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Asymptotic Transmission Conditions for Steady-State Potential in a High Contrast Medium. A Uniform Variational Formulation for Resistive Thin Layers.

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Asymptotic Transmission Conditions for Steady-State Potential in a High Contrast Medium. A Uniform Variational Formulation for Resistive Thin Layers.

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Abstract: We study the steady-state potential in a high contrast medium with a resistive thin layer. We provide asymptotic expansion of the potential at any order. Transmission conditions at any order and the corresponding variational formulation are given. We prove uniform estimates with respect to the thickness of the layer and with respect to its resistivity. The main insight consists in the uniform variational formulation whatever small the layer conductivity is. Numerical simulations illustrate the theoretical results.

Key-words: Asymptotic analysis, Finite Element Method, Laplace equations

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Conditions de transmission approchées pour des couches minces résistives.

Résumé :

Mots-clés : Analyse Asymptotique, Méthode des Eléments Finis, Equations de Laplace

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1 Introduction

We present transmission conditions for the electro-quasistatic approximation of the Maxwell equations in a high contrast medium with an insulating thin layer. We aim at providing asymptotic expansion at any order, whatever small the conductivity of the layer is. The asymptotic for the thickness tending to zero are very different compared with the soft contrast medium [7, 8], since the effect of the thin layer appears at the zeroth order in the high contrast case. Therefore, unlike the result of Vogelius and Nguyen [13] for small diameter inhomogeneities, which provides a unique asymptotic formulation for both high and soft contrast media, such a result does not exist for thin layers.

1.1 Motivation

The distribution of the steady-state potential in a biological cell is important for bio-electromagnetic investigations. A sufficiently large amplitude of the transmembranar potential (TMP), which is the difference of the electric potentials between both sides of the cell membrane, leads to an increase of the membrane permeability [9, 12]. Molecules such as bleomycin can then diffuse across the plasma membrane. This phenomenon, called electropermeabilization, has been already used in oncology and holds promises in gene therapy [6, 11], justifying precise assessments of the TMP. Since the experimental measurements of the TMP on living cells are limited — mainly due to the membrane thinness, which is a few nanometers thick — a numerical approach is often chosen [9, 10]. However, these computations are confronted with the heterogeneous parameters of the biological cells. Therefore in this paper we derive a rigorous asymptotic analysis to tackle these numerical difficulties.

We consider the three-dimensional model of biological cell given by Schwan [4, 5] for different frequency ranges. This model is studied under the electro-quasistatic approximation¹ of the Maxwell equations. This approximation is usually considered to describe the behavior of a cell submitted to an electric field of frequency smaller than few giga Hertz. Depending on the frequency, the modulus of the complex conductivity of the thin layer is either very small (for the frequency range under 10KHz), or of the same order (for the frequency range between 100KHz and 100MHz) compared with the membrane thickness. *We aim to give a uniform formulation, that is valid whatever the frequency is, and that avoid to mesh the thin layer.* More precisely, we provide uniform approximate transmission conditions, that describe the effect of the layer without meshing it. This uniform estimate over the proposed range of frequency (up to 100MHz) enables us to apply safely the transmission conditions also for time-transient and non-linear problem, which are more relevant for modeling the electropermeabilization process.

1.2 The studied problem

Denote by Ω a bounded domain with a smooth boundary $\partial\Omega$. The domain \mathcal{O}_c^δ is a subdomain of Ω . It is surrounded by a thin layer \mathcal{O}_m^δ with thickness δ such

¹The electro-quasistatic approximation consists of neglecting the curl part of the electric field, which therefore derives from a potential so-called electric potential. This amounts to considering the steady-state potential equations with complex coefficients.

that $\mathcal{O}_m^\delta \cup \mathcal{O}_c^\delta$ is compactly embedded in Ω . We denote by $\mathcal{O}_e = \Omega \setminus \overline{\mathcal{O}_m^\delta \cup \mathcal{O}_c^\delta}$, which is supposed δ -independent. Let σ_c be the inner complex conductivity² of \mathcal{O}_c^δ and we define similarly σ_e and σ_m . We suppose that both imaginary and real parts of σ_e , σ_c , and σ_m are positive. We define $\mathcal{O}_c = \Omega \setminus \overline{\mathcal{O}_e}$ with smooth boundary Γ and

$$\sigma = \begin{cases} \sigma_c, & \text{in } \mathcal{O}_c^\delta, \\ \sigma_m, & \text{in } \mathcal{O}_m^\delta, \\ \sigma_e, & \text{in } \mathcal{O}_e, \end{cases} \quad \text{and } \tilde{\sigma} = \begin{cases} \sigma_c, & \text{in } \mathcal{O}_c, \\ \sigma_e, & \text{in } \mathcal{O}_e. \end{cases}$$

The electro-quasistatic formulation is given by:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & \text{in } \Omega \\ u|_{\partial\Omega} = g, & \text{on } \partial\Omega; \end{cases} \quad (1)$$

g is the electric potential imposed on the boundary of Ω . We suppose that g is as regular as we need. We aim at providing the rigorous asymptotic expansion of the potential u for δ tending to zero and $|\sigma_m| \leq \delta$. The main insight of the paper is to consider the thin layer highly insulating in the sense that

$$\sigma_m = \delta \xi S_m,$$

where $|S_m| = 1$ and $|\xi| \leq 1$. Depending on the frequency of the studied phenomenon, the parameter $|\xi|$ can be of the order 1, δ , or even smaller. For each frequency, it is possible to provide a precise asymptotic expansion by solving specific partial differential equations. However, in this paper we aim at giving a rigorous variational formulation, that allows to approach the steady-state potential accurately and for all the different cases in a same maneer. The method used to derive the effective transmission conditions has been extensively described previously. We refer to [3] for the original paper and to [8, 7] for a more general description. It consists in a suitable change of variables in the thin layer in order to make appear the small parameter in the equations.

Multiply (1) by \bar{u} and integrate by parts. There exists a constant C such that

$$\|u\|_{H^1(\mathcal{O}_e)} \leq C|g|_{H^{1/2}(\partial\Omega)}, \quad \|\nabla u\|_{L^2(\mathcal{O}_e^\delta)} \leq C|g|_{H^{1/2}(\partial\Omega)},$$

and

$$\|\nabla u\|_{L^2(\mathcal{O}_m^\delta)} \leq \frac{C}{\sqrt{\xi\delta}}|g|_{H^{1/2}(\partial\Omega)}.$$

According to the last inequality, the more resistive the thin layer is, the worse the estimate of the electric field ∇u in the membrane is. This estimate makes us think that “the electric field is trapped into the membrane”. Therefore, unlike the case of a soft contrast medium, we can assume that the resistive thin layer has an influence on the zeroth approximation of the potential. We aim at justifying this conjecture by performing a rigorous asymptotic expansion of the potential at any order, that are valid from the resistive case to the very resistive case.

²The complex conductivity σ of a given material is defined by

$$\sigma = i\omega\varepsilon + s,$$

where s and ε denote respectively the (real) conductivity and permittivity of the material, and ω is the frequency of the time-harmonic field.

1.3 Main result

The main result of the present paper can be summarized in the following Theorem.

Theorem 1.1. *Let g belong to $H^{1+n+2}(\partial\Omega)$, for $n \in \mathbb{N}$. For all $0 \leq k \leq n$, there exists (g_D^k, g_N^k) in $H^{1/2+n+2-k}(\Gamma) \times H^{1/2+n+2-k}(\Gamma)$ and u^k such that:*

$$\begin{cases} \Delta u^k = 0 & \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^k|_{\partial\Omega} = \delta_0^k g, \\ [\tilde{\sigma} \partial u^k]_{\Gamma} = g_N^k & \text{on } \Gamma, \\ S_m \xi [u^k]_{\Gamma} - \sigma_c \partial u_{\Gamma-}^k = \xi g_D^k & \text{on } \Gamma. \end{cases}$$

The function (g_D^k, g_N^k) are defined by induction in equalities (17). Denote by

$$u_{app}^n = \sum_{k=0}^n (-\delta)^k u^k,$$

then there exists a constant C_n independent on δ and ξ such that

$$\|u - u_{app}^n\|_{H^1(\mathcal{O}_e)} \leq C_n \xi \delta^n |g|_{H^{5/2+n}(\partial\Omega)}, \quad (2)$$

and for all $\omega \Subset \mathcal{O}_c$

$$\|u - u_{app}^n\|_{H^1(\omega)} \leq C_n \delta^n |g|_{H^{5/2+n}(\partial\Omega)}, \quad (3)$$

$$\|\nabla(u - u_{app}^n)\|_{L^2(\omega)} \leq C_n \xi \delta^n |g|_{H^{5/2+n}(\partial\Omega)}. \quad (4)$$

Observe that the previous conjecture is justified in the approximate transmission conditions of the jump of u^k across Γ . Actually, even at the zeroth order, the thin membrane has an influence on the potential, since the specific membrane conductivity S_m appears. This is the main insight of this paper: for resistive thin layer, the membrane influence appears at the zeroth order, and the electric field is trapped into the membrane. The variational formulation of the above problem is given by equation (19).

From the electromagnetic point of view, the estimates (2)–(4) are the more relevant since they provide precise estimates on the electric field. Moreover, these estimates are optimal since they are valid even if the complex conductivity of the thin layer dramatically decreases.

In Section 2, we perform the change of variables, that will be useful to derive the asymptotic expansion. We then perform the formal identification of the coefficients in Section 3, and we provide preliminary estimates in Section 4. Section 5 is devoted to the proof of our main result, and in the last section, numerical simulations with finite element method illustrate the theoretical results.

2 Geometry

Let $\mathbf{x}_\Gamma = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathbf{x}_\Gamma)\}$. Denote by n the normal vector³ of Γ outwardly directed to \mathcal{O}_c and define the map Φ by

$$\forall(\mathbf{x}_\Gamma, x_3) \in \Gamma \times \mathbb{R}, \quad \Phi(\mathbf{x}_\Gamma, x_3) = \psi(\mathbf{x}_\Gamma) + x_3 n(\mathbf{x}_\Gamma).$$

The thin layer \mathcal{O}_m^δ is then parameterized by

$$\mathcal{O}_m^\delta = \{\Phi(\mathbf{x}_\Gamma, x_3), \quad (\mathbf{x}_\Gamma, x_3) \in \Gamma \times (-\delta, 0)\}.$$

The Euclidean metric in (\mathbf{x}_Γ, x_3) is given by the 3×3 -matrix $(g_{ij})_{i,j=1,2,3}$ where $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$. According to [1], $g_{33} = 1$, $g_{\alpha,3} = g_{3,\alpha} = 0$ for $\alpha \in \{1, 2\}$ and for $(\alpha, \beta) \in \{1, 2\}^2$

$$g_{\alpha\beta}(\mathbf{x}_\Gamma, x_3) = g_{\alpha\beta}^0(\mathbf{x}_\Gamma) - 2x_3 b_{\alpha\beta}(\mathbf{x}_\Gamma) + x_3^2 c_{\alpha\beta}(\mathbf{x}_\Gamma), \quad (5)$$

where

$$g_{\alpha\beta}^0 = \langle \partial_\alpha \psi, \partial_\beta \psi \rangle, \quad b_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta \psi \rangle, \quad c_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta n \rangle.$$

We denote by (g^{ij}) the inverse matrix of (g_{ij}) , and by g the determinant of (g_{ij}) .

2.1 Laplace operator for functions

The Laplace-Beltrami operator acting on functions for the metric $(g_{ij})_{i,j=1,2,3}$ is given by⁴ [2]

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1,2,3} \partial_i (\sqrt{g} g^{ij} \partial_j). \quad (6)$$

According to (5) and using the compactness of Γ , the functions g_{ij} are analytic on the cylinder $\Gamma \times [-\delta, 0]$, for δ small enough. Define for all $l \in \mathbb{N}$,

$$\begin{cases} a_{ij}^l = \partial_3^l \left(\frac{\partial_i (\sqrt{g} g^{ij})}{\sqrt{g}} \right) \Big|_{x_3=0}, & \text{for } (i, j) \in \{1, 2, 3\}^2, \\ A_{\alpha\beta}^l = \partial_3^l (g^{\alpha\beta}) \Big|_{x_3=0}, & \text{for } (\alpha, \beta) \in \{1, 2\}^2. \end{cases}$$

Observe that the above functions a_{ij}^l and $A_{\alpha\beta}^l$ are smooth functions defined on Γ . Therefore we infer

$$\forall(\mathbf{x}_\Gamma, x_3) \in \Gamma \times (-\delta, 0),$$

$$\Delta_g = \partial_3^2 + \sum_{l \geq 0} \frac{x_3^l}{l!} \left(a_{33}^l \partial_3 + \sum_{\alpha, \beta=1,2} a_{\alpha\beta}^l \partial_\beta + A_{\alpha\beta}^l \partial_\alpha \partial_\beta \right). \quad (7)$$

³The normal vector n is defined by

$$n = \frac{\partial_1 \psi \wedge \partial_2 \psi}{|\partial_1 \psi \wedge \partial_2 \psi|}.$$

⁴ ∂_i denotes the partial derivative with respect to the variable x_i , for $i = 1, 2, 3$.

For all $l \in \mathbb{N}$, we denote by \mathcal{S}_Γ^l the differential operator of order 2 defined on Γ by

$$\mathcal{S}_\Gamma^l = \sum_{\alpha, \beta=1,2} a_{\alpha\beta}^l \partial_\beta + A_{\alpha\beta}^l \partial_\alpha \partial_\beta, \quad (8)$$

and we perform the change of variable $\eta = x_3/\delta$ to obtain the Laplacian in the local $(\mathbf{x}_\Gamma, \eta)$ -coordinates

$$\Delta_g = \frac{1}{\delta^2} \partial_\eta^2 + \frac{1}{\delta} a_{33}^0 \partial_\eta + \sum_{l \geq 0} \delta^l \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta + \mathcal{S}_\Gamma^l \right), \quad \forall (\mathbf{x}_\Gamma, \eta) \in \Gamma \times (-1, 0). \quad (9)$$

The following proposition will be useful in order to derive the asymptotics.

Proposition 2.1. *Let u be a smooth harmonic function in \mathcal{O}_c . We still denote by u the function $u \circ \Phi$ in the tubular neighborhood of Γ . For all $p \in \mathbb{N}$ there exists two surfacic operators on Γ denoted by R_N^p and R_D^p respectively of order $p-1$ and p such that*

$$\partial_3^p u|_{\Gamma^-} = R_N^p(\partial_n u|_{\Gamma^-}) + R_D^p(u|_{\Gamma^-}). \quad (10)$$

The operators R_N^p and R_D^p are defined by the following recurrence formulae:

$$R_N^0 = 0, R_D^0 = \text{Id}, \quad R_N^1 = \text{Id}, R_D^1 = 0,$$

and for all $p \geq 0$

$$R_N^{p+2} = - \sum_{l=0}^p C_p^l \left(a_{33}^l R_N^{p+1-l} + R_N^{p-l} \mathcal{S}_\Gamma^l \right),$$

$$R_D^{p+2} = - \sum_{l=0}^p C_p^l \left(a_{33}^l R_D^{p+1-l} + R_D^{p-l} \mathcal{S}_\Gamma^l \right),$$

where $C_p^l = p!/(l!(p-l)!)$.

Proof. The proof is an application of the Leibniz rule for the differentiation applied to (7). \square

The above proposition leads to the following corollary by applying the Taylor formula with integral remainder term in the variable η .

Corollary 2.2. *Let u be a smooth harmonic function in \mathcal{O}_c , and denote by $\Gamma_{-\delta}$ the boundary of \mathcal{O}_c^δ . There exists a sequence of operators $(T^p)_{p \geq 0}$ of order p such that for all $p \in \mathbb{N}$*

$$u|_{\Gamma_{-\delta}^-} = \sum_{k=0}^p \frac{(-\delta)^k}{k!} (R_N^k(\partial_n u|_{\Gamma^-}) + R_D^k(u|_{\Gamma^-})) + \frac{(-\delta)^{p+1}}{(p+1)!} T^{p+1}(u). \quad (11)$$

$$\partial_n u|_{\Gamma_{-\delta}^-} = \sum_{k=0}^p \frac{(-\delta)^k}{k!} (R_N^{k+1}(\partial_n u|_{\Gamma^-}) + R_D^{k+1}(u|_{\Gamma^-})) + \frac{(-\delta)^{p+1}}{(p+1)!} T^{p+2}(u), \quad (12)$$

where there exists a constant C such that for all p

$$|T^{p+i}(u)|_{H^{1/2-i}(\Gamma)} \leq C \|u\|_{H^{p+2}(\mathcal{O}_c)}, \quad i = 0, 1.$$

3 Formal asymptotics

Define the so-called profile function u_m in $\Gamma \times (-1, 0)$ by

$$\forall (\mathbf{x}_\Gamma, \eta) \in \Gamma \times (-1, 0), \quad u_m(\mathbf{x}_\Gamma, \eta) = u \circ \Phi(\mathbf{x}_\Gamma, \delta\eta).$$

Since $\sigma_m/\delta = \xi S_m$, the transmission conditions inherent to (1) write now:

$$\sigma_c \partial_n u|_{\Gamma_{-\delta}^-} \circ \Phi(\cdot, -\delta) = \xi S_m \partial_\eta u_m|_{\eta=-1}, \quad u|_{\Gamma_{-\delta}^-} \circ \Phi(\cdot, -\delta) = u_m|_{\eta=-1}, \quad (13)$$

$$\sigma_e \partial_n u|_{\Gamma^+} \circ \psi = \xi S_m \partial_\eta u_m|_{\eta=0}, \quad u|_{\Gamma^+} \circ \psi = u_m|_{\eta=0}. \quad (14)$$

To obtain our transmission conditions, we set

$$\begin{cases} u = u^0 - \delta u^1 + \delta^2 u^2 - \dots, & \text{in } \mathcal{O}_c^\delta \cup \mathcal{O}_e, \\ u_m = u_m^0 - \delta u_m^1 + \delta^2 u_m^2 - \dots, & \text{in } \Gamma \times (-1, 0), \end{cases} \quad (15)$$

and we perform the identification of the terms with the same power in δ . We emphasize that the coefficients u^j will be defined in the whole domain Ω even if $\sum_{j=0}^p (-\delta)^j u^j$ approximates u only in $\mathcal{O}_e \cup \mathcal{O}_c^\delta$. Observe that equations (13) are written on $\Gamma_{-\delta}$, while we want to write transmission conditions on Γ . Insert the ansatz (15) into equalities (11)–(12) of Corollary 2.2 to infer

$$\begin{aligned} u|_{\Gamma_{-\delta}^-} &= \sum_{p \geq 0} (-\delta)^p \left(u^p|_{\Gamma^-} + \sum_{l=1}^p \frac{1}{l!} \left(R_N^l (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l (u^{p-l}|_{\Gamma^-}) \right) \right), \\ \partial_n u|_{\Gamma_{-\delta}^-} &= \sum_{p \geq 0} (-\delta)^p \left(\partial_n u^p|_{\Gamma^-} + \sum_{l=1}^p \frac{1}{l!} \left(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} (u^{p-l}|_{\Gamma^-}) \right) \right). \end{aligned}$$

3.1 Recurrence formulae

Let us now identify the terms with the same power in δ . We necessarily have:

$$\Delta u^p = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^0|_{\partial\Omega} = \delta_0^p g,$$

where δ_0^p is the Kronecker symbol equal to 1 if $p = 0$ and 0 otherwise. Write now the equations satisfied by u_m^p in $\Gamma \times (-1, 0)$:

$$\partial_\eta^2 u_m^p = -a_{33}^0 \partial_\eta u_m^{p-1} - \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right). \quad (16)$$

According to transmission conditions (13)–(14), by integrating the previous equality, we infer:

$$\begin{aligned} \xi S_m \partial_n u_m^p|_{\eta=-1} &= \sigma_c \left(\partial_n u^p|_{\Gamma^-} + \sum_{l=1}^p \frac{1}{l!} \left(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} (u^{p-l}|_{\Gamma^-}) \right) \right), \\ &= -\xi S_m \int_{-1}^0 \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) d\eta + \sigma_e \partial_n u^p|_{\Gamma^+}. \end{aligned}$$

This leads to the following Neumann transmission condition for u^p :

$$\begin{aligned} [\tilde{\sigma} \partial_n u^p]_\Gamma &= \sigma_c \sum_{l=1}^p \frac{1}{l!} \left(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} (u^{p-l}|_{\Gamma^-}) \right) \\ &+ \xi S_m \int_{-1}^0 \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) d\eta. \end{aligned}$$

To obtain the Dirichlet transmission condition, we use the Taylor equality

$$\forall \eta \in (-1, 0), \quad u_m^p(\cdot, \eta) = u_m^p|_{\eta=0} + \eta \partial_\eta u_m^p|_{\eta=0} + \int_0^\eta (\eta - s) \partial_\eta^2 u_m^p(\cdot, s) ds.$$

According to (16)–(13)–(14) and applying the above equality with $\eta = -1$, we infer

$$\begin{aligned} \xi S_m [u^p]_\Gamma - \sigma_e \partial_n u^p|_{\Gamma^+} &= \xi S_m \sum_{l=1}^p \frac{1}{l!} \left(R_N^l (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l u^{p-l}|_{\Gamma^-} \right) \\ &+ \xi S_m \int_{-1}^0 (1 + \eta) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) d\eta. \end{aligned}$$

Define u^k and u_m^k identically equal to zero for $k = -2, -1$. Therefore the sequences $(u^p)_{p \geq 0}$ and $(u_m^p)_{p \geq 0}$ are defined by:

$$\Delta u^p = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad u^0|_{\partial\Omega} = \delta_0^p \phi, \quad (17a)$$

$$\begin{aligned} \xi S_m [u^p]_\Gamma - \sigma_e \partial_n u^p|_{\Gamma^+} &= \xi S_m \sum_{l=1}^p \frac{1}{l!} \left(R_N^l (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^l u^{p-l}|_{\Gamma^-} \right) \\ &+ \xi S_m \int_{-1}^0 (1 + \eta) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) d\eta, \end{aligned} \quad (17b)$$

$$\begin{aligned} [\tilde{\sigma} \partial_n u^p]_\Gamma &= \sigma_c \sum_{l=1}^p \frac{1}{l!} \left(R_N^{l+1} (\partial_n u^{p-l}|_{\Gamma^-}) + R_D^{l+1} u^{p-l}|_{\Gamma^-} \right) \\ &+ \xi S_m \int_{-1}^0 \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{\eta^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) d\eta. \end{aligned} \quad (17c)$$

In the cylinder $\Gamma \times (-1, 0)$ we have

$$\begin{aligned} u_m^p &= u^p|_{\Gamma^+} + \eta \frac{\sigma_e}{\xi S_m} \partial_n u^p|_{\Gamma^+} \\ &+ \int_\eta^0 (\eta - s) \left(a_{33}^0 \partial_\eta u_m^{p-1} + \sum_{l=0}^{p-2} \frac{s^l}{l!} \left(\frac{\eta}{l+1} a_{33}^{l+1} \partial_\eta u_m^{p-2-l} + \mathcal{S}_\Gamma^l u_m^{p-2-l} \right) \right) ds. \end{aligned} \quad (17d)$$

4 Preliminary estimates

Let $s \geq 0$. Let g , g_N , and g_D belong respectively to $H^{1/2+s}(\partial\Omega)$, $H^{-1/2+s}(\Gamma)$, and to $H^{1/2+s}(\Gamma)$. Let v satisfy

$$\Delta v = 0, \quad \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \quad v|_{\partial\Omega} = g, \quad (18a)$$

$$[\tilde{\sigma}\partial_n v]_{\Gamma} = g_N, \quad (18b)$$

$$\xi[v]_{\Gamma} - \frac{\sigma_c}{S_m}\partial_n v|_{\Gamma^-} = \xi g_D. \quad (18c)$$

4.1 Variational formulation

Define $PH^1(\Omega)$ the space of function in Ω , which have the H^1 -regularity in \mathcal{O}_e and \mathcal{O}_c :

$$PH^1(\Omega) = \left\{ u : u|_{\mathcal{O}_e} \in H^1(\mathcal{O}_e), \quad u|_{\mathcal{O}_c} \in H^1(\mathcal{O}_c) \right\},$$

and let $PH_0^1(\Omega)$ the space of functions in $PH^1(\Omega)$, which vanish on $\partial\Omega$. The variational formulation of Problem (18) is:

$$\begin{aligned} & \text{find } v \in g + PH_0^1(\Omega) \text{ such that for all } \phi \in PH_0^1(\Omega) \\ & \int_{\Omega} \tilde{\sigma}\nabla v \nabla \phi dx + \xi \int_{\Gamma} S_m[v]_{\Gamma}[\phi]_{\Gamma} ds = \xi \int_{\Gamma} S_m g_D [\phi]_{\Gamma} ds + \int_{\Gamma} g_N \phi|_{\Gamma^+} ds. \end{aligned} \quad (19)$$

Observing that the bilinear form $a(\cdot, \cdot)$ defined by

$$\forall(\phi, \psi) \in [PH^1(\Omega)]^2, \quad a(\phi, \psi) = \int_{\Omega} \tilde{\sigma}\nabla \phi \nabla \psi dx + \xi \int_{\Gamma} S_m [\phi]_{\Gamma} [\psi]_{\Gamma} ds,$$

is an hermitian product on $PH^1(\Omega)$ we infer the following proposition.

Proposition 4.1. *Let $s \geq 0$. Let g , g_N , and g_D belong respectively to $H^{1/2+s}(\partial\Omega)$, $H^{-1/2+s}(\Gamma)$, and to $H^{1/2+s}(\Gamma)$. There exists a unique solution v to problem (18), which belongs to $PH^1(\Omega)$. More precisely, using elliptic regularity*

$$v|_{\mathcal{O}_e} \in H^{1+s}(\mathcal{O}_e), \quad \text{and} \quad v|_{\mathcal{O}_c} \in H^{2+s}(\mathcal{O}_c).$$

4.2 Uniform estimates in ξ for the generic problem

In this section, we aim at providing uniform estimates for ξ tending to zero, for the solution v to (18). Let \mathcal{I} be defined by

$$\begin{cases} \Delta \mathcal{I} = 0, & \text{in } \mathcal{O}_e, \\ \mathcal{I}|_{\partial\Omega} = 0, & \mathcal{I}|_{\Gamma} = 1. \end{cases}$$

Multiply (18) by \mathcal{I} in \mathcal{O}_e and by 1 in \mathcal{O}_c , and integrate by parts twice in \mathcal{O}_e , and once in \mathcal{O}_c . We obtain the following equality:

$$\int_{\Gamma} \sigma_e v|_{\Gamma^+} \partial_n \mathcal{I}|_{\Gamma^+} ds = \int_{\partial\Omega} \sigma_e g \partial_n \mathcal{I} ds + \int_{\Gamma} g_N ds.$$

Using the above equality, and multiplying (18) by \mathcal{I} in \mathcal{O}_e and by 0 in \mathcal{O}_c integration by parts leads to

$$\int_{\Gamma} [v]_{\Gamma} ds = \xi \int_{\Gamma} g_D ds.$$

We define the sequence $(v_e^l, v_c^l)_{l \geq 0}$ by

$$\begin{cases} \Delta v_e^0 = 0, & \text{in } \mathcal{O}_e, \\ v_e^0|_{\partial\Omega} = g, & \sigma_e \partial_n v_e^0|_{\Gamma} = g_N, \end{cases}$$

and

$$v_c^0 = \frac{1}{\text{mes}(\Gamma)} \int_{\Gamma} v_e^0 ds,$$

and for $l \geq 1$

$$\begin{cases} \Delta v_e^l = 0, & \text{in } \mathcal{O}_e, \\ v_e^l|_{\partial\Omega} = 0, & \sigma_e \partial_n v_e^l|_{\Gamma} = S_m(v_e^{k-1} - v_c^{k-1})|_{\Gamma} - S_m \delta_1^l g_D, \end{cases} \quad (20a)$$

and

$$\begin{cases} \Delta v_c^l = 0, & \text{in } \mathcal{O}_c, \\ \sigma_c \partial_n v_c^l|_{\Gamma} = S_m(v_e^{k-1} - v_c^{k-1})|_{\Gamma} - S_m \delta_1^l g_D, \\ \int_{\Gamma} v_c^l ds = \int_{\Gamma} (v_e^l - \delta_1^l g_D) ds. \end{cases} \quad (20b)$$

Proposition 4.2. *For all $k \geq 0$, $(v_e^k, v_c^k) \in H^{1+s+k}(\mathcal{O}_e) \times H^{2+s+k}(\mathcal{O}_c)$, and v_c^0 is constant. Moreover there exists a constant $C_{\mathcal{O}_e, \mathcal{O}_c}$, which only depends on the domain \mathcal{O}_e and \mathcal{O}_c such that for all $k \geq 0$*

$$\begin{aligned} \|v_e^k\|_{H^{1+s}(\mathcal{O}_e)} + \|v_c^k\|_{H^{2+s}(\mathcal{O}_c)} &\leq C_{\mathcal{O}_e, \mathcal{O}_c}^k \left(|g|_{H^{1/2+s}(\partial\Omega)} \right. \\ &\quad \left. + |g_D|_{H^{1/2+s}(\Gamma)} + |g_N|_{H^{-1/2+s}(\Gamma)} \right). \end{aligned} \quad (21)$$

Proof. According to (20) there exists a constant $C_{\mathcal{O}_e, \mathcal{O}_c}$ such that for all $k \geq 1$

$$\|v_e^{k+1}\|_{H^1(\mathcal{O}_e)} + \|v_c^{k+1}\|_{H^1(\mathcal{O}_c)} \leq C_{\mathcal{O}_e, \mathcal{O}_c} (\|v_e^k\|_{H^1(\mathcal{O}_e)} + \|v_c^k\|_{H^1(\mathcal{O}_c)}),$$

hence

$$\begin{aligned} \|v_e^{k+1}\|_{H^1(\mathcal{O}_e)} + \|v_c^{k+1}\|_{H^1(\mathcal{O}_c)} &\leq C_{\mathcal{O}_e, \mathcal{O}_c}^k (\|v_e^1\|_{H^1(\mathcal{O}_e)} + \|v_c^1\|_{H^1(\mathcal{O}_c)}), \\ &\leq C_{\mathcal{O}_e, \mathcal{O}_c}^{k+1} \left(|g|_{H^{1/2}(\partial\Omega)} \right. \\ &\quad \left. + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)} \right). \end{aligned}$$

Then we use the elliptic regularity satisfied by v_e^k and v_c^k to conclude. \square

Proposition 4.3. *For ξ small enough, the series $\sum_{l \geq 0} \xi^l v_e^l$ and $\sum_{l \geq 0} \xi^l v_c^l$ normally converge in $H^{1+s}(\mathcal{O}_e)$ and in $H^{1+s}(\mathcal{O}_c)$ respectively. Moreover, the sums satisfy Problem (18), hence by uniqueness*

$$v|_{\mathcal{O}_e} = \sum_{l \geq 0} \xi^l v_e^l, \quad v|_{\mathcal{O}_c} = \sum_{l \geq 0} \xi^l v_c^l.$$

Particularly, since v_c^0 is constant, we infer that all the derivatives of $v|_{\mathcal{O}_c}$ are of order ξ .

Proof. Choose $\xi \in (0, 1/(2C_{\mathcal{O}_e, \mathcal{O}_c}))$ and Proposition 4.2 leads to Proposition 4.3. \square

4.3 A priori estimates for the asymptotic coefficients

The following proposition is an application of both propositions 4.2–4.3. The proof performed using the induction process is left to the reader.

Proposition 4.4. *Let $k \in \mathbb{N}$ and $s \geq 0$. We suppose that g , g_D and g_N satisfy*

$$g \in H^{1/2+s+k}(\partial\Omega), \quad g_D \in H^{1/2+s+k}(\Gamma), \quad g_N \in H^{-1/2+s+k}(\Gamma).$$

For $l \in \{0, \dots, k\}$ denote u_e^l and u_c^l the respective restrictions of u^l to \mathcal{O}_e and \mathcal{O}_c . The following regularities hold:

$$u_e^l \in H^{1+s+k-l}(\mathcal{O}_e), \quad u_c^l \in H^{2+s+k-l}(\mathcal{O}_c).$$

In the cylinder $\Gamma \times (-1, 0)$, the coefficients u_m^l equals

$$\forall (\mathbf{x}_\Gamma, \eta) \in \Gamma \times (-1, 0), \quad u_m^l(\mathbf{x}_\Gamma, \eta) = \sum_{n=0}^l \eta^n f^n(\mathbf{x}_\Gamma),$$

where $f^n \in H^{1/2+s+k-l}(\Gamma)$ for $n \in 0, \dots, l$. The following estimates hold

$$\begin{aligned} \|u_e^l\|_{H^{1+s+k-l}(\mathcal{O}_e)} &\leq C_{\Omega, k} \left(|g|_{H^{1/2}(\partial\Omega)} + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)} \right), \\ \|u_c^l\|_{H^{1+s+k-l}(\mathcal{O}_c)} &\leq C_{\Omega, k} \left(|g|_{H^{1/2}(\partial\Omega)} + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)} \right), \\ \max_{k=0, \dots, l} \|f^k\|_{H^{1/2+s+k-l}(\Gamma)} &\leq C_{\Omega, k} \left(|g|_{H^{1/2}(\partial\Omega)} + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)} \right). \end{aligned}$$

Moreover, we have the following estimate uniformly in ξ :

$$\|\nabla u_c^l\|_{H^{s+k-l}(\mathcal{O}_c)} \leq C_{\Omega, k} \xi \left(|g|_{H^{1/2}(\partial\Omega)} + |g_D|_{H^{1/2}(\Gamma)} + |g_N|_{H^{-1/2}(\Gamma)} \right).$$

5 Error estimates

For $p \geq 0$, we denote by \mathcal{L}_p the operator on $\Gamma \times (-1, 0)$ of order 1 in the η -variable and 2 in the \mathbf{x}_Γ -variables defined by:

$$\mathcal{L}_p = \Delta_g - \frac{1}{\delta^2} \left(\partial_\eta^2 + a_{33}^0 \delta \partial_\eta + \sum_{l=2}^p \delta^l \frac{\eta^{l-2}}{(l-2)!} \left(\frac{\eta}{l-1} a_{33}^{l-1} \partial_\eta + \mathcal{S}_\Gamma^{l-2} \right) \right).$$

According to (9), for any function $u \in C^\infty([0, 1]; H^{s+3/2}(\Gamma))$:

$$\|\mathcal{L}_p(u)\|_{H^{-1}(\Gamma \times (-1, 0))} \leq C_p \delta^{p-1} \max_{\eta \in [0, 1]} |u(\eta, \cdot)|_{H^{s+3/2}(\Gamma)}.$$

According to Proposition 4.3, for all $k \geq 0$, the derivatives of $u^k|_{\mathcal{O}_c}$ are of order ξ . Define U^k as follows:

$$U^k = \begin{cases} u^0 - \delta u^1 + \dots + (-\delta)^k u^k, & \text{in } \mathcal{O}_e \cup \mathcal{O}_c^\delta, \\ u_0^m - \delta u_1^m + \dots + (-\delta)^k u_k^m, & \text{in } \Gamma \times (-1, 0). \end{cases}$$

Our main result provides uniform estimates of the error performed by the approximate expansion at any order, for $|\xi|$ tending to zero. The following theorem straightforwardly leads to Theorem 1.1.

Theorem 5.1. *Let $k \geq 0$. Suppose that g belongs to $H^{5/2+k}(\Gamma)$.*

Let $w^k = u - U^k$. There exists a constant C_k such that

$$\|w^k\|_{H^1(\mathcal{O}_e)} \leq C_k \xi \delta^k |g|_{H^{5/2+k}(\partial\Omega)}, \quad (22)$$

and for all $\omega \in \mathcal{O}_c$

$$\|w^k\|_{H^1(\omega)} \leq C_k \delta^k |g|_{H^{5/2+k}(\partial\Omega)}, \quad (23)$$

$$\|\nabla w^k\|_{L^2(\omega)} \leq C_k \xi \delta^k |g|_{H^{5/2+k}(\partial\Omega)}. \quad (24)$$

Proof. By definition of the functions $(u^l)_{l \geq 0}$ and $(u_l^m)_{l \geq 0}$, we infer

$$\begin{aligned} \sigma \Delta w^k &= 0, & \text{in } \mathcal{O}_e \cup \mathcal{O}_c^\delta, \\ \sigma_m \Delta_g w^k &= -\sigma_m \mathcal{L}_k(U^k), & \text{in } \Gamma \times (-1, 0), \\ w^k|_{\partial\Omega} &= 0, \end{aligned}$$

with the following transmission conditions

$$\begin{aligned} \sigma_c \partial_n w^k|_{\Gamma_{-\delta}^-} - \sigma_m \partial_n w^k|_{\Gamma_{-\delta}^+} &= \sigma_m \partial_n U^k|_{\Gamma_{-\delta}^+} - \sigma_c \partial_n U^k|_{\Gamma_{-\delta}^-}, \\ \sigma_m \partial_n w^k|_{\Gamma^-} - \sigma_e \partial_n w^k|_{\Gamma^+} &= 0, \\ w^k|_{\Gamma_{-\delta}^-} - w^k|_{\Gamma_{-\delta}^+} &= U^k|_{\Gamma_{-\delta}^+} - U_{\Gamma_{-\delta}^-}^k, \\ w^k|_{\Gamma^+} - w^k|_{\Gamma^-} &= 0. \end{aligned}$$

Moreover, by definition we have

$$\begin{aligned} \sigma_m \partial_n U^k|_{\Gamma_{-\delta}^+} - \sigma_c \partial_n U^k|_{\Gamma_{-\delta}^-} &= -\sigma_c \delta^{k+1} T^{k+2}(U^k), \\ U^k|_{\Gamma_{-\delta}^+} - U^k|_{\Gamma_{-\delta}^-} &= -\delta^{k+1} T^{k+1}(U^k). \end{aligned}$$

According to Proposition 4.4 and to Corollary 2.2 we infer

$$|T^{k+1+i}(U^k)|_{H^{1/2-i}(\Gamma)} \leq C \xi, \quad i = 0, 1.$$

Multiply by ϕ in $H^1(\Omega)$ and integrate by parts with the help of Green's formula. Taking $\phi = w^k$ and using the trace theorem on Γ we infer that

$$\|\sigma \nabla w^k\|_{L^2(\Omega)} = O(\xi \delta^{k+1/2}).$$

Then defining $\tilde{w}^k = w^k - (-\delta)^{k+1}u_{k+1}^m$ in $\Gamma \times (-1, 0)$ and $\tilde{w}^k = w^k$ in $\mathcal{O}_e^\delta \cup \mathcal{O}_c$ we infer

$$\|\sigma \nabla \tilde{w}^k\|_{L^2(\Omega)} = O(\xi \delta^{k+1}),$$

hence the theorem. \square

6 Numerical simulations for a 3D-resistive thin layer

6.1 Transmission conditions for the two first orders

To obtain the two first order terms, we first observe that according to definition (8), S_Γ^0 is the Laplace-Beltrami operator on Γ , therefore we denote it by $S_\Gamma^0 = \Delta_\Gamma$. Hence, applying recurrence formula (17), we get the first two orders of the asymptotics.

The order 0

$$\begin{cases} \Delta u^0 = 0, & \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \\ [\tilde{\sigma} \partial_n u^0]_\Gamma = 0, & \xi S_m [u^0]_\Gamma - \sigma_e \partial_n u^0|_{\Gamma^+} = 0. \\ u^0|_{\partial\Omega} = \phi. \end{cases} \quad (25)$$

In the membrane:

$$v_0^m = \eta [u^0]_\Gamma \circ \Psi + u_0|_{\Gamma^+}$$

The order 1

$$\begin{cases} \Delta u^1 = 0, & \text{in } \mathcal{O}_e \cup \mathcal{O}_c, \\ [\tilde{\sigma} \partial_n u^1]_\Gamma = \sigma_c \Delta_\Gamma u^0|_{\Gamma^-}, \\ \xi S_m [u^1]_\Gamma - \sigma_e \partial_n u^1|_{\Gamma^+} = \xi S_m \left(1 + \frac{a_{33}^0}{2} \frac{\sigma_c}{\xi S_m} \right) \partial_n u^0|_{\Gamma^-}. \\ u^1|_{\partial\Omega} = 0. \end{cases} \quad (26)$$

In the membrane:

$$v_1^m = u^1|_{\Gamma^+} + \eta \frac{\sigma_e}{\xi S_m} \partial_n u^1|_{\Gamma^+} - \frac{\eta^2}{2} a_{33}^0 [u^0]_\Gamma$$

Observe that the effect of the geometry appears at the first order term, while in the case of soft contrast material, it appears at the second order term [7, 8], since we have

$$\mathcal{H} = -\frac{a_{33}^0}{2},$$

where \mathcal{H} is the mean curvature of the surface Γ . If we were considering an infinite cylinder, and still denoting by Γ the curve of the cross section, a_{33}^0 would be the curvature of Γ .

We recall that the zeroth order approximation is u^0 and the first order approximation is $u^0 - \delta u^1$. In the next part we illustrate numerically the theoretical results by studying the rate of convergence of the errors $u - u^0$ and $u - (u^0 - \delta u^1)$ in H^1 -norm in the outer medium.

6.2 Simulations

Domains Ω and \mathcal{O}_i are two disks centered in 0 of respective radius equal to 2 and 1. Parameters σ_c and S_m are taken equal to 1 and σ_e to 2. Function ϕ is $\exp(i3\theta)$.

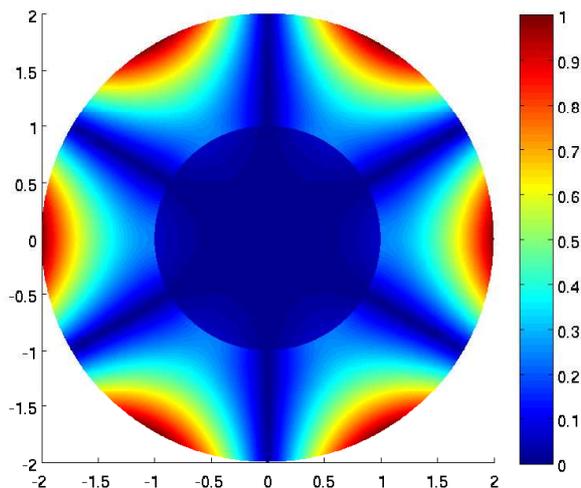


Figure 1: Real part of $u^0 + \delta u^1$ computed by the finite element method with $\delta = 0.05$.

From the analytic solutions the expected rates of convergence with δ are verified since the numerical estimated exponents are respectively 1 for the order 0 and 2 for the order 1; see Figure 2. The green line corresponds to the zeroth order for the soft contrast media. In the soft contrast case, the effect of the layer appears only at the first order term, meaning that the influence of the layer is of order δ . We observe that this potential does not provide an accurate approximation of the potential, since the error estimate does not converge, when δ tends to zero.

As predicted by Theorem 5.1 and Theorem 1.1, the rate of convergence of the zeroth order term u^0 defined by (25) increases when the membrane conductivity decreases, justifying the use of such approximate conditions whatever small the membrane conductivity is; see Figure 3. This behavior is also recovered when using the finite element method based on the variational formulation (19) for solving our model problem; see Figure 4.

We conclude by observing that the convergence rates are asymptotic meaning that if the membrane is not thin enough, it is not sure that our approximate transmission conditions are relevant. For instance, in the configurations we chose, the accuracy of the transmission conditions seems to increase for a thickness around 10^{-1} but if we were choosing other configurations, the accuracy could have decreased as well. Therefore, the shape of the error curves are not significative for a thickness larger than 10^{-2} .

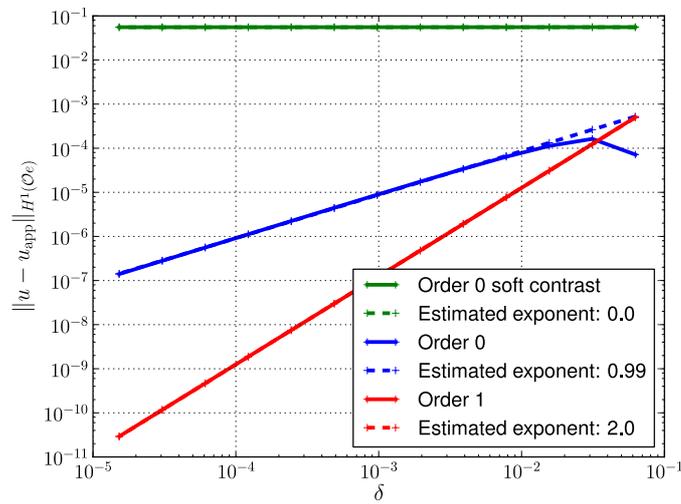


Figure 2: Log-log diagram of the error versus the membrane thickness.

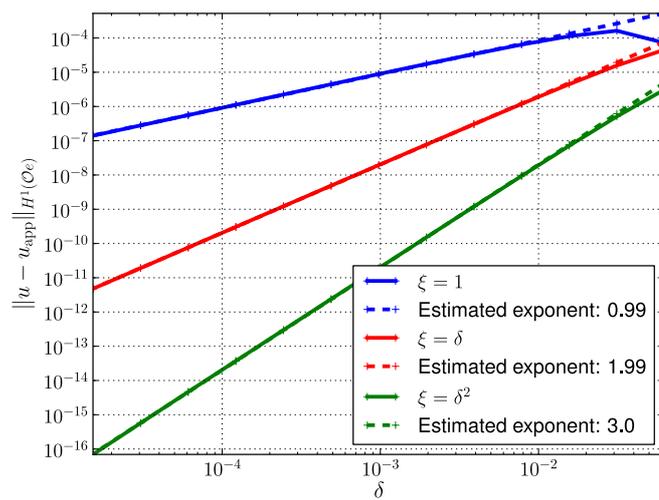


Figure 3: Log-log diagram of the error of the zeroth order for different small conductivities.

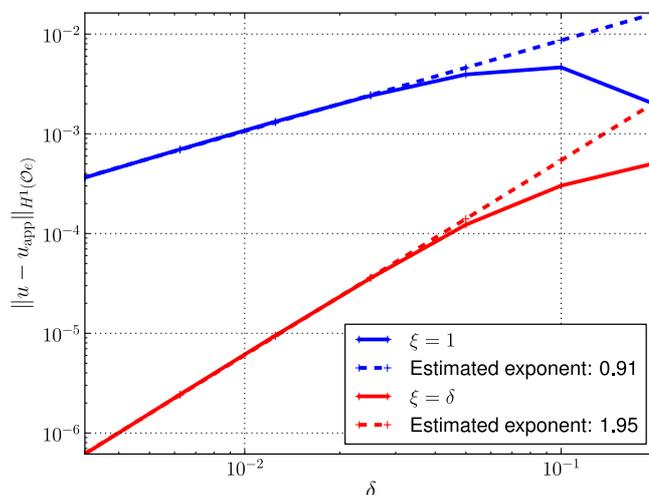


Figure 4: Log-log diagram of the error of the zeroth order for different small conductivities. Computations are performed with the finite element method.

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