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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Pedro Machado Manhães de Castro — Olivier Devillers

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*Rapport  
de recherche*



## On the Size of Some Trees Embedded in $\mathbb{R}^d$

Pedro Machado Manhães de Castro , Olivier Devillers

Thème : Computational Geometry, Geometric Graphs  
Équipe-Projet Geometrica

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**Abstract:** This paper extends the result of Steele [6, 5] on the worst-case length of the Euclidean minimum spanning tree  $EMST$  and the Euclidean minimum insertion tree  $EMIT$  of a set of  $n$  points  $S \subset \mathbb{R}^d$ . More precisely, we show that, if the weight  $w$  of an edge  $e$  is its Euclidean length to the power of  $\alpha$ , the following quantities  $\sum_{e \in EMST} w(e)$  and  $\sum_{e \in EMIT} w(e)$  are both worst-case  $O(n^{1-\alpha/d})$ , where  $d$  is the dimension and  $\alpha$ ,  $0 < \alpha < d$ , is the weight. Also, we analyze and compare the value of  $\sum_{e \in T} w(e)$  for some trees  $T$  embedded in  $\mathbb{R}^d$  which are of interest in (but not limited to) the point location problem [2].

**Key-words:** Computational Geometry, Geometric Graphs

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# Sûr la Taille de Quelques Arbres Plongées dans $\mathbb{R}^d$

**Résumé :** Ce papier étend le résultat de Steele [6, 5] sûr la taille au pire des cas de la plus petite arbre de couverture minimal Euclidienne et l'arbre d'insertion minimal d'un ensemble de  $n$  points  $S \subset \mathbb{R}^d$ . Plus précisément, nous démontrons que si le poids  $w$  d'une arête  $e$  est sa longueur Euclidienne à la puissance  $\alpha$ , les quantités suivantes  $\sum_{e \in EMST} w(E)$  et  $\sum_{e \in EMIT} w(E)$  valent au pire des cas  $O(n^{1-\alpha/d})$ , où  $d$  est la dimension et  $\alpha$ ,  $0 < \alpha < d$ , est le poid. Nous déterminons et comparons aussi la valeur de  $\sum_{e \in T} w(E)$  pour des arbres  $T$  plongées dans  $\mathbb{R}^d$ , qui sont d'intérêt au problème de la localisation des points [2].

**Mots-clés :** Géométrie Algorithmique, Graphes Géométriques

## 1 Introduction

This paper extends the result of Steele [6, 5] on the worst-case length of the Euclidean minimum spanning tree  $EMST$  and the Euclidean minimum insertion tree  $EMIT$  of a set of  $n$  points  $S \subset \mathbb{R}^d$ . More precisely, we show that, if the weight  $w$  of an edge  $e$  is its Euclidean length to the power  $\alpha$ , the following quantities  $\sum_{e \in EMST} w(e)$  and  $\sum_{e \in EMIT} w(e)$  are both worst-case  $O(n^{1-\alpha/d})$ , where  $d$  is the dimension and  $\alpha$ ,  $0 < \alpha < d$ , is the weight. Also, we analyze and compare the value of  $\sum_{e \in T} w(e)$  for some trees  $T$  embedded in  $\mathbb{R}^d$  which are of interest in (but not limited to) the point location problem [2].

Let  $S = \{p_i, 1 \leq i \leq n\}$  be a set of points in  $\mathbb{R}^d$  and  $G = (V, E)$  be the complete graph such that the vertex  $v_i \in V$  is embedded on the point  $p_i \in S$ ; the edge  $e_{ij} \in E$  linking two vertices  $v_i$  and  $v_j$  is weighted by  $|p_i - p_j|^\alpha$ , its Euclidean length to the power of  $\alpha$ .  $G$  is usually referred to as the *geometric graph* of  $S$ . We will denote the sum of the weight of the edges of  $G$  by  $|G|_\alpha$  (or  $|G|$  if  $\alpha = 1$ ). We will also refer to  $|G|_\alpha$  as the *weighted-length* of  $G$ .

Consider the following trees:

- (i) A *star* is a tree having one vertex that is linked to all others (see Figure 1a).
- (ii) A *path* is a tree having all vertices of degree 2 but two with degree 1 (see Figure 1b).
- (iii) Among all the trees spanning  $S$ , a tree with the minimal length is called an *Euclidean minimum spanning tree* of  $S$  and denoted  $EMST(S)$  (see Figure 1c).
- (iv) Consider that an ordering is given by a permutation  $\sigma$ , vertices are inserted in the order  $v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}$ . We build incrementally a spanning tree  $T_i$  for  $S_i = \{p_{\sigma(j)} \in S, i \leq j\}$  with  $T_1 = \{v_{\sigma(1)}\}$ ,  $T_i = T_{i-1} \cup \{v_{\sigma(i)}v_{\sigma(j)}\}$  and a fixed  $k$ , with  $1 \leq k < n$ , such that  $v_{\sigma(i)}v_{\sigma(j)}$  has the shortest length for any  $\max(1, i-k) \leq j < i$ . This tree is called the *Euclidean minimum  $k$ -insertion tree*, and will be denoted by  $EMIT_k(S)$  (see Figure 1e); when  $k = n - 1$ , we will write  $EMIT(S)$  (see Figure 1d).  $|EMIT(S)|$  depends on  $\sigma$  and for some permutations it coincides with  $|EMST(S)|$ .

It is noteworthy that both the combinatorics of  $EMST(S)$  and of  $EMIT(S)$  are invariant to  $\alpha$  (since  $f(\lambda) = \lambda^\alpha$ , is monotonically increasing for positive  $\alpha$  and  $\lambda$ ). What changes is the sum of the weights associated with the edges. Also,  $EMST$  assumes that the position of all the points in  $S$  is available (static) whereas  $EMIT$  clearly does not (dynamic).

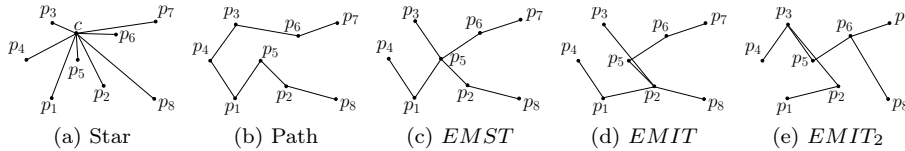


Figure 1: **Trees embedded in the plane.**

Steele proves [4] that if  $p_i$  are i.i.d. random variables with compact support, then  $|EMST(S)|_\alpha = O(n^{1-\alpha/d})$  with probability 1. For the extreme case of  $\alpha = d$ , Aldous and Steele [1] show that  $|EMST(S)|_d = O(1)$  if the variables

above are evenly distributed in the unit cube. Without any dependence on probabilistic hypotheses, Steele proves [6] that the complexity of  $|EMST(S)|$  is bounded by  $O(n^{1-1/d})$  in the worst case. Finally, the asymptotic length of  $|EMIT(S)|$  is shown to be the same as the one of  $|EMST(S)|$  [5]. In other words,  $|EMIT(S)| = O(n^{1-1/d})$ . This result is surprising because it means that *a priori* knowledge of  $S$  does not affect the asymptotic length of trees following the greedy strategy of an *EMST*. This fact has application in the dynamic point location problem [2].

This work is presented as follow: First, in Section 2, we extend the result of Steele [5] stating that  $|EMIT|$  and  $|EMST|$  have the same asymptotical behavior to the case of  $|EMIT|_\alpha$  and  $|EMST|_\alpha$  for  $0 < \alpha < d$  as well. Second, in Section 3, we obtain the expected weighted-length of some stars of interest inside the unit ball. Then in Section 4 we obtain some ratios between the expected weighted-length of such stars and random paths in the unit ball. Finally, in Section 5 we obtain bounds on the expected weighted-length of  $EMIT_k$ .

## 2 Weighted Euclidean Minimum Insertion Tree

We extend the result of Steele [5] for  $|EMIT|_\alpha$  (and consequently  $|EMST|_\alpha$ ) with the following theorem:

**Theorem 1.** *Let  $S$  be a sequence of  $n$  points in  $[0, 1]^d$ , then  $|EMST(S)|_\alpha \leq |EMIT(S)|_\alpha \leq \gamma_{d,\alpha} n^{1-\alpha/d}$ , with  $d \geq 2$  and  $0 < \alpha < d$ . Where,  $\gamma_{d,\alpha} = 1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}$ .*

The proof of Theorem 1 follows exactly the same line as Steele [5], and starts with the two lemmas below. Given a fixed sequence  $S = \{p_1, p_2, \dots, p_n\}$  of points in  $\mathbb{R}^d$ , then we can build a spanning tree for  $S$  by sequentially joining  $x_i$  to the tree formed by  $\{p_1, p_2, \dots, p_{i-1}\}$  for  $1 < i \leq n$ . Let  $w_i \in \mathbb{R}$  be defined as follow:

$$w_i = \min_{1 \leq j < i} |p_i - p_j|^\alpha, \quad (1)$$

then  $w_i$  is the minimal cost of joining  $p_i$  to a vertex of a spanning tree of  $\{p_1, p_2, \dots, p_{i-1}\}$ . Now, we have that  $|EMIT(S)|_\alpha = \sum_{1 < i \leq n} w_i$ .

**Lemma 2.** *If  $\{p_1, p_2, \dots, p_n\} \subset [0, 1]^d$  and  $w_i = \min_{1 \leq j < i} |p_i - p_j|^\alpha$ , for  $1 < i \leq n$ ,  $d \geq 2$  and  $0 < \alpha < d$ , then for any  $0 < \lambda < \infty$  we have*

$$\sum_{\lambda \leq w_i < 2^\alpha \lambda} w_i^{d/\alpha} \leq 8^d d^{d/2}. \quad (2)$$

*Proof.* Let  $C = \{i : \lambda \leq w_i < 2^\alpha \lambda\}$  and for each  $i \in C$  let  $B_i$  be a ball of radius  $r_i = \frac{1}{4} w_i^{1/\alpha}$  with center  $p_i$ . We will argue by contradiction that  $B_i \cap B_j = \emptyset$  for all  $i < j$ . If  $B_i \cap B_j \neq \emptyset$ , then the bounds  $r_i \leq 2^\alpha \lambda$  and  $r_j \leq 2^\alpha \lambda$  gives us

$$|p_i - p_j| \leq \frac{1}{4} (w_i^{1/\alpha} + w_j^{1/\alpha}) < \lambda^{1/\alpha}. \quad (3)$$

But, by definition of  $w_j$  we have  $|p_i - p_j|^\alpha \geq w_j$  for all  $i < j$ , which implies  $|p_i - p_j| \geq w_j^{1/\alpha}$  for all  $i < j$ ; and, by the lower bound on the summands in Eq.(2)

we have  $\lambda \leq w_j$ , which means  $\lambda^{1/\alpha} \leq w_j^{1/\alpha}$ , so we also see  $|p_i - p_j| \geq \lambda^{1/\alpha}$ . Since  $|p_i - p_j| \geq \lambda^{1/\alpha}$  contradicts Eq.(3), we have  $B_i \cap B_j = \emptyset$ .

Now, since all of the balls  $B_i$  are disjoint and contained in a sphere with radius  $2\sqrt{d}$ , the sum of their volumes is bounded by the volume of the sphere of radius  $2\sqrt{d}$ . Thus, if  $\omega_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ , we have the bound  $\sum_{i \in C} \omega_d w_i^{d/\alpha} 4^{-d} \leq \omega_d 2^d d^{d/2}$  from which Eq.(2) follows.  $\square$

**Lemma 3.** *Let  $\Psi$  be a positive and non-increasing function on the interval  $(0, \sqrt{d}]$ , then for any  $0 < a < b \leq \sqrt{d}$ , with  $d \geq 2$  and  $0 < \alpha < d$ ,*

$$\sum_{a \leq w_i \leq b} w_i^{(d/\alpha)+1} \Psi(w_i) \leq \frac{2^\alpha}{2^\alpha - 1} \cdot 8^d d^{d/2} \int_{a/2^\alpha}^b \Psi(\lambda) d\lambda. \quad (4)$$

*Proof.* By Lemma 2 we have for any  $0 < \lambda < \infty$ ,

$$\sum_{a \leq w_i < b} w_i^{d/\alpha} I(\lambda \leq w_i < 2^\alpha \lambda) \leq 8^d d^{d/2},$$

where

$$I(\lambda \leq w_i < 2^\alpha \lambda) = I\left(\frac{1}{2^\alpha} w_i \leq \lambda < w_i\right)$$

is the indicator function. If we multiply by  $\Psi(\lambda)$  and integrate over  $[\frac{1}{2^\alpha} a, b]$ , we find

$$\sum_{a \leq w_i \leq b} w_i^{(d/\alpha)} \int_{w_i/2^\alpha}^{w_i} \Psi(\lambda) d\lambda \leq 8^d d^{d/2} \int_{a/2^\alpha}^b \Psi(\lambda) d\lambda. \quad (5)$$

Since  $\Psi$  is non-increasing, the integrand on the left-hand side of Eq.(5) is bounded from below by  $\Psi(w_i)$ , so  $\Psi(w_i) w_i (1 - \frac{1}{2^\alpha}) \leq \int_{w_i/2^\alpha}^{w_i} \Psi(\lambda) d\lambda$ , and Eq.(4) follows from Eq(5).  $\square$

*Proof of Theorem 1.* Divide the set  $\{w_2, w_3, \dots, w_n\}$  in two sets  $R_1 = \{w_i : w_i \leq n^{-\alpha/d}\}$  and  $R_2 = \{w_i : w_i > n^{-\alpha/d}\}$ . We have the trivial bound  $\sum_{w_i \in R_1} w_i = \sum_{w_i \leq n^{-\alpha/d}} w_i \leq n \cdot n^{-\alpha/d} = n^{1-\alpha/d}$ . Now, let  $[a, b] = [n^{-\alpha/d}, \sqrt{d}]$ ,  $\Psi(\lambda) = \lambda^{-d/\alpha}$  in Eq.(4), then we have:

$$\sum_{w_i \geq n^{-\alpha/d}} w_i \leq \frac{2^\alpha}{2^\alpha - 1} \cdot 8^d d^{d/2} (d/\alpha - 1)^{-1} \cdot \left(2^{d-\alpha} n^{1-\alpha/d} - d^{-(d/\alpha-1)/2}\right).$$

And hence,  $\sum_{w_i \in R_2} w_i \leq \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)} \cdot n^{1-\alpha/d}$ . Now, we have that

$$\sum_{i=2}^n w_i = \sum_{w_i \in R_1} w_i + \sum_{w_i \in R_2} w_i \leq \left(1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}\right) \cdot n^{1-\alpha/d},$$

from which Theorem 1 follows. The constant is  $\gamma_{d,\alpha} = 1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}$ .  $\square$



### 3 Two Stars Embedded in $\mathbb{R}^d$

Assume a set of points  $S = \{p_i, 1 \leq i \leq n\}$ , evenly distributed inside the unit ball  $\mathcal{B}$ . Let  $c$  be a point in  $\mathcal{B}$ , and  $E(|cp|^\alpha, p \in \mathcal{B})$  be the expected value of the distance between  $c$  and a point inside the unit ball to the power of  $\alpha$ , with  $\alpha > 0$ , as  $n \rightarrow \infty$ . The following theorem distinguish two stars of particular interest.

**Theorem 4.** *The shortest and largest expected weighted-length stars inside the ball are respectively: the star centered at  $O$ , the center of the ball; and the star centered at  $\Omega$ , a point on the boundary of the ball, denoted by  $\mathcal{H}$ .*

We have that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{|p_i - c|^\alpha}{n} = E(|cp|^\alpha, p \in \mathcal{B}).$$

In other words,  $\lim_{n \rightarrow \infty} E(|cp|^\alpha, p \in \mathcal{B})$  is the weighted-length of an edge of the star centered in  $c$  as  $n \rightarrow \infty$ . Now, let us turn the attention to the so-called *weighted Steiner star* of  $S$ , the star with the smallest weighted-length we can imagine of  $S$ . Consider that the weighted Steiner star of the first  $n$  points is centered at  $c_n^*$ , then we have

$$\sum_{i=1}^n |p_i - c_n^*|^\alpha \leq \sum_{i=1}^n |p_i - c|^\alpha, \quad (6)$$

for any  $n > 0$  and  $c \in \mathcal{B}$ . Let  $A_n(c) = \sum_{i=1}^n |p_i - c|^\alpha / n$ . The proof of Theorem 4 is divided into several lemmas. First, we show that  $A_n(c_n^*)$  converges. Then we show that there is a point  $\bar{c} \in \mathcal{B}$  such that  $A_n(\bar{c})$  and  $A_n(c_n^*)$  converge to the same limit. Finally, we argue that  $\bar{c}$  must be the center of the ball, the point  $O$ .

**Lemma 5.** *Let  $A_n(c) = \sum_{i=1}^n |p_i - c|^\alpha / n$  and  $c_n^*$  be the center of the weighted Steiner star of the first  $n$  points  $A_n(c_n^*)$ . Then  $A_n(c_n^*)$  converges.*

*Proof.* As points are evenly distributed in  $\mathcal{B}$ ,  $c_n^*$  is a non-constant sequence with probability 1, and thus we can assume that  $0 < A_n(c_n^*) < 2^\alpha$  for any  $n > 1$  and  $d \geq 1$ . We have from Eq.(6)

$$\begin{aligned} A_{n+1}(c_{n+1}^*) &\leq A_{n+1}(c_n^*) = \sum_{i=1}^n \frac{|p_i - c_n^*|^\alpha}{n+1} + \frac{|p_{n+1} - c_n^*|^\alpha}{n+1} \\ &\leq \sum_{i=1}^n \frac{|p_i - c_n^*|^\alpha}{n} + \frac{|p_{n+1} - c_n^*|^\alpha}{n}. \end{aligned}$$

But  $|p_{n+1} - c_n^*| \leq 2$ , and thus  $A_{n+1}(c_{n+1}^*) \leq A_n(c_n^*) + 2^\alpha/n$ , which means that  $\lim_{n \rightarrow \infty} \frac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} \leq 1$ . Analogously, we have, from the fact that  $|p_{n+1} - c_{n+1}^*| \leq 2$  and Eq.(6),  $A_n(c_n^*) \leq \left(\frac{n+1}{n}\right) A_{n+1}(c_{n+1}^*) - \frac{2^\alpha}{n}$ . This means that  $\lim_{n \rightarrow \infty} \frac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} \geq 1$ , and thus  $\lim_{n \rightarrow \infty} \frac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} = 1$ . As  $0 < A_n(c_n^*) < 2^\alpha$  for any  $n > 1$  and  $d \geq 1$ ,  $A_n(c_n^*)$  converges.  $\square$

Now, we need the following easy lemma.

**Lemma 6.** Assume  $a, b \in \mathbb{R}$ ,  $0 \leq a, b \leq 2$ , for each  $\alpha > 0$  there is a function  $f_\alpha(b)$  such that  $(a+b)^\alpha \leq a^\alpha + f_\alpha(b)$  and  $\lim_{b \rightarrow 0} f_\alpha(b) = 0$ .

*Proof.* For  $\alpha \leq 1$ , we have  $f_\alpha(b) = b^\alpha$ , as  $(a+b)^\alpha \leq a^\alpha + b^\alpha$ .  
For  $\alpha \geq 2$  and  $\alpha \in \mathbb{N}$ , we have  $f_\alpha(b) = 2^{2\alpha-1}b$ , as

$$(a+b)^\alpha = a^\alpha + b \sum_{i=1}^{\alpha} \binom{\alpha}{i} a^{\alpha-i} b^{i-1} \leq a^\alpha + 2^{2\alpha-1}b.$$

Let,  $\{x\}$  signify  $x - \lfloor x \rfloor$ , then for  $\alpha > 1$  and  $\alpha \notin \mathbb{N}$ , we have  $f_\alpha(b) = 2^{2\lfloor \alpha \rfloor} b^{\{\alpha\}} + 2^{2\alpha}b$ , as

$$\begin{aligned} (a+b)^\alpha &= (a+b)^{\lfloor \alpha \rfloor} (a+b)^{\{\alpha\}} \\ &\leq \left( a^{\lfloor \alpha \rfloor} + 2^{2\lfloor \alpha \rfloor-1}b \right) \left( a^{\{\alpha\}} + b^{\{\alpha\}} \right) \\ &\leq a^\alpha + 2^{2\lfloor \alpha \rfloor} b^{\{\alpha\}} + 2^{2\alpha}b. \end{aligned}$$

□

**Lemma 7.** There is a point  $\bar{c} \in \mathcal{B}$  such that  $A_n(\bar{c})$  and  $A_n(c_n^*)$  converge to the same limit.

*Proof.* If a topological space  $X$  is compact, then every infinite subset of  $X$  has an accumulation point. Assume  $\bar{c}$  is an accumulation point of the sequence  $\{c_i^*\}_{i=1,2,\dots,\infty}$ , then we have a subsequence of indices  $\zeta : \mathbb{N}^* \rightarrow \mathbb{N}^*$  such that  $\{c_{\zeta(i)}^*\}_{i=1,2,\dots,\infty}$  converges to  $\bar{c}$ . Because of the triangulation inequality and by direct application of Lemma 6, for any  $i > 0$ , we have that

$$A_{\zeta(i)}(\bar{c}) \leq A_{\zeta(i)}(c_{\zeta(i)}^*) + f_\alpha(|\bar{c}c_{\zeta(i)}^*|).$$

As  $|\bar{c}c_{\zeta(i)}^*|$  converges to 0, then  $f_\alpha(|\bar{c}c_{\zeta(i)}^*|)$  also converges to 0 (see Lemma 6). And thus  $\lim_{i \rightarrow \infty} A_{\zeta(i)}(\bar{c}) = \lim_{i \rightarrow \infty} A_{\zeta(i)}(c_{\zeta(i)}^*)$ . Therefore, as  $A_n(c_n^*)$  converges,  $\lim_{i \rightarrow \infty} A_{\zeta(i)}(c_{\zeta(i)}^*) = \lim_{n \rightarrow \infty} A_n(c_n^*)$  and  $\lim_{n \rightarrow \infty} A_n(\bar{c}) = \lim_{i \rightarrow \infty} A_{\zeta(i)}(\bar{c}) = \lim_{n \rightarrow \infty} A_n(c_n^*)$ . □

*Proof of Theorem 4.* By symmetry,  $\lim_{n \rightarrow \infty} A_n(O) \leq \lim_{n \rightarrow \infty} A_n(c)$  for any  $c \in \mathcal{B}$ , and from Eq.(6) and Lemma 7, we have that  $\lim_{n \rightarrow \infty} A_n(O) \geq \lim_{n \rightarrow \infty} A_n(c_n^*) = \lim_{n \rightarrow \infty} A_n(\bar{c}) \geq \lim_{n \rightarrow \infty} A_n(O)$ . Therefore, at the limit, we have that the length of the weighted Steiner star is equivalent to the length of the star centered at  $O$ .

With analogous arguments, we have that the largest weighted-star and a star centered at the boundary of the ball have equivalent length. □

Denote the shortest and largest expected weighted-length stars inside the ball by  $\mathcal{S}$  and  $\mathcal{H}$  respectively. Let  $E(|Op|^\alpha, p \in \mathcal{B})$  and  $E(|\Omega p|^\alpha, p \in \mathcal{B})$  be the expected value of an edge of  $\mathcal{S}$  and  $\mathcal{H}$  respectively, then as  $n \rightarrow \infty$  the size of  $\mathcal{S}$  and  $\mathcal{H}$  are given accordingly by  $n \cdot E(|Op|^\alpha, p \in \mathcal{B})$  and  $n \cdot E(|\Omega p|^\alpha, p \in \mathcal{B})$ .

We analyze in the sequel the values of  $E(|Op|^\alpha, p \in \mathcal{B})$  and  $E(|\Omega p|^\alpha, p \in \mathcal{B})$ .

**Theorem 8.** *When points are uniformly i.i.d in a ball, the expected size  $E(|Op|^\alpha, p \in \mathcal{B})$  of and edge of the star centered at the center of the unit ball, with positive  $\alpha$ , is given by:*

$$\left( \frac{d}{d + \alpha} \right).$$

*Proof.* Let  $\mathcal{B}_l$  be a ball with radius  $l$  centered at the origin, we have

$$\begin{aligned} E(|Op|^\alpha, p \in \mathcal{B}) &= \int_0^1 l^\alpha \text{Prob}(p \in \mathcal{B}_{l+dl} \setminus \mathcal{B}_l) = \int_0^1 l^\alpha \frac{dV_d(l)/l}{V_d(1)} dl \\ &= \int_0^1 dl^{d-1+\alpha} = \frac{d}{d + \alpha}, \end{aligned}$$

where  $V_d(l)$  is the volume of a ball of radius  $l$  (and  $dV_d(l)/l$  is its area).  $\square$

**Theorem 9.** *When points are uniformly i.i.d in a ball, the expected size  $E(|\Omega p|^\alpha, p \in \mathcal{B})$  of and edge of the star centered at the boundary of the unit ball, with positive  $\alpha$ , is given by:*

$$2^{d+\alpha} \left( \frac{2d + \alpha}{2d + 2\alpha} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)},$$

where  $B(x, y) = \int_0^1 \lambda^{x-1} (1 - \lambda)^{y-1} d\lambda$  is the so-called Beta function.

The computation of the average is more involved than in Theorem 8, and we split the computation into several lemmas.

**Lemma 10.** *Consider the spherical cap  $\mathcal{H}_h$  formed by crossing a ball  $\mathcal{B}_R$  with radius  $R$  centered at the origin, with the plane  $x = R - h$ . Denote  $h$  the height of the cap. The volume of  $\mathcal{H}_h$  is the volume of the intersection between the half-space  $x \geq R - h$  and  $\mathcal{B}_R$ . This volume is given by:*

$$R^d \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \int_0^{\arccos(\frac{R-h}{R})} \sin^d(\lambda) d\lambda. \quad (7)$$

*Proof.* The volume  $V_d(r)$  of a ball with radius  $r$  in dimension  $d$  is given by  $r^d \cdot \pi^{\frac{d}{2}} / \Gamma(1 + \frac{d}{2})$ . Each cross-section  $x = R - h + \delta$ ,  $0 \leq \delta \leq h$  is a  $(d-1)$ -dimensional ball. If we integrate all of those balls along the  $x$  axis, we have  $\int_{R-h}^R V_{d-1}(\sqrt{R^2 - t^2}) dt$ . Eq.(7) follows from replacing  $t$  by  $\lambda = R \cos(t)$ .  $\square$

**Lemma 11.** *Let  $\Omega$  be a point on the boundary of the unit ball  $\mathcal{B}_{unit}$ , and  $P_{\mathcal{H}}(l) = \text{Prob}(|\Omega p| \leq l; p \in \mathcal{B}_{unit})$  be the cumulative distribution function of distances between an uniformly distributed random point inside  $\mathcal{B}_{unit}$  and  $\Omega$ , then*

$$P_{\mathcal{H}}(l) = \frac{1}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \left( \int_0^{\arccos(1-l^2/2)} \sin^d(\lambda) d\lambda + l^d \int_0^{\arccos(l/2)} \sin^d(\lambda) d\lambda \right),$$

where  $B(x, y) = \int_0^1 \lambda^{x-1} (1 - \lambda)^{y-1} d\lambda$  is the Beta function.

*Proof.* If we denote  $\mathcal{B}_l$  the ball of radius  $l$  centered in  $\Omega$ , the desired probability is clearly  $\text{volume}(\mathcal{B}_l \cap \mathcal{B}_{unit}) / \text{volume}(\mathcal{B}_{unit})$ .  $\mathcal{B}_l \cap \mathcal{B}_{unit}$  is the union of two spherical caps limited by the plane  $x = 1 - l^2/2$  which can be computed using Lemma 10.  $\square$

*Proof of Theorem 9.* The theorem follows from:

$$\begin{aligned}
E(|\Omega p|^\alpha, p \in \mathcal{B}) &= \int_0^2 l^\alpha P'_\mathcal{H}(l) dl \\
&= \int_0^2 l^\alpha \left( \frac{\frac{1}{2} l^d \left(1 - \frac{l^2}{4}\right)^{\frac{d-1}{2}}}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} + dl^{d-1} \frac{\int_0^{\arccos(l/2)} \sin^d(\lambda) d\lambda}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \right) dl \\
&= \frac{1}{2} \int_0^2 \frac{l^{d+\alpha} \left(1 - \frac{l^2}{4}\right)^{\frac{d-1}{2}} dl}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} + \frac{1}{2} \int_0^2 2dl^{d-1+\alpha} \frac{\int_0^{\arccos(l/2)} \sin^d(\lambda) d\lambda}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} dl
\end{aligned}$$

The right part of the expression above corresponds exactly to the expected value of  $l^\alpha$  where  $l$  is the length of a random segment determined by two evenly distributed points in the unit ball [3, 7]. Its value is given by:

$$\int_0^2 2dl^{d-1+\alpha} \frac{\int_0^{\arccos(l/2)} \sin^d(\lambda) d\lambda}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} dl = 2^{d+\alpha} \left( \frac{d}{d+\alpha} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)}.$$

The left part of the expression can be obtained as follows:

$$\begin{aligned}
\int_0^2 \frac{l^{d+\alpha} \left(1 - \frac{l^2}{4}\right)^{\frac{d-1}{2}} dl}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} &= 2 \int_0^1 \frac{2^{d+\alpha} y^{d+\alpha} (1-y^2)^{\frac{d-1}{2}} dy}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \\
&= \int_0^1 \frac{2^{d+\alpha} z^{\frac{d}{2} + \frac{\alpha}{2} - \frac{1}{2}} (1-z)^{\frac{d-1}{2}} dz}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \\
&= 2^{d+\alpha} \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)}.
\end{aligned}$$

Finally, we have

$$E(|\Omega p|^\alpha, p \in \mathcal{B}) = 2^{d+\alpha} \left( \frac{2d+\alpha}{2d+2\alpha} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)}.$$

□

## 4 Some Ratios

We may ask now what is the value of the ratio  $\rho(d, \alpha)$  between  $E(|\Omega p|^\alpha, p \in \mathcal{B})$  and  $E(|Op|^\alpha, p \in \mathcal{B})$ . It is an easy exercise to verify that  $\rho(1, \alpha) = 2^\alpha$ . In Corollary 12, we compute  $\lim_{d \rightarrow \infty} \rho(d, \alpha)$ .

**Corollary 12.** *The ratio  $\rho(d, \alpha) = E(|\Omega p|^\alpha, p \in \mathcal{B}) / E(|Op|^\alpha, p \in \mathcal{B})$  when  $d \rightarrow \infty$  is given by  $2^{\alpha/2}$ .*

*Proof.* Computing  $\rho(d, \alpha)$  with Theorems 8 and 9 gives:

$$\rho(d, \alpha) = 2^{d+\alpha} \left( \frac{2d+\alpha}{2d} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \quad (8)$$

Using the Stirling's identities:

$$B(a, b) \sim \sqrt{2\pi} \frac{a^{a-\frac{1}{2}} b^{b-\frac{1}{2}}}{(a+b)^{a+b-\frac{1}{2}}}, a, b \gg 0, \quad (9)$$

$$B(a, b) \sim \Gamma(b) a^{-b}, a \gg b > 0, \quad (10)$$

we have:

$$\begin{aligned} \lim_{d \rightarrow \infty} \rho(d, \alpha) &= \lim_{d \rightarrow \infty} \left\{ 2^{d+\alpha} \left( \frac{2d+\alpha}{2d} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}, \frac{d}{2} + \frac{1}{2}\right)}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \right\} \\ &= \lim_{d \rightarrow \infty} \left\{ \frac{2^{d+\alpha} \sqrt{2\pi} \left(\frac{d}{2} + \frac{1}{2}\right)^{\frac{d}{2}} \left(\frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)^{\frac{d}{2} + \frac{\alpha}{2}}}{\sqrt{\pi} \left(d + 1 + \frac{\alpha}{2}\right)^{d + \frac{1}{2} + \frac{\alpha}{2}} \left(\frac{d}{2} + \frac{1}{2}\right)^{-\frac{1}{2}}} \right\} \\ &= 2^{\alpha/2} \cdot \lim_{d \rightarrow \infty} \left\{ \frac{(d+1)^{\frac{d+1}{2}} (d+1+\alpha)^{\frac{d+\alpha}{2}}}{\left(d + 1 + \frac{\alpha}{2}\right)^{\frac{d+1}{2}} \left(d + 1 + \frac{\alpha}{2}\right)^{\frac{d+\alpha}{2}}} \right\} \\ &= 2^{\alpha/2} \cdot e^{-\alpha/4} \cdot e^{\alpha/4} = 2^{\alpha/2} \end{aligned}$$

□

If we consider a tree which is a random path in the unit ball, then the average size of its edge is given by the expected weighted-length of a random segment determined by two evenly distributed points in the unit ball. This is given by:

$$2^{d+\alpha} \left( \frac{d}{d+\alpha} \right) \frac{B\left(\frac{d}{2} + \frac{1}{2}, \frac{d}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)}{B\left(\frac{d}{2} + \frac{1}{2}, \frac{1}{2}\right)} \quad (11)$$

The reader may refer to Tu and Fischbach [7] for a proof.

From Theorem 8, Corollary 12 and Eq.(11) we obtain the following corollary:

**Corollary 13.** *The ratio between the weighted-length of a random path in the unit ball and the weighed-length of a star centered at the center of the unit ball is  $\frac{2^\alpha}{1+\alpha/2}$  when  $d = 1$  and  $2^{\alpha/2}$  when  $d \rightarrow \infty$ .*

From Theorem 9 and Eq.(11) we obtain the following corollary:

**Corollary 14.** *The ratio between the weighted-length of a star centered at the boundary of the unit ball and the weighted-length of a random path in the unit ball is given by  $\frac{2d+\alpha}{2d}$ .*

## 5 Weighted Euclidean Minimum $k$ -Insertion Tree for Random Points

Now, we will compute a bound on the expected weighted-length of an edge of  $|EMIT_k|_\alpha$  for points evenly distributed in the unit ball.

**Theorem 15.** *When points are uniformly i.i.d in a ball, the expected length  $E(\text{length})$  of an edge of  $|EMIT_k|_\alpha$ , with positive  $\alpha$ , verifies:*

$$\left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right) \leq E(\text{length}) \leq 2^\alpha \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right), \quad (12)$$

where  $B(x, y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda$  is the Beta function.

First, we will evaluate the weighted-distance between the origin and the closest amongst  $k$  points  $\{p_1, p_2, \dots, p_k\}$  evenly distributed in the unit ball. This provides the lower-bound. Then, we will find an upper-bound on the weighted-distance between any point inside the ball and the closest amongst  $k$  points  $\{p_1, p_2, \dots, p_k\}$  evenly distributed in the unit ball.

**Lemma 16.** *Let  $c$  be a point inside the unit ball,  $\text{Prob}(|cp| \leq l) = P_c(l)$  be the probability that the distance between a point  $p \in \mathcal{B}$  and  $c$  is less or equal to  $l$ , and  $P_{c,k}(l) = \text{Prob}(\min(|cp_j|)_{1 \leq j \leq k} \leq l)$  be the cumulative distribution function of the minimum distance among  $k$  points following a uniformly i.i.d inside the unit ball and  $c$ , then*

$$P_{c,k}(l) = 1 - (1 - P_c(l))^k.$$

*Proof.*

$$\begin{aligned} P_{c,k}(l) &= \text{Prob}(\min(|cp_j|)_{1 \leq j \leq k} \leq l) \\ &= 1 - \text{Prob}(|cp_j| > l, 1 \leq j \leq k) \\ &= 1 - \text{Prob}(|cp_1| > l)^k \\ &= 1 - (1 - P_c(l))^k. \end{aligned}$$

□

A direct consequence of Lemma 16 is the following corollary.

**Corollary 17.** *Let  $P_{\mathcal{B},k}(l) = \text{Prob}(\min(|Op_j|)_{1 \leq j \leq k} \leq l)$  be the cumulative distribution function of the minimum distance among  $k$  points following a uniformly i.i.d inside the unit ball, and the center of the unit ball, then*

$$P_{\mathcal{B},k}(l) = 1 - (1 - l^d)^k.$$

**Lemma 18.** *The expected value  $E(\min(|Op_j|^\alpha)_{1 \leq j \leq k})$  of the minimum weighted-distance among  $k$  points following a uniformly i.i.d inside the unit ball and the center of the unit ball, with positive  $\alpha$ , is given by*

$$E(\min(|Op_j|^\alpha)_{1 \leq j \leq k}) = \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right).$$

*Proof.* Using Corollary 17, we have:

$$\begin{aligned} E(\min(|Op_j|^\alpha)_{1 \leq j \leq k}) &= \int_0^1 l^\alpha P'_{\mathcal{B},k}(l) dl = kd \int_0^1 l^{d-1+\alpha} (1 - l^d)^{k-1} dl \\ &= k \int_0^1 \lambda^{\alpha/d} (1 - \lambda)^{k-1} d\lambda = kB\left(k, 1 + \frac{\alpha}{d}\right) \\ &= \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right). \end{aligned}$$

□

Now, we shall obtain the upper-bound, which is more involved. First we will obtain a general expression for the expected value of  $\min(|cp_j|^\alpha)_{1 \leq j \leq k}$ . Assume  $\delta(c) = 1 + |Oc|$  in what follows.

**Lemma 19.** *The expected value  $E(\min(|cp_j|^\alpha)_{1 \leq j \leq k})$  of the minimum weighted-distance among  $k$  points following a uniformly i.i.d inside the unit ball and  $c$ , with positive  $\alpha$ , is given by*

$$E(\min(|cp_j|^\alpha)_{1 \leq j \leq k}) = \int_0^{\delta(c)} \alpha l^{\alpha-1} (1 - P_c(l))^k dl.$$

*Proof.* As  $P_c(0) = 0$  and  $P_c(\delta(c)) = 1$ , integration by parts gives us the following identity:

$$\int_0^{\delta(c)} l^\alpha P'_c(l) P_c^{i-1}(l) dl = \frac{\delta(c)^\alpha - \int_0^{\delta(c)} \alpha l^{\alpha-1} P_c^i(l) dl}{i}, i > 0. \quad (13)$$

From Lemma 16, we also have the following expression for  $E(\min(|cp_j|^\alpha)_{1 \leq j \leq k})$ :

$$\begin{aligned} E(\min(|cp_j|^\alpha)_{1 \leq j \leq k}) &= \int_0^{\delta(c)} k l^\alpha P'_c(l) (1 - P_c(l))^{k-1} dl \\ &= \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \int_0^{\delta(c)} k l^\alpha P'_c(l) P_c^i(l) dl. \end{aligned} \quad (14)$$

Replacing Eq.(13) in Eq.(14) leads to:

$$\begin{aligned} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \int_0^{\delta(c)} k l^\alpha P'_c(l) P_c^i(l) dl &= \sum_{i=0}^k (-1)^i \binom{k}{i} \int_0^{\delta(c)} \alpha l^{\alpha-1} P_c^i(l) dl \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1} \sum_{i=0}^k (-1)^i \binom{k}{i} P_c^i(l) dl \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1} (1 - P_c(l))^k dl. \end{aligned}$$

□

*Proof of Theorem 15.* Lemma 18 gives us the lower-bound in Theorem 15. Now, if we take a function  $\Psi(l)$  such that  $\Psi(l) \leq P_c(l)$  for  $0 \leq l \leq \delta(c)$ , it upper-bounds the integral in Lemma 19. Take  $\Psi(l) = (l/\delta(c))^d$ , then we have:

$$\begin{aligned} E(\min(|cp_j|^\alpha)_{1 \leq j \leq k}) &= \int_0^{\delta(c)} \alpha l^{\alpha-1} (1 - P_c(l))^k dl \\ &\leq \int_0^{\delta(c)} \alpha l^{\alpha-1} (1 - \Psi(l))^k dl \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1} \left(1 - \frac{l^d}{\delta(c)^d}\right)^k dl \\ &= \int_0^1 \alpha \delta(c)^\alpha \lambda^{\alpha-1} (1 - \lambda^d)^k d\lambda \\ &= \delta(c)^\alpha \left(\frac{\alpha}{d}\right) \int_0^1 y^{(\alpha/d)-1} (1 - y)^k dy \\ &= \delta(c)^\alpha \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right). \end{aligned}$$

And thus,  $\delta(c) = 2$  ( $c$  on the boundary) maximizes the value above. This completes the proof.  $\square$

**Remark:** The results obtained here for evenly distributed points in the unit ball (expected values, ratios, bounds) so far are valid for any positive  $\alpha$ . Then, we have that the expected weighted-length of the  $i$ -th step  $s_i$  of  $|EMIT|_d$  in the unit ball for evenly distributed point is

$$\frac{1}{i+1} = B(i+1, 1) \leq s_i \leq 2^d B(i+1, 1) = \frac{2^d}{i+1}.$$

Evaluating for  $n$  steps leads to

$$\Omega(\log n) = \sum_{i=2}^{n+1} \frac{1}{i} \leq |EMIT|_d \leq 2^d \sum_{i=2}^{n+1} \frac{1}{i} = O(\log n).$$

Unlike  $|EMST|_d$  of points evenly distributed inside the unit cube, which converges to a constant as  $n \rightarrow \infty$  [1], the expected value of  $|EMIT|_d$  of points evenly distributed inside the unit ball is  $\Theta(\log n)$ . With the same argument, for  $k > 0$  we have that the expected  $|EMIT_k|_d = \Theta(\log k + \frac{n}{k})$ .  $\blacksquare$

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