



**HAL**  
open science

# High order transmission conditions for thin conductive sheets in magneto-quasistatics

Kersten Schmidt, Sébastien Tordeux

► **To cite this version:**

Kersten Schmidt, Sébastien Tordeux. High order transmission conditions for thin conductive sheets in magneto-quasistatics. ESAIM: Mathematical Modelling and Numerical Analysis, 2011, 45, pp.1115-1140. 10.1051/m2an/2011009 . inria-00473213v1

**HAL Id: inria-00473213**

**<https://inria.hal.science/inria-00473213v1>**

Submitted on 14 Apr 2010 (v1), last revised 24 Nov 2011 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

***High order transmission conditions for thin  
conductive sheets in magneto-quasistatics***

Kersten Schmidt\* — Sébastien Tordeux†

**N° 7254**

April 2010

Domaine 1



***Rapport  
de recherche***



## High order transmission conditions for thin conductive sheets in magneto-quasistatics

Kersten Schmidt\* , Sébastien Tordeux†

Domaine : Mathématiques appliquées, calcul et simulation  
Équipes-Projets POEMS

Rapport de recherche n° 7254 — April 2010 — 28 pages

**Abstract:** We propose transmission conditions of order 1, 2 and 3 approximating the shielding behaviour of thin conducting curved sheets for the magneto-quasistatic eddy current model in 2D. This model reduction applies to sheets whose thicknesses  $\varepsilon$  are essentially smaller or at the order of the skin depth. The sheet has itself not to be resolved, only its midline is represented by an interface. The computation is directly in one step with almost no additional cost. We prove the well-posedness w.r.t. to the small parameter  $\varepsilon$  and obtain optimal bound for the modelling error outside the sheet of order  $\varepsilon^{N+1}$  for the condition of order  $N$ . Numerical experiments with high order finite elements for sheets with varying curvature verify the theoretical findings.

**Key-words:** Asymptotic Expansions, Transmission Condition, Thin Conducting Sheets.

\* Project POEMS, INRIA Paris-Rocquencourt, 78153 Le Chesnay, France, e-mail: [kersten.schmidt@inria.fr](mailto:kersten.schmidt@inria.fr)

† Institut de Mathématiques de Toulouse, Université de Toulouse, France, e-mail: [sebastien.tordeux@insa-toulouse.fr](mailto:sebastien.tordeux@insa-toulouse.fr)

## **Conditions de transmission d'ordre élevé pour les plaques minces conductrices en magnéto-quasistatique**

**Résumé :** En domaine magnéto-quasistatique, nous nous intéressons à la modélisation des plaques minces conductrices dont l'épaisseur  $\varepsilon$  est de l'ordre de leur épaisseur de peau. Nous les modélisons par des conditions de transmission d'ordre 1, 2 et 3 posées sur une interface localisée en leur centre. La prise en compte dans les codes de calcul de ces modèles est aisée et ne génère qu'un coût de calcul marginal. Nous démontrons le caractère bien posé des modèles approchés associés et obtenons des estimations d'erreurs optimales d'ordre  $\varepsilon^{N+1}$  pour le modèle d'ordre  $N$ . Quelques expériences numériques réalisées avec des éléments finis d'ordre élevé sont en adéquation avec les résultats théoriques.

**Mots-clés :** Développements asymptotiques, Condition de transmission, plaques minces conductrices.

## CONTENTS

Introduction	3
1. Problem definition	4
1.1. The geometrical setting	4
1.2. Two-dimensional magneto-quasistatic with eddy current modelling	5
1.3. Asymptotic expansion with respect to the thickness of the sheet	6
2. Hierarchical asymptotic expansions	7
3. Derivation of the transmission conditions for the exterior fields	8
3.1. The definition of the exterior approximation	8
3.2. The definition of the interior approximation	9
4. Weak formulation, Uniqueness, and Stability	10
4.1. Preliminary material: function spaces and admissible boundary conditions	10
4.2. The model of order 1	11
4.3. The models of order 2 and 3	12
4.4. Regularity	17
5. Estimates of the modelling error	18
6. Numerical examples	20
Concluding remarks	25
Appendix A.	26
A.1. The surface operators	26
A.2. The interior operators	27
References	27

## INTRODUCTION

In many practical applications, electronic devices are surrounded by casings or other sheets of a highly conductive material to protect them from external electromagnetic fields (*e. g.*, data cables) or to protect the environment from the electromagnetic fields generated by devices (*e. g.*, transformer or bushings). To minimize the cost, size and weight, these sheets have to be thin. This leads to a non-perfect shielding where the electromagnetic fields partly penetrate the shields and, *e. g.*, external fields have a small but significant effect on the encased electronic devices. The large ratio of characteristic lengths (width of the device against thickness of the sheet) leads to serious numerical problems. Indeed the classical numerical methods such as finite differences or finite elements require a small mesh size and are consequently very costly or simply not able (due to limited memory) to compute a numerical approximation of the solution of such a problem. Another important issue for such problems is related to mesh generation. It is very time consuming to take into account small details in the geometry. Moreover, most of commercial mesh generators generate meshes with poor quality when the geometrical characteristic lengths are too different. Designing methods where the sheet needs not be included in the mesh is really a crucial objective. These two considerations point out the necessity of an appropriate modelling of the shielding behaviour by thin sheets. This is the problem that we address in this paper.

So called impedance boundary conditions (IBC) for thin layers of low order have been proposed by several authors, *e. g.*, [14,16,20], and also for transient analysis [19]. We prefer the notation “transmission condition” to distinguish from IBCs, originally proposed for solid conductors by Shchukin [28] and Leontovich [17], and derived for higher orders [1,9,13,27], and for perfect conductors with thin coatings [2,3,7]. Asymptotic expansion to any order have been derived for the electro-quasistatic equations [22] and time-harmonic Maxwell equations [21] in biological cells with isolating membranes. In a previous article [26] we derived an asymptotic expansion at any order for the eddy current problem in 2D with thin conducting sheets. We have chosen an asymptotical framework of constant shielding when the thickness  $\varepsilon$  of the sheet tends to zero leading to a non-trivial limit

problem. These derivations lead to the definition of a limit solution and correctors of higher orders, which are computed iteratively.

In this article we introduce approximate transmission conditions for the eddy current problem in 2D with thin conducting sheets which define approximate solutions of order 1, 2 and 3. Like for the IBCs the original problem is equipped with conditions to take into account the shielding behaviour of the thin sheet.

The problem we investigate is to find the electrical field  $e^\varepsilon$  satisfying

$$\begin{cases} -\Delta e^\varepsilon(\underline{x}) = f(\underline{x}), & \text{in } \Omega_{\text{ext}}^\varepsilon \\ -\Delta e^\varepsilon(\underline{x}) + \frac{c_0}{\varepsilon} e^\varepsilon(\underline{x}) = 0, & \text{in } \Omega_{\text{int}}^\varepsilon, \\ e^\varepsilon = e_{\text{imp}} & \text{on } \Gamma_e, \\ \nabla e^\varepsilon \cdot \underline{n} - \beta e^\varepsilon = \iota_{\text{imp}} & \text{on } \Gamma_i, \end{cases} \quad (1)$$

where  $\Omega$  is the whole domain,  $\Omega_{\text{int}}^\varepsilon$  is domain of the thin sheet of thickness  $\varepsilon$  and  $\Omega_{\text{ext}}^\varepsilon := \Omega \setminus \overline{\Omega_{\text{int}}^\varepsilon}$  (see detailed definition in Sec. 1). The relative conductivity is  $c_0$ ,  $\beta$  is some operator related to an impedance boundary condition and the remaining symbols stands for sources and boundary data. We have indexed the function  $e$  by  $\varepsilon$  which shall vary to 0.

In what follows we will design a one step procedure to compute a numerical approximation of  $e^\varepsilon$  which does require neither mesh refinement nor meshing of the thin sheet  $\Omega_{\text{int}}^\varepsilon$ . This technique is based on the asymptotic expansion of  $e^\varepsilon$  obtained in [25] and consists in modelling the thin sheet by two approximate transmission conditions which are derived and justified in three steps:

- (i) We derive formally an approximate models whose solutions  $\tilde{e}^{\varepsilon, N}$  are candidates to approximate the exact solution  $e^{\varepsilon, N}$ .
- (ii) We prove that the approximate problems are well posed for small  $\varepsilon$  and asymptotically stable.
- (iii) We prove that  $\tilde{e}^{\varepsilon, N}$  is an approximation of  $e^\varepsilon$  of order  $N$ , *i. e.*,  $\tilde{e}^{\varepsilon, N} - e^\varepsilon = \underset{\varepsilon \rightarrow 0}{o}(\varepsilon^N)$ .

In order to have a presentation as clear as possible, this article will only carry on the cases corresponding to approximation order less than 4, *i. e.*,  $N \leq 3$ . These results can be extended to  $N > 3$  even if one has to deal with higher derivative operators on the midline  $\Gamma$  of the sheet which introduces extra difficulties. The role of steps (ii) and (iii) is to give a mathematical background to the formal computations of step (i). In Section 2 the result of [25] will be shortly summarized. The steps (i) will be carried out in Section 3. Section 4 is devoted to step (ii). Here, we observe that with the chosen asymptotical framework of constant shielding the approximate problems are stable for thicknesses not exceeding some multiple of the skin depth  $d_{\text{skin}} := \sqrt{2/\omega\mu\sigma}$ . This transfers to the modelling error (step (iii)) with which we will deal in Section 5. Finally, in Section 6 numerical experiments verifying the theoretical results will be shown.

## 1. PROBLEM DEFINITION

### 1.1. The geometrical setting

Let us denote by  $\vec{x} = (x, y, z)$  a parametrization of  $\mathbb{R}^3$  and by  $\vec{e}_x$ ,  $\vec{e}_y$  and  $\vec{e}_z$  the associated orthogonal unit vectors. To avoid difficulties mostly related to differential geometry, we will be concerned in this article with a  $z$ -invariant configuration. To take care of the two-dimensional phenomenon, we introduce the vector  $\underline{x} = (x, y)$  composed of the two first coordinates of  $\vec{x}$ .

The computational domain  $\Omega \times \mathbb{R}$  is decomposed into a highly conducting sheet  $\Omega_{\text{int}}^\varepsilon \times \mathbb{R}$  and a domain  $\Omega_{\text{ext}}^\varepsilon \times \mathbb{R}$  filled with air which satisfies

- (i)  $\Omega$  is a bounded domain of  $\mathbb{R}^2$  with Lipschitz boundary.
- (ii) The conducting sheet

$$\Omega_{\text{int}}^\varepsilon = \{ \underline{x} \in \Omega : \exists y \in \Gamma \quad \|\underline{x} - \underline{y}\|_2 < \frac{\varepsilon}{2} \} \quad (2)$$

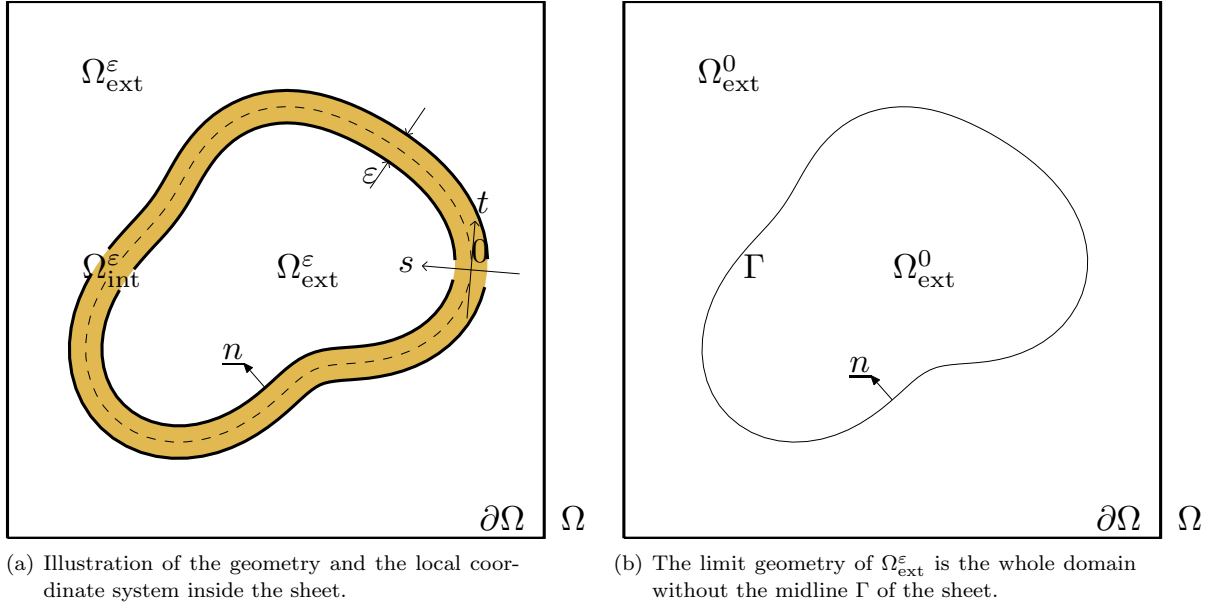


FIGURE 1. The two-dimensional geometrical setting for a sheet of thickness  $\varepsilon$  and the limit geometry for  $\varepsilon \rightarrow 0$ .

has constant thickness  $\varepsilon > 0$  and is centered around  $\Gamma$  a regular closed curve of  $\Omega$  with no cross point.

To each point of  $\Gamma$  can be associated a curvature  $\kappa(t)$  and a left normal unit vector  $\underline{n}(t)$  (see Fig. 1). Moreover it can be parameterized, for  $\varepsilon$  small enough, by a local coordinate system as follows. Let  $\hat{\Gamma}$  be a one-dimensional torus of the same length than  $\Gamma$ . Let denote by  $\underline{x}_m : \hat{\Gamma} \rightarrow \Omega$  an injective  $C^\infty$  mapping<sup>1</sup> whose range is  $\Gamma$  and satisfying  $\|\underline{x}'_m\| = 1$ . The domain  $\Omega_{\text{int}}^\varepsilon$  can then be seen as the range of the injective mapping

$$\begin{cases} \hat{\Gamma} \times ]-\varepsilon/2, \varepsilon/2[ & \longrightarrow & \Omega \\ (t, s) & \longmapsto & \underline{x}(t, s) = \underline{x}_m(t) + s\underline{n}(t). \end{cases} \quad (3)$$

This sub-domain is filled with material of constant conductivity  $\sigma$  and permeability  $\mu_0$ .

(iii) The exterior of the sheet  $\Omega_{\text{ext}}^\varepsilon = \Omega \setminus \overline{\Omega_{\text{int}}^\varepsilon}$  has constant permeability  $\mu_0$  and is not conductive.

## 1.2. Two-dimensional magneto-quasistatic with eddy current modelling

When the geometric characteristic length are all much smaller than the wave length, the electromagnetic fields are accurately described by the eddy current model, a quasi-static approximation to the Maxwell equations [5,24],

$$\begin{cases} \operatorname{div}(\vec{E}) = 0, \\ \operatorname{rot}(\vec{E}) = -\partial_t \vec{B}, \\ \operatorname{rot}(\vec{B}) = \mu_0 \vec{J}, \end{cases} \quad (4)$$

<sup>1</sup>A  $C^2$  mapping would be enough to have a continuous normal vector. We assume  $C^\infty$  for simplicity.



where the current is impressed, *i. e.*,  $\vec{J} = \vec{J}_0$  is known, in a sub-domain of  $\Omega_{\text{ext}}^\varepsilon$  (see Fig. 1) and it is by Ohms law proportional to the electric field  $\vec{E}$ , *i. e.*,  $\vec{J} = \sigma \vec{E}$  inside  $\Omega_{\text{int}}^\varepsilon$ .

We consider a time-harmonic excitation

$$\vec{J}_0(\vec{x}, t) = \exp(-i\omega t) j_0(\underline{x}) \vec{e}_z. \quad (5)$$

Due to  $z$ -invariance, the electromagnetic fields has to be sought in frequency domain with the form

$$\vec{E}(\vec{x}, t) = e(\underline{x}) \exp(-i\omega t) \vec{e}_z \quad \text{and} \quad \vec{B}(\vec{x}, t) = b_x(\underline{x}) \exp(-i\omega t) \vec{e}_x + b_y(\underline{x}) \exp(-i\omega t) \vec{e}_y. \quad (6)$$

Inside  $\Omega_{\text{int}}^\varepsilon$  and  $\Omega_{\text{ext}}^\varepsilon$  it reads for the out-of-plane electric field [23]

$$-\Delta e(\underline{x}) = -i\omega\mu_0 j_0(\underline{x}) =: f(\underline{x}), \quad \text{in } \Omega_{\text{ext}}^\varepsilon, \quad (7)$$

$$-\Delta e(\underline{x}) + i\omega\mu_0\sigma e(\underline{x}) = 0, \quad \text{in } \Omega_{\text{int}}^\varepsilon. \quad (8)$$

Furthermore, let the electric field satisfies the standard transmission conditions at the two interfaces between  $\Omega_{\text{int}}^\varepsilon$  and  $\Omega_{\text{ext}}^\varepsilon$ : the function  $e$  and its normal are continuous across the interfaces or equivalently

$$e \in H_\Delta^1(\Omega) := \left\{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \right\}. \quad (9)$$

Finally, let equations (7) and (8) be supplemented with suitable boundary conditions on  $\partial\Omega$  such it provides a unique solution in  $H^1(\Omega)$ : Let be given a prescribed electric field  $e^{\text{imp}}$  on  $\Gamma_e \subset \partial\Omega$  (Dirichlet condition) and a general impedance boundary condition on  $\Gamma_i := \partial\Omega \setminus \Gamma_e$ , which are given as

$$\begin{aligned} e &= e_{\text{imp}} \quad \text{on } \Gamma_e, \\ \nabla e \cdot \underline{n} - \beta e &= \iota_{\text{imp}} \quad \text{on } \Gamma_i, \end{aligned} \quad (10)$$

with a source term  $\iota^{\text{imp}}$  of the impedance condition and a impedance operator  $\beta$ . The latter could represent for example an integral representation of the electric field in the homogeneous exterior of  $\Omega$  or a simple Neumann boundary condition ( $\beta = 0$ ).

### 1.3. Asymptotic expansion with respect to the thickness of the sheet

In this article, we aim in designing a numerical method which does not require any meshing of the sheet. This method will be based on the asymptotic expansions of [25] which were derived for small thickness  $\varepsilon$  and a conductivity scaled reciprocal to the thickness  $\varepsilon$ , *i. e.*, inside  $\Omega_{\text{int}}^\varepsilon$  it holds

$$-\Delta e(\underline{x}) + \frac{c_0}{\varepsilon} e(\underline{x}) = 0, \quad (11)$$

where  $c_0 := i\varepsilon\omega\mu_0\sigma$  is independent of  $\varepsilon$ .

In real configuration, a thin sheet has a given thickness ( $\varepsilon = 1$  mm for example). Consequently looking for an asymptotic expansion with respect to the thickness of the sheet, *i. e.*, varying  $\varepsilon$  to 0, does not have at first glance a clear meaning. However, this point of view is known for similar problems to be rather efficient [3, 13] to design numerical methods.

Moreover, a real thin sheet has a given conductivity ( $\sigma = 5.9 \cdot 10^7 \frac{\text{A}}{\sqrt{\text{m}}}$  for copper). However, scaling  $\omega\mu_0\sigma$  like  $1/\varepsilon$  corresponds to a borderline case where the sheet is neither impenetrable (this will happen for  $|\omega\mu_0\sigma|^{-1} = o(\varepsilon)$ ) nor transparent ( $|\omega\mu_0\sigma| = o(\varepsilon^{-1})$ ) in the limit for  $\varepsilon$  going to 0. Thus, with the scaling  $\omega\mu_0\sigma \sim 1/\varepsilon$  already the limit model for  $\varepsilon \rightarrow 0$  is physically relevant and an asymptotic expansion is expected to be accurate already with a few terms.

## 2. HIERARCHICAL ASYMPTOTIC EXPANSIONS

We aim in this section in summarizing the results of [25] obtained for Dirichlet boundary conditions and adapting them to other boundary conditions. More precisely, we derived the complete asymptotic expansion with respect to the width of the sheet  $\varepsilon$  of the solution of problem (1) for a regular sheet.

We have shown that the use two points of view is necessary in order to describe sharply  $e^\varepsilon$ . The first point of view consists in considering the restriction of  $e^\varepsilon$  to the exterior of the sheet and in looking for a Taylor expansion of this restriction

$$e_{\text{ext}}^\varepsilon(\underline{x}) = e^\varepsilon|_{\Omega_{\text{ext}}^\varepsilon}(\underline{x}) = e_{\text{ext}}^{\varepsilon,N}(\underline{x}) + r_{\text{ext}}^{\varepsilon,N}(\underline{x}) \quad \text{with} \quad e_{\text{ext}}^{\varepsilon,N}(\underline{x}) := \sum_{j=0}^N \varepsilon^j u_{\text{ext}}^j(\underline{x}), \quad \forall N \in \mathbb{N}. \quad (12)$$

A contrario, the second point of view considers the restriction to the interior of the sheet of  $e^\varepsilon$ . We do not look for a Taylor expansion of  $e^\varepsilon$  in the original curvilinear coordinates  $(s, t)$ , but in the normalised curvilinear coordinates  $(S, t) = (\frac{s}{\varepsilon}, t)$

$$e_{\text{int}}^\varepsilon(\underline{x}) = e^\varepsilon|_{\Omega_{\text{int}}^\varepsilon}(\underline{x}) = e_{\text{int}}^{\varepsilon,N}(\underline{x}) + r_{\text{int}}^{\varepsilon,N}(\underline{x}) \quad \text{with} \quad e_{\text{int}}^{\varepsilon,N}(s, t) := \sum_{j=0}^N \varepsilon^j u_{\text{int}}^j\left(\frac{s}{\varepsilon}, t\right), \quad \forall N \in \mathbb{N}. \quad (13)$$

The coefficients  $u_{\text{ext}}^j(\underline{x})$  and  $u_{\text{int}}^j(S, t)$  of these Taylor expansions are functions not depending on  $\varepsilon$  which are defined on the limit domain  $\Omega_{\text{ext}}^0$  of  $\Omega_{\text{ext}}^\varepsilon$  for  $\varepsilon \rightarrow 0$

$$\Omega_{\text{ext}}^0 = \Omega \setminus \Gamma \quad (14)$$

and on the normalised sub-domain of the sheet  $\hat{\Gamma} \times [-\frac{1}{2}, \frac{1}{2}]$ , respectively. They are defined hierarchically order by order by a coupled problem (that will not be detailed here). Moreover, the following two estimates make clear what we mean by Taylor expansion

$$\|e_{\text{ext}}^\varepsilon - e_{\text{ext}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{ext}}^\varepsilon)} = \|r_{\text{ext}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{ext}}^\varepsilon)} \leq C_N \varepsilon^{N+1}, \quad \forall N \in \mathbb{N}, \quad (15)$$

$$\|e_{\text{int}}^\varepsilon - e_{\text{int}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{int}}^\varepsilon)} = \|r_{\text{int}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{int}}^\varepsilon)} \leq C_N \varepsilon^{N+\frac{1}{2}}, \quad \forall N \in \mathbb{N}. \quad (16)$$

Analysing deeply the coupled system that is solved by the exterior and interior asymptotic expansions, one can remark — this has been done up to order 2 in [25] — that one can define the exterior coefficients with problems involving only the exterior coefficients of lower order and not the interior coefficients. These hierarchical decouple problems take the form

$$-\Delta u_{\text{ext}}^j(\underline{x}) = f_j(\underline{x}), \quad \text{in } \Omega_{\text{ext}}^0, \quad (17a)$$

$$\left[ u_{\text{ext}}^j \right] (t) = \sum_{\ell=2}^j (\gamma_\ell u_{\text{ext}}^{j-\ell})(t), \quad \text{on } \Gamma, \quad (17b)$$

$$\left[ \partial_n u_{\text{ext}}^j \right] (t) - c_0 \left\{ u_{\text{ext}}^j \right\} (t) = \sum_{\ell=1}^j (\zeta_\ell u_{\text{ext}}^{j-\ell})(t), \quad \text{on } \Gamma, \quad (17c)$$

with the two transmission operators  $[\cdot]$  and  $\{\cdot\}$  defined on the midline  $\Gamma$  by

$$[u](t) := u(t, 0^+) - u(t, 0^-), \quad \{u\}(t) := \frac{1}{2} (u(t, 0^+) + u(t, 0^-)),$$

with the differential operators  $\gamma_\ell$  and  $\zeta_\ell$ . that are explicitly given for  $\ell \leq 3$  in Appendix A.1, and with the source terms  $f_j$  and boundary conditions which are inherited from the original problem and consequently satisfy ( $j > 0$ )

$$\begin{aligned} f_0(\underline{x}) &= f(\underline{x}) & \text{and} & & f_j(\underline{x}) &= 0, & \text{in } \Omega_{\text{ext}}^0, \\ u_{\text{ext}}^0 &= e_{\text{imp}} & \text{and} & & u_{\text{ext}}^j &= 0 & \text{on } \Gamma_e, \\ \nabla u_{\text{ext}}^0 \cdot \underline{n} - \beta u_{\text{ext}}^0 &= \iota_{\text{imp}}, & \text{and} & & \nabla u_{\text{ext}}^j \cdot \underline{n} - \beta u_{\text{ext}}^j &= 0, & \text{on } \Gamma_i, \end{aligned} \quad (18)$$

Once the exterior coefficients are defined the interior coefficients can be computed in the following way. They are polynomials in the normal coordinate  $S$  and result by the exterior fields of the same and the previous orders. More precisely, they can be written

$$U_{\text{int}}^j(t, S) = \sum_{\ell=0}^j (\eta_\ell u_{\text{ext}}^{j-\ell})(t, S) \quad (19)$$

with for  $\ell \leq 3$  the  $\eta_\ell$  given in Appendix A.2.

**Remark 2.1.** *The latter asymptotic expansion can directly be used to obtain a numerical approximation of  $e^\varepsilon$ . Indeed one has just to compute  $e_{\text{ext}}^{\varepsilon, N}$ , and  $e_{\text{int}}^{\varepsilon, N}$ , with  $N$  fixed by the desired precision. These computations do require neither mesh refinement nor the meshing of the thin sheet. However, this method suffer from a major drawback: For relatively large  $\varepsilon$  the model of order 0 does quite possibly not reach the desired precision and one has to compute more further terms of the asymptotic expansion in order to obtain a sharp approximation of  $e^\varepsilon$ . The multi-step procedure is not standard and may discourage to be implemented in a numerical library.*

### 3. DERIVATION OF THE TRANSMISSION CONDITIONS FOR THE EXTERIOR FIELDS

In this section, we show how one can derive the approximate problems for a regular sheet ( $\Gamma$  is  $C^\infty$ ).

#### 3.1. The definition of the exterior approximation

We adopt the point of view of formal series. Due to (12), the formal Taylor series of  $e_{\text{ext}}^\varepsilon$  takes the form

$$e_{\text{ext}}^\varepsilon(\underline{x}) \sim \sum_{j=0}^{+\infty} \varepsilon^j u_{\text{ext}}^j(\underline{x}). \quad (20)$$

where we have adopted the symbol “ $\sim$ ” to mention that this series may diverge or converge but not toward  $e_{\text{ext}}^\varepsilon$ . Therefore multiplying for all  $j \in \mathbb{N}$  system (17b), (17c) by  $\varepsilon^j$  and summing we get with

$$\left[ \sum_{j=0}^{+\infty} \varepsilon^j u_{\text{ext}}^j \right](t) = \sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^j (\gamma_\ell u_{\text{ext}}^{j-\ell})(t), \quad \text{on } \Gamma, \quad (21a)$$

$$\left[ \partial_n \sum_{j=0}^{+\infty} \varepsilon^j u_{\text{ext}}^j \right](t) - c_0 \left\{ \sum_{j=0}^{+\infty} \varepsilon^j u_{\text{ext}}^j \right\}(t) = \sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^j (\zeta_\ell u_{\text{ext}}^{j-\ell})(t), \quad \text{on } \Gamma, \quad (21b)$$

with the convention  $\gamma_0 = \gamma_1 = \zeta_0 = 0$ . Interchanging the two sums and identifying  $e_{\text{ext}}^\varepsilon$ , we find

$$[e_{\text{ext}}^\varepsilon](t) \sim (\gamma^\varepsilon e_{\text{ext}}^\varepsilon)(t), \quad \text{on } \Gamma, \quad (22a)$$

$$[\partial_n e_{\text{ext}}^\varepsilon](t) - c_0 \{e_{\text{ext}}^\varepsilon\}(t) \sim (\zeta^\varepsilon e_{\text{ext}}^\varepsilon)(t), \quad \text{on } \Gamma, \quad (22b)$$

with the two formal operator series  $\gamma^\varepsilon$  and  $\zeta^\varepsilon$  given by

$$\gamma^\varepsilon = \sum_{j=0}^{+\infty} \varepsilon^j \gamma_j, \quad \text{and} \quad \zeta^\varepsilon = \sum_{j=0}^{+\infty} \varepsilon^j \zeta_j. \quad (23)$$

These two transmission conditions appear to be perfect. However, the question of convergence of the series (23) remains (we think they diverge potentially). Moreover, it seems not possible to get a simple formula for the sum if it exists. Consequently, these two perfect transmission conditions could not directly be used for numerical computations. However, truncating these two series at a given order  $N$

$$\gamma^{\varepsilon,N} = \sum_{j=0}^N \varepsilon^j \gamma_j, \quad \text{and} \quad \zeta^{\varepsilon,N} = \sum_{j=0}^N \varepsilon^j \zeta_j, \quad (24)$$

we get well defined transmission conditions to model the highly conductive thin sheet.

This series truncation leads to approximate problems for the approximate solutions  $\tilde{e}_{\text{ext}}^{\varepsilon,N} \in H^1(\Omega_{\text{ext}}^0)$

$$-\Delta \tilde{e}_{\text{ext}}^{\varepsilon,N}(\underline{x}) = f(\underline{x}), \quad \text{in } \Omega_{\text{ext}}^0, \quad (25a)$$

$$\left[ \tilde{e}_{\text{ext}}^{\varepsilon,N} \right](t) - (\gamma^{\varepsilon,N} \tilde{e}_{\text{ext}}^{\varepsilon,N})(t) = 0, \quad \text{on } \Gamma, \quad (25b)$$

$$\left[ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N} \right](t) - c_0 \left\{ \tilde{e}_{\text{ext}}^{\varepsilon,N} \right\}(t) - (\zeta^{\varepsilon,N} \tilde{e}_{\text{ext}}^{\varepsilon,N})(t) = 0, \quad \text{on } \Gamma, \quad (25c)$$

$$\tilde{e}_{\text{ext}}^{\varepsilon,N} = e_{\text{imp}} \quad \text{on } \Gamma_e, \quad (25d)$$

$$\nabla \tilde{e}_{\text{ext}}^{\varepsilon,N} \cdot \underline{n} - \beta \tilde{e}_{\text{ext}}^{\varepsilon,N} = t_{\text{imp}} \quad \text{on } \Gamma_i. \quad (25e)$$

The approximate solutions are indexed by  $N$  which is related to the order of the approximation. For  $N > 1$ , the reader can note that this approximate solution is no more continuous across  $\Gamma$  and therefore does not belong to  $H^1(\Omega)$ .

### 3.2. The definition of the interior approximation

The same strategy can be applied to the derivation of an interior approximation. Due to (13), the Taylor series of  $e_{\text{int}}^\varepsilon$  reads  $e_{\text{int}}^\varepsilon(t, s) \sim \sum_{j=0}^{+\infty} \varepsilon^j U_{\text{int}}^j(t, \frac{s}{\varepsilon})$ . Inserting (19) we have

$$e_{\text{int}}^\varepsilon(t, s) \sim \sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^j (\eta_\ell w_{\text{ext}}^{j-\ell})(t, \frac{s}{\varepsilon}) = \left( \sum_{\ell=0}^{+\infty} \varepsilon^\ell \eta_\ell \right) \left( \sum_{j=0}^{+\infty} \varepsilon^j w_{\text{ext}}^j \right)(t, \frac{s}{\varepsilon}) \sim \boldsymbol{\eta}^\varepsilon e_{\text{ext}}^\varepsilon(t, \frac{s}{\varepsilon}).$$

with the formal series operator  $\boldsymbol{\eta}^\varepsilon$  and its associated partial sums defined by

$$\boldsymbol{\eta}^\varepsilon = \sum_{\ell=0}^{+\infty} \varepsilon^\ell \eta_\ell \quad \text{and} \quad \boldsymbol{\eta}^{\varepsilon,N} := \sum_{\ell=0}^N \varepsilon^\ell \eta_\ell. \quad (26)$$

It leads to the introduction of the interior approximation of order  $N$

$$\tilde{e}_{\text{int}}^{\varepsilon,N}(t, s) = (\boldsymbol{\eta}^{\varepsilon,N} \tilde{e}_{\text{ext}}^{\varepsilon,N})(t, \frac{s}{\varepsilon}). \quad (27)$$

**Remark 3.1.** *In the continuation we will not carry on the justification of these approximations. Note, however, that it can be proved that  $\tilde{e}_{\text{int}}^{\varepsilon,N}$  is an approximation of order  $N$  of  $e_{\text{int}}^\varepsilon$ . More precisely, it holds*

$$\|e_{\text{int}}^\varepsilon - \tilde{e}_{\text{int}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{int}}^\varepsilon)} = \|\boldsymbol{\eta}^{\varepsilon,N}\|_{H^1(\Omega_{\text{int}}^\varepsilon)} \leq C_N \varepsilon^{N+\frac{1}{2}}. \quad (28)$$

#### 4. WEAK FORMULATION, UNIQUENESS, AND STABILITY

The question of existence and uniqueness of  $\tilde{e}^{\varepsilon, N}$  is not only an interesting mathematical question. Very often problems arising from an asymptotic expansion are not well posed. Consequently, it is crucial to check the existence and uniqueness of solution of problem (25). Moreover, the convergence proof needs a stability result that is closely related to the existence and uniqueness of the solution of the collected models.

Since offset functions allow to exchange Dirichlet data  $e_{\text{imp}}$  against source terms  $f$ , we will only deal with homogeneous Dirichlet boundary conditions *i. e.*,  $e_{\text{imp}} = 0$ .

As the jump of the external field  $\tilde{e}_{\text{ext}}^{\varepsilon, 1}$  in contrast to the higher orders is vanishing in general due to  $\gamma^{\varepsilon, 1} = 0$  we will propose for order 1 an own weak formulation in  $H^1(\Omega)$ . For the models of order 2 and higher we will impose the jump condition (25b) weakly with an additional equation.

##### 4.1. Preliminary material: function spaces and admissible boundary conditions

In what follows, homogeneous Dirichlet boundary conditions at  $\Gamma_e$  will be incorporated in the trial and test spaces which are

$$\text{for } N = 1 \quad H_{\Gamma_e}^1(\Omega) = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_e \right\}, \quad (29a)$$

$$\text{for } N = 2 \text{ or } 3 \quad H_{\Gamma_e}^1(\Omega_{\text{ext}}^0) = \left\{ v \in H^1(\Omega_{\text{ext}}^0) : v = 0 \text{ on } \Gamma_e \right\}. \quad (29b)$$

Moreover, we consider impedance operators that provides coercive variational weak formulations. This leads to the notion of admissible boundary conditions.

**Definition 4.1** (Admissible boundary conditions). *Let  $V = H_{\Gamma_e}^1(\Omega)$  or  $H_{\Gamma_e}^1(\Omega_{\text{ext}}^0)$ . A boundary condition (10) is  $V$ -admissible if for all  $v \in V$*

$$\text{Im} \left( \int_{\Gamma_i} \beta |v|^2 dS \right) \geq 0, \quad \text{Re} \left( \int_{\Gamma_i} \beta |v|^2 dS \right) \geq 0,$$

and if for any constant  $\delta = |\delta|e^{i\phi} \in \mathbb{C} \setminus \{0\}$  with  $\phi \in [0, \pi)$ , the bilinear form

$$\mathbf{b}_\delta(u, v) := \int_{\Omega_{\text{ext}}^0} \nabla u \cdot \bar{v} d\mathbf{x} + \int_{\Gamma_i} \beta u \bar{v} dS + \delta \int_{\Gamma} \{u\} \{\bar{v}\} + [u][\bar{v}] dt. \quad (30)$$

is  $V$ -continuous and  $V$ -elliptic with an ellipticity constant  $\gamma \geq h(\delta)$  and  $h$  a non-negative continuous function defined for  $\delta = |\delta|e^{i\phi} \neq 0$  with  $\phi \in [0, \pi)$ .

**Remark 4.2.** *In  $H_{\Gamma_e}^1(\Omega)$ , ones has  $\{u\} = u$  and  $[u] = 0$  on  $\Gamma$ . Consequently, the bilinear form  $\mathbf{b}_\delta$  can be simplified into*

$$\mathbf{b}_\delta(u, v) := \int_{\Omega_{\text{ext}}^0} \nabla u \cdot \bar{v} d\mathbf{x} + \int_{\Gamma_i} \beta u \bar{v} dS + \delta \int_{\Gamma} u \bar{v} dt. \quad (31)$$

**Remark 4.3.** *It is easy to show, that either Dirichlet (prescribed electric field) or Neumann, or general impedance boundary conditions with  $\text{Im} \langle \beta v, v \rangle_{\Gamma_i} \geq 0$ ,  $\text{Re} \langle \beta v, v \rangle_{\Gamma_i} \geq 0$ , or any boundary condition mixed out of these three are  $H_{\Gamma_e}^1(\Omega)$ - and  $H_{\Gamma_e}^1(\Omega_{\text{ext}}^0)$ -admissible. This follows by testing the bilinear form with  $v = e^{i\frac{\phi}{2}} u$ , taking the real part and applying the Poincaré-Friedrich inequality [10]. The general impedance boundary conditions with purely real  $\beta$ -operator are included.*

Moreover, due to the presence of the differential operator  $\partial_t$  in (25c) for  $N = 2$  and 3 the following function space will be introduced

$$H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0) := \{v \in H_{\Gamma_e}^1(\Omega_{\text{ext}}^0) : \{v\} \in H^1(\Gamma)\}$$

with its associated norm defined by

$$\|v\|_{H^{1,1}(\Omega_{\text{ext}}^0)}^2 := \|v\|_{H^1(\Omega_{\text{ext}}^0)}^2 + \varepsilon^2 \|\partial_t \{v\}\|_{L^2(\Gamma)}^2 = \|v\|_{H^1(\Omega_{\text{ext}}^0)}^2 + \varepsilon^2 |\{v\}|_{H^1(\Gamma)}^2, \quad (32)$$

where the norm has been weighted by  $\varepsilon$  in order to simplify the proofs that follow.

**Definition 4.4.** *The topological dual space of a space  $V$  is denoted by  $V'$ .*

**Definition 4.5.** *Let  $\mathbf{A}$  be a bounded or unbounded operator of  $L^2$  equipped with the inner product  $\langle \cdot, \cdot \rangle$ . We denote by  $\overline{\mathbf{A}}$  the adjoint operator of  $\mathbf{A}$  meaning that  $\langle \mathbf{A}u, v \rangle = \langle u, \overline{\mathbf{A}}v \rangle$  and with  $\bar{c} = \text{Re}(c) - i \text{Im}(c)$  the complex conjugate of a complex number  $c \in \mathbb{C}$ .*

Note, that due to the definition of the scalar product we have  $\langle cu, v \rangle = \langle u, \bar{c}v \rangle$  for  $c$  a multiplication operator.

## 4.2. The model of order 1

### 4.2.1. The weak formulation

The first order approximation is given by (25) with  $N = 1$ . Since  $\gamma^{\varepsilon,1} = 0$ ,  $\tilde{e}_{\text{ext}}^{\varepsilon,1}$  belongs to  $H_{\Gamma_e}^1(\Omega)$ . The system of order 1 is given by

$$\tilde{e}_{\text{ext}}^{\varepsilon,1} \in H_{\Gamma_e}^1(\Omega), \quad (33a)$$

$$-\Delta \tilde{e}_{\text{ext}}^{\varepsilon,1}(\underline{x}) = f(\underline{x}), \quad \text{in } \Omega_{\text{ext}}^0, \quad (33b)$$

$$\left[ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,1} \right](t) - c_0 \left( 1 + \varepsilon \frac{c_0}{6} \right) \left\{ \tilde{e}_{\text{ext}}^{\varepsilon,1} \right\}(t) = 0, \quad \text{on } \Gamma. \quad (33c)$$

$$\nabla \tilde{e}_{\text{ext}}^{\varepsilon,1} \cdot \underline{n} - \beta \tilde{e}_{\text{ext}}^{\varepsilon,1} = \iota_{\text{imp}}, \quad \text{on } \Gamma_i, \quad (33d)$$

where one can replace  $\left\{ \tilde{e}_{\text{ext}}^{\varepsilon,1} \right\}$  by  $\tilde{e}_{\text{ext}}^{\varepsilon,1}$  if it is necessary. Moreover, this system is equivalent to the variational formulation: Seek  $\tilde{e}_{\text{ext}}^{\varepsilon,1} \in H_{\Gamma_e}^1(\Omega)$  such that for all  $e' \in H_{\Gamma_e}^1(\Omega)$

$$\mathbf{a}_1(\tilde{e}_{\text{ext}}^{\varepsilon,1}, e') = \langle \ell, e' \rangle, \quad (34)$$

with the bilinear form  $\mathbf{a}_1$  and the linear form  $\ell$  defined by

$$\mathbf{a}_1(e, e') = \int_{\Omega_{\text{ext}}^0} \nabla e \cdot \nabla \bar{e}' \, d\underline{x} + \int_{\Gamma_i} \beta e \bar{e}' \, dS + \int_{\Gamma} c_0 \left( 1 + \varepsilon \frac{c_0}{6} \right) \{e\} \{\bar{e}'\} \, dt, \quad (35)$$

$$\langle \ell, e' \rangle = \int_{\Omega_{\text{ext}}^0} f \bar{e}' \, d\underline{x} - \int_{\Gamma_i} \iota_{\text{imp}} \bar{e}' \, dS. \quad (36)$$

### 4.2.2. Well-posedness and stability of the order 1 model

**Lemma 4.6.** *Let  $\varepsilon_m := \frac{m}{|c_0|}$  for any  $m > 0$ , let the boundary conditions be  $H_{\Gamma_e}^1(\Omega)$ -admissible, and  $\ell \in (H_{\Gamma_e}^1(\Omega))'$ . Then, there exists a unique solution  $\tilde{e}_{\text{ext}}^{\varepsilon,1} \in H_{\Gamma_e}^1(\Omega)$  of (34) and a constant  $C > 0$  such that*

$$\|\tilde{e}_{\text{ext}}^{\varepsilon,1}\|_{H^1(\Omega)} \leq C_m \|\ell\|_{(H_{\Gamma_e}^1(\Omega))'}, \quad \forall \varepsilon \in (0, \varepsilon_m)$$

*Proof.* Let  $\delta_\varepsilon := c_0 \left( 1 + \varepsilon \frac{c_0}{6} \right)$ . Since  $c_0 = i|c_0| \neq 0$  we have  $\text{Im}(\delta_\varepsilon) = |c_0|$ ,  $\text{Re}(\delta_\varepsilon) \in (-\varepsilon_m |c_0|^2/6, 0)$  and so  $\delta_\varepsilon = |\delta_\varepsilon| e^{i\phi_m}$  with  $\phi_m$  only depending on  $m$ . Since the bilinear form  $\mathbf{a}_1$  is  $H_{\Gamma_e}^1(\Omega)$ -elliptic (see Def. 4.1) we get the uniform coercivity of  $\mathbf{a}_1$  ( $h$  is continuous)

$$\exists \gamma_m = \min_{\varepsilon \in ]0, \varepsilon_m[} h(\delta_\varepsilon) > 0 : \forall \varepsilon \in ]0, \varepsilon_m[ \quad |\mathbf{a}_1(e, e)| \geq \gamma_m \|e\|_{H^1(\Omega)}^2 \quad \forall e \in H_{\Gamma_e}^1(\Omega). \quad (37)$$

Application of the Lax-Milgram lemma [6] completes the proof with  $C_m = \frac{1}{\gamma_m}$ .  $\square$

**Remark 4.7.** For sheet thicknesses up to some multiple  $m$  of  $\frac{1}{|c_0|}$ , e. g.,  $m = 10$ , uniform stability holds with a stability constant well bounded away from 0. Only if the sheet is much thicker than  $\frac{1}{|c_0|}$  and so the skin depth  $d_{\text{skin}}$  the stability constant approaches 0.

### 4.3. The models of order 2 and 3

In this section, a weak formulation for the models of order 2 and 3 will be derived. Their solutions solve transmission problems (25) that involve the operators  $\gamma_j$  and  $\zeta_j$  given in (69). They are differential operators in  $t$  acting on the mean trace and the mean normal trace on  $\Gamma$ . Hence, we can rewrite the operators  $\gamma^{\varepsilon,N}$  and  $\zeta^{\varepsilon,N}$  as follows

$$(\gamma^{\varepsilon,N} \tilde{e}_{\text{ext}}^{\varepsilon,N})(t) := \gamma_0^{\varepsilon,N}(t) \{ \tilde{e}_{\text{ext}}^{\varepsilon,N} \}(t) + \gamma_1^{\varepsilon,N}(t) \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N} \}(t), \quad (38a)$$

$$(\zeta^{\varepsilon,N} \tilde{e}_{\text{ext}}^{\varepsilon,N})(t) := \zeta_0^{\varepsilon,N}(t) \{ \tilde{e}_{\text{ext}}^{\varepsilon,N} \}(t) + \zeta_1^{\varepsilon,N}(t) \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N} \}(t). \quad (38b)$$

The multiplication operators  $\gamma_0^{\varepsilon,N}$  (\*),  $\gamma_1^{\varepsilon,N}$  (\*\*),  $\zeta_1^{\varepsilon,N}$  (\*\*\*\*), and the second order differential operator  $\zeta_0^{\varepsilon,N}$  (\*\*\*) can be read from the transmission conditions (25b) and (25c), which are given for  $N = 2$  by

$$[\tilde{e}_{\text{ext}}^{\varepsilon,2}](t) - \underbrace{\left( -\varepsilon^2 \frac{c_0}{24} \kappa(t) \right)}_{(*)} \{ \tilde{e}_{\text{ext}}^{\varepsilon,2} \}(t) - \underbrace{\left( -\varepsilon^2 \frac{c_0}{12} \right)}_{(**)} \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,2} \}(t) = 0, \quad (39a)$$

$$[\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,2}](t) - \underbrace{\left( c_0 + \varepsilon \frac{c_0^2}{6} + \varepsilon^2 \frac{c_0}{12} \left( \frac{7}{20} c_0^2 - \partial_t^2 \right) \right)}_{(***)} \{ \tilde{e}_{\text{ext}}^{\varepsilon,2} \}(t) - \underbrace{\left( -\varepsilon^2 \frac{c_0}{24} \kappa(t) \right)}_{(****)} \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,2} \}(t) = 0 \quad (39b)$$

and for  $N = 3$

$$[\tilde{e}_{\text{ext}}^{\varepsilon,3}](t) - \underbrace{\left( -\varepsilon^2 \frac{c_0}{24} \kappa(t) \left( 1 - \varepsilon \frac{c_0}{10} \right) \right)}_{(*)} \{ \tilde{e}_{\text{ext}}^{\varepsilon,3} \}(t) - \underbrace{\left( -\varepsilon^2 \frac{c_0}{12} \left( 1 - \varepsilon \frac{c_0}{10} \right) \right)}_{(**)} \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,3} \}(t) = 0, \quad (40a)$$

$$[\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,3}](t) - \underbrace{\left( c_0 + \varepsilon \frac{c_0^2}{6} + \varepsilon^2 \frac{c_0}{12} \left( \frac{7}{20} c_0^2 - \partial_t^2 \right) + \varepsilon^3 \frac{c_0^2}{40} \left( \frac{17}{84} c_0^2 - \frac{1}{3} \kappa^2(t) - \partial_t^2 \right) \right)}_{(***)} \{ \tilde{e}_{\text{ext}}^{\varepsilon,3} \}(t) \quad (40b)$$

$$- \underbrace{\left( -\varepsilon^2 \frac{c_0}{24} \kappa(t) \left( 1 - \varepsilon \frac{c_0}{10} \right) \right)}_{(****)} \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,3} \}(t) = 0.$$

#### 4.3.1. The weak formulation: a mixed formulation

The original problem (1) is posed in  $H^1(\Omega)$ . Consequently it is natural to intend to find a formulation posed in  $H^1(\Omega_{\text{ext}}^0)$ . However, this function space does not impose a sufficient regularity to deal with derivative of mean trace and with mean normal traces. As a remedy, the mean normal trace will be considered as a new unknown function of  $L^2(\Gamma)$

$$\tilde{\lambda}_{\text{ext}}^{\varepsilon,N} = \{ \partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N} \}, \quad (41)$$

and the solution  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  will be searched in the trial space  $H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$ . A variational formulation of (25) will be derived as a problem coupling  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  and  $\tilde{\lambda}_{\text{ext}}^{\varepsilon,N}$ .

We will use the notation  $\langle u, v \rangle_{\Gamma}$  for the integral  $\int_{\Gamma} u \bar{v} dS$ . Integration by parts is applied to the tangential derivatives such that the highest tangential derivative is one  $(-\langle \partial_t^2 u, v \rangle_{\Gamma} = \langle \partial_t u, \partial_t v \rangle_{\Gamma})$ .

To derive the variational formulation we multiply (25a) with a test function  $e' \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  and using Green's formula we get

$$\int_{\Omega_{\text{ext}}^0} \nabla \tilde{e}_{\text{ext}}^{\varepsilon,N} \cdot \nabla \bar{e}' \, d\mathbf{x} + \int_{\partial\Omega} (\nabla \tilde{e}_{\text{ext}}^{\varepsilon,N} \cdot \mathbf{n}) \bar{e}' \, dS + \int_{\Gamma} [\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N}] \{\bar{e}'\} + \{\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N}\} [\bar{e}'] \, dt = \int_{\Omega_{\text{ext}}^0} f \bar{e}' \, d\mathbf{x}.$$

Inserting the boundary conditions (10) we obtain

$$\int_{\Omega_{\text{ext}}^0} \nabla \tilde{e}_{\text{ext}}^{\varepsilon,N} \cdot \nabla \bar{e}' \, d\mathbf{x} + \int_{\Gamma_i} \beta \tilde{e}_{\text{ext}}^{\varepsilon,N} \bar{e}' \, dS + \int_{\Gamma} [\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N}] \{\bar{e}'\} + \{\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N}\} [\bar{e}'] \, dt = \langle \ell, e' \rangle \quad (42)$$

with  $\ell$  defined by (36). Inserting the second transmission condition (25c) with the expression (38) into (42) and multiplying the first transmission condition (25b) with a test function  $\lambda'$  integrating over  $\Gamma$  we obtain the variational formulation: Seek  $\tilde{e}_{\text{ext}}^{\varepsilon,N} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  and  $\tilde{\lambda}_{\text{ext}}^{\varepsilon,N} \in L^2(\Gamma)$  such that

$$\mathbf{a}_N \left( \begin{pmatrix} \tilde{e}_{\text{ext}}^{\varepsilon,N} \\ \tilde{\lambda}_{\text{ext}}^{\varepsilon,N} \end{pmatrix}, \begin{pmatrix} e' \\ \lambda' \end{pmatrix} \right) = \langle \ell, e' \rangle, \quad \forall e' \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0), \forall \lambda' \in L^2(\Gamma) \quad (43)$$

with

$$\begin{aligned} \mathbf{a}_N \left( \begin{pmatrix} e \\ \lambda \end{pmatrix}, \begin{pmatrix} e' \\ \lambda' \end{pmatrix} \right) &= \int_{\Omega_{\text{ext}}^0} \nabla e \cdot \nabla \bar{e}' \, d\mathbf{x} + \int_{\Gamma_i} \beta e \bar{e}' \, dS + \langle (c_0 + \zeta_0^{\varepsilon,N}) \{e\}, \{e'\} \rangle_{\Gamma} \\ &\quad + \langle \zeta_1^{\varepsilon,N} \lambda, \{e'\} \rangle_{\Gamma} + \langle \lambda, [e'] \rangle_{\Gamma} + \langle [e], \lambda' \rangle_{\Gamma} - \langle \gamma_0^{\varepsilon,N} \{e\}, \lambda' \rangle_{\Gamma} - \langle \gamma_1^{\varepsilon,N} \lambda, \lambda' \rangle_{\Gamma}. \end{aligned} \quad (44)$$

Conversely, it is easy to check if  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  and  $\tilde{\lambda}_{\text{ext}}^{\varepsilon,N}$  are solutions of (43) that  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  satisfies (25) and (41).

#### 4.3.2. Well-posedness and stability

In the following lemma we collect some bounds on the operators defined in (38).

**Lemma 4.8** (Bounds on the operators of the models of order 2 and 3). *Let  $N = 2, 3$  and  $\varepsilon \leq \frac{5}{|c_0|}$ . Then,*

- (i)  $\|\gamma_0^{\varepsilon,N}\|_{L^\infty(\Gamma)} = \|\zeta_1^{\varepsilon,N}\|_{L^\infty(\Gamma)} \leq \varepsilon^2 \frac{|c_0|}{20} \|\kappa\|_{L^\infty(\Gamma)},$
- (ii)  $\|\gamma_1^{\varepsilon,N}\|_{L^\infty(\Gamma)} \leq \varepsilon^2 \frac{|c_0|}{10}, \quad \|(\gamma_1^{\varepsilon,N})^{-1}\|_{L^\infty(\Gamma)} \leq \frac{12}{|c_0|} \varepsilon^{-2},$
- (iii)  $\text{Im} \langle \overline{\gamma_1^{\varepsilon,N}} u, u \rangle_{\Gamma} = -\text{Im} \langle \gamma_1^{\varepsilon,N} u, u \rangle_{\Gamma} = \frac{\varepsilon^2}{12} |c_0| \|u\|_{L^2(\Gamma)}^2,$
- (iv)  $|\langle \zeta_0^{\varepsilon,N} u, u \rangle_{\Gamma}| \leq \frac{5}{16} \varepsilon |c_0| \left( \|u\|_{L^2(\Gamma)}^2 + \frac{3}{2} \varepsilon |u|_{H^1(\Gamma)}^2 \right),$
- (v)  $\text{Im} \langle (c_0 + \zeta_0^{\varepsilon,N}) u, u \rangle_{\Gamma} = -\text{Im} \langle \overline{(c_0 + \zeta_0^{\varepsilon,N})} u, u \rangle_{\Gamma} \geq \frac{|c_0|}{4} \left( \|u\|_{L^2(\Gamma)}^2 + \frac{\varepsilon^2}{3} |u|_{H^1(\Gamma)}^2 \right).$

*Proof.* The proof will be divided into the cases of the lemma.

- (i) With the assumption  $\varepsilon \leq \frac{5}{|c_0|}$  and  $c_0 = i|c_0|$  we have  $|1 - \varepsilon \frac{c_0}{10}| \leq \frac{6}{5}$ , and the inequalities follow.
- (ii) The first inequality is a consequence of  $|1 - \varepsilon \frac{c_0}{10}| \leq \frac{6}{5}$  and the second of  $|1 - \varepsilon \frac{c_0}{10}|^{-1} \leq 1$ .
- (iii) Since  $c_0 = i|c_0|$  and  $\gamma_1^{\varepsilon,N}$  are multiplication operators the equality holds.



(iv) With  $|1 - \frac{\varepsilon^2}{20} \|\kappa\|_{L^\infty(\Gamma)}^2| < 1$  and  $\varepsilon \leq \frac{5}{|c_0|}$  we get

$$\begin{aligned} |\operatorname{Re}\langle \zeta_0^{\varepsilon, N} u, u \rangle_\Gamma| &\leq \varepsilon \frac{|c_0|^2}{6} \|u\|_{L^2(\Gamma)}^2 + \varepsilon^3 \frac{|c_0|^2}{40} |u|_{H^1(\Gamma)}^2 \leq \varepsilon \frac{|c_0|^2}{6} \|u\|_{L^2(\Gamma)}^2 + \varepsilon^2 \frac{|c_0|}{8} |u|_{H^1(\Gamma)}^2, \\ |\operatorname{Im}\langle \zeta_0^{\varepsilon, N} u, u \rangle_\Gamma| &\leq \frac{7}{240} \varepsilon^2 |c_0|^3 \|u\|_{L^2(\Gamma)}^2 + \varepsilon^2 \frac{|c_0|}{12} |u|_{H^1(\Gamma)}^2 \leq \frac{7}{48} \varepsilon |c_0|^2 \|u\|_{L^2(\Gamma)}^2 + \varepsilon \frac{|c_0|}{12} |u|_{H^1(\Gamma)}^2, \end{aligned}$$

and with the triangle inequality the desired bound results.

(v) With the assumption  $\varepsilon \leq \frac{5}{|c_0|}$  it follows

$$\begin{aligned} \operatorname{Im}\langle (c_0 + \zeta_0^{\varepsilon, N})u, u \rangle_\Gamma &= -\operatorname{Im}\langle \overline{c_0 + \zeta_0^{\varepsilon, N}} u, u \rangle_\Gamma = |c_0| \left(1 - \frac{7}{240} \varepsilon^2 |c_0|^2\right) \|u\|_{L^2(\Gamma)}^2 + \frac{\varepsilon^2}{12} |c_0| |u|_{H^1(\Gamma)}^2, \\ &\geq \frac{|c_0|}{4} \left( \|u\|_{L^2(\Gamma)}^2 + \frac{\varepsilon^2}{3} |u|_{H^1(\Gamma)}^2 \right). \quad \square \end{aligned}$$

**Remark 4.9.** *The optimal bound for  $\varepsilon_0$  is of order  $\frac{1}{|c_0|}$  and consequently in the order of the skin depth  $d_{\text{skin}}$ . This is the typical setting where we would like to apply the model where  $\varepsilon \ll d_{\text{skin}}$  or  $\varepsilon \sim d_{\text{skin}}$ . For thicker sheets than the skin depth we refer to Chap. 7 in [23] where the author consider an optimal basis approach.*

The next lemma is in terms of mathematics the most technical of the paper but it is also the key argument that makes the second and third order models work.

**Lemma 4.10** (Well-posedness and stability of the models of order 2 and 3). *Let  $N = 2, 3$ , the boundary conditions be  $H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$ -admissible (see Def. 4.1)  $f \in (H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'$ ,  $g \in L^2(\Gamma)$  and  $\varepsilon \leq \min(\frac{5}{|c_0|}, 2\|\kappa\|_{L^\infty(\Gamma)}^{-1})$ . Then, there exists a unique solution  $(e, \lambda) \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0) \times L^2(\Gamma)$  of*

$$\langle f, e' \rangle + \langle g, \lambda' \rangle_\Gamma = \mathbf{a}_N \left( \begin{pmatrix} e \\ \lambda \end{pmatrix}, \begin{pmatrix} e' \\ \lambda' \end{pmatrix} \right), \quad \forall (e', \lambda') \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0) \times L^2(\Gamma) \quad (45)$$

and it holds

$$\|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)} + \varepsilon \|\lambda\|_{L^2(\Gamma)} \leq C \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right) \quad (46)$$

with a constant  $C$  independent of  $\varepsilon$ .

*Proof.* The continuity of  $\mathbf{a}_N(\cdot, \cdot)$  follows by the Cauchy-Schwarz inequality. For convenience we will omit  $N$  in the superscripts of the operators. The proof of the stability result (46) falls into four steps.

(i) Testing (45) with  $e' = e$  and  $\lambda'$  satisfying

$$[e] - \overline{\gamma_1^\varepsilon} \lambda' + \overline{\zeta_1^\varepsilon} \{e\} = 0 \quad (47)$$

we obtain

$$\begin{aligned} |e|_{H^1(\Omega_{\text{ext}}^0)}^2 + \langle \beta e, e \rangle_{\Gamma_i} + \langle (c_0 + \zeta_0^\varepsilon) \{e\}, \{e\} \rangle_\Gamma + \langle [e], \lambda' \rangle_\Gamma \\ + \langle \lambda, [e] \rangle_\Gamma + \langle \zeta_1^\varepsilon \lambda, \{e\} \rangle_\Gamma - \langle \gamma_0^\varepsilon \{e\}, \lambda' \rangle_\Gamma - \langle \gamma_1^\varepsilon \lambda, \lambda' \rangle_\Gamma = \langle f, e \rangle + \langle g, \lambda' \rangle_\Gamma. \end{aligned}$$

Due to (47),  $\langle \zeta_1^\varepsilon \lambda, \{e\} \rangle_\Gamma + \langle \lambda, [e] \rangle_\Gamma - \langle \gamma_1^\varepsilon \lambda, \lambda' \rangle_\Gamma = 0$ . Therefore, we have

$$\begin{aligned} |e|_{H^1(\Omega_{\text{ext}}^0)}^2 + \langle \beta e, e \rangle_{\Gamma_i} + \langle (c_0 + \zeta_0^\varepsilon) \{e\}, \{e\} \rangle_\Gamma + \langle \overline{\gamma_1^\varepsilon} \lambda', \lambda' \rangle_\Gamma \\ = \langle f, e \rangle + \langle (\gamma_1^\varepsilon)^{-1} g, [e] \rangle_\Gamma - \langle (\gamma_1^\varepsilon)^{-1} \zeta_1^\varepsilon g, \{e\} \rangle_\Gamma + \langle (\overline{\zeta_1^\varepsilon} + \gamma_0^\varepsilon) \{e\}, \lambda' \rangle_\Gamma. \end{aligned}$$

Now, taking the imaginary part and bounding with the estimates (i) of Lemma 4.8 we get

$$\begin{aligned} & \operatorname{Im}\langle \beta e, e \rangle_{\Gamma_i} + \operatorname{Im}\langle (c_0 + \zeta_0^\varepsilon)\{e\}, \{e\} \rangle_{\Gamma} + \operatorname{Im}\langle \overline{\gamma_1^\varepsilon} \lambda', \lambda' \rangle_{\Gamma} \\ & \leq |\langle f, e \rangle| + |\langle (\gamma_1^\varepsilon)^{-1} g, [e] \rangle_{\Gamma}| + |\langle (\gamma_1^\varepsilon)^{-1} \zeta_1^\varepsilon g, \{e\} \rangle_{\Gamma}| + \varepsilon^2 \frac{|c_0|}{20} \|\kappa\|_{L^\infty(\Gamma)} \|\{e\}\|_{L^2(\Gamma)} \|\lambda'\|_{L^2(\Gamma)}. \end{aligned} \quad (48)$$

Since  $\operatorname{Im}\langle \beta e, e \rangle \geq 0$  (see Definition 4.1) and due to (v) of Lemma 4.8 we obtain for the left hand side

$$\begin{aligned} & \frac{|c_0|}{4} \left( \|\{e\}\|_{L^2(\Gamma)}^2 + \frac{\varepsilon^2}{3} \|\{e\}\|_{H^1(\Gamma)}^2 + \frac{\varepsilon^2}{3} \|\lambda'\|_{L^2(\Gamma)}^2 \right) \\ & \leq |\langle f, e \rangle| + |\langle (\gamma_1^\varepsilon)^{-1} g, [e] \rangle_{\Gamma}| + |\langle (\gamma_1^\varepsilon)^{-1} \zeta_1^\varepsilon g, \{e\} \rangle_{\Gamma}| + C \varepsilon^2 \|\{e\}\|_{L^2(\Gamma)} \|\lambda'\|_{L^2(\Gamma)}. \end{aligned}$$

Since for all  $\alpha > 0$  we have  $2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2$  it holds

$$\|\kappa\|_{L^\infty(\Gamma)} 2 \|\{e\}\|_{L^2(\Gamma)} \|\lambda'\|_{L^2(\Gamma)} \leq 8 \|\kappa\|_{L^\infty(\Gamma)}^2 \|\{e\}\|_{L^2(\Gamma)}^2 + \frac{1}{8} \|\lambda'\|_{L^2(\Gamma)}^2.$$

So, we can incorporate the last term of the right hand side of (48) to its left hand side

$$\begin{aligned} & \frac{|c_0|}{2} \left( \underbrace{\left(1 - \frac{2}{5} \varepsilon^2 \|\kappa\|_{L^\infty(\Gamma)}^2\right)}_{\leq \frac{1}{5}} \|\{e\}\|_{L^2(\Gamma)}^2 + \frac{\varepsilon^2}{24} \|\{e\}\|_{H^1(\Gamma)}^2 + \frac{\varepsilon^2}{12} \|\lambda'\|_{L^2(\Gamma)}^2 \right) \leq \\ & |\langle f, e \rangle| + |\langle (\gamma_1^\varepsilon)^{-1} g, [e] \rangle_{\Gamma}| + |\langle (\gamma_1^\varepsilon)^{-1} \zeta_1^\varepsilon g, \{e\} \rangle_{\Gamma}| =: rhs. \end{aligned} \quad (49)$$

The right hand side of (49) can then be bounded with the help of (i) and (ii) of Lemma 4.8

$$rhs \leq \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} \|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)} + C \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \|[e]\|_{L^2(\Gamma)} + C \|g\|_{L^2(\Gamma)} \|\{e\}\|_{L^2(\Gamma)}.$$

The trace theorems leads to

$$rhs \leq C \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right) \|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}. \quad (50)$$

Thus, due to (49) and (50), we obtain with an  $\varepsilon$  independent constant  $C > 0$

$$\|\{e\}\|_{L^2(\Gamma)}^2 + \varepsilon^2 \|\{e\}\|_{H^1(\Gamma)}^2 \leq C \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right) \|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}, \quad (51)$$

$$\|\lambda'\|_{L^2(\Gamma)}^2 \leq \frac{C}{\varepsilon^2} \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right) \|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}. \quad (52)$$

Using (47), (51), (52), Youngs inequality and the upper bounds for  $\gamma_1^\varepsilon$  and  $\zeta_1^\varepsilon$  given in (i) and (ii) of Lemma 4.8 we get

$$\begin{aligned} \|[e]\|_{L^2(\Gamma)}^2 & \leq 2 \|\overline{\gamma_1^\varepsilon} \lambda'\|_{L^2(\Gamma)}^2 + 2 \|\overline{\zeta_1^\varepsilon} \{e\}\|_{L^2(\Gamma)}^2 \\ & \leq 2 \|\gamma_1^\varepsilon\|_{L^\infty(\Gamma)}^2 \|\lambda'\|_{L^2(\Gamma)}^2 + 2 \|\zeta_1^\varepsilon\|_{L^\infty(\Gamma)}^2 \|\{e\}\|_{L^2(\Gamma)}^2 \\ & \leq C \varepsilon^2 \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right) \|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)} \end{aligned} \quad (53)$$

with an  $\varepsilon$  independent constant  $C > 0$ .

(ii) Let us bound  $\langle \tilde{f}, e \rangle$  with  $\tilde{f} \in (H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'$  defined by

$$\langle \tilde{f}, e' \rangle := \langle f, e' \rangle - \langle \zeta_1^\varepsilon \lambda, \{e'\} \rangle_\Gamma - \langle \lambda, [e'] \rangle_\Gamma - \langle \zeta_0^\varepsilon \{e\}, \{e'\} \rangle_\Gamma - \langle c_0 [e], [e'] \rangle_\Gamma.$$

Testing (45) with  $e' = 0$  and  $\lambda' = (\overline{\gamma_1^\varepsilon})^{-1} ([e] + \overline{\zeta_1^\varepsilon} \{e\})$ , we get

$$\langle \zeta_1^\varepsilon \lambda, \{e\} \rangle_\Gamma + \langle \lambda, [e] \rangle_\Gamma = \langle \lambda, [e] + \overline{\zeta_1^\varepsilon} \{e\} \rangle_\Gamma = \langle (\gamma_1^\varepsilon)^{-1} ([e] - \gamma_0^\varepsilon \{e\} - g), [e] + \overline{\zeta_1^\varepsilon} \{e\} \rangle_\Gamma.$$

Using the bounds of Lemma 4.8 and the Cauchy-Schwarz inequality we find

$$\begin{aligned} |\langle \zeta_1^\varepsilon \lambda, \{e\} \rangle_\Gamma + \langle \lambda, [e] \rangle_\Gamma| &\leq C\varepsilon^{-2} \left( \| [e] \|_{L^2(\Gamma)} + C\varepsilon^2 \| \{e\} \|_{L^2(\Gamma)} + \| g \|_{L^2(\Gamma)} \right) \left( \| [e] \|_{L^2(\Gamma)} + C\varepsilon^2 \| \{e\} \|_{L^2(\Gamma)} \right) \\ &\leq C\varepsilon^{-2} \left( \| [e] \|_{L^2(\Gamma)}^2 + \varepsilon^4 \| \{e\} \|_{L^2(\Gamma)}^2 + \| g \|_{L^2(\Gamma)}^2 \right). \end{aligned}$$

By (i) of Lemma 4.8, we have

$$|\langle \zeta_0^\varepsilon \{e\}, \{e\} \rangle_\Gamma + \langle c_0 [e], [e] \rangle_\Gamma \leq C \left( \varepsilon \| \{e\} \|_{L^2(\Gamma)}^2 + \varepsilon^2 \| \{e\} \|_{H^1(\Gamma)}^2 + \| [e] \|_{L^2(\Gamma)}^2 \right).$$

Inserting (51), (52), (53) leads to

$$\begin{aligned} |\langle \tilde{f}, e \rangle| &\leq |\langle f, e \rangle| + C \left( \varepsilon \| \{e\} \|_{L^2(\Gamma)}^2 + \varepsilon^2 \| \{e\} \|_{H^1(\Gamma)}^2 + \varepsilon^{-2} \| [e] \|_{L^2(\Gamma)}^2 + \varepsilon^{-2} \| g \|_{L^2(\Gamma)}^2 \right) \\ &\leq C \left( \| f \|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \| g \|_{L^2(\Gamma)} \right) \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)} + C\varepsilon^{-2} \| g \|_{L^2(\Gamma)}^2. \end{aligned}$$

Since  $2ab \leq \alpha a^2 + b^2/\alpha$  for all  $\alpha > 0$ , we have

$$\forall \alpha > 0, \quad \exists C_\alpha > 0: \quad |\langle \tilde{f}, e \rangle| \leq C_\alpha \left( \| f \|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \| g \|_{L^2(\Gamma)} \right)^2 + \alpha \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}^2 \quad (54)$$

(iii) Now, we bound the traces of  $e$  on  $\Gamma$ . Testing (45) with  $\lambda' = 0$  we obtain a variational formulation

$$\begin{aligned} \mathbf{b}_{c_0}(e, e') &:= \int_{\Omega_{\text{ext}}^0} \nabla e \cdot \nabla \overline{e'} \, d\mathbf{x} + \langle \beta e, e' \rangle_{\Gamma_i} + c_0 (\langle \{e\}, \{e'\} \rangle_\Gamma + \langle [e], [e'] \rangle_\Gamma) \\ &= \langle f, e' \rangle - \langle \zeta_1^\varepsilon \lambda, \{e'\} \rangle_\Gamma - \langle \lambda, [e'] \rangle_\Gamma - \langle \zeta_0^\varepsilon \{e\}, \{e'\} \rangle_\Gamma - \langle c_0 [e], [e'] \rangle_\Gamma = \langle \tilde{f}, e' \rangle. \end{aligned} \quad (55)$$

The left hand side of (55) is  $H_{\Gamma_e}^1(\Omega_{\text{ext}}^0)$ -elliptic (see Definition 4.1) due to the assumption on the boundary conditions, and it exists a constant  $\gamma > 0$  such that

$$\begin{aligned} \forall \alpha > 0, \quad \exists C_\alpha > 0: \quad \gamma \| e \|_{H_{\Gamma_e}^1(\Omega_{\text{ext}}^0)}^2 &\leq |\langle \tilde{f}, e \rangle| \\ &\leq C_\alpha \left( \| f \|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \| g \|_{L^2(\Gamma)} \right)^2 + \alpha \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}^2. \end{aligned}$$

Due to (51), for all  $\alpha > 0$  there exists  $C_\alpha > 0$  such that

$$\varepsilon^2 \gamma \| \{e\} \|_{H^1(\Gamma)}^2 \leq C_\alpha \left( \| f \|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \| g \|_{L^2(\Gamma)} \right)^2 + \alpha \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}^2,$$

We sum the two last equations and get

$$\forall \alpha > 0, \quad \exists C_\alpha > 0: \quad \gamma \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}^2 \leq C_\alpha \left( \| f \|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \| g \|_{L^2(\Gamma)} \right)^2 + 2\alpha \| e \|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}^2.$$

Picking  $\alpha = \frac{\gamma}{4}$  it follows

$$\|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)} \leq C \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right). \quad (56)$$

Inserting this bound into (51) and (53) yields

$$\|\{e\}\|_{L^2(\Gamma)} + \varepsilon \|\{e\}\|_{H^1(\Gamma)} + \varepsilon^{-1} \|[e]\|_{L^2(\Gamma)} \leq C \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right). \quad (57)$$

(iv) Now, testing (45) with  $e' = 0$  and  $\lambda' = (\overline{\gamma_1^\varepsilon})^{-1} \lambda$  we obtain

$$\langle \lambda, \lambda \rangle_\Gamma = \langle (\gamma_1^\varepsilon)^{-1} ([e] - \gamma_0^\varepsilon \{e\} - g), \lambda \rangle_\Gamma.$$

Thus, with the Cauchy-Schwarz inequality, cancelling a term  $\|\lambda\|_{L^2(\Gamma)}$ , squaring and using the Young inequality we have

$$\|\lambda\|_{L^2(\Gamma)} \leq \|(\gamma_1^\varepsilon)^{-1}\|_{L^\infty(\Gamma)} \left( \|[e]\|_{L^2(\Gamma)} + \|\gamma_0^\varepsilon\|_{L^\infty(\Gamma)} \|\{e\}\|_{L^2(\Gamma)} + \|g\|_{L^2(\Gamma)} \right).$$

It follows with the bounds on  $(\gamma_1^\varepsilon)^{-1}$  and  $\gamma_0^\varepsilon$  given in (i) and (ii) of Lemma 4.8

$$\|\lambda\|_{L^2(\Gamma)} \leq C \varepsilon^{-2} \left( \|[e]\|_{L^2(\Gamma)} + \varepsilon^2 \|\{e\}\|_{L^2(\Gamma)} + \|g\|_{L^2(\Gamma)} \right).$$

Finally, estimate (57) allows then to get

$$\|\lambda\|_{L^2(\Gamma)} \leq C \varepsilon^{-1} \left( \|f\|_{(H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0))'} + \varepsilon^{-2} \|g\|_{L^2(\Gamma)} \right). \quad (58)$$

with an  $\varepsilon$  independent constant  $C > 0$ . Summing the bounds for  $\|e\|_{H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)}$  in (56) and for  $\|\lambda\|_{L^2(\Gamma)}$  in (58) we have the desired bound and the solution is unique.

To show surjectivity we can prove injectivity of the adjoint formulation

$$\begin{aligned} \langle f, e' \rangle + \langle g, e' \rangle_\Gamma &= \int_{\Omega_{\text{ext}}^0} \nabla e \cdot \nabla \bar{e}' \, d\mathbf{x} + \int_{\Gamma_i} \bar{\beta} e \bar{e}' \, dS + \langle (c_0 + \zeta_0^\varepsilon) \{e\}, \{e'\} \rangle_\Gamma \\ &\quad - \langle \overline{\gamma_0^\varepsilon} \lambda, \{e'\} \rangle_\Gamma + \langle \lambda, [e'] \rangle_\Gamma + \langle [e], \lambda' \rangle_\Gamma + \langle \overline{\zeta_1^\varepsilon} \{e\}, \lambda' \rangle_\Gamma - \langle \overline{\gamma_1^\varepsilon} \lambda, \lambda' \rangle_\Gamma \end{aligned} \quad (59)$$

similarly to the proof of the original formulation following the items (i-iii), where the assumption of  $\beta$  and the bounds on the respective adjoint operators in Lemma 4.8 are to be used. However, we will not give the proof in detail.  $\square$

#### 4.4. Regularity

The variational formulation (43) provides the unique solution  $\tilde{e}_{\text{ext}}^{\varepsilon,N} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  and  $\tilde{\lambda}_{\text{ext}}^{\varepsilon,N} = \{\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,N}\} \in L^2(\Gamma)$  of (25) with the concrete transmission conditions in (39) and (40). This means by elliptic regularity theory [18] that  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  is actually in  $H^k(\Omega_\Gamma)$  for any  $k \in \mathbb{N}_0$  in a neighbourhood  $\Omega_\Gamma$  of the midline. We will not give a proof of higher regularity, as we do not need it in the following of this article, but refer to [23, Lemma 6.14].

## 5. ESTIMATES OF THE MODELLING ERROR

The derivation of the problems defining  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  has been formally done. In this section we will prove that  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  is indeed an approximation of order  $N$  of the solution  $e_{\text{ext}}^{\varepsilon}$  of the original problem, *i. e.*,  $\tilde{e}_{\text{ext}}^{\varepsilon,N} - e_{\text{ext}}^{\varepsilon} = o_{\varepsilon \rightarrow 0}(\varepsilon^N)$ . This will be done into two steps. First, we will derive the asymptotic expansion of  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$ . Then, we will remark that the asymptotic expansions  $\tilde{e}_{\text{ext}}^{\varepsilon,N}$  and  $e_{\text{ext}}^{\varepsilon,N}$  coincide up to order  $N$  and obtain the final result with triangular inequalities.

**Lemma 5.1** (Asymptotic expansion of the approximate models). *Let  $N = 1, 2, 3$  and  $\Omega_{\Gamma}$  a neighbourhood of the midline of the sheet. There exist families of functions of  $(v_{\text{ext}}^{N,j} \in H_{\Gamma_e}^1(\Omega_{\text{ext}}^0) \cap H^2(\Omega_{\Gamma}))_{j=0}^{\infty}$  with  $v_{\text{ext}}^{N,j} = u_{\text{ext}}^j$  for  $j \leq N$  satisfying for all  $m \in \mathbb{N}$*

$$\left\| \tilde{e}_{\text{ext}}^{\varepsilon,N} - \sum_{j=0}^m \varepsilon^j v_{\text{ext}}^{N,j} \right\|_{H^1(\Omega_{\text{ext}}^0)} \leq C_m \varepsilon^{m+1}, \quad (60a)$$

and for  $N = 2, 3$

$$\left\| \tilde{\lambda}_{\text{ext}}^{\varepsilon,N} - \sum_{j=0}^m \varepsilon^j \{ \partial_n v_{\text{ext}}^{N,j} \} \right\|_{L^2(\Gamma)} \leq C_m \varepsilon^{m+1}, \quad (60b)$$

with constants  $C_m$  independent of  $\varepsilon$ .

*Proof.* Let  $v_{\text{ext}}^{N,j} = 0$  for  $j < 0$  and  $v_{\text{ext}}^{N,j}$  for  $j \in \mathbb{N}$  defined by the following system

$$\begin{aligned} v_{\text{ext}}^{1,j} &\in H_{\Gamma_e}^1(\Omega_{\text{ext}}^0) \text{ and } v_{\text{ext}}^{N,j} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0) \text{ for } N = 2, 3 \\ -\Delta v_{\text{ext}}^{N,j}(\underline{x}) &= f_j(\underline{x}), \quad \text{in } \Omega_{\text{ext}}^0, \\ \left[ v_{\text{ext}}^{N,j} \right](t) &= \sum_{\ell=1}^N (\gamma_{\ell} v_{\text{ext}}^{N,j-\ell})(t) =: \gamma^{N,j}(t), \quad \text{on } \Gamma, \\ \left[ \partial_n v_{\text{ext}}^{N,j} \right](t) - c_0 \left\{ v_{\text{ext}}^{N,j} \right\}(t) &= \sum_{\ell=1}^N (\zeta_{\ell} v_{\text{ext}}^{N,j-\ell})(t) =: \delta^{N,j}(t), \quad \text{on } \Gamma, \end{aligned} \quad (61a)$$

completed with the source term and boundary conditions

$$\begin{aligned} f_0(\underline{x}) &= f(\underline{x}) \quad \text{and} \quad f_j(\underline{x}) = 0, \quad \text{in } \Omega_{\text{ext}}^0, \\ v_{\text{ext}}^{N,0} &= 0, \quad \text{and} \quad v_{\text{ext}}^{N,j} = 0, \quad \text{on } \Gamma_e, \\ \nabla v_{\text{ext}}^{N,0} \cdot \underline{n} - \beta v_{\text{ext}}^{N,0} &= \iota_{\text{imp}}, \quad \text{and} \quad \nabla v_{\text{ext}}^{N,j} \cdot \underline{n} - \beta v_{\text{ext}}^{N,j} = 0, \quad \text{on } \Gamma_i. \end{aligned} \quad (61b)$$

This system uniquely defines the functions  $v_{\text{ext}}^{N,0} \in H_{\Gamma_e}^1(\Omega_{\text{ext}}^0)$  [26, Lemma 4.2] which do not depend on  $\varepsilon$ , and similarly to [23, Proposition 2.8] we conclude higher regularity and  $v_{\text{ext}}^{N,0} \in H^k(\Omega_{\Gamma})$  for any  $k \in \mathbb{N}$ . Hence, the right hand sides of the transmission conditions (61a) for  $N = 1$  are given in terms of  $v_{\text{ext}}^{N,0}$  and so  $\gamma^{N,1}(t) \in H^{k-3/2}(\Omega_{\Gamma})$  and  $\delta^{N,1}(t) \in H^{k-5/2}(\Omega_{\Gamma})$ . Repeating these steps we find the well-posedness of (61) for  $v_{\text{ext}}^{N,j}$  and the regularity  $v_{\text{ext}}^{N,j} \in H_{\Gamma_e}^1(\Omega_{\text{ext}}^0) \cap H^k(\Omega_{\Gamma})$  for any  $k \in \mathbb{N}$  and any  $j \in \mathbb{N}$ . Comparing the above system with (17) leads to the agreement with the terms of the asymptotics of  $e_{\text{ext}}^{\varepsilon,N}$ , *i. e.*,  $v_{\text{ext}}^{N,j} = u_{\text{ext}}^j$  for  $j = 0, \dots, N$ .

Now, let us bound the residual

$$r_{\text{ext}}^{\varepsilon,N,m} := \tilde{u}_{\text{ext}}^{\varepsilon,N} - \sum_{j=0}^m \varepsilon^j v_{\text{ext}}^{N,j}, \quad \lambda_{\text{ext}}^{\varepsilon,N,m} := \tilde{\lambda}_{\text{ext}}^{\varepsilon,N} - \sum_{j=0}^m \varepsilon^j \{\partial_n v_{\text{ext}}^{N,j}\}. \quad (62)$$

(i) Let  $N = 1$ . The residual  $r_{\text{ext}}^{\varepsilon,1,m} \in H_{\Gamma_e}^1(\Omega)$  solves

$$-\Delta r_{\text{ext}}^{\varepsilon,1,m}(\underline{x}) = 0, \quad \text{in } \Omega_{\text{ext}}^0, \quad (63a)$$

$$[\partial_n r_{\text{ext}}^{\varepsilon,1,m}](t) - c_0 \left(1 + \varepsilon \frac{c_0}{6}\right) \{r_{\text{ext}}^{\varepsilon,1,m}\}(t) = -\varepsilon^{m+1} \frac{c_0^2}{6} \{v_{\text{ext}}^{1,m}\}(t), \quad \text{on } \Gamma, \quad (63b)$$

$$\nabla r_{\text{ext}}^{\varepsilon,1,m} \cdot \underline{n} - \beta r_{\text{ext}}^{\varepsilon,1,m} = 0, \quad \text{on } \Gamma_i. \quad (63c)$$

Equivalently we can write

$$\mathbf{a}_1(r_{\text{ext}}^{\varepsilon,1,m}, e') = \langle \ell_\varepsilon^m, e' \rangle := -\varepsilon^{m+1} \frac{c_0^2}{6} \int_\Gamma \{v_{\text{ext}}^{1,m}\}(t) e' \, dS, \quad \forall e' \in H_{\Gamma_e}^1(\Omega) \quad (64)$$

Since  $v_{\text{ext}}^{1,m} \in H^1(\Omega)$  does not depend on  $\varepsilon$  we get

$$\left| \int_\Gamma \{v_{\text{ext}}^{1,m}\} e' \, dS \right| \leq \| \{v_{\text{ext}}^{1,m}\} \|_{L^2(\Gamma)} \| e' \|_{L^2(\Gamma)} \leq C \| \{v_{\text{ext}}^{1,m}\} \|_{H^1(\Omega)} \| e' \|_{H^1(\Omega)} \leq C_m \| e' \|_{H^1(\Omega)}.$$

Therefore, we get  $\| \ell_\varepsilon^m \|_{(H_{\Gamma_e}^1(\Omega))'} \leq C \varepsilon^{m+1}$ , and the  $\varepsilon$ -independent stability of (63) by Lemma 4.6 leads to the desired result.

(ii) For  $N = 2, 3$ , the residual  $r_{\text{ext}}^{\varepsilon,N,m} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  satisfies

$$\left\{ \begin{array}{ll} -\Delta r_{\text{ext}}^{\varepsilon,N,m} = 0 & \text{in } \Omega_{\text{ext}}^0, \\ \left[ r_{\text{ext}}^{\varepsilon,N,m} \right](t) - \sum_{\ell=1}^N (\varepsilon^\ell \gamma_\ell r_{\text{ext}}^{\varepsilon,N,m})(t) = g^{\varepsilon,N,m}(t) = \sum_{(j,\ell) \in \mathbf{J}_{m,N}} (\varepsilon^{j+\ell} \gamma_\ell v_{\text{ext}}^{N,j})(t), & \text{on } \Gamma \\ \left[ \partial_n r_{\text{ext}}^{\varepsilon,N,m} \right](t) - c_0 \left\{ r_{\text{ext}}^{\varepsilon,N,m} \right\}(t) - \sum_{\ell=1}^N (\varepsilon^\ell \zeta_\ell r_{\text{ext}}^{\varepsilon,N,m})(t) = -f^{\varepsilon,N,m}(t) = \sum_{(j,\ell) \in \mathbf{J}_{m,N}} (\varepsilon^{j+\ell} \zeta_\ell v_{\text{ext}}^{N,j})(t), & \text{on } \Gamma \\ r_{\text{ext}}^{\varepsilon,N,m} = 0, & \text{on } \Gamma_e, \\ \nabla r_{\text{ext}}^{\varepsilon,N,m} \cdot \underline{n} - \beta r_{\text{ext}}^{\varepsilon,N,m} = 0, & \text{on } \Gamma_i. \end{array} \right.$$

with

$$\mathbf{J}_{m,N} = \{(j, \ell) \in \mathbb{N}^2 : j \leq m, \ell \leq N \text{ and } j + \ell > m\}.$$

Equivalently, the residual solves the variational problem

$$\mathbf{a}_N \left( \begin{pmatrix} r_{\text{ext}}^{\varepsilon,N,m} \\ \lambda_{\text{ext}}^{\varepsilon,N,m} \end{pmatrix}, \begin{pmatrix} e' \\ \lambda' \end{pmatrix} \right) = \langle f^{\varepsilon,N,m}, \{e'\} \rangle_\Gamma + \langle g^{\varepsilon,N,m}, \lambda' \rangle_\Gamma, \quad \forall e' \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0), \forall \lambda' \in L^2(\Gamma). \quad (65)$$

The two linear forms of the right hand side can then be bounded as follows. First we remark that the  $v_{\text{ext}}^{N,j} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  does not depend on  $\varepsilon$ . Consequently, we get by integration by part due to the weight

in the definition of the  $H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$ -norm in (32)

$$\left\langle \zeta_\ell v_{\text{ext}}^{N,j}, \{e'\} \right\rangle_\Gamma \leq \frac{C_{\ell,j}}{\varepsilon} \|e'\|_{H^{1,1}(\Omega_{\text{ext}}^0)} \quad \text{and} \quad \|\gamma_\ell v_{\text{ext}}^{N,j}\|_{L^2(\Gamma)} \leq C_{\ell,j}$$

where we have used the Cauchy-Schwartz inequality with  $v_{\text{ext}}^{N,j} \in H_{\Gamma_e}^{1,1}(\Omega_{\text{ext}}^0)$  and  $\{\partial_n v_{\text{ext}}^{N,j}\} \in L^2(\Gamma)$ .

Multiplying by  $\varepsilon^{j+\ell}$  and summing all these expressions over  $J_{m,N}$  we get

$$\left\langle f^{\varepsilon,N,m}, \{e'\} \right\rangle_\Gamma \leq C_m \varepsilon^m \|e'\|_{H^{1,1}(\Omega_{\text{ext}}^0)} \quad \text{and} \quad \|g^{\varepsilon,N,m}\|_{L^2(\Gamma)} \leq C_m \varepsilon^{m+1}.$$

Now, Lemma 4.10 leads to

$$\|r_{\text{ext}}^{\varepsilon,N,m}\|_{H^1(\Omega_{\text{ext}}^0)} \leq C_m \varepsilon^{m-1}, \quad \|\lambda_{\text{ext}}^{\varepsilon,N,m}\|_{L^2(\Gamma)} \leq C_m \varepsilon^{m-2}. \quad (66)$$

This estimate can be made optimal by considering  $r_{\text{ext}}^{\varepsilon,N,m+2} = r_{\text{ext}}^{\varepsilon,N,m} - \varepsilon^{m+1} v_{\text{ext}}^{N,m+1} - \varepsilon^{m+2} v_{\text{ext}}^{N,m+2}$  and using the triangular inequality

$$\|r_{\text{ext}}^{\varepsilon,N,m}\|_{H^1(\Omega_{\text{ext}}^0)} \leq \|r_{\text{ext}}^{\varepsilon,N,m+2}\|_{H^1(\Omega_{\text{ext}}^0)} + \varepsilon^{m+1} \|v_{\text{ext}}^{N,m+1}\|_{H^1(\Omega_{\text{ext}}^0)} + \varepsilon^{m+2} \|v_{\text{ext}}^{N,m+2}\|_{H^1(\Omega_{\text{ext}}^0)} \leq C_m \varepsilon^{m+1},$$

since each of these terms is bounded by  $C_m \varepsilon^{m+1}$ . Repeating the same arguments with  $\lambda_{\text{ext}}^{\varepsilon,N,m+3} = \lambda_{\text{ext}}^{\varepsilon,N,m} - \varepsilon^{m+1} v_{\text{ext}}^{N,m+1} - \varepsilon^{m+2} v_{\text{ext}}^{N,m+2} - \varepsilon^{m+3} v_{\text{ext}}^{N,m+3}$  we get the desired estimate for  $\tilde{\lambda}_{\text{ext}}^{\varepsilon,N}$ , which completes the proof.  $\square$

The stability bound for the Lagrange multiplier of the variational formulation in Lemma 4.10 is not uniform w.r.t. to  $\varepsilon$ . With the boundness of  $\{\partial_n v_{\text{ext}}^{N,0}\}$  and Lemma 5.1 we now reveal the uniform stability of  $\tilde{\lambda}^{\varepsilon,N}$ .

**Corollary 5.2** (Uniform stability of the models of order 2 and 3). *For  $N = 2, 3$  for the solutions  $(\tilde{e}^{\varepsilon,N}, \tilde{\lambda}^{\varepsilon,N})$  of (43) it holds with an  $\varepsilon$  independent constant  $C > 0$*

$$\|\tilde{\lambda}^{\varepsilon,N}\|_{L^2(\Gamma)} \leq C.$$

**Theorem 5.3** (Modelling error). *Let  $N = 1, 2, 3$ . Then, there exists a constant  $C$  independent of  $\varepsilon$  such that*

$$\|\tilde{e}_{\text{ext}}^{\varepsilon,N} - e_{\text{ext}}^\varepsilon\|_{H^1(\Omega_{\text{ext}}^\varepsilon)} \leq C \varepsilon^{N+1}. \quad (67)$$

*Proof.* Inserting  $e_{\text{ext}}^{\varepsilon,N} = \sum_{j=0}^N \varepsilon^j u_{\text{ext}}^j$ , we get by the triangle inequality since  $\Omega_{\text{ext}}^\varepsilon \subset \Omega_{\text{ext}}^0$

$$\|\tilde{e}_{\text{ext}}^{\varepsilon,N} - e_{\text{ext}}^\varepsilon\|_{H^1(\Omega_{\text{ext}}^\varepsilon)} \leq \|\tilde{e}_{\text{ext}}^{\varepsilon,N} - e_{\text{ext}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{ext}}^0)} + \|e_{\text{ext}}^{\varepsilon,N} - e_{\text{ext}}^\varepsilon\|_{H^1(\Omega_{\text{ext}}^\varepsilon)}.$$

Due to Lemma 5.1, we have  $e_{\text{ext}}^{\varepsilon,N} = \sum_{j=0}^N \varepsilon^j v_{\text{ext}}^{N,j}$  and the final estimate follows by (60) and (15).  $\square$

## 6. NUMERICAL EXAMPLES

This section is devoted to the numerical validation of the approximate models of order 1, 2 and 3. The experiments will be performed with the numerical C++ library *Concepts* [8, 11] using exactly curved elements of high order which easily permits discretisation error lying below the modelling error.

The geometrical setting of the experiments is an ellipsoidal thin sheet, a sheet with varying curvature, with two live circular conductors in the middle (with opposite direction of the currents). The problem is completed by perfect magnetic conductor (PMC) boundary condition on the circular outer boundary which turns out to

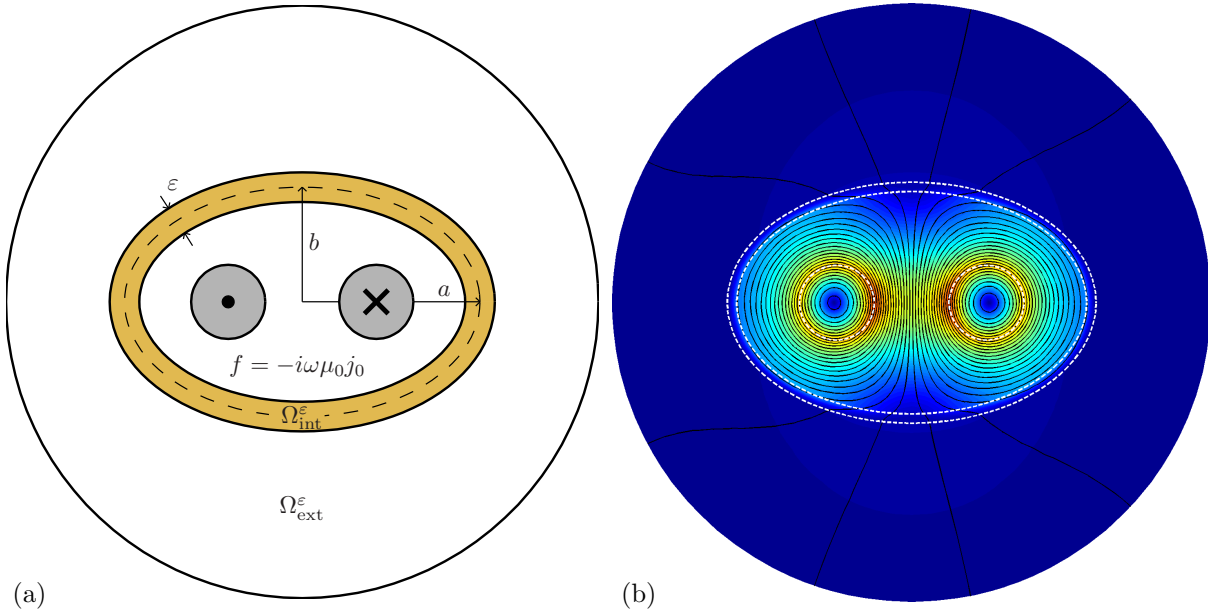


FIGURE 2. (a) Geometrical setting with elliptic mid-line (dashed line) with the semi-major axis  $a = 1.2$  and semi-minor axis  $b = \sqrt{0.6}$ . The boundary is a circle of radius  $R = 2$ . The live wires are circles of radius 0.25 and midpoints  $(\pm 0.5, 0)$ . (b) The magnitude and the flux lines of the in-plane magnetic field for  $\varepsilon = 1/16$ ,  $c_0 = 10$  and  $f = 1$  in the left wire and  $f = -1$  in the right one – corresponding to an alternating currents  $j_0$  with opposite direction. The flux lines of the magnetic field compass the wires and are almost trapped in the interior area enclosed by the thin sheet.

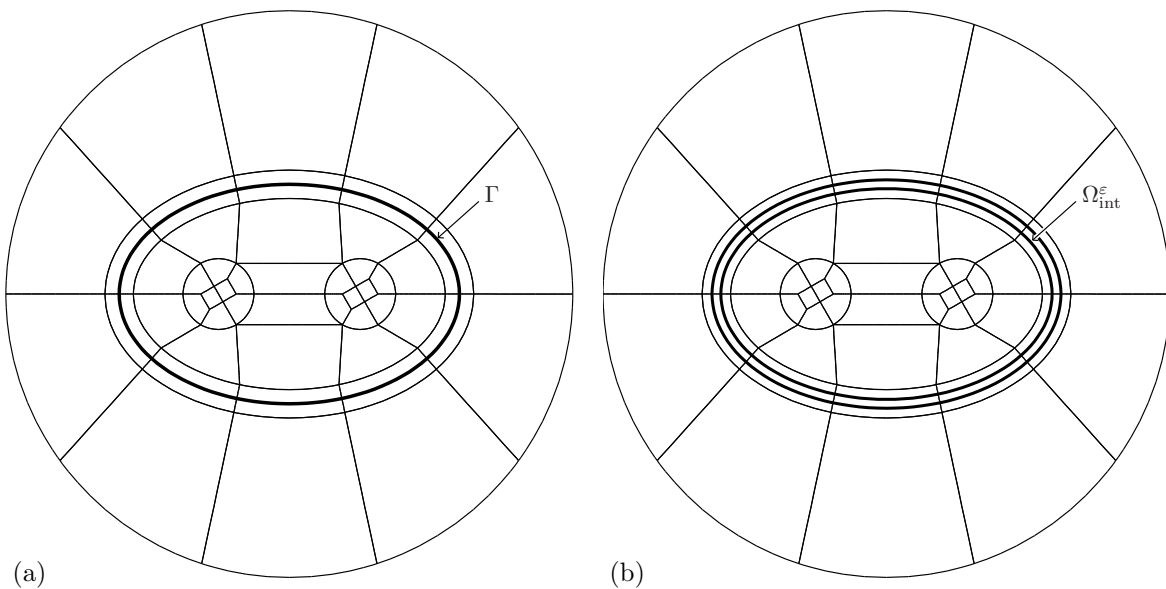


FIGURE 3. (a) Mesh  $\mathcal{M}^0$  for the finite element solution of the asymptotic expansion models. The mid-line  $\Gamma$  is labelled. (b) Associate mesh  $\mathcal{M}^\varepsilon$  for the finite element solution of the exact model with the cells in the sheet, here of thickness  $\varepsilon = 1/16$ .



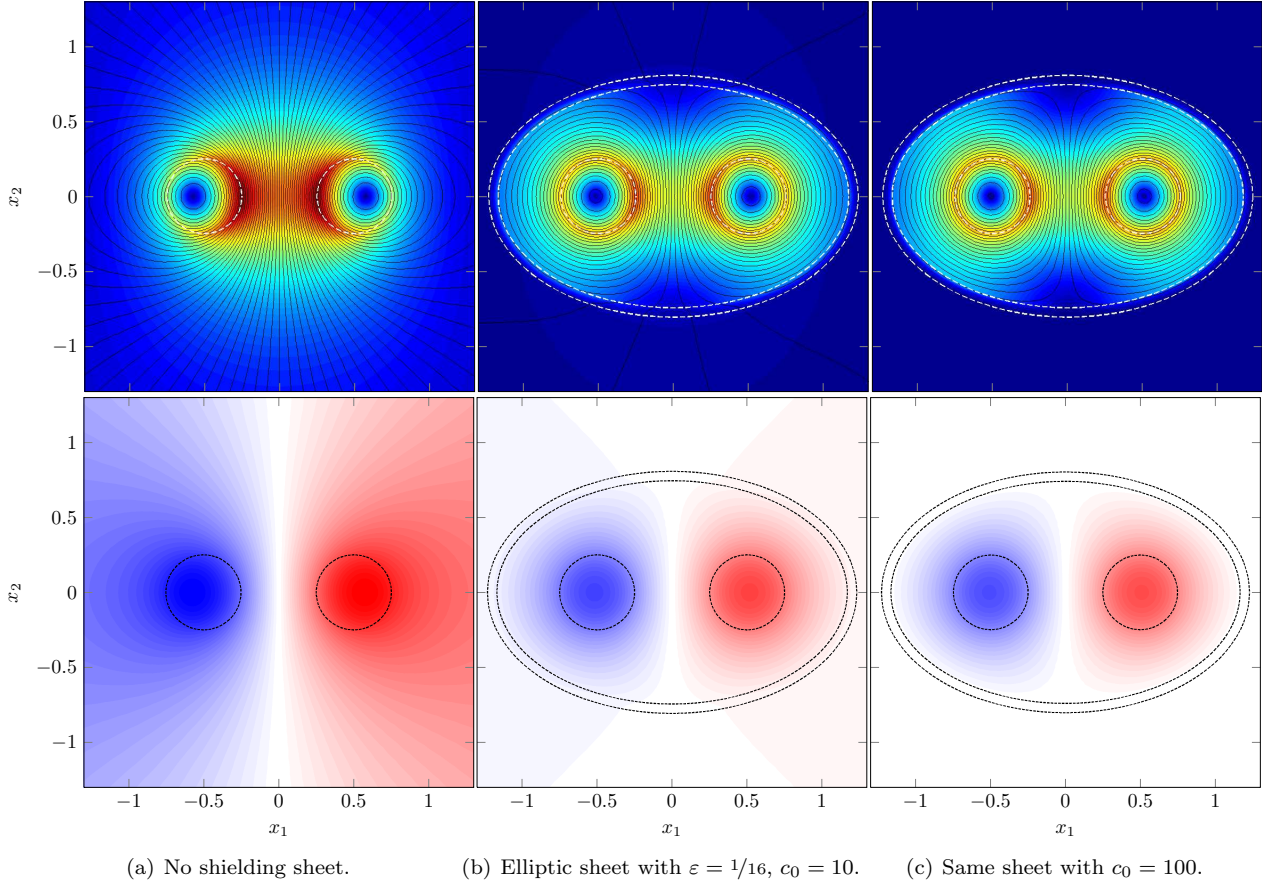


FIGURE 4. Illustration of the increasing shielding of two live wires by an elliptic thin sheet of increasing conductivity. Visible is always the same part of the geometric domain. In the top row the magnetic field intensity and direction (flux lines) are shown and in the bottom line the in plane electric field. The scaling of the colours and the distances of the flux lines correspond in all the plots of each row. The parameters are the same as in Fig. 2.

be a homogeneous Neumann boundary condition. See Fig. 2(a) for a sketch of the geometry and Fig. 2(b) for the flux lines and the absolute value of the magnetic field induced by the two wires and shielded by the thin sheet (computed with the exact model).

In Fig. 4 we illustrate the shielding behaviour more explicitly. In the first row the magnetic field is shown for different (relative) conductivities  $c_0$  and the same thickness of the sheet, where in the first figure no sheet is present. The geometrical setting is like in Fig. 2, where just a part is shown. In the second row the corresponding electric field is plotted. To compare the results the same color scaling is used. In the case of no shielding sheet the fields decay slowly away from the two wires, in or inbetween, respectively, they are mainly concentrated. In the presence of the thin sheets the fields are to some degree trapped in the enclosed area, especially pronounced for most right pictures. The skin depth is  $d_{\text{skin}} = 0.079 = 1.26 \varepsilon$  for  $c_0 = 10$  and  $d_{\text{skin}} = 0.025 = 0.4 \varepsilon$  for  $c_0 = 100$ .

The above computations have been done on meshes  $\mathcal{M}^\varepsilon$  resolving the sheet (see Fig. 3(b)) using curved cells with polynomial degree  $p = 10$ . For simulation with our transmission conditions we use a limit mesh  $\mathcal{M}^0$  in which the thin sheet is represented by the midline  $\Gamma$ . The purpose of the two layers of cells around  $\Gamma$  in  $\mathcal{M}^0$  and

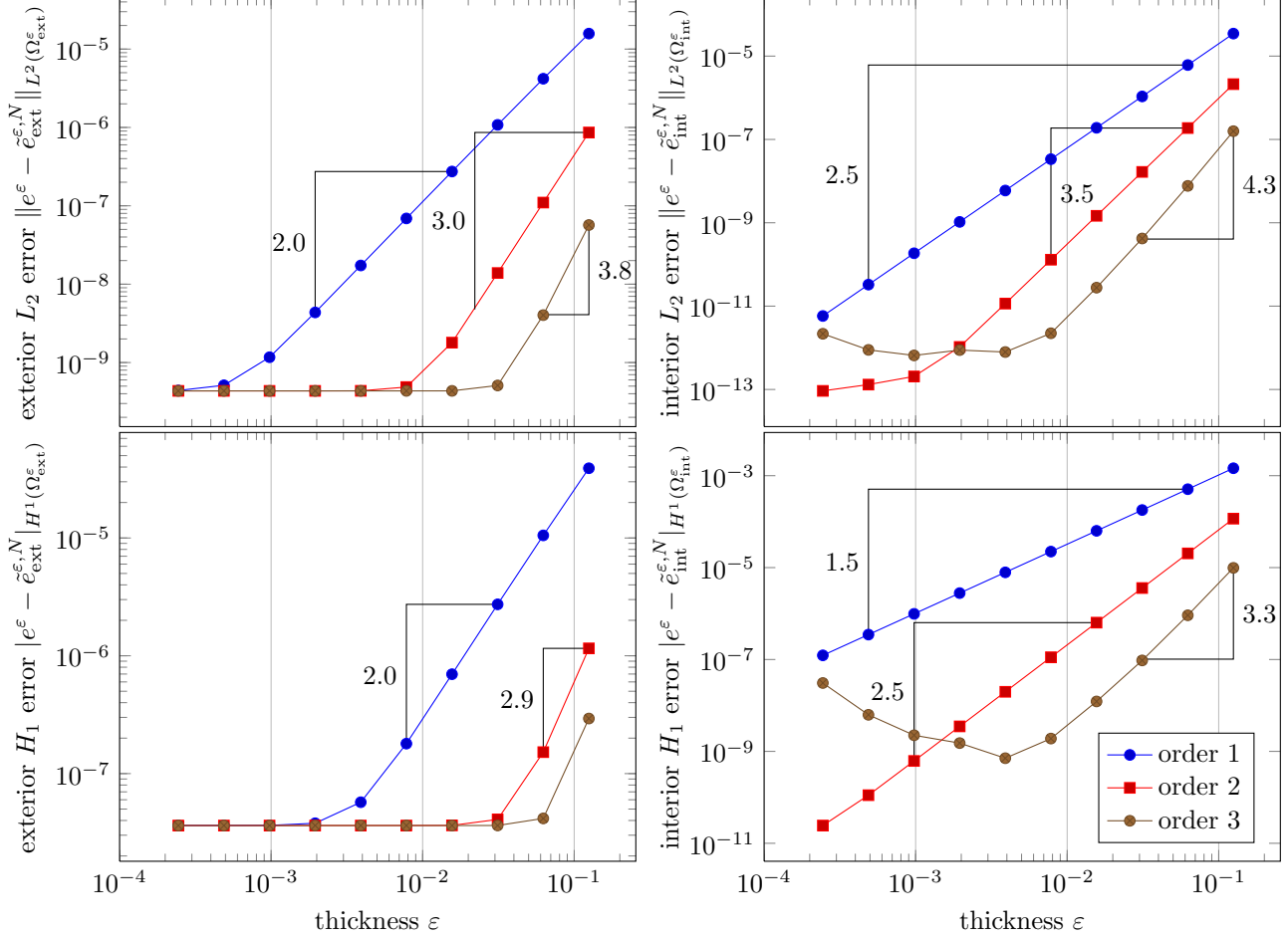


FIGURE 5. Convergence of the error of the transmission conditions of order 1, 2 and 3 for the geometry shown in Fig. 2 with  $c_0 = 1$  and varying thickness  $\varepsilon$ . The solution with the transmission condition is computed with  $p = 18$  and subtracted from a numerical approximation ( $p = 20$ ) to the exact solution to get the error. The error is measured in the  $L^2(\Omega_{\text{ext}}^\varepsilon)$ -norm (*top left*), in the  $L^2(\Omega_{\text{int}}^\varepsilon)$ -norm (*top right*), in the  $H^1(\Omega_{\text{ext}}^\varepsilon)$ -seminorm (*bottom left*), and in the  $H^1(\Omega_{\text{int}}^\varepsilon)$ -seminorm (*bottom right*). The numerically observed convergence rates verify the estimates in Theorem 5.3 and Remark 3.1. Note, that the  $H^1$ -norm of the exact solution in  $\Omega_{\text{ext}}^\varepsilon$  is of order 1, so the given absolute errors corresponds nearly to the relative errors.

around  $\Omega_{\text{int}}^\varepsilon$  in  $\mathcal{M}^\varepsilon$  are practicable representation of the solution on respective other mesh, which we need here for the computation of norms of the error, but which is no needed for simulation only using the transmission conditions.

The Figures 5 and 6 show the convergence of the error of the transmission conditions of order 1, 2 and 3 w.r.t. the sheet thickness for the geometry 2 and relative conductivities  $c_0 = 1$  and  $c_0 = 250$ . The error is shown in the  $L^2$ -norm and the  $H^1$ -seminorm in the exterior and the interior of the sheet. For the reference solution a polynomial degree  $p = 20$  was chosen higher than that for the approximative models ( $p = 18$ ) to observe in the convergence plot when the modelling error falls below the discretisation error. The Lagrange multiplier was modelled as piecewise continuous elements on the edges of the interface  $\Gamma$ . The polynomial degree has to be chosen as high as  $p = 18$  to make the modelling error visible. For the computation of the interior solution we

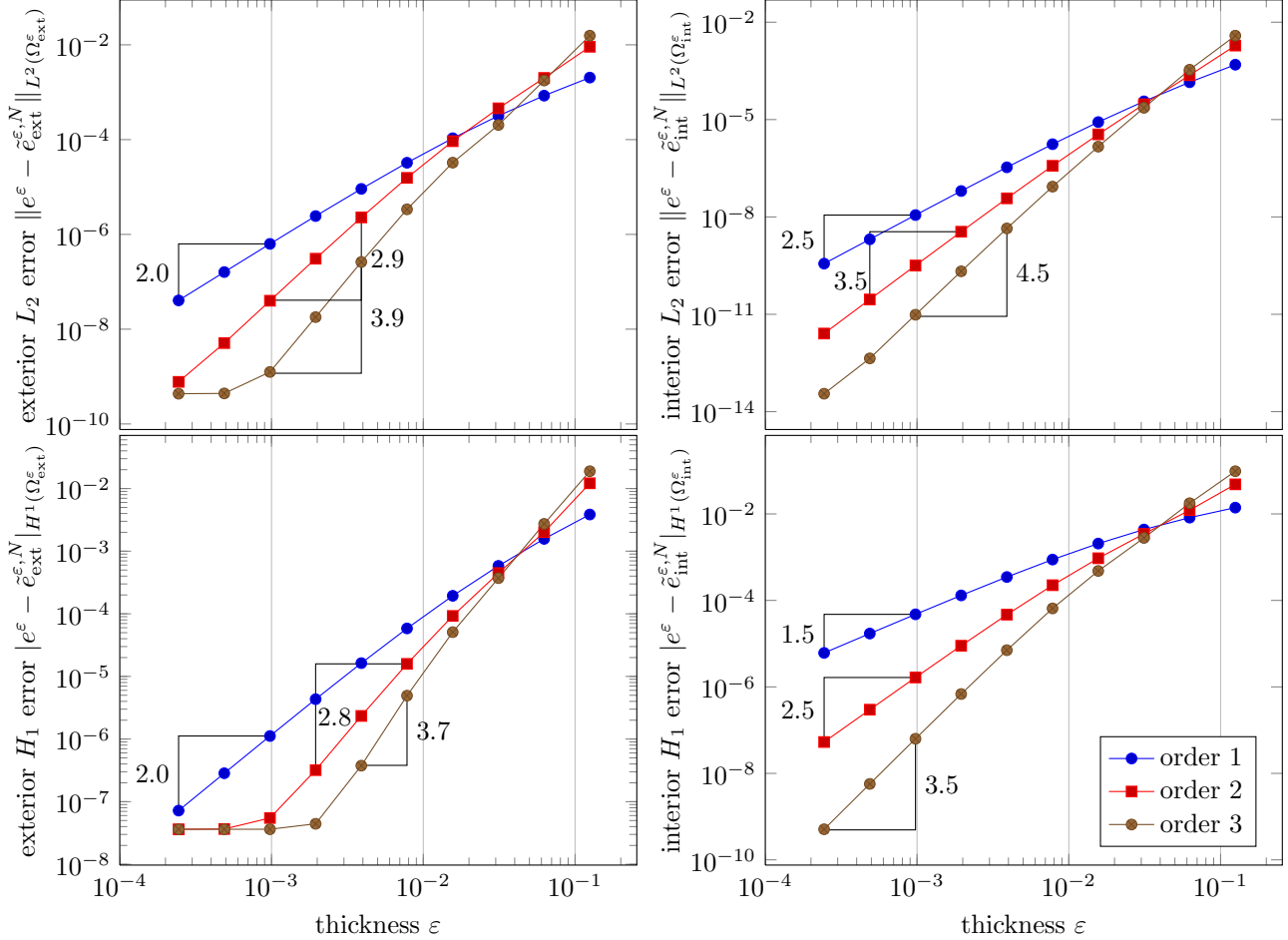


FIGURE 6. Convergence of the error as in Fig. 5, where only the relative conductivity is changed to  $c_0 = 250$ . The numerically observed rates in  $\varepsilon$  correspond again to theoretically predicted. At about  $\varepsilon = 4 \cdot 10^{-2} = \frac{10}{c_0}$  the curves of different orders have a crossing point ( $d_{\text{skin}} \approx 3\varepsilon$ ). For larger thicknesses  $\varepsilon$  the model of order 1 achieves the best results.

used the Lagrange multiplier for the mean value of the normal derivative and computed locally its second derivative which is present in the model of order 3.

With  $c_0 = 1$  (Fig. 5) the range of investigated thicknesses is well below  $\frac{5}{c_0} = 5$ . We observe very low error levels for all three models and convergence rates which coincide with the theoretically predicted ones. The convergence stops when reaching the discretisation error gets dominant. For the model of order 3 and the error in the interior this point is reached earlier due to inexact evaluation of  $\partial_t^2 \tilde{\lambda}_{\text{ext}}^{\varepsilon,3} = \partial_t^2 \{\partial_n \tilde{e}_{\text{ext}}^{\varepsilon,3}\}$ .

For the case  $c_0 = 250$  (Fig. 5) we see the same convergence rates as for  $c_0 = 1$ . In this case a part of the investigated thicknesses is above the skin depth (above  $\varepsilon = 4 \cdot 10^{-3}$ ), and for about  $\varepsilon = 4 \cdot 10^{-2} = \frac{10}{c_0}$  which correspond to about 3 times the skin depth ( $d_{\text{skin}} = 1/\sqrt{c} = \sqrt{\varepsilon}/\sqrt{c_0} \approx 1.3 \cdot 10^{-3}$ ) the convergence curves intersect. Below this point it is worthwhile to use transmission conditions of higher orders where above the point the model of order 1 achieves the best results. Although the last thickness  $\varepsilon = \frac{1}{8} = \frac{31}{c_0}$  exceeds largely the proven range of stability (Lemma 4.6 and Lemma 4.10) no stability problem was observed in the presented numerical experiments.

The error distribution of the electric field in the exterior of the sheet is shown in Fig. 7 for three examples and the transmission conditions of order 1, 2 and 3. The error levels differ largely which are indicated by the individual colorbars. The first and second examples with sheet thickness  $\varepsilon = 1/16 = \frac{0.026}{c_0}$  (relative conductivity  $c_0 = 10$ ) and  $\varepsilon = 1/256 = \frac{3.91}{c_0}$  (relative conductivity  $c_0 = 1000$ ) are well and tight, respectively, in the range of validity of the models. We observe also a considerable decrease of the error for the first example when increasing the order, whereas the decrease is lesser in the second example. The thickness  $\varepsilon = 1/16 = \frac{62.5}{c_0}$  (relative conductivity  $c_0 = 1000$ ) in the third example lies clearly above the range of validity. The error in this example is smallest for order 1 and increases for higher model orders. In all the examples the error for the order 1 is mainly located in inner area where the electric field itself has highest values. By increasing the model order the error field is more distributed.

## CONCLUDING REMARKS

In the context of magneto-quasistatic, we derived three approximate models of order 1, 2 and 3 to take into account the far field behaviour of thin and highly conducting sheets. With these models the sheet has not be discretised, but it is represented by an interface in place of the mid-line on which local transmission conditions are added. Therefore the models can easily be implemented in most of the finite element libraries or codes based on a Galerkin approximation. Once the field outside the sheet (far field) is computed the internal field follows as a polynomial in thickness direction. Our few numerical simulations verify the theoretically achieved estimates for the modelling error and confirm also its range of validity which are thickness up to the order of the skin depth. We end this article with some remarks and open problems.

**About regularity of the solution.** When the midline  $\Gamma$  of the sheet is regular and the boundary conditions do not create singularities (we think for example to an exterior boundary  $\partial\Omega$  with corners), the coefficients of the Taylor expansions are also regular. Moreover, a stronger modelling error estimate similar to Theorem 5.3 can be obtained.

It takes the form

**Theorem 6.1.** *Let  $N = 1, 2, 3$ . Then, there exist  $\varepsilon_0 > 0$  and a constant  $C$  independent of  $\varepsilon < \varepsilon_0$  such that*

$$\|\tilde{e}_{\text{ext}}^{\varepsilon, N} - e_{\text{ext}}^{\varepsilon}\|_{H^p(\Omega_{\text{ext}}^{\varepsilon})} \leq C_p \varepsilon^{N+1}, \quad \forall p \in \mathbb{N}. \quad (68)$$

We refer to [23] for most of the required ingredients.

**About regularity of the boundary.** A lot of industrial casings are polygonal or polyhedral and so contains edges. Our first order approximate model can be applied to such geometries. Anyway its justifications require more advance arguments based on a multiscale analysis similar to the one of [7]. The same type of problematic appears for open sheets.

**About full Maxwell systems in 2D.** When the displacement current is not neglected, one has to face a Helmholtz equation inside the exterior domain. This equation has to be supplemented with radiation condition in order to obtain a well posed problem. This leads to a non-coercive variational formulation and the stability result of Section 4 requires an important modification (one has to act by contradiction). However, the extra ingredients are now classical. One can refer to [15] for a similar problem.

**About three-dimensional thin sheets.** When the sheet is not  $z$ -invariant but completely three-dimensional, the relevant problem is vectorial. However, in the context of IBC, many authors, see [4, 12] for example, have proposed approximate models. One can think to adapt their approach to highly conducting sheets.

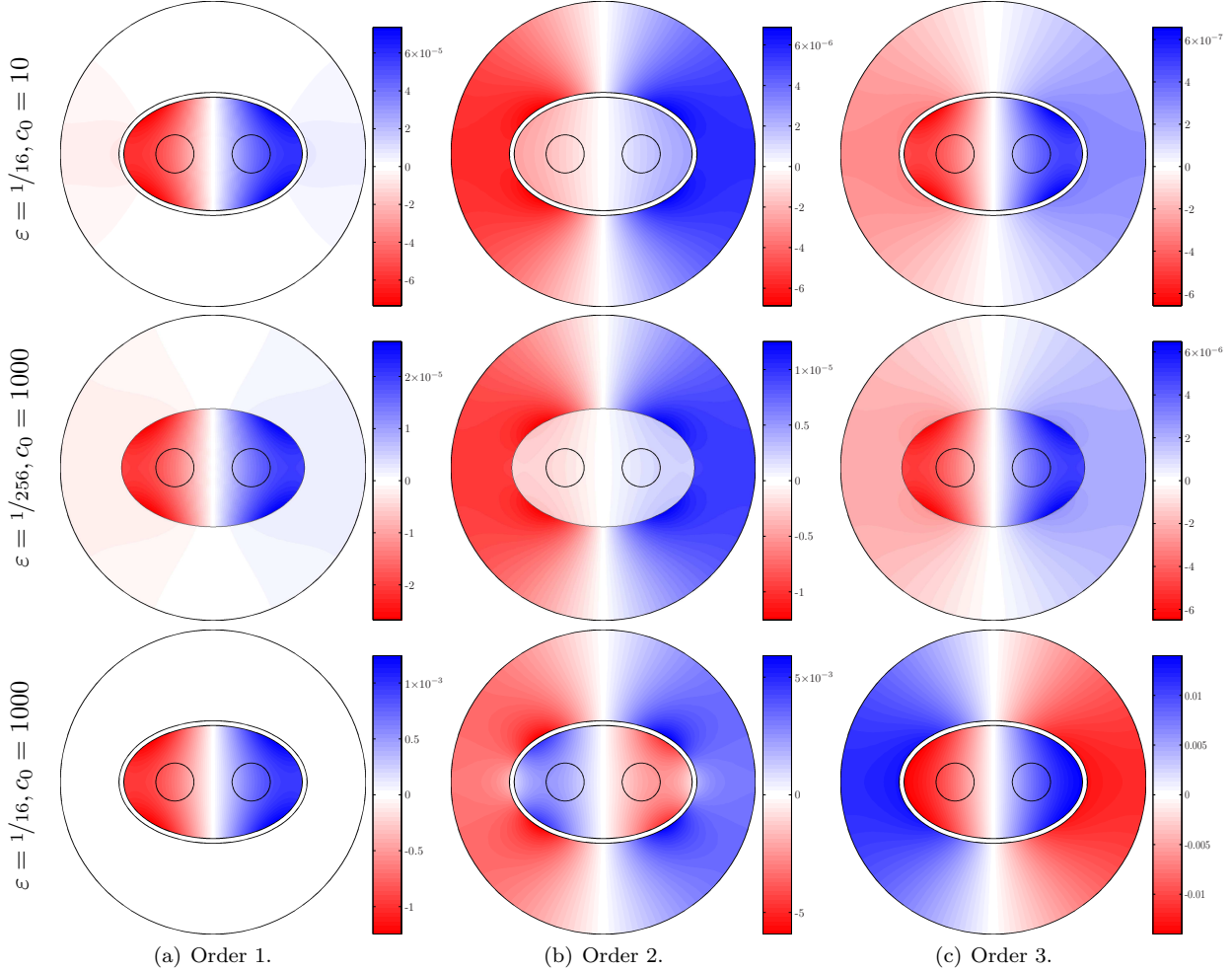


FIGURE 7. Error of the electric field in the exterior of the thin conducting sheet for the asymptotic models of order 1, 2 and 3 (note the different scalings of the color representation). The configuration in the first row is with a sheet of thickness  $\varepsilon = 1/16$  and relative conductivity  $c_0 = 10$ . The exact solution for this configuration is shown in Fig. 4(b). With this setting the error increases by about one order of magnitude when increasing the order of the model by one ( $\varepsilon = \frac{0.625}{c_0}$ ). In the second row the error is plotted for  $\varepsilon = 1/256$  and  $c_0 = 1000$  and decreases by a factor of 2 per order ( $\varepsilon = \frac{3.91}{c_0}$ ). In the third row the configuration is  $\varepsilon = 1/16$  and  $c_0 = 1000$  and the error increases with increasing order ( $\varepsilon = \frac{62.5}{c_0}$ ). Here the skin depth fall below the valid value of the proposed models.

## APPENDIX A.

### A.1. The surface operators

The approximate solutions  $\tilde{e}_{\text{ext}}^{\varepsilon, N}$  of order  $N$  are defined with the help of two differential operators that consists in the truncation of two formal series of operators, see (25),

$$\gamma^{\varepsilon, N} := \sum_{j=2}^N \varepsilon^j \gamma_j \quad \text{and} \quad \zeta^{\varepsilon, N} := \sum_{j=1}^N \varepsilon^j \zeta_j$$

where  $\gamma_0 = \gamma_1 = \zeta_0 = 0$  and the differential operators  $\gamma_\ell$  and  $\zeta_\ell$  that are explicitly given for  $\ell \leq 3$  by

$$\begin{aligned}
(\gamma_2 u)(t) &:= -\frac{c_0}{24} (\kappa(t) \{u\}(t) + 2 \{\partial_n u\}(t)), \\
(\gamma_3 u)(t) &:= \frac{c_0^2}{240} (\kappa(t) \{u\}(t) + 2 \{\partial_n u\}(t)), \\
(\zeta_1 u)(t) &:= \frac{c_0^2}{6} \{u\}(t), \\
(\zeta_2 u)(t) &:= \frac{c_0}{12} \left( \frac{7}{20} c_0^2 - \partial_t^2 \right) \{u\}(t) + \frac{c_0}{24} \kappa(t) \{\partial_n u\}(t) \\
(\zeta_3 u)(t) &:= \frac{c_0^2}{40} \left( \frac{17}{84} c_0^2 - \frac{1}{3} \kappa^2(t) - \partial_t^2 \right) \{u\}(t) - \frac{c_0^2}{240} \kappa(t) \{\partial_n u\}(t).
\end{aligned} \tag{69}$$

## A.2. The interior operators

The interior approximation of order  $N$  involves the differential operators

$$\boldsymbol{\eta}^{\varepsilon, N} := \sum_{j=0}^N \varepsilon^j \boldsymbol{\eta}_j$$

with

$$\begin{aligned}
(\boldsymbol{\eta}_0 u)(t, S) &= \{u\}(t), \\
(\boldsymbol{\eta}_1 u)(t, S) &= \frac{c_0}{2} \{u\}(t) (S^2 + \frac{1}{4}) + \{\partial_n u\}(t) S, \\
(\boldsymbol{\eta}_2 u)(t, S) &= \frac{c_0^2}{24} \{u\}(t) (S^2 + \frac{3}{4})^2 + \frac{c_0}{6} \{\partial_n u\}(t) (S^3 - \frac{3}{4} S) - \frac{c_0}{6} \kappa(t) \{u\}(t) (S^3 + \frac{3}{4} S) \\
&\quad - \frac{1}{2} (\kappa(t) \{\partial_n u\}(t) + \partial_t^2 \{u\}(t)) S^2, \\
(\boldsymbol{\eta}_3 u)(t, S) &= \frac{c_0^3}{720} \{u\}(t) \left( (S^2 + \frac{5}{4})^3 + \frac{15}{4} (S^2 + \frac{1}{4}) \right) + \frac{c_0^2}{120} \{\partial_n u\}(t) (S^2 - \frac{5}{4})^2 S \\
&\quad - \frac{c_0^2}{60} \kappa(t) \{u\}(t) \left( S^4 + \frac{5}{2} S^2 + \frac{5}{16} \right) S - \frac{c_0}{12} \kappa(t) \{\partial_n u\}(t) (S^2 - \frac{3}{4}) S^2 \\
&\quad - \frac{c_0}{12} \partial_t^2 \{u\}(t) (S^2 + \frac{1}{2}) (S^2 + \frac{1}{4}) + \frac{c_0}{8} \kappa^2(t) \{u\}(t) (S^4 + \frac{1}{2} S^2 - \frac{1}{48}) \\
&\quad + \frac{1}{2} (\kappa(t) \partial_t^2 + \frac{1}{3} \kappa'(t) \partial_t) \{u\}(t) S^3 + \frac{1}{3} (\kappa^2(t) - \frac{1}{2} \partial_t^2) \{\partial_n u\}(t) S^3.
\end{aligned}$$

## REFERENCES

- [1] ANTOINE, X., BARUCQ, H., AND VERNHET, L. High-frequency asymptotic analysis of a dissipative transmission problem resulting in generalized impedance boundary conditions. *Asymptot. Anal.* 26, 3-4 (2001), 257–283.
- [2] BARTOLI, N., AND BENDALI, A. Robust and high-order effective boundary conditions for perfectly conducting scatterers coated by a thin dielectric layer. *IMA J. Appl. Math.* 67 (2002), 479–508.
- [3] BENDALI, A., AND LEMRABET, K. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.* 6 (1996), 1664–1693.
- [4] BENDALI, A., AND LEMRABET, K. Asymptotic analysis of the scattering of a time-harmonic electromagnetic wave by a perfectly conducting metal coated with a thin dielectric shell. *Asymptot. Anal.* 57, 3-4 (2008), 199–227.
- [5] BOSSAVIT, A. *Computational Electromagnetism. Variational Formulation, Complementarity, Edge Elements*. No. 2 in Academic Press Electromagnetism Series. Academic Press, San Diego, 1998.
- [6] BRAESS, D. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, 3th ed. Cambridge University Press, 2007.

- [7] CALOZ, G., COSTABEL, M., DAUGE, M., AND VIAL, G. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptot. Anal.* 50, 1 (2006), 121–173.
- [8] CONCEPTS DEVELOPMENT TEAM. *Webpage of Numerical C++ Library Concepts 2*. <http://www.concepts.math.ethz.ch>, 2008.
- [9] DURUFLÉ, M., HADDAR, H., AND JOLY, P. Higher order generalized impedance boundary conditions in electromagnetic scattering problems. *C. R. Physique* 7, 5 (2006), 533–542.
- [10] EVANS, L. C. *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*. AMS, 1998.
- [11] FRAUENFELDER, P., AND LAGE, C. Concepts – an object-oriented software package for partial differential equations. *M2AN Math. Model. Numer. Anal.* 36, 5 (September 2002), 937–951.
- [12] HADDAR, H., JOLY, P., AND NGUYEN, H. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the case of Maxwell's equations. *Math. Models Methods Appl. Sci.* 18, 10 (2008), 1787–1827.
- [13] HADDAR, H., JOLY, P., AND NGUYEN, H.-M. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case. *Math. Models Meth. Appl. Sci.* 15, 8 (2005), 1273–1300.
- [14] IGARASHI, H., KOST, A., AND HONMA, T. A boundary element analysis of magnetic shielding for electron microscopes. *COMPEL* 17, 5/6 (1998), 585–594.
- [15] JOLY, P., AND TORDEUX, S. Asymptotic analysis of an approximate model for time harmonic waves in media with thin slots. *M2AN Math. Model. Numer. Anal.* 40, 1 (2006), 63–97.
- [16] KRÄHENBÜHL, L., AND MULLER, D. Thin layers in electrical engineering. Example of shell models in analysing eddy-currents by boundary and finite element methods. *IEEE Trans. Magn.* 29 (1993), 1450–1455.
- [17] LEONTOVICH, M. A. On approximate boundary conditions for electromagnetic fields on the surface of highly conducting bodies (in russian). *Research in the propagation of radio waves* (1948), 5–12. Moscow, Academy of Sciences.
- [18] MCLEAN, W. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.
- [19] MIRI, A. M., RIEGEL, N. A., AND MEINECKE, C. FE calculation of transient eddy currents in thin conductive sheets using dynamic boundary conditions. *Int. J. Numer. Model.* 11 (1998), 307–316.
- [20] NAKATA, T., TAKAHASHI, N., FUJIWARA, K., AND SHIRAKI, Y. 3D magnetic field analysis using special elements. *IEEE Trans. Magn.* 26, 5 (1990), 2379–2381.
- [21] PÉRON, V., AND POIGNARD, C. Approximate transmission conditions for time-harmonic Maxwell equations in a domain with thin layer. Research Report RR-6775, INRIA, 2008.
- [22] PERRUSSEL, R., AND POIGNARD, C. Asymptotic Transmission Conditions for Steady-State Potential in a High Contrast Medium. A Uniform Variational Formulation for Resistive Thin Layers. Research Report RR-7163, INRIA, Jan 2010.
- [23] SCHMIDT, K. *High-order numerical modeling of highly conductive thin sheets*. PhD thesis, ETH Zürich, July 2008.
- [24] SCHMIDT, K., STERZ, O., AND HIPTMAIR, R. Estimating the eddy-current modelling error. *IEEE Trans. Magn.* 44 (June 2008), 686–689.
- [25] SCHMIDT, K., AND TORDEUX, S. Asymptotic modelling of conductive thin sheets. SAM Report 2008-28, ETH Zürich, Seminar for Applied Mathematics, Sep 2008.
- [26] SCHMIDT, K., AND TORDEUX, S. Asymptotic modelling of conductive thin sheets. *Z. Angew. Math. Phys.* (2010). in print.
- [27] SENIOR, T., AND VOLAKIS, J. *Approximate Boundary Conditions in Electromagnetics*. Institution of Electrical Engineers, 1995.
- [28] SHCHUKIN, A. N. *Propagation of Radio Waves (in russian)*. Svyazizdat, Moscow, 1940.



---

Centre de recherche INRIA Paris – Rocquencourt  
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex  
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier  
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq  
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique  
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex  
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex  
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex  
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

---

Éditeur  
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399