Bayesian Multi-Task Reinforcement Learning

Alessandro Lazaric, Mohammad Ghavamzadeh

Bayesian Multi-Task Reinforcement Learning

Alessandro Lazaric **Mohammad Ghavamzadeh**
INRIA Lille - Nord Europe, Team SequeL, France
{alessandro.lazaric,mohammad.ghavamzadeh}@inria.fr

Abstract

We consider the problem of multi-task reinforcement learning where the learner is provided with a set of tasks, for which only a small number of samples can be generated for any given policy. As the number of samples may not be enough to learn an accurate evaluation of the policy, it would be necessary to identify classes of tasks with similar structure and to learn them jointly. We consider the case where the tasks share structure in their value functions, and model this by assuming that the value functions are all sampled from a common prior. We adopt the Gaussian process temporal-difference value function model and use a hierarchical Bayesian approach to model the distribution over the value functions. We study two cases, where all the value functions belong to the same class and where they belong to an undefined number of classes. For each case, we present a hierarchical Bayesian model, and derive inference algorithms for (i) joint learning of the value functions, and (ii) efficient transfer of the information gained in (i) to assist learning the value function of a newly observed task.

1 Introduction

Multi-task learning (MTL) is an important learning paradigm and has recently been an area of active research in machine learning (e.g., [4; 1; 17; 16; 3]). A common setup is that there are multiple related tasks for which we are interested in improving the performance over individual learning by sharing information across the tasks. This transfer of information is particularly important when we are provided with only a limited number of data to learn each task. Exploiting data from related problems provides more training samples for the learner and can improve the performance of the resulting solution.

Most reinforcement learning (RL) algorithms [2; 13] often need a large number of samples to solve a problem and cannot directly take advantage of the information coming from other similar tasks. Nonetheless, recent work has shown that transfer and multi-task learning techniques can be employed in RL to reduce the number of samples needed to achieve nearly-optimal solutions. All approaches to multi-task RL (MTRL) assume that the tasks share similarity in some components of the problem such as dynamics, reward structure, or value function. While some methods explicitly assume that the shared components are drawn from a common generative model [15; 10], this assumption is more implicit in others [14; 9]. In [10], tasks share the same dynamics and reward features, and only differ in the weights of the reward function. The proposed method initializes the value function for a new task using the previously learned value functions as a prior. In [15], the distribution over the dynamics and the reward functions of the tasks is drawn from a hierarchical Bayesian model (HBM). Due to some similarity to our work, we discuss this method in more details in Section 5. In [9], the authors implicitly assume that the tasks are drawn from a common distribution. They propose a method to selectively transfer samples from source tasks to a target task based on the likelihood of the target samples being generated by the models built for the source tasks. Finally, in [14], learning the value function of the target task is expedited using the solution learned in a source task with related, but different, state and action spaces.

In this paper, we study the MTRL scenario in which the learner is provided with a number of tasks with common state and action spaces. For any given policy, only a small number of samples can be generated in each task, which may not be enough to accurately evaluate the policy. In such a MTRL problem, it is necessary to identify classes of tasks with similar structure and to learn them jointly. In our work, we consider a particular class of MTRL problems in which the tasks share structure in their value functions. To allow the value functions to share a common structure, one way would be to assume that they are all sampled from a common prior. We adopt the Gaussian process temporal-difference (GPTD) value function model [6] for each task, model the distribution over the value functions using a HBM, and develop solutions to the following problems: (i) joint learning of the value functions, and (ii) efficient transfer of the information acquired in (i) to facilitate learning the value function of a newly observed task. We refer to the above problems as *symmetric* and *asymmetric* multi-task learning, respectively. In Section 3, we present a HBM for the case in which all the value functions belong to the same class, and derive an EM algorithm to find MAP estimates of the value functions and the model's hyper-parameters. However, as pointed out in [4; 1], if the functions do not belong to the same class, simply learning them together can be detrimental (*negative transfer*). It is therefore important to have models that will generally benefit from related tasks and will not hurt performance when the tasks are unrelated. This is particularly important in RL as changing the policy at each step of the policy iteration algorithm (this is true even for the fitted value iteration algorithm) can change the way tasks are clustered together. This means that even if we start with value functions that all belong to the same class, after one iteration the new value functions may be clustered into several classes. In Section 4, we introduce a Dirichlet process (DP) based HBM for the case that the value functions belong to an undefined number of classes, and derive inference algorithms for both the symmetric and asymmetric scenarios. In Section 5, we discuss the similarities and differences with closely related work. In Section 6, we report and analyze experimental results.

2 Preliminaries

The agent-environment interaction in RL is conventionally modelled as a Markov Decision Process (MDP). A MDP is a tuple $\mathcal{M} = \langle \mathcal{X}, \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ where $\mathcal{X}$ and $\mathcal{A}$ are the state and action spaces, respectively; $\mathcal{R}$ is the probability distribution over rewards R; and $\mathcal{P}$ is the transition probability distribution. A *stationary* policy $\pi : \mathcal{X} \times \mathcal{A} \to [0, 1]$ is a mapping from states to action selection probabilities. The MDP controlled by a policy π induces a Markov chain with transition probability distribution $P^\pi(x'|x) = \int_{\mathcal{A}} P(x'|x, a)\pi(a|x)da$. Given a policy π, the (possibly discounted, $\gamma \in [0, 1)$) return for a state x, $D^\pi(x)$, is a random process defined by $D^\pi(x) = \sum_{t=0}^{\infty} \gamma^t R(x_t)|x_0 = x$, with $x_{t+1} \sim P^\pi(\cdot|x_t)$. The *value function* $V^\pi(x)$ is the expected value of $D^\pi(x)$ where the expectation is over all possible trajectories and rewards collected along them.

A key problem in RL is to learn the value function of a given policy, which is called *policy evaluation* [2; 13]. Loosely speaking, in policy evaluation the goal is to find a "close enough" approximation V of the value function V^π. Unlike in *supervised learning*, the target function V^π is not known in advance and its values have to be inferred from the observed rewards. Therefore, it is required to define a stochastic generative model connecting the underlying hidden value function with the observed rewards. In this paper, we adopt the GPTD value function model proposed in [6], in which the discounted return D is decomposed into its mean V and a random zero-mean residual ΔV, $D(x) = V(x) + \Delta V(x)$. Combining this decomposition with the Bellman equation, we obtain

$$R(x) = V(x) - \gamma V(x') + \epsilon(x, x'), \qquad x' \sim P^\pi(\cdot|x), \tag{1}$$

where $\epsilon(x, x') \stackrel{\text{def}}{=} \Delta V(x) - \gamma \Delta V(x')$. Suppose we are provided with a set of samples $\mathcal{D} = \{(x_n, x'_n, r_n)\}_{n=1}^N$, where r_n and x'_n are the reward and the next state observed by following policy π in state x_n, respectively. By writing the model of Eq. (1) w.r.t. these samples, we obtain $R = \mathbf{H}V + \mathcal{E}$, where $\mathbf{H} \in \mathbb{R}^{N \times 2N}$ and

$$R^\top = (r_n)_{n=1}^N;\ \mathcal{E}^\top = \big(\epsilon(x_n, x'_n)\big)_{n=1}^N;\ V^\top = \big(V(x_n), V(x'_n)\big)_{n=1}^N$$

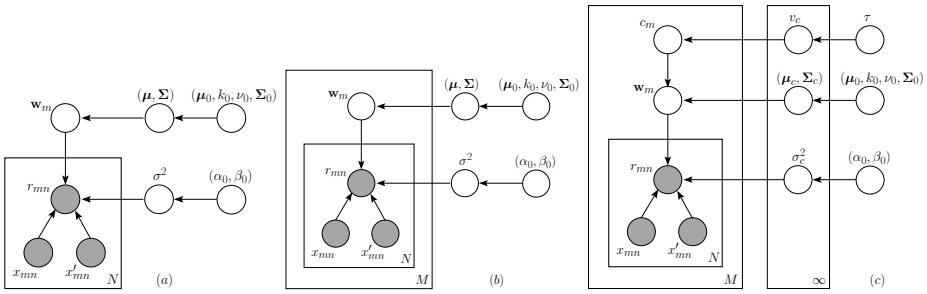


Figure 1: Graphical representations for *(a)* the single-task model — an extension of GPTD by defining a hyper-prior over the parameters, *(b)* the single-class multi-task model of Section 3, and *(c)* the multi-class multi-task model of Section 4.

$$\mathbf{H} = \begin{bmatrix} 1 & -\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -\gamma & \dots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -\gamma \end{bmatrix}.$$

Note that if the samples are generated from a single trajectory then $x'_n = x_{n+1}$ and $\mathbf{H}$ will be of the form defined by Eq. (2.7) in [6]. In order to specify a complete probabilistic generative model connecting values and rewards, we need to define a prior distribution for the value function V and the distribution of the noise ϵ. Similar to [6], we model the value function as a Gaussian process (GP), and the noise vector as $\mathcal{E} \sim \mathcal{N}(\mathbf{0}, \mathbf{S})$, where $\mathbf{S}$ is the noise covariance matrix modelling the correlation of the noise between different states. In the following we write $\mathbf{S} = \sigma^2 \mathbf{P}$, where σ^2 and $\mathbf{P}$ are the variance and the correlation matrix of the noise, respectively. For a more extended discussion about different models of noise we refer readers to [5] and Section 8.4. The value function V may be represented either in parametric or non-parametric form. In this paper we use the parametric representation to make the formulation easier to follow, but all the results can be extended to the non-parametric case following similar steps as in Section 5.2 of [17]. In the parametric form, the value function is represented by a finite set of d features $\boldsymbol{\phi}(\cdot) = \big(\phi_1(\cdot), \dots, \phi_d(\cdot)\big)^\top$ and a weight vector $\mathbf{w} = (w_1, \dots, w_d)^\top$ as $V(\cdot) = \boldsymbol{\phi}(\cdot)^\top \mathbf{w}$. The randomness in V is now due to $\mathbf{w}$ being a random vector with Gaussian prior $\mathbf{w} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The model equation now becomes $R = \mathbf{H}\boldsymbol{\Phi}^\top \mathbf{w} + \mathcal{E}$, where $\boldsymbol{\Phi} = [\boldsymbol{\phi}(x_1), \boldsymbol{\phi}(x'_1), \dots, \boldsymbol{\phi}(x_N), \boldsymbol{\phi}(x'_N)]$ is a $d \times 2N$ matrix. Fig. 1*(a)* shows the graphical representation of this model. Note that the model shown in Fig. 1*(a)* is the model that we use for single-task learning (STL) in the experiments of this paper. It is an extension of the original GPTD model by defining a Normal-inverse-Wishart and an inverse-Gamma hyper-priors parametrized by $\psi_0 = (\boldsymbol{\mu}_0, k_0, \nu_0, \boldsymbol{\Sigma}_0, \alpha_0, \beta_0)$ over the model parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \sigma^2)$. This allows us to optimize the model parameters given the data.

In the MTRL setting of this paper, the learner is provided with M tasks or MDPs with common state and action spaces $\mathcal{M}_m = \langle \mathcal{X}, \mathcal{A}, \mathcal{R}_m, \mathcal{P}_m \rangle$, $m = 1, \dots, M$. Given a fixed policy, N samples are generated in each task, i.e., $\mathcal{D}_m = \{(x_{mn}, x'_{mn}, r_{mn})\}_{n=1}^N$, which may not be enough to have an accurate evaluation of the policy. We consider the case in which the tasks share structure in their value functions. In the parametric value function model discussed above, this can be interpreted as the value functions share the same feature space and their weight vectors are sampled independently from a common prior, i.e., $V_m(\cdot) = \boldsymbol{\phi}(\cdot)^\top \mathbf{w}_m; \mathbf{w}_m \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. In the next two sections, we study two different scenarios: 1) when all the tasks belong to the same class, i.e., they share the same prior, and 2) when they can be clustered into an undefined number of classes.

3 Single-class Multi-task Learning

In this section, we consider the case where all the tasks belong to the same class, i.e., they share the same distribution over their value functions $\mathbf{w}_m \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),\ m = 1, \dots, M$; and the same observation noise σ^2. The goal in the symmetric form of this problem is to estimate $\{\mathbf{w}_m\}_{m=1}^M$ from

the data $\{\mathcal{D}_m\}_{m=1}^M$, whereas in the asymmetric case we are interested in estimating the parameters $\theta = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \sigma^2)$ from the data in order to use them as a prior for a newly observed task (e.g., task $\mathcal{M}_{M+1}$). We use a parametric HBM for this problem. HBMs allow us to model both the individuality of the tasks and the correlation between them. In HBMs, individual models with task specific parameters are usually located at the bottom, and at the layer above, tasks are connected together via a common prior placed over those parameters. Learning the common prior is a part of the training process in which data from all the tasks contribute to learning, thus making it possible to share information between the tasks usually via sufficient statistics. Then given the learned prior, individual models are learned independently. As a result, learning at each task is affected by both its own data and by data from the other tasks related through the common prior.

3.1 The Model

We assume a normal-inverse-Wishart and an inverse-Gamma hyper-priors for $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and σ^2, respectively.

$$p(\theta|\psi_0) = p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \;\times\; p(\sigma^2) = \mathcal{N}(\boldsymbol{\mu}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}/k_0)\, \mathcal{IW}(\boldsymbol{\Sigma}; \nu_0, \boldsymbol{\Sigma}_0) \;\times\; \mathcal{IG}(\sigma^2; \alpha_0, \beta_0). \quad (2)$$

These distributions are the conjugate priors for multivariate Gaussian distributions $p(\mathbf{w}_m|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $p(R_m|\mathbf{w}_m, \sigma^2) = \mathcal{N}(\mathbf{H}\boldsymbol{\Phi}_m^\top \mathbf{w}_m, \sigma^2\mathbf{P})$, respectively. This leads to the following generative model for the data, $\{\mathcal{D}_m\}$. Fig. 1*(b)* shows the graphical representation of this model. The details of the model can be found in Section 8.

Single-Class Model: *Given the hyper-parameters* $\psi_0 = (\boldsymbol{\mu}_0, k_0, \nu_0, \boldsymbol{\Sigma}_0, \alpha_0, \beta_0)$,

1. *The parameters* $\theta = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \sigma^2)$ *are sampled once from the hyper-prior as in Eq.* (2),
2. *For each task* $\mathcal{M}_m$ *(value function* V_m*), the weight vector is sampled as* $\mathbf{w}_m \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,
3. *Given* $\{(x_{mn}, x'_{mn})\}_{n=1}^N$, *we have* $R_m = \boldsymbol{H}\boldsymbol{\Phi}_m^\top \mathbf{w}_m + \mathcal{E}$, *where* $\mathcal{E} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{P})$, $m = 1, \ldots, M$.

3.2 Inference

This model can be learned by optimizing the penalized likelihood $p\big(\{R_m\}|\{(x_{mn}, x'_{mn})\}, \theta\big)p(\theta)$ w.r.t. the parameters $\theta = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \sigma^2)$ using an EM algorithm. In the rest of the paper, we refer to this algorithm as SCMTL, for single-class multi-task learning.

E-step: Since the posterior distribution of the latent variables $p(\{\mathbf{w}_m\}|\{\mathcal{D}_m\}, \theta)$ is a product of M Gaussian posterior distributions $p(\mathbf{w}_m|\mathcal{D}_m, \theta) = \mathcal{N}(\boldsymbol{\mu}'_{0m}, \boldsymbol{\Sigma}'_{0m})$, for each task m, we compute the mean and covariance as

$$\boldsymbol{\mu}'_{0m} = \boldsymbol{\Sigma}'_{0m}\left[\frac{1}{\sigma^2}\boldsymbol{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}R_m + \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right], \qquad \boldsymbol{\Sigma}'_{0m} = \left[\frac{1}{\sigma^2}\boldsymbol{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}\mathbf{H}\boldsymbol{\Phi}_m^\top + \boldsymbol{\Sigma}^{-1}\right]^{-1}.$$

M-step: We optimize θ to maximize the penalized expected log-likelihood of complete data $\log p\big(\{\mathcal{D}_m\}, \{\mathbf{w}_m\}|\theta\big)$ over the posterior distribution estimated in the E-step and obtain the new parameters

$$\boldsymbol{\mu}_{\text{new}} = \frac{1}{M + k_0}\left(k_0\boldsymbol{\mu}_0 + \sum_{m=1}^M \boldsymbol{\mu}'_{0m}\right),$$

$$\boldsymbol{\Sigma}_{\text{new}} = \frac{1}{M + \nu_0 + d + 2}\left\{k_0(\boldsymbol{\mu} - \boldsymbol{\mu}_0)(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^\top + \boldsymbol{\Sigma}_0 + \sum_{m=1}^M \left[(\boldsymbol{\mu}'_{0m} - \boldsymbol{\mu})(\boldsymbol{\mu}'_{0m} - \boldsymbol{\mu})^\top + \boldsymbol{\Sigma}'_{0m}\right]\right\},$$

$$\sigma^2_{\text{new}} = \frac{1}{MN + 2(1 + \alpha_0)}\left\{2\beta_0 + \sum_{m=1}^M \left[\text{tr}\left(\mathbf{P}^{-1}\mathbf{H}\boldsymbol{\Phi}_m^\top\boldsymbol{\Sigma}'_{0m}\boldsymbol{\Phi}_m\mathbf{H}^\top\right) + \left(R_m - \mathbf{H}\boldsymbol{\Phi}_m^\top\boldsymbol{\mu}'_{0m}\right)\mathbf{P}^{-1}\left(R_m - \mathbf{H}\boldsymbol{\Phi}_m^\top\boldsymbol{\mu}'_{0m}\right)^\top\right]\right\}.$$

4 Multi-class Multi-task Learning

In this section, we consider the case where the tasks belong to an undefined number of classes. Tasks in the same class $\{\mathcal{M}_m|\ c_m = c\}$ share the same distribution over their value functions $\mathbf{w}_m \sim \mathcal{N}(\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c)$, and the same observation noise σ_c^2. We use a nonparametric HBM for this problem. In the HBM proposed in this paper, the common prior is drawn from a Dirichlet process (DP). As addressed in the statistics literature (see, e.g., [11]), DP is powerful enough to model the parameters of individual classes, to fit them well without any assumption about the functional form of the prior, and to automatically learn the number of underlying classes.

4.1 The Model

We place a DP(τ, G_0) prior over the class assignment and the class parameters. The concentration parameter τ and the base distribution G_0 can be considered as priors over the number of classes and the class parameters $\theta_c = (\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c, \sigma_c^2)$, respectively. G_0 is specified as the product of a d-dimensional normal-inverse-Wishart and a 1-dimensional inverse-Gamma distributions, with parameters $\psi_0 = (\boldsymbol{\mu}_0, k_0, \nu_0, \boldsymbol{\Sigma}_0, \alpha_0, \beta_0)$, (see Eq. 2). We employ the stick-breaking representation of the DP prior [7], and define a task-to-class assignment variable $(c_{m1}, \ldots, c_{m\infty})$ for each task m, whose elements are all zero except that the cth element is equal to one if task m belongs to class c. Given the above, the data $\{\mathcal{D}_m\}$ can be seen as drawn from the following generative model, whose graphical representation is shown in Fig. 1*(c)*.

Multi-Class Model: *Given the hyper-parameters* (τ, ψ_0)*,*

1. Stick-breaking view: *Draw* v_c *from the Beta distribution* $\mathcal{B}e(1, \tau)$*, compute* $\pi_c = v_c \prod_{i=1}^{c-1}(1 - v_i)$*, and independently draw* $\theta_c \sim G_0$*,* $c = 1, \ldots, \infty$*,*
2. Task-to-class assignment: *Draw the indicator* $(c_{m1}, \ldots, c_{m\infty})$ *from a multinomial distribution* $\mathcal{M}_\infty(1; \pi_1, \ldots, \pi_\infty)$*,* $m = 1, \ldots, M$*,*
3. *The weight vector is sampled as* $\mathbf{w}_m \sim \mathcal{N}(\boldsymbol{\mu}_{c_m}, \boldsymbol{\Sigma}_{c_m})$*,* $m = 1, \ldots, M$*,*
4. *Given* $\{(x_{mn}, x'_{mn})\}_{n=1}^N$*, we have* $R_m = \boldsymbol{H}\boldsymbol{\Phi}_m^\top \mathbf{w}_m + \mathcal{E}$*, where* $\mathcal{E} \sim \mathcal{N}(\mathbf{0}, \sigma_{c_m}^2 \mathbf{P})$*,* $m = 1, \ldots, M$*.*

4.2 Inference

We are interested in the posterior distribution of the latent variables $\mathcal{Z} = \{\{\mathbf{w}_m\}, \{c_m\}, \{\theta_c\}\}$ given the observed data and the hyper-parameters τ and ψ_0, i.e., $p\big(\mathcal{Z}|\{\mathcal{D}_m\}, \tau, \psi_0\big) \propto p\big(\{\mathcal{D}_m\}|\mathcal{Z}, \tau, \psi_0\big)p(\mathcal{Z}|\tau, \psi_0)$. In the following we outline the main steps of the algorithm used to solve this inference problem, which we refer to as MCMTL, for multi-class multi-task learning (see Fig. 2). MCMTL combines the SCMTL algorithm of Sec. 3.2 for class parameters estimation, with a Gibbs sampling algorithm for learning the class assignments [12]. The main advantage of such combination is that at each iteration, given the current estimate of the weights, we take advantage of the conjugate priors to derive an efficient Gibbs sampling procedure.

More formally, given an arbitrary initial class assignment $\{c_m\}$, a distinct EM algorithm is run on each class $c = 1, \ldots, C$ (with C the current estimate of the number of classes) and returns M distributions $\mathcal{N}(\boldsymbol{\mu}'_{0m}, \boldsymbol{\Sigma}'_{0m})$. Given the weights estimated at the previous step, $\widehat{\mathbf{w}}_m = \boldsymbol{\mu}'_{0m}$, the Gibbs sampling solves the DP inference by drawing samples from the posterior distribution $p\big(\{c_m\}|\{R_m\}, \{\widehat{\mathbf{w}}_m\}, \tau, \psi_0\big)$. In particular, the state of the Markov chain simulated in the Gibbs sampling is the class assignment $\{c_m\}$, i.e., the vector of the classes each task belongs to. At each iteration, each component c_m of the state is updated by sampling from the following distribution

$$\text{If } c = c_{m'}, m' \neq m\text{: } p\big(c_m = c|\{c_{m'}\}, R_m, \widehat{\mathbf{w}}_m, \tau, \psi_0\big) = b\frac{M_{-m,c}}{M - 1 + \tau}\int p\big(R_m, \widehat{\mathbf{w}}_m|\theta_c\big)p\big(\theta_c|\{c_{m'}\}, \psi_0\big)d\theta_c,$$

$$\text{else: } p\big(c_m \neq c_{m'}, m' \neq m|\{c_{m'}\}, R_m, \widehat{\mathbf{w}}_m, \tau, \psi_0\big) = b\frac{\tau}{M - 1 + \tau}\int p(R_m, \widehat{\mathbf{w}}_m|\theta)p(\theta|\psi_0)d\theta, \qquad (3)$$

MCMTL$(\{R_m\}, \tau, \psi_0)$
Initialize $\{c_m\}$
repeat
  for $c = 1, \ldots, C$ **do**
    Initialize θ_c
    repeat
      for $m : c_m = c$ **do**
        $p(\mathbf{w}_m|R_m, \theta_c) = \mathcal{N}(\boldsymbol{\mu}'_{0m}, \boldsymbol{\Sigma}'_{0m})$ **(E-step)**
      end for
      Optimize θ_c **(M-step)**
    until convergence
  end for
  Set $\widehat{\mathbf{w}}_m = \boldsymbol{\mu}'_{0m}, \ \ m = 1, \ldots, M$
  $p(\{c_m\}|\{\widehat{\mathbf{w}}_m\}, \{R_m\}, \tau, \psi_0)$
until convergence
return $\{\widehat{\mathbf{w}}_m\}$ and $\{c_m\}$

Figure 2: The inference algorithm for the multi-class multi-task learning (MCMTL) scenario.

where $M_{-m,c}$ is the number of tasks in class c except task m, and b is a normalizing constant. While the first term in Eq. (3) is the probability of task m to belong to an existing class c, the second term returns the probability of assigning task m to a new class. Thanks to the conjugate base distribution G_0, the integrals in Eq. (3) can be solved analytically. In fact

$$\begin{aligned} p(R_m, \widehat{\mathbf{w}}_m|\theta)p(\theta|\psi_0) &= p(R_m|\widehat{\mathbf{w}}_m, \sigma^2)p(\widehat{\mathbf{w}}_m|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \times p(\boldsymbol{\mu}, \boldsymbol{\Sigma}|\boldsymbol{\mu}_0, k_0, \nu_0, \boldsymbol{\Sigma}_0)p(\sigma^2|\alpha_0, \beta_0) \\ &\propto \mathcal{N}(\boldsymbol{\mu}; \boldsymbol{\mu}'_0, \boldsymbol{\Sigma}/k'_0)\mathcal{IW}(\boldsymbol{\Sigma}; \nu'_0, \boldsymbol{\Sigma}'_0) \times \mathcal{IG}(\sigma^2; \alpha'_0, \beta'_0), \end{aligned} \tag{4}$$

where $\psi'_0 = (\boldsymbol{\mu}'_0, k'_0, \nu'_0, \boldsymbol{\Sigma}'_0, \alpha'_0, \beta'_0)$ are the posterior parameters of G_0 given the weight $\widehat{\mathbf{w}}_m$ and the rewards R_m (see Section 8 for their definition). Using the posterior hyper-parameters, the second integral in Eq. (3) can be written as

$$\int p(R_m, \widehat{\mathbf{w}}_m|\theta)p(\theta|\psi_0)d\theta = \left(\frac{k_0}{\pi k'_0}\right)^{\frac{d}{2}} \frac{|\boldsymbol{\Sigma}_0|^{\nu_0/2}}{|\boldsymbol{\Sigma}'_0|^{\nu'_0/2}} \frac{\Gamma\left(\frac{\nu'_0}{2}\right)}{\Gamma\left(\frac{\nu'_0-d}{2}\right)} \times (2\pi|\mathbf{P}|)^{-\frac{N}{2}} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{\Gamma(\alpha'_0)}{\beta_0'^{\alpha'_0}}. \tag{5}$$

In the first integral of Eq. (3), the density function $p(\theta_c|\{c_{m'}\}, \psi_0)$ is the posterior probability over the class parameters θ_c given the data from all the tasks belonging to c_m according to the current class assignment $\{c_{m'}\}$. Similar to Eq. (4), we compute the posterior hyper-parameters ψ_{0c} of the normal-inverse-Wishart and inverse-Gamma distributions given $\{\widehat{\mathbf{w}}_{m'}\}$ and $\{R_{m'}\}$, with $m' \neq m$ and $c_{m'} = c_m$. Finally, the integral can be analytically calculated as in Eq. (5), where the hyper-parameters ψ_0 and the posterior hyper-parameters ψ'_0 are replaced by ψ_{0c} and ψ'_{0c}, respectively.

4.3 Symmetric vs. Asymmetric Learning

The MCMTL algorithm returns both the distribution over the weights for each task and the learned hierarchical model (task-class assignments). While the former can be used to evaluate the learning performance in the symmetric case, the latter provides a prior for learning a new task in the asymmetric scenario.

Symmetric Learning. According to the generative model in Section 4.1, the task weights are distributed according to the normal distribution $\mathcal{N}(\boldsymbol{\mu}'_{0m}, \boldsymbol{\Sigma}'_{0m})$, where $\boldsymbol{\mu}'_{0m}$ and $\boldsymbol{\Sigma}'_{0m}$ are the posterior mean and covariance of the weight vector $\mathbf{w}_m$ returned by the MCMTL algorithm. Since $V_m(x) = \phi(x)^\top \mathbf{w}_m$, the value of V_m at a test state x_* is distributed as

$$p\big(V_m(x_*)|x_*, \boldsymbol{\mu}'_{0m}, \boldsymbol{\Sigma}'_{0m}\big) = \mathcal{N}\big(\phi(x_*)^\top \boldsymbol{\mu}'_{0m}, \phi(x_*)^\top \boldsymbol{\Sigma}'^{-1}_{0m} \phi(x_*)\big).$$

If MCMTL successfully clusters the task, we expect the value function prediction to be more accurate than learning each task independently.

Asymmetric Learning. In the asymmetric setting the class of the new task is not known in advance. The inference problem is formalized as $p\left(\mathbf{w}_{M+1}|R_{M+1},\psi_0,\{c_m\}_{m=1}^{M}\right)$, where $\mathbf{w}_{M+1}$ and R_{M+1} are the weight vector and rewards of the new task $\mathcal{M}_{M+1}$, respectively. Similar to Section 4.2, this inference problem cannot be solved in closed form, thus, we must apply the MCMTL algorithm to the new task. The main difference with the symmetric learning is that the class assignments $\{c_m\}$ and weights $\{\widehat{\mathbf{w}}_m\}$ for all the previous tasks are kept fixed, and are used as a prior over the new task learned by the MCMTL algorithm. As a result, the Gibbs sampling reduces to a one-step sampling process assigning the new task either to one of the existing classes or to a new class. If $\mathcal{M}_{M+1}$ belongs to a new class, then the inference problem becomes $p\left(\mathbf{w}_{M+1}|R_{M+1},\psi_0\right)$, that is exactly the same as in STL. On the other hand, if $\mathcal{M}_{M+1}$ belongs to class c, the rewards and weights $\{R_{m'}\}$, $\{\mathbf{w}_{m'}\}$ of the tasks in class c can be used to compute the posterior hyper-parameters ψ'_{0c} as in Eq. (4), and to solve the inference problem $p(\mathbf{w}_{M+1}|R_{M+1},\psi'_{0c})$.

5 Related Work

In RL, the approach of this paper is mainly related to [15]. Although we both use a DP-based HBM to model the distribution over the common structure of the tasks, in [15] the tasks share structure in their dynamics and reward function, while we consider the case that the similarity is in the value function. There are scenarios in which significantly different MDPs and policies may lead to very similar value functions. In such scenarios, the method proposed in this paper would still be able to leverage on the commonality of the value functions, thus performing better than single-task learning. Moreover in [15], the setting is incremental, i.e., the tasks are observed as a sequence, and there is no restriction on the number of samples generated by each task. The focus is not on joint learning with finite number of samples, it is on using the information gained from the previous tasks to facilitate learning in a new one. This setting is similar to the asymmetric learning considered in our work.

In supervised learning, our work is related to [17] and [16]. In [17], the authors present a single-class HBM for learning multiple related functions using GPs. Our single-class model of Section 3 is an adaptation of this work for RL using GPTD. Besides, our multi-class model of Section 4 extends this method to the case with an undefined number of classes. In [16], a DP-based HBM is used to learn the extent of similarity between classification problems. The problem considered in our paper is regression, the multi-class model of Section 4 is more complex than the one used in [16], and the inference algorithms of Section 4 are based on Gibbs sampling, where a variational method is used for inference in [16].

6 Experiments

In this section, we report empirical results applying the Bayesian multi-task learning (BMTL) algorithms presented in this paper to a regression problem and a benchmark RL problem, inverted pendulum. We compare the performance of single-task learning (STL) with single-class multi-task learning (SCMTL), i.e., all tasks are assumed to belong to the same class, and multi-class multi-task learning (MCMTL), i.e., tasks belong to a number of classes not known in advance. By STL, we refer to running the EM algorithm of Section 3.2 for each task separately. The reason to use the regression problem in our experiments is that it allows us to evaluate our BMTL algorithms when the tasks are generated exactly according to the generative models of Sections 3 and 4.

6.1 A Regression Problem

In this problem, tasks are functions in the linear space spanned by a feature space $\boldsymbol{\phi}(x) = \left(1,x,x^2,x^3,x^4,x^5\right)^\top$ on the domain $\mathcal{X}=[-1,1]$. The weights for the tasks are drawn from four different classes, i.e., four 6-dim multivariate Gaussian distributions, with the parameters shown in Fig. 3*(a)*. The noise covariance matrix $\mathbf{S}=\text{diag}(\sigma^2)$ for all the algorithms. We evaluate the performance of each BMTL algorithm by computing its relative mean squared error (MSE) improvement over STL : $(MSE_{\text{STL}}-MSE_{\text{BMTL}})/MSE_{\text{STL}}$. The MSEs are computed over $N'=1000$ test samples. All the reported results are averaged over 200 independent runs.

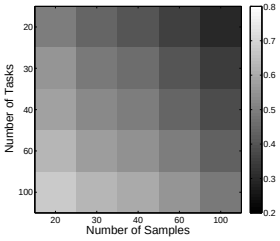
	$\boldsymbol{\mu}_c$	σ_c^2
c_1	(50 10 5 0 0 0)	1.0
c_2	(0 0 0 0.5 0.1 0.05)	3.0
c_3	(0 -20 0 -0.5 0 -0.05)	5.0
c_4	(-50 0 -5 0 -0.1 0)	7.5

	$\boldsymbol{\Sigma}_c$
c_1	diag(20.0 10.0 5.0 0.0 0.0 0.0)
c_2	diag(0.0 0.0 0.0 0.5 0.1 0.05)
c_3	diag(0.0 5.0 0.0 0.5 0.0 0.01)
c_4	diag(20.0 0.0 5.0 0.0 0.1 0.0)

(a)

	Min	Max
c_1	$37.35\% \pm 1.2$	$64.78\% \pm 0.9$
c_2	$29.86\% \pm 0.9$	$67.64\% \pm 0.8$
c_3	$66.69\% \pm 0.8$	$91.81\% \pm 0.6$
c_4	$56.47\% \pm 1.2$	$75.78\% \pm 0.3$

(b)



(c)

Figure 3: *(a)* class parameters, *(b)* minimum and maximum improvement of SCMTL over STL in each class, *(c)* relative MSE improvement of SCMTL over STL when all the tasks are drawn from class c_2.

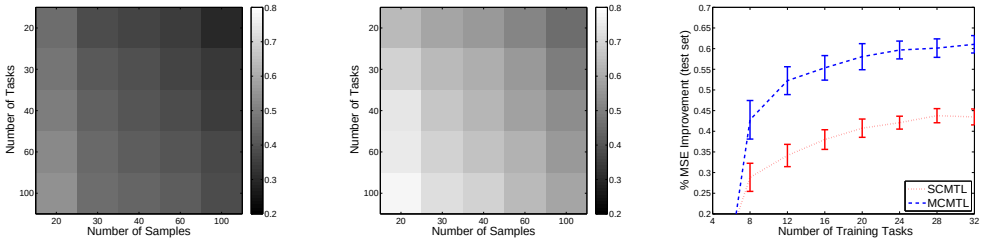
In the first experiment, we draw all the tasks from class c_2. Fig. 3*(c)* shows the performance of SCMTL for different number of tasks (M) and samples per task (N). SCMTL achieves an improvement over STL that varies from $29.86\% \pm 0.9$ for $N = 100$ and 20 tasks to $67.64\% \pm 0.8$ for 100 tasks with only 20 samples each. The results indicate that SCMTL successfully takes advantage of the samples coming from all the tasks to build a more accurate prior than the one obtained by considering each task separately as in STL. However, the advantage of SCMTL over STL declines as N is increased. In fact, as STL converges, SCMTL cannot make further improvement. We repeated the same experiment for the other classes. The minimum and maximum performance of SCMTL for all the classes (all obtained for $N = 100, M = 20$ and $N = 20, M = 100$, respectively) are summarized in Fig. 3*(b)*.

In the second experiment, we draw the tasks randomly from the four classes. We first apply SCMTL to this problem. Fig. 4*(a)* shows the SCMTL's performance. As it can be seen, the results are worse than those in the first experiment (Fig. 3*(c)*), varying from $30.15\% \pm 4.8$ to $54.05\% \pm 1.2$. By clustering all the tasks together, SCMTL takes advantage of all the available samples, thus, performs better than STL. However, when the tasks are drawn from significantly different distributions, it learns a very general prior which does not allow a significant improvement over STL. We then apply MCMTL to this problem. MCMTL's performance (Fig. 4*(b)*) is significantly better than SCMTL's (Fig. 4*(a)*), and it varies from $45.64\% \pm 5.6$ to $77.65\% \pm 0.8$. In order to evaluate how well MCMTL classifies the tasks, we also compare its performance to a version of MCMTL in which each task is assigned to the right class in advance. The difference between the two algorithms is statistically significant only for $N = 20$ (with the maximum of 5.08% for $M = 20$), in which the noise on the samples makes it more difficult to discriminate between the distributions generating the tasks, and thus, to classify them correctly.

Finally, we compare the performance of SCMTL and MCMTL in the asymmetric setting. At the end of each run of the symmetric problem, we draw 100 new test tasks at random from the same four classes used to generate the training tasks. We run the asymmetric algorithm described in Section 4.3 on each of the test tasks separately. Fig. 4*(c)* shows the performance of SCMTL and MCMTL for different number of training tasks and N fixed to 20. The results indicate that MCMTL performs relatively better than SCMTL as the number of training tasks increases.

6.2 Inverted Pendulum

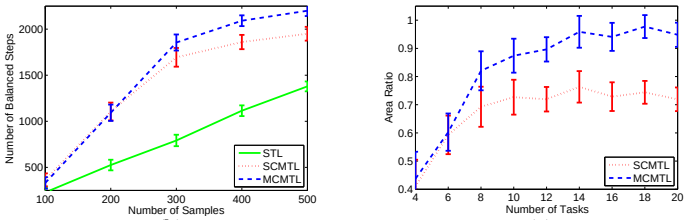
The experiments of Section 6.1 indicate that when the tasks are generated exactly according to the generative models of Sections 3 and 4, the BMTL methods can significantly improve the performance of a regression problem w.r.t. STL. As discussed in Section 2, the policy evaluation step of policy iteration can be casted as a regression problem, thus, similar improvement can be expected. In this section, we compare our BMTL algorithms with STL in the problem of learning a control policy for balancing an inverted pendulum. Dynamics, reward function, and basis functions are the same as in [8]. Each task is generated by drawing the parameters of the dynamics (pole mass, pole length, cart mass, and noise on the actions) from Gaussian distributions with means and variances summarized in Fig. 5*(a)*. The distribution over the two classes is uniform. It is worth noting that, unlike the regression experiments, here we have no guarantee that the weights of the value functions



(a) (b) (c)

Figure 4: Results for the case that the tasks are drawn randomly from the four classes: *(a)* relative MSE improvement of SCMTL over STL, *(b)* relative MSE improvement of MCMTL over STL, *(c)* asymmetric performance of MCMTL and SCMTL for $N = 20$.

	c_1	c_2
pole mass	1.0, 0.2	3.0, 0.0
pole length	0.5, 0.0	2.0, 0.2
cart mass	6.0, 0.5	10.0, 0.5
noise	12, 0.1	12, 0.1



(a) (b) (c)

Figure 5: Results for the inverted pendulum problem: *(a)* distributions of the parameters of the dynamics, *(b)* comparing the performance of STL, SCMTL, and MCMTL in terms of the number of balanced steps for $M = 10$, *(c)* comparing the performance of SCMTL and MCMTL in terms of the area ratio on the first 500 samples.

will follow the generative models assumed by the BMTL methods. We use policy iteration with 10 iterations and the noise correlation matrix $\mathbf{P}^{-1} = {\mathbf{\Phi}_m}^\top(\mathbf{\Phi}_m {\mathbf{\Phi}_m}^\top)^{-1}\mathbf{\Phi}_m$ for all the algorithms.[1] In STL, each policy evaluation step is solved using the EM algorithm of Section 3.2 for each task separately, where in BMTL, it is solved by running SCMTL or MCMTL over all the tasks. All the results are averaged over 150 independent runs.

Fig. 5*(b)* shows the performance of the policy learned by STL, SCMTL, and MCMTL for $M = 10$ tasks and different (up to 500) number of samples per task. Note that STL converges at about 1200 samples per task with an average performance of 2473 ± 61.9 balanced steps. As it can be seen, both BMTL methods outperform STL, and MCMTL achieves a better performance than SCMTL as the number of samples is increased. Since SCMTL forces all the tasks to have weights generated from a common distribution, it learns a very general prior, and thus, it cannot approximate the value functions as accurate as MCMTL, which is able to correctly discriminate between the tasks in classes c_1 and c_2. In order to show how the performance changes with different number of tasks, we compute the area ratio [14] on the first 500 samples as $\rho_{\text{BMTL}} = \frac{A_{\text{BMTL}} - A_{\text{STL}}}{A_{\text{STL}}}$, where A_{STL} (A_{BMTL}) is the area under the learning curve of STL (BMTL) from 100 to 500 samples. Fig. 5*(c)* shows that MCMTL has significantly better area ratio than SCMTL for all values of M except very small ones.

7 Conclusions

We presented hierarchical Bayesian models (HBMs) and inference algorithms for multi-task reinforcement learning (RL) where the tasks share structure in their value functions. To the best of our knowledge, this is the first work that models value function similarity using HBMs. In particular, we considered two cases, where all the value functions belong to the same class, and where they belong to an undefined number of classes. In these cases, we modelled the distribution over the value functions using a parametric HBM and a Dirichlet process (DP) based non-parametric HBM,

[1] This is the noise correlation matrix of LSTD(0) in the parametric GPTD form (see Section 8.4).

respectively. For each case, we derived inference algorithms for learning the value functions jointly and to transfer the knowledge acquired in the joint learning to improve the performance of learning the value function of a new task.

We first applied our proposed Bayesian multi-task learning (BMTL) algorithms to a regression problem, in which the tasks are drawn from the generative models used by the BMTL methods. The results indicate that BMTL algorithms achieve significant improvement over single-task learning (STL) in both symmetric and asymmetric settings. We then applied our BMTL algorithms to a benchmark RL problem, inverted pendulum. Although the tasks are no longer generated according to the models used by the BMTL algorithms, they still outperform STL. In our DP-based model we used Gibbs sampling, the most common simulation tool for Bayesian inference. We plan to look into variational techniques for Bayesian inference as an alternative approach.

8 Appendix

8.1 Details of the Single-class Multi-task Model

- R_m

$$p(R_m|\mathbf{w}_m, \theta) = p(R_m|\mathbf{w}_m, \sigma^2) = \mathcal{N}(\mathbf{H}\mathbf{\Phi}_{mn}^\top \mathbf{w}_m, \sigma^2 \mathbf{P})$$

- $\mathbf{w}_m$

$$p(\mathbf{w}_m|\theta) = p(\mathbf{w}_m|\boldsymbol{\mu}, \mathbf{\Sigma}) = \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma}) = (2\pi)^{-d/2}|\mathbf{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{w}_m - \boldsymbol{\mu})^\top \mathbf{\Sigma}^{-1}(\mathbf{w}_m - \boldsymbol{\mu})\right)$$

- θ

$$\begin{aligned}
p(\theta|\psi_0) &= p(\boldsymbol{\mu}, \mathbf{\Sigma}; \boldsymbol{\mu}_0, k_0, \nu_0, \mathbf{\Sigma}_0) \ \times \ p(\sigma^2; \alpha_0, \beta_0) \\
&= \mathcal{N}(\boldsymbol{\mu}; \boldsymbol{\mu}_0, \mathbf{\Sigma}/k_0)\, \mathcal{IW}(\mathbf{\Sigma}; \nu_0, \mathbf{\Sigma}_0) \ \times \ \mathcal{IG}(\sigma^2; \alpha_0, \beta_0) \\
&= (2\pi/k_0)^{-d/2}|\mathbf{\Sigma}|^{-1/2} \exp\left(-\frac{k_0}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^\top \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)\right) && \text{(normal)} \\
&\times B|\mathbf{\Sigma}_0|^{\nu_0/2}|\mathbf{\Sigma}|^{-(\nu_0+d+1)/2} \exp\left(-\frac{1}{2}\mathrm{tr}\left(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1}\right)\right) && \text{(inverse-Wishart)} \\
&\times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\left(\frac{1}{\sigma^2}\right)^{\alpha_0+1} \exp\left(-\frac{\beta_0}{\sigma^2}\right) && \text{(inverse-Gamma)}
\end{aligned}$$

where

$$B^{-1} = 2^{\nu_0 d/2}\pi^{d(d-1)/4} \prod_{j=1}^{d} \Gamma\left(\frac{\nu_0 + 1 - j}{2}\right).$$

8.2 Posterior Distribution of the Parameters with the Normal-Inverse-Wishart $\times$ Inverse-Gamma Prior

Taking advantage of the conjugate prior the posterior distribution over $\theta = (\boldsymbol{\mu}, \mathbf{\Sigma}, \sigma^2)$ given observations $\{(\mathbf{w}_m, R_m)\}_{m=1}^M$ and hyperprior ψ_0 is

$$p(\boldsymbol{\mu}, \mathbf{\Sigma}, \sigma^2|\mathbf{w}_m, R_m, \psi_0) = \mathcal{N}(\boldsymbol{\mu}; \boldsymbol{\mu}_0', \mathbf{\Sigma}/k_0')\, \mathcal{IW}(\mathbf{\Sigma}; \nu_0', \mathbf{\Sigma}_0') \times \mathcal{IG}(\sigma^2; \alpha_0', \beta_0') \tag{6}$$

where the posterior hyper-parameters $\psi_0' = (\boldsymbol{\mu}_0', k_0', \nu_0', \boldsymbol{\Sigma}_0', \alpha_0', \beta_0')$ are

$$\begin{aligned}
\boldsymbol{\mu}_0' &= \frac{M}{k_0 + M}\bar{\mathbf{w}} + \frac{k_0 \boldsymbol{\mu}_0}{k_0 + M} , && (7)\\
k_0' &= k_0 + M , && (8)\\
\nu_0' &= \nu_0 + M , && (9)\\
\boldsymbol{\Sigma}_0' &= \boldsymbol{\Sigma}_0 + Q_0 + \frac{k_0 M}{k_0 + M_{-m,c}}(\bar{\mathbf{w}} - \boldsymbol{\mu}_0)(\bar{\mathbf{w}} - \boldsymbol{\mu}_0)^\top , && (10)\\
\alpha_0' &= \alpha_0 + \frac{NM}{2} , && (11)\\
\beta_0' &= \beta_0 + \frac{1}{2}\sum_{m=1}^{M} (\mathbf{H}\boldsymbol{\Phi}_m^\top \mathbf{w}_m - R_m)^\top \mathbf{P}^{-1}(\mathbf{H}\boldsymbol{\Phi}_m^\top \mathbf{w}_m - R_m) , && (12)
\end{aligned}$$

where $\bar{\mathbf{w}} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{w}_m$, $Q_0 = \sum_{m=1}^{M}(\mathbf{w}_m - \bar{\mathbf{w}})(\mathbf{w}_m - \bar{\mathbf{w}})^\top$.

8.3 Gibbs Sampling

In this section, we report the equations used in the Gibbs sampling described in Section 4.2 of the paper. At each iteration of the MCMTL algorithm of Figure 2 in the paper, the Gibbs sampling is fed with observations $(\{\widehat{\mathbf{w}}_m\}, \{R_m\})$, where $\widehat{\mathbf{w}}_m = \boldsymbol{\mu}_{0m}'$. In particular, we use the the Gibbs sampling with conjugate-priors (Algorithm 3 in [12]).

We begin with the probability of task m belonging to a new class. Given observation $(\widehat{\mathbf{w}}_m, R_m)$ and hyper-prior ψ_0, the non-normalized probability can be written as

$$\int p(R_m, \mathbf{w}_m|\theta)p(\theta|\psi_0)d\theta = \left(\frac{k_0}{\pi k_0'}\right)^{d/2} \frac{|\boldsymbol{\Sigma}_0|^{\nu_0/2}}{|\boldsymbol{\Sigma}_0'|^{\nu_0'/2}} \frac{\Gamma\left(\frac{\nu_0'}{2}\right)}{\Gamma\left(\frac{\nu_0'-d}{2}\right)} \times (2\pi|P|)^{-N/2} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{\Gamma(\alpha_0')}{\beta_0'^{\alpha_0'}} ,$$

where hyper-parameters $\psi_0' = (\boldsymbol{\mu}_0', k_0', \nu_0', \boldsymbol{\Sigma}_0', \alpha_0', \beta_0')$ are computed as in Section 8.2 for observation $(R_m, \mathbf{w}_m)$.

Similarly, let $m' : c_{m'} = c, m' \neq m$, then the posterior over parameters for class c is

$$p(\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c, \sigma_c^2|\{\widehat{\mathbf{w}}_{m'}\}, \{R_{m'}\}, \psi_0) = \mathcal{N}(\boldsymbol{\mu}_c; \boldsymbol{\mu}_{0c}, \boldsymbol{\Sigma}_c/k_0)\mathcal{IW}(\boldsymbol{\Sigma}_c; \nu_{0c}, \boldsymbol{\Sigma}_{0c}) \times \mathcal{IG}(\sigma_c^2; \alpha_{0c}, \beta_{0c}) \quad (13)$$

with the following posterior hyper-parameters (which play the role of prior hyper-parameters for the class)

$$\begin{aligned}
\boldsymbol{\mu}_{0c} &= \frac{M_{-m,c}}{k_0 + M_{-m,c}}\bar{\mathbf{w}} + \frac{k_0 \boldsymbol{\mu}_0}{k_0 + M_{-m,c}} , && (14)\\
k_{0c} &= k_0 + M_{-m,c} , && (15)\\
\nu_{0c} &= \nu_0 + M_{-m,c} , && (16)\\
\boldsymbol{\Sigma}_{0c} &= \boldsymbol{\Sigma}_0 + Q_0 + \frac{k_0 M_{-m,c}}{k_0 + M_{-m,c}}(\bar{\mathbf{w}} - \boldsymbol{\mu}_0)(\bar{\mathbf{w}} - \boldsymbol{\mu}_0)^\top , && (17)\\
\alpha_{0c} &= \alpha_0 + \frac{NM_{-m,c}}{2} , && (18)\\
\beta_{0c} &= \beta_0 + \frac{1}{2}\sum_{m'} (\mathbf{H}\boldsymbol{\Phi}_m^\top \widehat{\mathbf{w}}_m - R_m)^\top \mathbf{P}^{-1}(\mathbf{H}\boldsymbol{\Phi}_m^\top \widehat{\mathbf{w}}_m - R_m) , && (19)
\end{aligned}$$

where $M_{-m,c}$ is the number of tasks belonging to class c except task m, $\bar{\mathbf{w}} = \frac{1}{M_{-m,c}}\sum_{m'} \widehat{\mathbf{w}}_{m'}$, and $Q_0 = \sum_{m'}(\widehat{\mathbf{w}}_{m'} - \bar{\mathbf{w}})(\widehat{\mathbf{w}}_{m'} - \bar{\mathbf{w}})^\top$. As a result, the integral for the probability of task m belong to class c becomes

$$\int p\big(R_m, \widehat{\mathbf{w}}_m|\theta\big)p\big(\theta|\{c_{m'}\},\psi_0\big)d\theta = \int p\big(R_m, \widehat{\mathbf{w}}_m|\theta\big)p\big(\theta|\psi_{0c}\big)d\theta$$
$$= \left(\frac{k_{0c}}{\pi k'_{0c}}\right)^{d/2} \frac{|\mathbf{\Sigma}_{0c}|^{\nu_{0c}/2}}{|\mathbf{\Sigma}'_{0c}|^{\nu'_{0c}/2}} \frac{\Gamma\left(\frac{\nu'_{0c}}{2}\right)}{\Gamma\left(\frac{\nu'_{0c}-d}{2}\right)} \times (2\pi|P|)^{-N/2} \frac{\beta_{0c}^{\alpha_{0c}}}{\Gamma(\alpha_{0c})} \frac{\Gamma(\alpha'_{0c})}{\beta_0'^{\alpha'_{0c}}},$$

where ψ'_{0c} is computed as in Equations (2)-(7) but using ψ_{0c} as prior instead of ψ_0.

8.4 Noise Correlation Models

We analyze the equations for $\widehat{\mathbf{w}}_m$ for two different noise correlation models. In particular we show that depending on the covariance matrix, both GPTD and BMTL can be seen as extensions of either Bellman residual minimization (BRM) or LSTD.

We first analyze the general formulation with $\mathbf{S} = \sigma^2\mathbf{P}$ be the covariance matrix of the noise.

Let $\theta = (\boldsymbol{\mu}, \mathbf{\Sigma}, \sigma^2)$ be the model parameters. The expected value of the weights in BMTL is written as

$$\widehat{\mathbf{w}}_m = \left[\frac{1}{\sigma^2}\mathbf{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}\mathbf{H}\mathbf{\Phi}_m^\top + \mathbf{\Sigma}^{-1}\right]^{-1}\left[\frac{1}{\sigma^2}\mathbf{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}R_m + \mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right], \tag{20}$$

where $\mathbf{S} = \sigma^2\mathbf{P}$ is the noise covariance matrix modeling the correlation of the noise at different states. We call σ^2 and $\mathbf{P}$ the noise variance and the noise correlation matrix, respectively.

In case $\boldsymbol{\mu} = \mathbf{0}$ and $\mathbf{\Sigma} = I$, we obtain a general form for the posterior mean of the weights in the parametric form of GPTD (Equation (4.4.41) in [5])

$$\widehat{\mathbf{w}}_m = \left[\mathbf{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}\mathbf{H}\mathbf{\Phi}_m^\top + \sigma^2 I\right]^{-1}\mathbf{\Phi}_m\mathbf{H}^\top\mathbf{P}^{-1}R_m. \tag{21}$$

The (parametric) GPTD of [5; 6] is obtained by setting $\mathbf{P} = \mathbf{H}\mathbf{H}^\top$ in Equation (21). As it was shown in Section 4.5 of [5], by setting $\sigma^2 \to 0$ and $\mathbf{P}^{-1} = {\mathbf{\Phi}_m}^\top G\mathbf{\Phi}_m$ in Equation (21), where G is an arbitrary $d \times d$ symmetric positive-definite matrix, we can derive a new set of GPTD algorithms that are based on the LSTD(0) algorithm. As it was discussed in Section 4.5.3 of [5], a reasonable choice for G is $G = \left(\mathbf{\Phi}_m{\mathbf{\Phi}_m}^\top\right)^{-1}$.

8.5 Experiment Setups

In the following we list all the details about the setup used in the experiments of the paper. We report the hyper-prior parameters (τ, ψ_0) and the parameters used in the inference algorithm. In particular, ϵ_{EM} is the threshold used in the stopping condition of the EM algorithm, n_{MCMTL} is the maximum number of iterations of the outer loop of MCMTL, and n_{Gibbs} is the number of steps of the Gibbs sampling. In none of the experiments the parameters have been systematically optimized.

8.5.1 The Regression Problem

In Figure 6 we report examples of the functions (tasks) and the generated samples for each of the four different classes used in the experiments of Section 6.1 of the paper. The parameters used in the experiments are reported in Tables 1. As it can be noticed the prior is not very informative and it has not been optimized for this specific problem. Since the four classes are quite well separated, the value of the concentration τ is not critical for the success of the MCMTL algorithm. Finally, the length of the Gibbs sampling changes with the number of tasks, so that when many tasks are involved, a longer MCMC simulation is performed and a more accurate estimation of $p(\{c_m\}|\{\widehat{\mathbf{w}}_m\}, \{R_m\}, \tau, \psi_0)$ is computed.

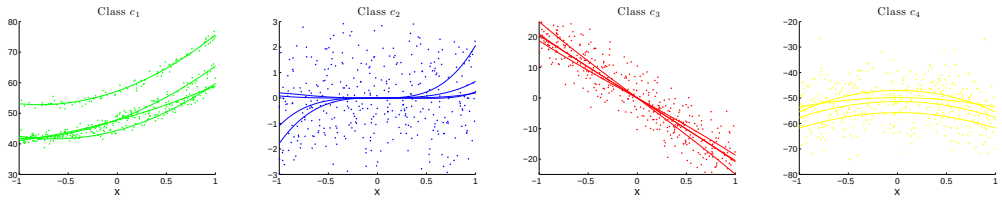


Figure 6: Examples of functions and samples drawn from each of the 4 classes ($N = 100$, $M = 4$).

Hyper-prior	Value
$\boldsymbol{\mu}_0$	[0 0 0 0 0 0]
k_0	1.0
ν_0	6
$\boldsymbol{\Sigma}_0$	$10I$
β_0	1.0
α_0	2.0
τ	30

Inference	Value
ϵ_{EM}	0.0001
n_{MCMTL}	10
n_{Gibbs}	$100\times M$

Table 1: Hyper-prior parameters and parameters of the inference algorithm in the regression problem.

8.5.2 Inverted Pendulum

In the inverted pendulum the state space $\mathcal{X} \in \mathbb{R}^2$ is a 2-dimensional space with variables $(\theta, \dot{\theta})$, the vertical angle and the angular velocity, respectively. The action space is $\mathcal{A} = \{-50, 0, 50\}$, where $a = 50$ means that a right force of 50 Newtons is applied to the cart. Each action is perturbed by a uniform noise in $[-\eta, \eta]$ (as reported in the paper for each task η is drawn from a Gaussian distribution $\mathcal{N}(12, 0.1)$). The discount factor is $\gamma = 0.9$. At each step of policy iteration, $Q(s, a)$ is approximated in a linear space spanned by 10 RBF features for each action as in [8]. The critical parameters controlling the nonlinear dynamics of the system are m the mass of the pendulum, M the mass of the cart, and l the length of the pendulum. To illustrate the impact of these parameters on the optimal value function and the corresponding optimal policy, we show the value functions and the policies for two sample tasks drawn from classes c_1 and c_2 in Figure 7. As it can be noticed different values for m, M, and l, induce significantly different value functions and policies. The parameters used in the experiments are reported in Tables 2.

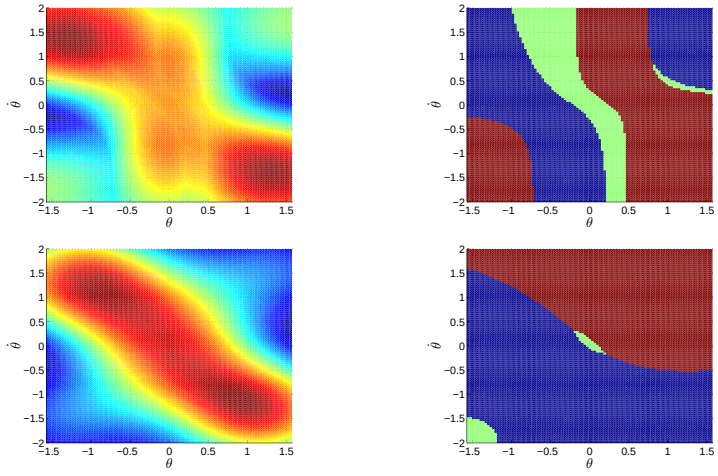


Figure 7: Examples of value functions and corresponding policies for two tasks with parameters $m_1 = 1.03$, $M_1 = 5.64$, $l_1 = 0.5$, $\eta_1 = 12.09$ and $m_2 = 3.0$, $M_2 = 10.04$, $l_2 = 2.01$, $\eta_2 = 11.91$, drawn from classes c_1 and c_2, respectively. The tasks are solved using STL with $N = 2,500$. In the policy plots blue, green, and red colors correspond to actions -50, 0, and 50, respectively.

Hyper-prior	Value
$\boldsymbol{\mu}_0$	$\underline{\mathbf{0}} \in \mathbb{R}^{30}$
k_0	1.0
ν_0	30
$\boldsymbol{\Sigma}_0$	$30I$
β_0	0.75
α_0	1.1
τ	20

Inference	Value
ϵ_{EM}	0.0001
n_{MCMTL}	10
n_{Gibbs}	$20\times M$

Table 2: Hyper-prior parameters and parameters of the inference algorithm in the inverted-pendulum problem.

At iteration $k = 1$ of policy iteration, the training set $\mathcal{D}_m^{(1)} = \{\langle x_n, a_n, x'_n, \pi_R(x'_n)\rangle\}_{n=1}^N$ is built by following a fully random policy π_R for each task $\mathcal{M}_m$. At iteration $k = 2, 3, \ldots$, all the samples in the training set $\mathcal{D}_m^{(k)}$ have the same x_n, a_n, x'_n components, while the fourth component is changed according to $\pi_{k-1}(x'_n)$. At the end of policy iteration, the performance of the learned policy is evaluated by taking the average over 30 episodes with a maximum of 3,000 steps each.

References

[1] J. Baxter. A model of inductive bias learning. *Journal of Artificial Intelligence Research*, 12: 149–198, 2000.

[2] D. Bertsekas and J. Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, 1996.

[3] E. Bonilla, K. Chai, and C. Williams. Multi-task Gaussian process prediction. In *Proceedings of NIPS 20*, pages 153–160, 2008.

[4] R. Caruana. Multitask learning. *Machine Learning*, 28(1):41–75, 1997.

[5] Y. Engel. *Algorithms and Representations for Reinforcement Learning*. PhD thesis, The Hebrew University of Jerusalem, Israel, 2005.

[6] Y. Engel, S. Mannor, and R. Meir. Reinforcement learning with Gaussian processes. In *Proceedings of ICML 22*, pages 201–208, 2005.

[7] H. Ishwaran and L. James. Gibbs sampling methods for stick-breaking priors. *Journal of the American Statistical Association*, 96:161–173, 2001.

[8] M. Lagoudakis and R. Parr. Least-squares policy iteration. *JMLR*, 4:1107–1149, 2003.

[9] A. Lazaric, M. Restelli, and A. Bonarini. Transfer of samples in batch reinforcement learning. In *Proceedings of ICML 25*, pages 544–551, 2008.

[10] N. Mehta, S. Natarajan, P. Tadepalli, and A. Fern. Transfer in variable-reward hierarchical reinforcement learning. *Machine Learning*, 73(3):289–312, 2008.

[11] S. Mukhopadhyay and A. Gelfand. Dirichlet process mixed generalized linear models. *Journal of the American Statistical Association*, 92(438):633–639, 1997.

[12] R. Neal. Markov chain sampling methods for Dirichlet process mixture models. *Journal of Computational and Graphical Statistics*, 9(2):249–265, 2000.

[13] R. Sutton and A. Barto. *An Introduction to Reinforcement Learning*. MIT Press, 1998.

[14] M. Taylor, P. Stone, and Y. Liu. Transfer learning via inter-task mappings for temporal difference learning. *JMLR*, 8:2125–2167, 2007.

[15] A. Wilson, A. Fern, S. Ray, and P. Tadepalli. Multi-task reinforcement learning: A hierarchical Bayesian approach. In *Proceedings of ICML 24*, pages 1015–1022, 2007.

[16] Y. Xue, X. Liao, L. Carin, and B. Krishnapuram. Multi-task learning for classication with dirichlet process priors. *JMLR*, 8:35–63, 2007.

[17] K. Yu, V. Tresp, and A. Schwaighofer. Learning Gaussian processes from multiple tasks. In *Proceedings of ICML 22*, pages 1012–1019, 2005.