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b-coloring of tight graphs *

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Abstract: A coloring c of a graph $G = (V,E)$ is a *b-coloring* if in every color class there is a vertex colored i whose neighborhood intersects every other color classes. The *b-chromatic number* of G, denoted $\chi_b(G)$, is the greatest integer k such that G admits a b-coloring with k colors. A graph G is *tight* if it has exactly $m(G)$ vertices of degree $m(G)-1$, where $m(G)$ is the largest integer m such that G has at least m vertices of degree at least $m-1$. Determining the b-chromatic number of a tight graph G is NP-hard even for a connected bipartite graph [9]. In this paper we show that it is also NP-hard for a tight chordal graph. We also show that the b-chromatic number of a split graph can be computed is polynomial. Then we define the b-closure and the partial b-closure of a tight graph, and use these concepts to give a characterization of tight graphs whose b-chromatic number is equal to $m(G)$. This characterization is used to develop polynomial time algorithms for deciding whether $\chi_b(G) = m(G)$, for tight graphs that are complement of bipartite graphs, P_4-sparse and block graphs. We generalize the concept of pivoted tree introduced by Irving and Manlove [6] and show its relation with the b-chromatic number of tight graphs. Finally, we give an alternative formulation of the Erdös-Faber-Lovász conjecture in terms of b-colorings of tight graphs.

Key-words: graph coloring, b-coloring, precoloring extension, tight graphs

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b-coloration des graphes étriqués

Résumé : Une k-coloration c d'un graphe G est une *b-coloration* si dans toute classe de couleur, il y a un sommet dont le voisinage intersecte toutes les autres classes de couleurs. The *nombre b-chromatique* d'un graphe est le plus grand entier k tel que G admette une b-coloration avec k couleurs. Un graphe est *étriqué* s'il a exactement $m(G)$ sommet de degré $m(G)-1$, avec $m(G)$ le plus grand entier m tel que G ait au moins m sommets de degré au moins $m-1$. Calculer le nombre b-chromatique d'un graphe étriqué est NP-dur même pour les graphes connexes bipartis [9]. Dans ce rapport, nous montrons que c'est également NP-difficile pour les graphes étriqués cordaux. Nous montrons également que le nombre b-chromatique d'un graphe split peut être calculé en temps polynomial. Ensuite nous définissons la b-clôture et la b-clôture partielle d'un graphe étriqué. Nous utilisons ces deux concepts pour concevoir des algorithmes en temps polynomial pour décider si $\chi_b(G) = m(G)$ pour les graphes étriqués qui sont bipartis, P_4-sparse ou des block-graphes. Nous généralisons également le concept d'arbre pivoté de Irving and Manlove [6] et montrons sa relation avec le nombre b-chromatique des graphes étriqués. Enfin, nous donnons une formulation alternative de la conjecture d'Erdös-Faber-Lovász en termes de b-coloration des graphes étriqués.

Mots-clés : coloration de graphe, b-coloration, extension de précoloration, graphes étriqués

1 Introduction

A *k-coloring* of a graph $G = (V,E)$ is a function $c : V \to \{1, 2, ..., k\}$, such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The *color class* c_i is the subset of vertices of G that are assigned to color i. The *chromatic number* of G, denoted $\chi(G)$, is the least integer k such that G admits a k-coloring. Given a k-coloring c, a vertex v is a *b-vertex* of color i, if $c(v) = i$ and v has at least one neighbor in every color class c_j, $j \neq i$. A coloring of G is a *b-coloring* if every color class has a b-vertex. The *b-chromatic number* of a graph G, denoted $\chi_b(G)$, is the largest integer k such that G admits a b-coloring with k colors. These concepts were first defined in [6]. In that paper, Irving and Manlove proved that the problem of determining the b-chromatic number of a graph is NP-Hard. In fact, it was shown in [9] that deciding whether a graph admits a b-coloring with a given number of colors is an NP-complete problem, even for connected bipartite graphs. The following upper bound for the b-chromatic number of a graph, presented in [6], has been proved to be very useful. If G admits a b-coloring with m colors, then G must have at least m vertices with degree at least $m-1$ (since each color class has one b-vertex). The *m-degree* of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m-1$. It is easy to see that $\chi_b(G) \leq m(G)$ for every graph G. A vertex of G with degree at least $m(G)$ is called a *dense* vertex. The preceding upper bound leads us to the definition of a class of graphs which are tight with respect to the number and degree of their dense vertices:

Definition 1 (tight graph)**.** A graph G is *tight* if it has exactly $m(G)$ dense vertices, each of which has degree $m(G)-1$.

In this paper, we mainly investigate the following decision problem:

TIGHT b-CHROMATIC PROBLEM
Instance: A tight graph G.
Question: Does $\chi_b(G)$ equals $m(G)$?

A direct consequence of the NP-completeness result shown in [9] is the following:

Theorem 2. *The* TIGHT b-CHROMATIC PROBLEM *is NP-complete for connected bipartite graphs.*

For any positive k, P_k denotes a path with k vertices. A graph G is *P_4-sparse* if every set of five vertices of G induces at most one P_4. Bonomo et al. [2] proved that the b-chromatic number of P_4-sparse graphs can be determined in polynomial time. They asked if this result could be extended to *distance-hereditary graphs*, that are graphs in which every induced path is a shortest path. We answer in the negative to this question by showing the following stronger result (Theorem 3). *The* TIGHT b-CHROMATIC PROBLEM *is NP-complete for chordal distance-hereditary graphs.* We recall that a graph is *chordal* if it does not contain any induced cycle of size greater than 3.

The proof of our NP-completeness result is a reduction from 3-EDGE-COLORABILITY. We reduce an instance of this problem to a graph which is slightly more than a *split graph*, i.e. a graph whose vertex set may be partitioned into a clique and an independent set. Hence a natural question is to ask about the complexity of finding the b-chromatic number of a given split graph. We show in Theorem 4 that it can be solved in polynomial time.

In Section 3, we introduce the b-closure G^* of a graph G. We show that for a tight graph G, $\chi_b(G) = m(G)$ if and only if $\chi(G^*) = m(G)$. Hence if one can determine the chromatic number of the closure in polynomial time, one can also solve the TIGHT b-CHROMATIC PROBLEM in polynomial time. We show that it is the case for (tight) complement of bipartite graphs. Indeed, we prove that the closures of such graphs are also complements of bipartite graphs and the chromatic number of the complement of a bipartite graph can be determined in polynomial time. This was unknown since the characterization of complements of bipartite graphs with $\chi_b(G) = k$ given by [7] does not lead to a polynomial algorithm for determining their b-chromatic number.

Moreover, we introduce the definition of pivoted tight graph and use this definition to give a sufficient condition for a tight graph to satisfy $\chi_b(G) < m(G)$.

The method of computing the b-closure of a graph and then the chromatic number of it does not yield polynomial-time algorithms to solve the TIGHT b-CHROMATIC PROBLEM for all classes of tight graphs. However, for some of them, we show in Section 4, that the TIGHT b-CHROMATIC PROBLEM may be solved in polynomial time using a slight modification of the closure, the *partial closure*. It is the case for block graphs and P_4-sparse graphs. It is already known that deciding if $\chi_b(G) = m(G)$ is polynomial time solvable for P_4-sparse graphs [2]. However, our linear-time algorithm for tight P_4-sparse graphs is faster than the $O(|V|^3)$ algorithm of [2]. It is also interesting to see how our general method can be used to solve these problems.

Finally, we give an alternative formulation of the Erdös-Faber-Lovász conjecture [1] in terms of b-colorings of tight graphs.

2 Chordal graphs

Theorem 3. *The* TIGHT b-CHROMATIC PROBLEM *is NP-complete for chordal distance-hereditary graphs.*

Proof. The problem belongs to NP since a b-coloring with $m(G)$ colors is a certificate. To show that it is also NP-complete, we present a reduction from 3-EDGE-COLORABILITY of 3-regular graphs, which is known to be NP-complete [4]. Let G be a 3-regular graph with n vertices. Set $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let I be the vertex-edge incidence graph of G, that is the bipartite graph with vertex set $V(I) = V(G) \cup E(G)$ in which an edge of G is adjacent to its two end-vertices. We construct from I a new graph H as follows. First, we add an edge between every pair of vertices in $V(G)$ and then, we add three disjoint copies of $K_{1,n+2}$. One can easily see that $d_H(v) = n-1+3 = n+2$, for $v \in V(G)$, and that $d_H(u) = 2$, for $u \in E(G)$. Moreover, each copy of $K_{1,n+2}$ has exactly one vertex with degree equal to $n+2$. Consequently, $m(H) = n+3$ and H is tight. In H, $V(G)$ is a clique and $E(G)$ is an independent set, so $H[A \cup B]$ is a split graph, and so it is chordal. As the disjoint copies of $K_{1,n+2}$ are themselves chordal graphs, we get that the entire graph H is chordal. One can easily check that H is also distance-hereditary. We now prove that G admits a 3-edge-coloring if and only if $\chi_b(H) = m(H) = n+3$.

Let c be a 3-edge-coloring of $E(G)$ that uses colors $\{1,2,3\}$. We shall construct a b-coloring c' of H with $n+2$ colors. Let $c'(u) = c(u)$, for $u \in E(G)$, and $c'(v_i) = i+3$, for $1 \leq i \leq n$. Note that in this partial coloring, the vertices in $V(G)$ are b-vertices of their respective colors. To obtain the remaining b-vertices, one just have to appropriately color the copies of $K_{1,n+2}$, which can be easily done. Then, c' is a b-coloring of H with $m(H) = n+3$ colors.

Now, let c' be a b-coloring of H that uses $n+3$ colors. Since $V(G)$ is a clique, we may assume that $c'(v_i) = i+3$, for $1 \leq i \leq n$. Since there are only $n+3$ vertices of degree $n+2$ in H, each vertex in $V(G)$ is an b-vertex. But then, since every vertex in $V(G)$ has degree exactly $n+2$ in H, all its neighbors must have distinct colors. As a consequence, since no vertex in $V(G)$ is colored with one of the colors in $\{1,2,3\}$, for every vertex in $V(G)$, its 3 neighbours in $E(G)$ are colored with distinct colors in $\{1,2,3\}$. This implies that G admits a 3-edge-coloring of G, and completes the proof. □

The three copies of $K_{1,n+2}$ play an important role in this reduction, since one can show the following

Theorem 4. *If G is a split graph then $\chi_b(G) = m(G)$. Hence, the b-chromatic number of a split graph can be determined in polynomial time.*

Proof. Let G be a split graph and (K,S) a partition of $V(G)$ with K a clique and S an independent set such that $|K|$ is maximum. Every vertex in K has degree at least $|K|-1$ and every vertex s in S has degree at most $|K|-1$ otherwise $(K \cup \{s\}, S \setminus \{s\})$ would contradict the maximality of $|K|$. Hence $m(G) = |K|$.

Coloring the vertices in K with $|K|$ distinct colors and then extend it greedily to the vertices of S (This is possible since every vertex in S has degree smaller than $|K|$.) gives a b-colouring of G with $m(G) = |K|$ colours. □

3 *b*-closure

Definition 5 (*b*-closure)**.** Let G be a tight graph. The *b-closure* of G, denoted by G^*, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are non-adjacent dense vertices}\} \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$.

The next theorem proves the relation, for a tight graph G, between the parameters $\chi_b(G)$ and $\chi(G^*)$:

Lemma 6. *Let G be a tight graph. Then $\chi_b(G) = m(G)$ if and only if $\chi(G^*) = m(G)$.*

Proof. Set $m = m(G)$. Suppose that $\chi_b(G) = m$, and let c be a b-coloring of G with m colors. It is easy to see that the m dense vertices form a clique in G^* and so $\chi(G^*) \geq m$. Let us show that c is a proper coloring for G^*. Let $uv \notin G$ be such that $uv \in E(G^*)$. If both u and v are dense, as there are exactly m dense vertices in G, they must have distinct colors in c. Now, suppose that u or v is not a dense vertex. By the definition of G^*, u and v have a common dense neighbor, say d, in G. Since all dense vertices of G have degree $m-1$ and c is a b-coloring, u and v must have been assigned distinct colors in c. Hence, $\chi(G^*) = m$.

Conversely, let c' be a proper coloring of G^* with m colors. In this case, since $E(G) \subseteq E(G^*)$, c' is also a proper coloring of G. It only remains to show that every color of c' has a b-vertex. As the dense vertices of G form a clique in G^*, they have distinct colors in c'. Moreover, for a dense vertex d of G, we have that $N_{G^*}(d)$ is a clique. As a consequence, d is a b-vertex. Therefore, $\chi_b(G) = m$. □

Since $\omega(G^*) > m$ implies that $\chi(G^*) > m$, it follows:

Corollary 7. *Let G be a tight graph. If $\chi_b(G) = m(G)$, then $\omega(G^*) = \chi(G^*) = m(G)$.*

3.1 Complement of bipartite graphs

By Lemma 6, it is interesting to consider the b-closure of a tight graph G if the chromatic number of its closure can be determined in polynomial time. Indeed if so, one can decide in polynomial time if $\chi_b(G) = m(G)$. We now show that it is the case if G is the complement of bipartite graph.

Lemma 8. *The b-closure of the complement of a bipartite graph is a complement of a bipartite graph.*

Proof. Let G be a tight complement of a bipartite graph. Let $V(G) = X \cup Y$ where X and Y are two disjoint cliques in G. As $V(G^*) = V(G)$, and since $E(G) \subseteq E(G^*)$, the sets X and Y are cliques in G^*. So they also form a partition of $V(G^*)$ into two cliques. □

Computing the chromatic number of the complement G of a bipartite graph $\overline{G}$ is equivalent to compute the maximum size of a matching in this bipartite graph. Hence it can be done in $O(\sqrt{|V(G)|} \cdot |E(\overline{G}|)$ by the algorithm of Hopcroft and Karp [5] and in $O(|V(G)|^{2.376})$ using an approach based on the fast matrix multiplication algorithm [11].

Corollary 9. *Let G be a tight complement of bipartite graph. It can be decided in $O(\max\{\sqrt{|V(G)|} \cdot |E(\overline{G})|, |V(G)|^{2.376}\})$ if $\chi_b(G) = m(G)$.*

3.2 Pivoted graphs

In the study of the b-chromatic number of trees, Irving and Manlove [6] introduced the notion of a *pivoted tree*, and showed that a tree T satisfies $\chi_b(T) < m(T)$ if and only if it is pivoted. We generalize this notion and show how our generalization is related to the b-chromatic number of tight graphs.

Definition 10 (Pivoted Graph)**.** Let G be a tight graph. We say that G is *pivoted* if there is a set N of non-dense vertices, with $|N| = k$, and a set of dense vertices D, with $|D| = m(G) - k + 1$, satisfying:

1. For every pair $u, v \in N$, u is adjacent to v, or there is a dense vertex w that is adjacent to both u and v.
2. For every pair $u \in N$, $d \in D$, either u is adjacent to d or u and d are both adjacent to a dense vertex w (not necessarily in D).

Theorem 11. *Let G be a tight graph. Then G is a pivoted graph if and only if $\omega(G^*) > m(G)$.*

Proof. First, assume that G is a pivoted graph. Then Definitions 5 and 10 immediately imply that $N \cup D$ is a clique of size $m+1$ in G^*.

Reciprocally, assume that $\omega(G^*) > m$. Let $S \subseteq V(G^*)$ be a clique of size $m+1$ in G^*. Let $N = \{v \in S \mid v$ is not dense in $G\}$ and $D = \{v \in S \mid v$ is dense in $G\}$. Let $u, v \in S$. If $u, v \in D$, there is nothing to show, since Definition 10 imposes no restrictions between dense vertices in G. If $u \in N, v \in D \cup N$, we have that either $uv \in E(G)$, or $ud, vd \in E(G)$, for a dense vertex $d \in V(G)$. So, it is easy to see that the sets N and D satisfy the requirements of Definition 10. □

Lemma 6 and Theorem 11 have the following corollary.

Corollary 12. *Let G be a tight graph. If G is a pivoted graph, then $\chi_b(G) < m(G)$.*

Proof. As G is pivoted, Theorem 11 implies that $\omega(G^*) > m(G)$, and therefore $\chi(G^*) > m(G)$. Then, by Lemma 6, $\chi_b(G) < m(G)$. □

There are graphs satisfying $\chi(G^*) > m(G)$ but not $\omega(G^*) > m(G)$. Figure 1 shows a chordal non-pivoted graph G with exactly $m(G) = 7$ dense vertices, each of degree 6, such that $\chi_b(G) < m(G)$.

In contrast to what happens with pivoted graphs, where a clique of size greater than m is formed in their b-closures, the graph of Figure 1 has clique number 7, but its b-closure produces an odd hole (by the five non-dense vertices in the bigger component) which causes $\chi(G^*) > 7$.

4 Partial b-closure

Definition 13 (partial b-closure)**.** Let G be a tight graph. The *partial b-closure* of G, denoted G^*_p, is the graph with vertex set $V(G^*) = V(G)$ and edge set $E(G^*) = E(G) \cup \{uv \mid u$ and v are vertices with a common dense neighbour$\}$.

Lemma 14. *Let G^*_p be the partial b-closure of a graph G, and let D be the set of $m(G)$ dense vertices of G. Then $\chi_b(G) = m(G)$ if and only if G^*_p admits a $m(G)$-coloring where all the vertices in D have distinct colors.*

Proof. The proof is similar to the one of Lemma 6. In this case, since we do not add edges between all the pairs of dense vertices in G^*_p, we need the requirement that the $m(G)$-colouring of G^*_p is such that all dense vertices have distinct colours. □

By Lemma 14, one can decide in polynomial time if $\chi_b(G) = m$ wherever it can be decided in polynomial time if the constrained coloring of its partial closure G^*_p exists. In particular, it is the case if the precoloring extension problem can be decided in polynomial time for G. We show that this is the case for block graphs and P_4-sparse graphs.

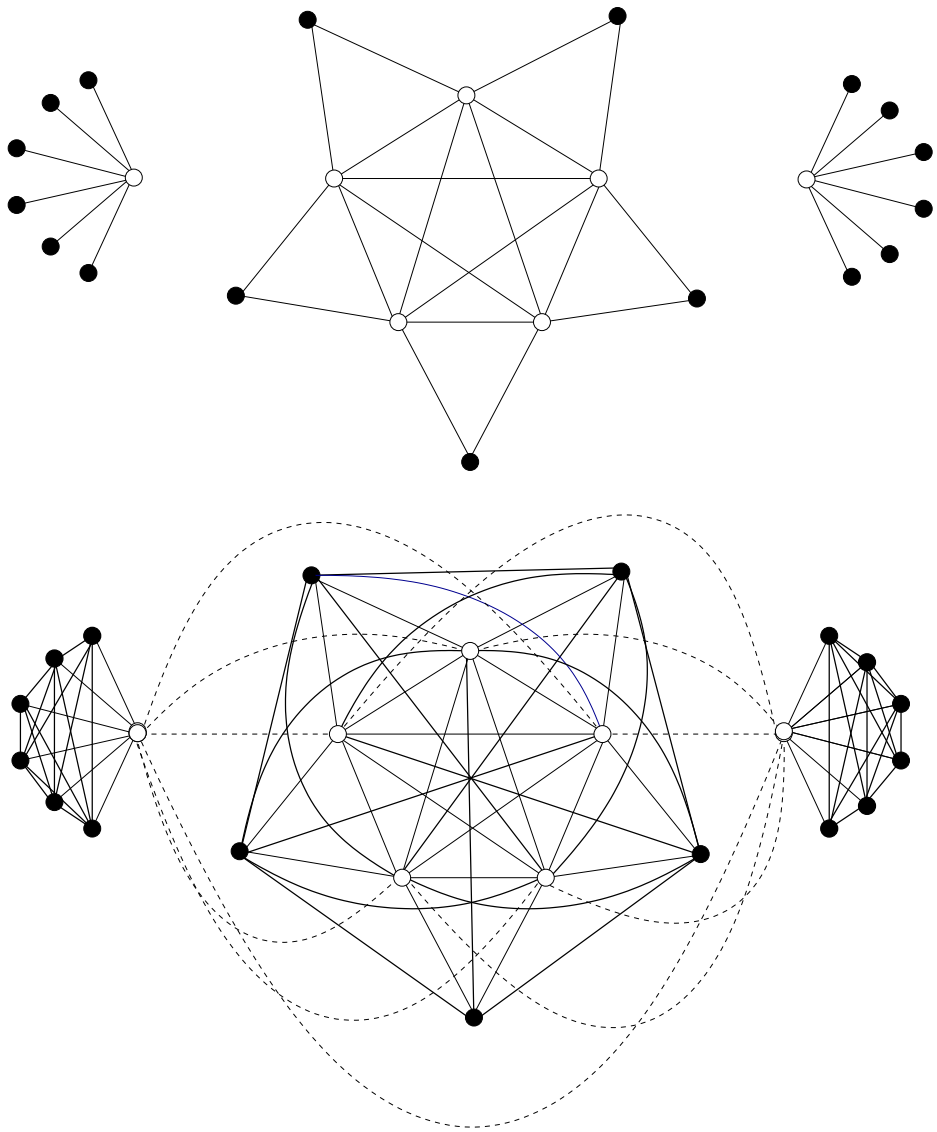

Figure 1: A non-pivoted chordal graph, satisfying $\chi_b(G) < m(G)$, and its b-closure G^*, satisfying $\chi(G^*) > \omega(G^*) = m(G)$
(the new edges between the dense vertices are dashed).

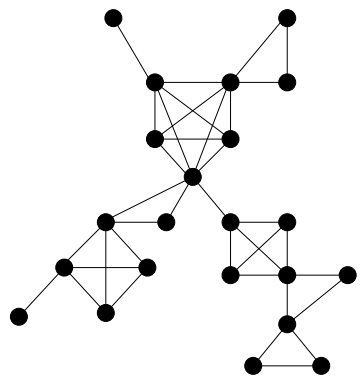

Figure 2: A block graph.

4.1 Block graphs

A graph $G = (V, E)$ is *a block graph* if every of its blocks (maximal 2-connected subgraphs) is a complete graph. For an example, see Figure 2.

Lemma 15. *The partial b-closure of a block graph is chordal.*

Proof. By contradiction, assume that the partial b-closure G_p^* of a block graph G is not chordal. Then it has an induced cycle $C = (v_1, v_2, ..., v_k)$ of length $k \geq 4$. For every edge $v_i v_{i+1}$ of C (indices must be taken modulo k) either $v_i v_{i+1} \in E(G)$ or there is a dense vertex $w_i \in V(G)$ such that $v_i w_i, w_i v_{i+1} \in E(G)$. In the latter case, the vertex w_i is adjacent to no v_j for $j \notin \{i, i+1\}$ in G, otherwise both $v_j v_i$ and $v_j v_{i+1}$ would be edges of G_p^* and C would not be induced. Furthermore, this implies that all the existing w_i's are distinct. Let C' be the cycle obtained from C by replacing each edge $v_i v_{i+1}$ by $v_i w_i v_{i+1}$ whenever $v_i v_{i+1} \notin E(G)$. Observe that C' is a cycle of G.

But, since G is a block graph, the vertices of any cycle (in particular, C') form a clique in G and thus also in G_p^*. Hence the vertices of C form a clique in G_p^*, a contradiction. □

Marx [10] showed that the precoloring extension problem when all the C colours are used at most once is solvable in time $O(C \cdot |V(G)|^3)$ for a chordal graph G. Hence,

Corollary 16. *the* TIGHT b-CHROMATIC PROBLEM *can be decided in time* $O(m(G)|V(G)|^3)$ *for tight block graphs.*

Remark 17. A tree is a block graph, so using the partial closure method the TIGHT b-CHROMATIC PROBLEM for tight trees can be solved in time $O(m(G)|V(G)|^3)$. However, Irving and Manlove [6] gave a linear time algorithm to compute the b-chromatic number of any tree. Hence the TIGHT b-CHROMATIC PROBLEM can be solved in linear time for trees.

4.2 P_4-sparse graphs

Lemma 18. *The partial b-closure of a P_4-sparse graph is P_4-sparse.*

Proof. Let G be a P_4-sparse graph. Suppose, by way of contradiction, that G_p^* is not P_4-sparse. Then there is at least one induced P_4 in G_p^* that is not in G. Let $P = (v_1, v_2, v_3, v_4)$ be such a P_4 in G_p^*. We will show that there are 5 vertices that induces two P_4's in G, thus getting a contradiction. By symmetry, it is enough to consider the following five cases.

Case 1 : $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$ and $v_3v_4 \notin E(G)$.
Then, v_3 and v_4 are both adjacent to a dense vertex $w \in V(G)$ (by the definition of the partial b-closure). Note that $v_1w \notin E(G)$ (resp. $v_2w \notin E(G)$) otherwise $v_1v_4 \in E(G_p^*)$ (resp. $v_2v_4 \in E(G_p^*)$) and P would not be an induced P_4 in G_p^*. Hence, $\{v_1, v_2, v_3, w, v_4\}$ induces a P_5 which contains two induced P_4.

Case 2 : $v_1v_2 \in E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \in E(G)$.
In this case, v_2 and v_3 are both adjacent to a dense vertex $w \in V(G)$ (again, by the definition of the b-closure). Note that $v_1w, v_4w \notin E(G)$, for otherwise, this would imply that $v_1v_3 \in E(G_p^*)$ ($v_2v_4 \in E(G_p^*)$), by the definition of the partial b-closure. But then, $\{v_1, v_2, w, v_3, v_4\}$ is an induced P_5 in G.

Case 3 : $v_1v_2 \notin E(G)$, $v_2v_3 \in E(G)$ and $v_3v_4 \notin E(G)$.
As $v_1v_2 \notin E(G)$, the vertices v_1 and v_2 are both adjacent to a dense vertex $w_1 \in V(G)$. Moreover, $w_1v_3 \notin E(G)$ (resp. $w_1v_4 \notin E(G)$), since for otherwise $v_1v_3 \in E(G_p^*)$ (resp. $v_1v_4 \in E(G_p^*)$) and P would not be an induced P_4 in G_p^*. By a similar argument, v_3 and v_4 are both adjacent to a dense vertex $w_2 \in V(G)$, which is not adjacent to v_1 and w_2. Note that w_1 and w_2 are distinct since $w_1v_4 \notin E(G)$. If $w_1w_2 \notin E(G)$, then $\{v_1, w_1, v_2, v_3, w_2\}$ is an induced P_5 in G. If $w_1w_2 \in E(G)$, then $\{v_1, w_1, v_2, w_2, v_4\}$ induces two P_4's in G.

Case 4 : $v_1v_2 \notin E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \in E(G)$.
Using arguments similar to the ones in the previous cases, we obtain that there are distinct dense vertices $w_1, w_2 \in V(G)$ satisfying $v_1w_1, v_2w_1, v_2w_2, v_3w_2 \in E(G)$, and $v_1w_2, v_4w_2, v_3w_1, v_4w_1 \notin E(G)$. If $w_1w_2 \in E(G)$ then $\{v_1, w_1, w_2, v_3, v_4\}$ induces a P_5 in G. If $w_1w_2 \notin E(G)$, then the set $\{v_1, w_1, v_2, w_2, v_3\}$ induces a P_5 in G.

Case 5 : $v_1v_2 \notin E(G)$, $v_2v_3 \notin E(G)$ and $v_3v_4 \notin E(G)$.
Again, by similar arguments to the ones used in the previous cases, there are distinct dense vertices $w_1, w_2, w_3 \in V(G)$ such that $v_1w_1, v_2w_1, v_2w_2, v_3w_2, v_3w_3, v_4w_3 \in E(G)$, and $v_3w_1, v_4w_1, v_1w_2, v_4w_2, v_1w_3, v_2w_3 \notin E(G)$. If $w_1w_3 \in E(G)$, the set $\{v_1, w_1, w_3, v_3, v_4\}$ induces two P_4's in G. Henceforth we may assume that $w_1w_3 \notin E(G)$. If $w_1w_2, w_2w_3 \in E(G)$, then the set $\{v_1, w_1, w_2, w_3, v_4\}$ induces a P_5 in G. Hence by symmetry, we may assume that $w_2w_3 \in E(G)$. If $w_1w_2 \in E(G)$, then the set $\{v_1, w_1, v_2, w_2, v_3\}$ induces two P_4's in G. If $w_1w_2 \notin E(G)$ the set $\{v_1, w_1, v_2, w_2, w_3\}$ induces two P_4's in G.

□

Babel et al. [8] showed that the precoloring extension problem is linear-time solvable for $(q, q-4)$-graphs, which are graphs where no set of at most q vertices induces more than $q-4$ different P_4's. Hence,

Corollary 19. *The* TIGHT b-CHROMATIC PROBLEM *can be decided in linear time for tight P_4-sparse graphs.*

Consequently, for tight P_4-sparse graphs, this algorithm is faster than the $O(|V|^3)$ algorithm given in [2], that solves the more general case where the input graph is not necessarily tight.

5 A relation with the Erdös-Faber-Lovász conjecture

As we saw in the previous sections, given a tight graph G, deciding if $\chi_b(G) = m(G)$ is equivalent to deciding if $\chi(G^*) = m(G)$. It is easy to see that the graph G^* can be seen as the union of $m(G)+1$ cliques each of size $m(G)$, where these cliques may intersect at some vertices. On the other hand, the well know Erdös-Faber-Lovász conjecture [1] states that:

Conjecture 20 (Erdös-Faber-Lovász)**.** *Any graph G obtained by the union of $K_m^1, K_m^2, \ldots, K_m^m$, where $|K_m^i \cap K_m^j| \leq 1$, $i \neq j$, has chromatic number m.*

Now, consider the conjecture:

Conjecture 21. *Let G be a tight graph such that:*

1. *For every edge $uv \in E(G)$, one of its endpoints is dense, and the other is non-dense, and*
2. *If d' and d'' are dense vertices in G, then $|N(d') \cap N(d'')| \leq 1$.*

Then, $\chi_b(G) = m(G)$.

We can prove that:

Theorem 22. *The Erdös-Faber-Lovász conjecture is true if and only if Conjecture 21 is true.*

Proof. Let G be a graph as in the Conjecture 21, and consider its b-closure G^*. It can be easily seen that G^* is the union of $m(G)+1$ cliques each of size $m(G)$. Notice that the restrictions imposed for the graph in Conjecture 21 imply that each pair of these cliques intersect in at most one vertex. Consider the graph H obtained from G^* by taking each maximal clique and adding a new vertex that it is adjacent to every other vertex in the clique. Then, H is the union of $m(G)+1$ cliques, each of size $m(G)+1$. Then, if the Erdös-Faber-Lovász conjecture is true, $\chi(H) = m(G)+1$. Since the vertices we added in the construction of H form a maximal independent set, we have that $\chi(G^*) = \chi(H) - 1 = m(G)$, and consequently, by Lemma 6, $\chi_b(G) = m(G)$.

Now, let G be a graph as described in the Erdös-Faber-Lovász conjecture. Then G is the union of $K_m^1, K_m^2, \ldots, K_m^m$, where $|K_m^i \cap K_m^j| \leq 1$, $i \neq j$. Since two cliques intersect in at most one vertex, and there are only m cliques, every clique, say K_i, has at least one vertex that is only contained in K_i. For each $1 \leq i \leq m$, let d_i be such a vertex. Consider the graph H obtained from G by taking, for each $1 \leq i \leq m$, the clique K_i and removing every edge on the clique such that d_i is not an endpoint of this edge. It can be easily seen that the resulting graph is a tight graph with the properties required by Conjecture 21. So, if this conjecture is true, we have that $\chi_b(H) = m(H)$. Observe that G is isomorphic to H^*. Then, by Lemma 6, $\chi(G) = \chi_b(H) = m(G)$. □

Once we proved this relationship between these problems, we can deduce results on one problem from results on the other. As an example, we can prove the following.

Theorem 23. *Let G be a tight graph, and let $G[D \cup N(D)]$ be the graph induced by its dense vertices and their neighbours. If $G[D \cup N(D)]$ is such that:*

1. *for every edge $uv \in E(G[D \cup N(D)])$, exactly one of its endpoints is dense, and*
2. *if d' and d'' are dense vertices in G, then $|N(d') \cap N(d'')| \leq 1$, and*
3. *every non-dense vertex is either adjacent to only one dense vertex or to at least $\sqrt{m(G)}$ dense vertices,*

then, $\chi_b(G) = m(G)$.

Proof. Since the vertices that are not in $G[D \cup N(D)]$ have degree lower than $m-1$, it can be easily seen that $\chi_b(G[D \cup N(D)]) = \chi_b(G)$. The result follows from the fact that $G[D \cup N(D)]^*$, the b-closure of $G[D \cup N(D)]$, is a graph as described in the Erdös-Faber-Lovász conjecture, with the additional property that every vertex in $G[D \cup N(D)]^*$ is either in exactly one clique or in at least $\sqrt{m(G)}$ cliques of size m. It was already proved that the conjecture of Erdös-Faber-Lovász is true in this case [12]. □

We conclude by observing that the problem of determining if a tight graph G satisfies $\chi_b(G) = m(G)$ is equivalent to the problem of determining if a graph composed of the union of m cliques of size m is m-colorable, where we have no restriction on the number of vertices that these cliques may share.

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