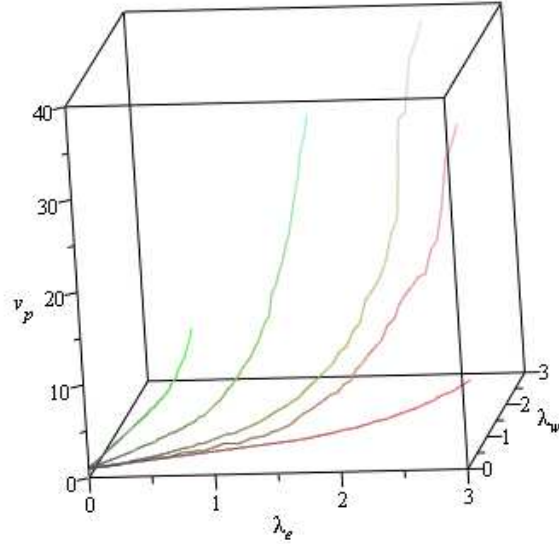


Figure 12: Maple simulations. Eastbound information propagation speed v_p for different values of the vehicle densities, λ_e and λ_w .

vehicle speed, and below which information propagation speed is on average equal to vehicle speed (in Theorem 1), and we computed the exact expression of this threshold. We exactly characterized the information speed near the threshold (Corollary 3), and we showed that, above the threshold, the information propagation speed increases quasi-exponentially with vehicle density (Corollary 4). We also analyzed in detail the way information propagates under the threshold, and we showed that DTN routing using bidirectional traffic provides a gain in the propagation distance, which follows a sub-linear power law with respect to the elapsed time (Theorems 2 and 3). Combining all these different situations, we obtain a complete image of the way information propagates in vehicular networks on roads and highways, which is useful in determining the performance limits and designing appropriate routing protocols for VANETs. All our theoretical results were validated with simulations in several environments (The One and Maple).

Our analysis can be extended to investigate other models of vehicle traffic and radio propagation. In future works, we intend to provide a detailed expression of this threshold in specific VANET models (e.g., intersections). Finally, an interesting direction for further research consists in collecting large traces of real traffic on roads and highways, and evaluating the information propagation properties in this context.

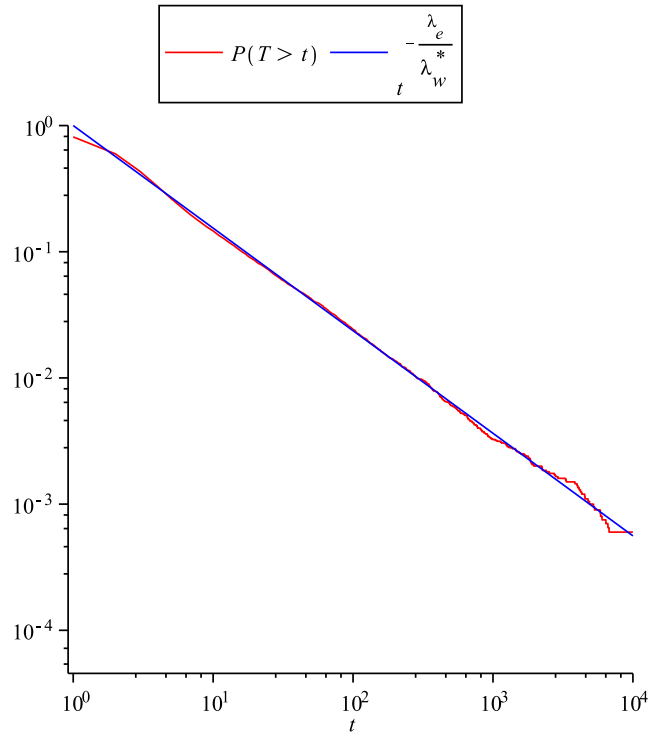


Figure 13: Maple simulations. Cumulative probability distribution $P(\mathbf{T} > t)$ of the waiting time \mathbf{T} , compared to the power law $t^{-\frac{\lambda_e}{\lambda_w^*}}$, for $\lambda_e = \lambda_w = 0.9$.

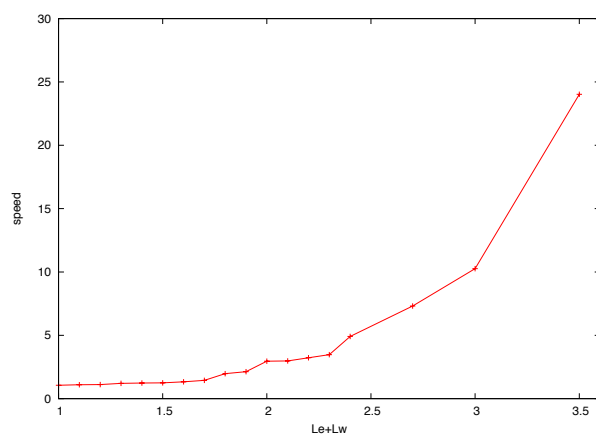


Figure 14: ONE simulations. Information propagation speed for $\lambda_e = \lambda_w$, with respect to $\lambda_e + \lambda_w$.

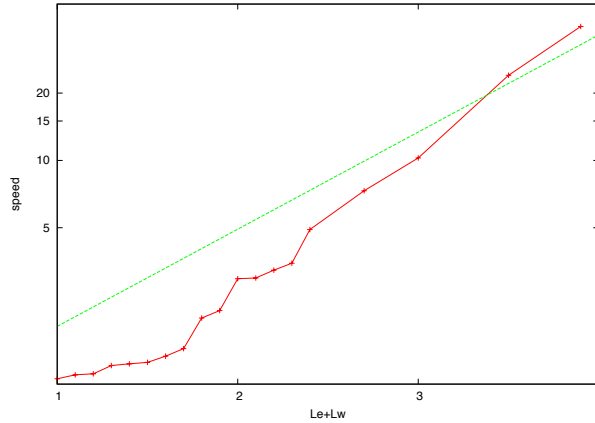


Figure 15: ONE simulations. Information propagation speed for $\lambda_e = \lambda_w$, with respect to $\lambda_e + \lambda_w$, in semi-log scale, compared to the theoretically predicted asymptotic exponential growth.

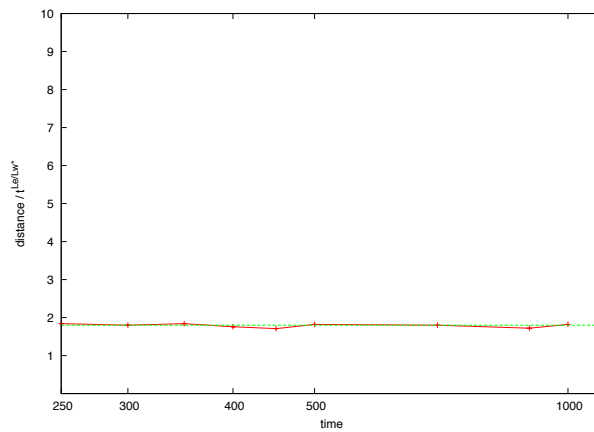


Figure 16: ONE simulations. Average distance traveled by information with respect to elapsed time $\mathbf{E}(L(t))$, divided by the predicted power law $t^{\lambda_e/\lambda_w^*}$ for $\lambda_e = 1$ and $\lambda_w = 0.9$, compared to the constant value 1.80 (dash).

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Appendix

8.1 Proof of Lemma 9

Proof. We want to find an evaluation of $P(\mathbf{T} > y)$ when $y \rightarrow \infty$. For this we will evaluate for x given $P(B_w(x) > y)$. We know that

$$\mathbf{E}(e^{\theta B_w(x+1)}) = \beta_w(\theta, x+1) = \frac{\lambda_w P(L_w > x)}{\theta - \lambda_w(f_w(\theta, x+1) - 1)}$$

Since $P(L_w > x) = \frac{\alpha}{\lambda_w^*} e^{-\lambda_w^* x} (1 + o(e^{-\varepsilon x}))$, with $\alpha = \frac{(\lambda_w - \lambda_w^*) e^{\lambda_w^* - \lambda_w}}{1 - \lambda_w^*}$, we have

$$f_w(\theta, x) = f_w(\theta) - \frac{\alpha e^{-\lambda_w^* x}}{\theta + \lambda_w^*} (1 + o(e^{-\varepsilon x})) \quad (41)$$

From here we drop the $o(e^{-\varepsilon x})$ for simplicity, since it will just bring an exponentially small factor. Therefore we have

$$\beta_w(\theta, x+1) = \frac{\lambda_w}{\lambda_w^* \theta - \lambda_w(f_w(\theta) - 1 - \frac{\alpha e^{-\lambda_w^* x}}{\theta + \lambda_w^*})} \alpha e^{-\lambda_w^* x} \quad (42)$$

We have

$$P(B_w(x+1) = y) = \frac{1}{2i\pi} \int \beta_w(\theta, x+1) e^{y\theta} d\theta .$$

Let $\theta(x)$ be the root of $\theta - \lambda_w(f_w(\theta) - 1 - \frac{\alpha e^{-\lambda_w^* x}}{\theta + \lambda_w^*})$. Straightforward analysis gives $\theta(x) = -\beta e^{-\lambda_w^* x} + O(e^{-2\lambda_w^* x})$, with $\beta = \frac{\lambda_w}{1 - \lambda_w \frac{\partial}{\partial \theta} f_w(0)} \frac{\alpha}{\lambda_w^*}$. Via singularity analysis we have

$$\begin{aligned} P(B_w(x+1) = y) &= \frac{\lambda_w \alpha e^{-\lambda_w^* x}}{\lambda_w^*} \\ &\quad \times \frac{e^{\theta(x)y}}{1 - \lambda_w \frac{\partial}{\partial \theta} f_w(\theta(x)) - \frac{\alpha e^{-\lambda_w^* x}}{(\theta(x) + \lambda_w^*)^2}} \\ &\quad + O(e^{(\theta(x) - \varepsilon)y}) \end{aligned}$$

Omitting the $O(\cdot)$ terms we get

$$\begin{aligned} P(B_w(x+1) = y) &= \frac{\lambda_w \alpha e^{-\lambda_w^* x}}{\lambda_w^*} \frac{e^{-\beta e^{-\lambda_w^* x} y}}{1 - \lambda_w \frac{\partial}{\partial \theta} f_w(0)} \\ &= \beta e^{-\theta x} \exp(-\beta e^{-\lambda_w^* x} y) , \end{aligned}$$

Or

$$P(B_w(x+1) > y) = \exp(-\beta e^{-\lambda_w^* x} y) . \quad (43)$$

Therefore stating $P(B_w > y) = \int p_e(x) P(B_w(x) > y) dx$ we get, omitting $O(\cdot)$ terms

$$P(B_w > y) = \int \frac{\lambda_e}{f_w(\lambda_e)} e^{-(x+1)\lambda_e} \exp(-\beta e^{-\lambda_w^* x} y) dx . \quad (44)$$

with the change of variable $u = e^{-\lambda_w^* x}$ we get

$$\begin{aligned} P(B_w > y) &= \int \frac{\lambda_e e^{-\lambda_e}}{\lambda_w^* f_w(\lambda_e)} u^{\frac{\lambda_e}{\lambda_w^*} - 1} \exp(-\beta y u) du \\ &= \frac{\lambda_e e^{-\lambda_e}}{\lambda_w^* f_w(\lambda_e)} \Gamma\left(\frac{\lambda_e}{\lambda_w^*}\right) (\beta y)^{-\frac{\lambda_e}{\lambda_w^*}}, \end{aligned}$$

which is in power law as claimed. \square

8.2 Proof of Lemma 11

We first prove the property for an hypothetic renewal process based on the B_i 's on the real line. Let $b(x)$ be this renewal process at time t :

$$\begin{aligned} \mathbf{E}(b(y)) &= P(B_1 < y) + P(B_1 + B_2 < y) \\ &\quad + \dots + P(B_1 + \dots + B_n < y) + \dots \end{aligned}$$

Let us define $\beta = 1 - \mathbf{E}(e^{-\theta B_w})$. We also define $N_B(\theta) = \int_0^\infty \mathbf{E}(b(y)) e^{-y\theta}$ the Laplace transform of $\mathbf{E}(b(y))$. We have:

$$N_B(\theta) = \frac{1 - B(\theta)}{\theta B(\theta)}.$$

Lemma 13. *When $\theta \rightarrow 0$, then $B(\theta) = b \frac{\pi \theta^a}{\sin(\pi a)}$ for some b and $a = \frac{\lambda_e}{\lambda_w^*}$.*

Proof. We start from

$$\beta(\theta, x+1) = \frac{\lambda_w P(L_w > x+1)}{\theta - \lambda_w (f_w(\theta, x+1) - 1)},$$

and

$$B(\theta) = 1 - \int_1^\infty p_e(x) \beta(\theta, x) dx = \int_1^\infty p_e(x) (1 - \beta(\theta, x)) dx.$$

We denote

$$1 - \beta(\theta, x) = \frac{\theta}{\theta + \theta(x)} g(\theta, x),$$

with $g(\theta, x)$ bounded and uniformly integrable when $\Re(\theta)$ remains in a compact set and $x \rightarrow \infty$. Indeed we have

$$g(\theta, x) = 1 + O(\theta(x))$$

Thus

$$B(\theta) = \int_1^\infty \frac{\theta}{\theta + \theta(x)} g(\theta, x) p_e(x) dx.$$

By change of variable $y = \theta(x)$ we get

$$B(\theta) = \int_0^{\theta(1)} \frac{\theta}{\theta + y} g(\theta, \theta^{-1}(y)) p_e(\theta^{-1}(y)) (\theta^{-1}(y))' dy.$$

We use together the following estimates $\theta(x) = -\beta e^{-\lambda_w^* x} + O(e^{-2\lambda_w^* x})$, and $p_e(x) = \frac{\lambda_e}{f_w(\lambda_e)} e^{-\lambda_e x} (1 + O(e^{-\varepsilon x}))$ to state

$$p_e(\theta^{-1}(y)) (\theta^{-1}(y))' = \frac{\lambda_e}{f_w(\lambda_e)} \left(\frac{y}{\beta}\right)^{\frac{\lambda_e}{\lambda_w^*} - 1} (1 + O(y^\varepsilon)),$$

for some $\varepsilon > 0$.

Then, we use the fact that

$$\int_0^\infty \frac{y^{a-1}}{\theta + y} dy = \frac{\pi\theta^{a-1}}{\sin(\pi a)},$$

and, when $\theta \rightarrow 0$,

$$\int_{\theta(1)}^\infty \frac{y^{a-1}}{\theta + y} dy = O(1)$$

Therefore, $B(\theta) = b \frac{\pi\theta^a}{\sin(\pi a)} + O(\theta)$ for some b and $a = \frac{\lambda_e}{\lambda_w^*}$ when $\theta \rightarrow 0$. This is also true for complex θ . \square

We can now prove Lemma 11 for the original renewal process.

Proof. From the previous lemma, we have that: $N_B(\theta) = \frac{\sin(\pi a)}{b\pi} \theta^{-a-1} + O(\theta^{-1} + \theta^{-2a})$, when $\theta \rightarrow 0$.

The inversion of the Laplace transform yields

$$\mathbf{E}(b(y)) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} N_B(\theta) e^{\theta y} d\theta .$$

for any $c > 0$. we bend the integration path so that it resemble the path described in figure 17 with $c' < 0$. Since far from its singularities on the line $-\theta(x)$ for $x > 0$ (which corresponds to the negative real axis), the functions $\beta(\theta, x)$ is uniformly in $\frac{1}{\theta}$, therefore $N_B(\theta) = O(\frac{1}{\theta^2})$ and the integral on the vertical part gives a bounded contribution (in fact exponentially decreasing in $\exp(c'y)$).

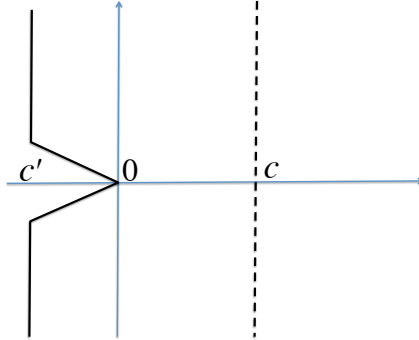


Figure 17: The Flajolet-Odlyzko cranted integral path.

The cranted part of the integral, using the Flajolet Odlyzko theorem on continuous functions [9], gives a contribution which is

$$\frac{\sin^2(\pi a)}{b\pi^2} \Gamma(-a) t^a + O(1 + t^{2a-1}).$$

Therefore, we obtain that, for $y \rightarrow \infty$:

$$\mathbf{E}(b(y)) = \frac{\sin^2(\pi a)}{b\pi^2} \Gamma(-a)y^a + O(1 + y^{2a-1}).$$

To terminate the argument we will prove that $\mathbf{E}(n(t)) = \mathbf{E}(b(2vt)) + o(1)$. We have $2v\mathbf{T} = B_w + H_w$, where $H_w = L_w + I_w - 1$, and $H_w \geq 0$ has distribution with an exponentially decreasing tail. However, we stress the fact that B_w and H_w are not independent. We denote H_i the successive values of the H_w . We have therefore $2v\mathbf{T}_i = B_i + H_i$.

Since $\mathbf{T}_i \geq 2vB_i$, we already have $\mathbf{E}(n(t)) \leq \mathbf{E}(b(2vt))$. We also have for any $t' \in [0, t]$ and for any integer i :

$$\begin{aligned} P(\mathbf{T}_1 + \dots + \mathbf{T}_i < t) &\geq P(B_1 + \dots + B_i < 2v(t - t')) \\ &\quad - P(H_1 + \dots + H_i > 2vt'). \end{aligned}$$

Using Chernoff bounds, we have for all $\theta > 0$ such that $\mathbf{E}(e^{\theta H})$ exists, $\forall i \leq k$ and $\forall x > 0$:

$$P(H_1 + \dots + H_i > x) \leq \mathbf{E}(\exp(k\theta H - x\theta))$$

Since $\log \mathbf{E}(e^{\theta H}) = \theta \mathbf{E}(H) + O(\theta^2)$, it is sufficient that $\frac{x}{k} - \mathbf{E}(H) > \gamma$, to find $\theta > 0$ such that $\log P(H_1 + \dots + H_i > x) < -k\gamma\theta$.

On the other side, we also have $\forall y > 0$:

$$\sum_{i \leq k} P(B_1 + \dots + B_i < y) = \mathbf{E}(b(y)) - \sum_{i > k} P(B_1 + \dots + B_i < y)$$

We have

$$P(B_1 + \dots + B_i < y) \leq (P(B < y))^i$$

We use the fact that $P(B > y) \leq Ay^{-a}$ for $a = \frac{\lambda_w}{\lambda_w^*}$ and for some $A > 0$ to state that

$$P(B_1 + \dots + B_i < y) \leq \exp(-iAy^{-a})$$

and, finally,

$$\sum_{i > k} P(B_1 + \dots + B_i < y) \frac{\exp(-kAy^{-a})}{Ay^{-a}}.$$

Collecting all results yields:

$$\mathbf{E}(n(t)) \geq \mathbf{E}(b(2v(t - t'))) - G_k(2vt, 2vt', \theta),$$

with

$$G_k(x, y, \theta) = \frac{\exp(-kA(y - x)^{-a})}{A(y - x)^{-a}} + k\mathbf{E}(\exp(k\theta H - x\theta)).$$

If we take $t' = t^{a'}$ with $1 > a' > a$, $k = \frac{2vt'}{\gamma + \mathbf{E}(H)} = \Omega(t^{a'})$, then we have

$$G_k(2vt', 2vt, \theta) \leq \frac{\exp(-k(2v(t - t'))^{-a})}{A(2v(t - t'))^{-a}} + k \exp(-k\gamma\theta)$$

which is exponentially small when $t \rightarrow \infty$. Therefore,

$$\mathbf{E}(n(t)) \geq \mathbf{E}(b(t)) + o(1).$$

□



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