



**HAL**  
open science

## A System of Interaction and Structure V: The Exponentials and Splitting

Alessio Guglielmi, Lutz Strassburger

► **To cite this version:**

Alessio Guglielmi, Lutz Strassburger. A System of Interaction and Structure V: The Exponentials and Splitting. *Mathematical Structures in Computer Science*, 2011, pp.1-22. 10.1017/S096012951100003X . inria-00441254v2

**HAL Id: inria-00441254**

**<https://inria.hal.science/inria-00441254v2>**

Submitted on 28 Jan 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A System of Interaction and Structure V: The Exponentials and Splitting

Alessio Guglielmi<sup>1†</sup> and Lutz Straßburger<sup>2‡</sup>

<sup>1</sup>*Department of Computer Science  
University of Bath  
Bath BA2 7AY  
United Kingdom  
<http://alessio.guglielmi.name/>*

<sup>2</sup>*INRIA Saclay–Île-de-France and École Polytechnique  
Laboratoire d'Informatique (LIX)  
Rue de Saclay  
91128 Palaiseau Cedex  
France  
<http://www.lix.polytechnique.fr/~lutz/>*

Received 22 January 2004; Revised 3 December 2010

System NEL is the mixed commutative/non-commutative linear logic BV augmented with linear logic's exponentials, or, equivalently, it is MELL augmented with the non-commutative self-dual connective *seq*. NEL is presented in deep inference, because no Gentzen formalism can express it in such a way that the cut rule is admissible. Other, recent works, show that system NEL is Turing-complete, it is able to directly express process algebra sequential composition and it faithfully models causal quantum evolution. In this paper, we show cut elimination for NEL, based on a technique that we call *splitting*. The splitting theorem shows how and to what extent we can recover a sequent-like structure in NEL proofs. Together with a ‘decomposition’ theorem, proved in the previous paper of this series, splitting yields a cut-elimination procedure for NEL.

## 1. Introduction

This is the fifth in a series of papers dedicated to the proof theory of a self-dual, non-commutative, linear connective, called *seq*, in the context of linear logic. The addition of *seq* to multiplicative linear logic yields a logic that we call BV. This logic is the main subject of study of this series of papers. BV is conjectured to be the same as pomset logic, studied by Retoré in (Ret97) and other papers.

<sup>†</sup> Guglielmi is supported by an ANR *Senior Chaire d'Excellence* titled ‘*Identity and Geometric Essence of Proofs*’ and by the INRIA ARC ‘*REDO: Redesigning Logical Syntax*’.

<sup>‡</sup> Straßburger is supported by the ANR project ‘*INFER: Theory and Application of Deep Inference*’ and by the INRIA ARC ‘*REDO: Redesigning Logical Syntax*’.

BV has been defined in (Gug07) (the first paper of this series), where a sound, complete and cut-free system for BV is given, together with a cut-elimination procedure. The proof system of BV radically departs from the traditional sequent calculus methodology, and adopts instead *deep inference* as the design principle. In short, this means that proofs can be freely composed by the same connectives used for formulae, or, equivalently, inference rules can be applied arbitrarily deep inside formulae. The cited paper provides an introduction to deep inference, which, as a matter of fact, has been originally conceived precisely for capturing BV.

The formalism adopted in the papers of the series so far, including this one, is called the *calculus of structures*, or *CoS*. This is, conceptually, the simplest formalism in deep inference, being just a special form of term rewriting. More sophisticated formalisms in deep inference are emerging, in particular *open deduction* (GGP10). This formalism improves on CoS because it has an increased algebraic flavour and allows for a speed-up in the size of proofs. However, these differences do not affect cut-elimination procedures, in the sense that from a cut-elimination procedure in CoS, one can trivially obtain a cut-elimination procedure in open deduction. For this reason, the results in this paper are broadly valid inside the deep-inference paradigm.

The use of deep inference is necessary: Alwen Tiu showed in (Tiu06) (the second paper of this series) that it is impossible for the sequent calculus to provide a sound, complete and cut-free proof system for BV. This is proved by exhibiting an infinite set of BV tautologies with a cleverly designed structure, such that any bounded-depth, cut-free inference system (in particular, any sequent calculus system) is either unsound or incomplete on the set of tautologies. The design of these tautologies exploits the ability of the seq connective, together with the usual ‘par’ disjunction of linear logic, to bury at arbitrary depths inside formulae certain key structures that have to be ‘unlocked’, by inference, before any other parts of the formula are touched. We stress the fact that this behaviour is independent of the logical formalism employed to describe it. So, deep inference appears to be the most natural choice of proof-system design methodology, because of its ability to apply inference at arbitrary depths inside formulae.

BV might be considered exotic as a logic, but it has a very natural algebraic nature, that already found applications in diverse fields. We mention here: 1) its ability to capture very precisely the sequential connective of Milner’s CCS (and so of other process algebras) (Bru02); 2) a better axiomatisation of causal quantum computation than linear logic provides (BPS08; BGI<sup>+</sup>10); and 3) a new class of categorical models (BPS09). We know that the proof system BV is NP-complete (Kah07), and its feasibility for proof search has been studied (Kah04).

This fifth paper, together with the fourth paper (SG09) in the series, is devoted to the proof theory of system BV when it is enriched with linear logic’s exponentials. We call NEL (non-commutative exponential linear logic) the resulting system. We can also consider NEL as MELL (multiplicative exponential linear logic (Gir87)) augmented with seq. NEL, which was first presented in (GS02), is conservative over both BV and over MELL augmented by the mix and nullary mix rules (FR94; Ret93; AJ94). Like BV, NEL cannot be expressed with a cut-free proof system outside of deep inference, because of

Tiu's counterexample in (Tiu06) (mentioned above), and because NEL is conservative over BV.

An important feature of NEL is that it is Turing-complete, as shown in (Str03c). This makes for an interesting comparison with MELL, whose complexity is currently unknown. MELL is expected to be decidable, but several years of research have not settled the question. If MELL turns out to be decidable, then seq would be the decisive factor that allows us to cross the border between decidability and Turing-completeness. This would have an intuitive explanation in the fact that seq can be employed to simulate the structure of a Turing machine tape, which is precisely what MELL, which is fully commutative, apparently cannot do.

Further interest in NEL comes from the possibility of enhancing, in a natural way, the range of applications of BV. Similarly to what happens when exponentials are added to multiplicative linear logic, the augmented expressivity of NEL over BV can be employed to better capture process algebras, for example, and this is indeed an active area of research. We might equally expect that NEL will further improve our ability to describe quantum evolution phenomena, and we expect to find enriched categorical models over those for BV.

Each of the two NEL papers in the series is devoted to a theorem: *splitting* in this paper and *decomposition* in the previous paper (SG09). Together, the two theorems immediately yield a cut-elimination procedure, and the cut-elimination result is claimed in this paper.

Splitting (first pioneered in (Gug07)) is, in a sense, a way of rebuilding into deep inference the structure of Gentzen sequent-calculus proofs, to the extent possible in the presence of par and seq. The technique consists in blocking the access of inference rules to a part of the formula to be proved, however deep; then, we remove from the context of this blocked part as much 'logical material' as possible. In other words, we prove as much as we can, of a given formula, in the presence of a part that has been blocked. The splitting theorem states properties of what is left of the context of the blocked part, in relation to the shape of the blocked part. It turns out that the splitting property is nothing else than a generalization of the shape of Gentzen calculi rules. It precisely coincides with them when we stipulate that the blocked part of a formula is at the shallowest possible level.

Splitting is, *a priori*, a hard theorem to prove, but, thanks to the decomposition theorem proved in (SG09), we only need to prove it for a fragment of NEL, and this is what we do in this paper. Once splitting is available, cut elimination follows immediately.

The main results of this paper have already been presented, without proof, in (GS02) (for several years, the proofs of the statements have been available in a manuscript on the web).

## 2. The System

Let us briefly recall system NEL as defined in (SG09); the reader can find in that paper more details and introductory comments.

<p>Associativity</p> $[\vec{R} \wp [\vec{T}] \wp \vec{U}] = [\vec{R} \wp \vec{T} \wp \vec{U}]$ $(\vec{R} \otimes (\vec{T} \otimes \vec{U})) = (\vec{R} \otimes \vec{T} \otimes \vec{U})$ $\langle \vec{R} \triangleleft (\vec{T} \triangleleft \vec{U}) \rangle = \langle \vec{R} \triangleleft \vec{T} \triangleleft \vec{U} \rangle$ <p>Commutativity</p> $[\vec{R} \wp \vec{T}] = [\vec{T} \wp \vec{R}]$ $(\vec{R} \otimes \vec{T}) = (\vec{T} \otimes \vec{R})$ <p>Unit</p> $[\circ \wp \vec{R}] = [\vec{R}]$ $(\circ \otimes \vec{R}) = (\vec{R})$ $\langle \circ \triangleleft \vec{R} \rangle = \langle \vec{R} \rangle$ $\langle \vec{R} \triangleleft \circ \rangle = \langle \vec{R} \rangle$	<p>Singleton</p> $[R] = (R) = \langle R \rangle = R$ <p>Negation</p> $\overline{\circ} = \circ$ $\overline{[R_1 \wp \dots \wp R_h]} = (\vec{R}_1 \otimes \dots \otimes \vec{R}_h)$ $\overline{(R_1 \otimes \dots \otimes R_h)} = [\vec{R}_1 \wp \dots \wp \vec{R}_h]$ $\overline{\langle R_1 \triangleleft \dots \triangleleft R_h \rangle} = \langle \vec{R}_1 \triangleleft \dots \triangleleft \vec{R}_h \rangle$ $\overline{?R} = !\vec{R}$ $\overline{!R} = ?\vec{R}$ $\overline{\vec{R}} = R$ <p>Contextual Closure</p> <p style="text-align: center;">if <math>R = T</math> then <math>S\{R\} = S\{T\}</math></p>
---	--

Fig. 1. Basic equations for the syntactic equivalence =

**Definition 2.1.** There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by  $a, b, \dots$ . *Structures* are denoted by  $S, P, Q, R, T, U, V, W, X$  and  $Z$ . The structures of the *language NEL* are generated by

$$S ::= a \mid \circ \mid \underbrace{[S \wp \dots \wp S]}_{>0} \mid \underbrace{(S \otimes \dots \otimes S)}_{>0} \mid \underbrace{\langle S \triangleleft \dots \triangleleft S \rangle}_{>0} \mid ?S \mid !S \mid \bar{S} \quad ,$$

where  $\circ$ , the *unit*, is not an atom and  $\bar{S}$  is the *negation* of the structure  $S$ . Structures with a hole that does not appear in the scope of a negation are denoted by  $S\{ \}$ . The structure  $R$  is a *substructure* of  $S\{R\}$ , and  $S\{ \}$  is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example,  $S[R \wp T]$  stands for  $S\{[R \wp T]\}$ . The structures of the language NEL are equivalent modulo the relation  $=$ , defined in Figure 1. There,  $\vec{R}, \vec{T}$  and  $\vec{U}$  stand for finite, non-empty sequences of structures (elements of the sequences are separated by  $\wp, \triangleleft$ , or  $\otimes$ , as appropriate in the context).

**Definition 2.2.** In Figure 2, *system SNEL* (*symmetric non-commutative exponential linear logic*) is shown. The rules  $\text{ai}\downarrow, \text{ai}\uparrow, \text{s}, \text{q}\downarrow, \text{q}\uparrow, \text{p}\downarrow, \text{p}\uparrow, \text{e}\downarrow, \text{e}\uparrow, \text{w}\downarrow, \text{w}\uparrow, \text{b}\downarrow, \text{b}\uparrow, \text{g}\downarrow$ , and  $\text{g}\uparrow$  are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *empty*, *coempty*, *weakening*, *coweakening*, *absorption*, *coabsorption*, *digging*, and *codigging*. The *down fragment* of SNEL is  $\{\text{ai}\downarrow, \text{s}, \text{q}\downarrow, \text{p}\downarrow, \text{e}\downarrow, \text{w}\downarrow, \text{b}\downarrow, \text{g}\downarrow\}$ , the *up fragment* is  $\{\text{ai}\uparrow, \text{s}, \text{q}\uparrow, \text{p}\uparrow, \text{e}\uparrow, \text{w}\uparrow, \text{b}\uparrow, \text{g}\uparrow\}$ . Then, *system NEL* is shown in Figure 3, where the rule  $\circ\downarrow$  is called *unit*.

All inference rules in SNEL are of the kind

$$\rho \frac{S\{T\}}{S\{R\}} \quad ,$$

$$\begin{array}{c}
 \text{ai}\downarrow \frac{S\{\circ\}}{S[a \wp \bar{a}]} \qquad \text{ai}\uparrow \frac{S(a \otimes \bar{a})}{S\{\circ\}} \\
 \\
 \text{s} \frac{S([R \wp U] \otimes T)}{S[(R \otimes T) \wp U]} \\
 \\
 \text{q}\downarrow \frac{S(\langle R \wp U \rangle \triangleleft [T \wp V])}{S[\langle R \triangleleft T \rangle \wp \langle U \triangleleft V \rangle]} \qquad \text{q}\uparrow \frac{S(\langle R \triangleleft U \rangle \otimes \langle T \triangleleft V \rangle)}{S[\langle R \otimes T \rangle \triangleleft (U \otimes V)]} \\
 \\
 \text{p}\downarrow \frac{S\{![R \wp T]\}}{S[!R \wp ?T]} \qquad \text{p}\uparrow \frac{S(?R \otimes !T)}{S\{?(R \otimes T)\}} \\
 \\
 \text{e}\downarrow \frac{S\{\circ\}}{S\{!\circ\}} \qquad \text{e}\uparrow \frac{S\{?\circ\}}{S\{\circ\}} \\
 \\
 \text{w}\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \text{w}\uparrow \frac{S\{!R\}}{S\{\circ\}} \\
 \\
 \text{b}\downarrow \frac{S[?R \wp R]}{S\{?R\}} \qquad \text{b}\uparrow \frac{S\{!R\}}{S\{!R \otimes R\}} \\
 \\
 \text{g}\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \text{g}\uparrow \frac{S\{!R\}}{S\{!!R\}}
 \end{array}$$

Fig. 2. System SNEL

$$\begin{array}{c}
 \circ\downarrow \frac{}{\circ} \qquad \text{ai}\downarrow \frac{S\{\circ\}}{S[a \wp \bar{a}]} \qquad \text{e}\downarrow \frac{S\{\circ\}}{S\{!\circ\}} \\
 \\
 \text{s} \frac{S([R \wp U] \otimes T)}{S[(R \otimes T) \wp U]} \qquad \text{q}\downarrow \frac{S(\langle R \wp U \rangle \triangleleft [T \wp V])}{S[\langle R \triangleleft T \rangle \wp \langle U \triangleleft V \rangle]} \qquad \text{p}\downarrow \frac{S\{![R \wp T]\}}{S[!R \wp ?T]} \\
 \\
 \text{w}\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \text{b}\downarrow \frac{S[?R \wp R]}{S\{?R\}} \qquad \text{g}\downarrow \frac{S\{??R\}}{S\{?R\}}
 \end{array}$$

Fig. 3. System NEL

saying that if a structure matches  $R$ , in a context  $S\{ \}$ , it can be rewritten as specified by  $T$ , in the same context  $S\{ \}$  (or vice versa if one reasons top-down). The unit rule of NEL is special in this respect, as it has no context; however, we only use it for convenience, and we could easily do without by slightly adapting the notion of proof.

A derivation  $\Delta$  is a chain of consecutive applications of instances of inference rules. A derivation with no premise is called a *proof*, denoted by  $\Pi$ . A system  $\mathcal{S}$  *proves*  $R$  if there

is in the system  $\mathcal{S}$  a proof  $\Pi$  whose conclusion is  $R$ . We use the notations

$$\mathcal{S} \parallel \begin{array}{c} T \\ \Delta \\ R \end{array} \quad \text{and} \quad \mathcal{S} \parallel \begin{array}{c} \Pi \\ R \end{array} .$$

to denote a derivation  $\Delta$  with premise  $T$  and conclusion  $R$ , whose rules are in  $\mathcal{S}$ , and a proof  $\Pi$  in  $\mathcal{S}$  whose conclusion is  $R$ , respectively.

**Definition 2.3.** A rule  $\rho$  is *derivable* in the system  $\mathcal{S}$  if  $\rho \notin \mathcal{S}$  and for every instance  $\rho \frac{T}{R}$  there exists a derivation from  $T$  to  $R$  in  $\mathcal{S}$ . We say that a rule  $\rho$  is *admissible* for the system  $\mathcal{S}$  if  $\rho \notin \mathcal{S}$  and for every proof in  $\mathcal{S} \cup \{\rho\}$  there is a proof in  $\mathcal{S}$  with the same conclusion. Two systems are *equivalent* if they prove the same structures. Two systems  $\mathcal{S}$  and  $\mathcal{S}'$  are *strongly equivalent* if for every derivation from  $T$  to  $R$  in  $\mathcal{S}$  there is a derivation from  $T$  to  $R$  in  $\mathcal{S}'$ , and vice versa.

Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

**Definition 2.4.** The following rules are called *interaction* and *cut*:

$$i\downarrow \frac{S\{\circ\}}{S[R \wp \bar{R}]} \quad \text{and} \quad i\uparrow \frac{S(R \otimes \bar{R})}{S\{\circ\}} ,$$

where  $R$  and  $\bar{R}$  are called *principal structures*.

The following two propositions have been shown in (SG09).

**Proposition 2.5.** The rule  $i\downarrow$  is derivable in  $\{ai\downarrow, s, q\downarrow, p\downarrow, e\downarrow\}$ , and, dually, the rule  $i\uparrow$  is derivable in the system  $\{ai\uparrow, s, q\uparrow, p\uparrow, e\uparrow\}$ .

**Proposition 2.6.** Each rule  $\rho\uparrow$  in SNEL is derivable in  $\{i\downarrow, i\uparrow, s, \rho\downarrow\}$ , and, dually, each rule  $\rho\downarrow$  in SNEL is derivable in the system  $\{i\downarrow, i\uparrow, s, \rho\uparrow\}$ .

As an immediate consequence of Propositions 2.5 and 2.6 we get:

**Proposition 2.7.** The systems  $NEL \cup \{i\uparrow\}$  and  $SNEL \cup \{o\downarrow\}$  are strongly equivalent.

In the remainder of this paper we will give the proof of the cut elimination theorem, which, as usual in deep inference, means proving that the up-fragment is admissible:

**Theorem 2.8 (Cut Admissibility).** System NEL is equivalent to  $SNEL \cup \{o\downarrow\}$ .

The following corollaries are immediate consequences of cut admissibility.

**Corollary 2.9.** The rule  $i\uparrow$  is admissible for system NEL.

$$\begin{array}{ccc}
 & & T \\
 & & \{g\uparrow\} \parallel \\
 & & T_1 \\
 & & \{b\uparrow\} \parallel \\
 & & T_2 \\
 & & \{w\uparrow\} \parallel \\
 & & T_3 \\
 & & \{e\downarrow\} \parallel \\
 & & T_4 \\
 & & \{ai\downarrow\} \parallel \\
 & & T_5 \\
 \text{SNEL} \parallel \Delta & \rightsquigarrow & \{s, q\downarrow, q\uparrow, p\downarrow, p\uparrow\} \parallel \\
 R & & R_5 \\
 & & \{ai\uparrow\} \parallel \\
 & & R_4 \\
 & & \{e\uparrow\} \parallel \\
 & & R_3 \\
 & & \{w\downarrow\} \parallel \\
 & & R_2 \\
 & & \{b\downarrow\} \parallel \\
 & & R_1 \\
 & & \{g\downarrow\} \parallel \\
 & & R
 \end{array}$$

Fig. 4. Decomposition of derivations (Theorem 2.11)

**Corollary 2.10.** For any two structures  $T$  and  $R$ , we have

$$\text{SNEL} \parallel \begin{array}{c} T \\ R \end{array} \quad \text{if and only if} \quad \text{NEL} \parallel \begin{array}{c} \\ [\bar{T} \otimes R] \end{array} .$$

Our cut elimination proof relies on the following theorem, which is a special case of a more general one, and whose proof can be found in (SG09).

**Theorem 2.11 (Decomposition).** Every derivation  $\Delta$  in SNEL can be rewritten as shown in Figure 4.

### 3. Splitting

Three approaches to cut elimination in deep inference have been explored in previous papers, but none of them can be applied in our case.

First, we could hope to rely on semantics, as Brünnler and Tiu did in (BT01) for



classical logic. However, so far, there is no provability semantics for NEL (in the sense, for example, of phase spaces (Oka99) or other model theoretic semantics) that we could use for a completeness argument.

Second, we could hope to do as in (Brü03), where Brünnler presents a simple syntactic method that employs the atomicity of cut together with certain proof theoretical properties of classical logic. More recent papers that refine his method by using *atomic flows* are (GG08; GGS10). However, this approach cannot be used for NEL, because contraction cannot be applied to arbitrary formulas as in classical logic.

Third, we could hope to do as in (GS01; Str03b), where we relied on permutations of rules. However, traditional techniques based on simple notions of rule-instance permutation cannot work. To use permutation arguments for NEL requires a more general notion of permutation than the usual one. The following remark shows an example that illustrates the point and gives some hints to anybody who would like to proceed in that direction.

**Remark 3.1.** Let  $P$  and  $Q$  be arbitrary provable structures with  $P \neq \circ \neq Q$ . Consider then the following proof, where two proofs of  $P$  and  $Q$  are stacked one on top of the other and then we interweave two structures over the atoms  $a$ ,  $b$  and  $c$  and their duals, and where the bottom rule instance is a  $\mathfrak{q}\uparrow$ :

$$\begin{array}{c}
\text{NEL} \parallel \\
P \\
\text{NEL} \parallel \\
(P \otimes Q) \\
2 \cdot \text{ai}\downarrow \frac{}{(P \otimes \langle [b \wp \bar{b}] \triangleleft [c \wp \bar{c}] \rangle \otimes Q)} \\
\mathfrak{q}\downarrow \frac{}{(P \otimes \langle [b \triangleleft c] \wp \langle \bar{b} \triangleleft \bar{c} \rangle \rangle \otimes Q)} \\
s \frac{}{(P \otimes \langle [b \triangleleft c] \wp \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle)} \\
\text{ai}\downarrow \frac{}{(P \otimes \langle [a \wp \bar{a}] \triangleleft [b \triangleleft c] \wp \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle)} \\
\mathfrak{q}\downarrow \frac{}{(P \otimes \langle [a \triangleleft b \triangleleft c] \wp \langle \bar{a} \triangleleft \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle \rangle)} \\
s \frac{}{[(P \otimes \langle a \triangleleft b \triangleleft c \rangle) \wp \langle \bar{a} \triangleleft \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle]} \\
\mathfrak{q}\uparrow \frac{}{[\langle (P \otimes \langle a \triangleleft b \rangle) \triangleleft c \rangle \wp \langle \bar{a} \triangleleft \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle]}
\end{array} \tag{1}$$

We can further suppose that the atoms  $a$ ,  $b$ ,  $c$  and their duals do not appear in  $P$  and  $Q$ . If so, any permutation of the bottom  $\mathfrak{q}\uparrow$  instance over the rule instances immediately above it leads to an unprovable structure, because some two dual atoms  $a/\bar{a}$  or  $b/\bar{b}$  or  $c/\bar{c}$  would become connected by a connective different from a  $\wp$ . In order for a proof to exist, any rule instance changing the mutual logical relations of the  $a/\bar{a}$ ,  $b/\bar{b}$  and  $c/\bar{c}$  atoms in the conclusion must be an instance of  $\mathfrak{q}\uparrow$ . As an example of the contrary, consider

$$\mathfrak{q}\downarrow \frac{\langle [(P \otimes \langle a \triangleleft b \rangle) \wp \langle \bar{a} \triangleleft \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle] \triangleleft c \rangle}{[\langle (P \otimes \langle a \triangleleft b \rangle) \triangleleft c \rangle \wp \langle \bar{a} \triangleleft \langle \langle \bar{b} \triangleleft \bar{c} \rangle \otimes Q \rangle \rangle]} \tag{2}$$

where  $c$  and  $\bar{c}$  are not able to be joined in any  $\text{ai}\downarrow$  going upwards. This represents a difficulty for any argument based on permuting up the  $\text{q}\uparrow$  rule.

However, we can observe that the entire proofs of  $P$  and  $Q$  could be permuted below the  $\text{q}\uparrow$  instance, step by step. The two structures  $P$  and  $Q$  act as sort of “locks” for the substructure made of  $a$ ,  $b$  and  $c$  atoms (and their duals). Once the locks are open, by independently proving them and so reducing them to the unit, the permuting up of the  $\text{q}\uparrow$  instance can take place.

So, an approach to proof normalisation based on permutations is not to be ruled out completely, but appears to be complicated and is not pursued in this paper.

By using the idea in this example, it is not difficult to construct another example (different from the one used in (Tiu06)) that shows the necessity of deep inference for designing cut-free systems for logics incorporating a self-dual non-commutative connective, like NEL.

Luckily, for proving cut admissibility for NEL we can use the technique that has been called *splitting* in (Gug07). As the cited paper explains, this technique establishes a clear connection with the sequent calculus, at least for the fragments of proof systems that allow for a sequent calculus presentation (in our case, the commutative fragment).

We refer the reader to (Gug07) for an intuitive explanation of splitting. We only repeat here that the technique relies on two separate phases:

- 1 Context reduction: If a structure  $S\{R\}$  is provable, then  $S\{\ \}$  can be reduced (by performing inference steps going upwards in the derivation) to the structure  $[\{\ \} \wp U]$ , for some  $U$ , such that  $[R \wp U]$  is provable (the hole can be filled by any structure that does not play an active part in inference steps).
- 2 Splitting: If  $[(R \otimes T) \wp P]$  is provable, then  $P$  can be reduced to  $[P_1 \wp P_2]$ , such that  $[R \wp P_1]$  and  $[T \wp P_2]$  are provable.

Context reduction is in turn proved by splitting, which is then at the core of the matter. In this section we will state and prove splitting proper, and in the next one we tackle context reduction.

For notational convenience, we define *system* NELc to be the system obtained from NEL by removing the rules for weakening, absorption, and digging (called *non-core* in (SG09):

$$\text{NELc} = \text{NEL} \setminus \{\text{w}\downarrow, \text{b}\downarrow, \text{g}\downarrow\} = \{\text{o}\downarrow, \text{ai}\downarrow, \text{s}, \text{q}\downarrow, \text{p}\downarrow, \text{e}\downarrow\} \quad .$$

The following lemma has been called ‘shallow splitting’ in (Gug07). The proof is very similar, and we do not give it in full here. But note that we organize the case analysis here in a different way than in (Gug07), which allows us to reuse it for the Lemmas 3.3 and 3.4 below.

**Lemma 3.2 (Splitting).** Let  $R, T, P$  be any NEL structures.

- (i) If  $[(R \otimes T) \wp P]$  is provable in NELc, then there are structures  $P_R$  and  $P_T$ , such that

$$\begin{array}{c} [P_R \wp P_T] \\ \text{NELc} \parallel \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [R \wp P_R] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [T \wp P_T] \end{array} \quad .$$

(ii) If  $[\langle R \triangleleft T \rangle \wp P]$  is provable in NELc, then there are structures  $P_R$  and  $P_T$ , such that

$$\text{NELc} \begin{array}{c} \parallel \\ \langle P_R \triangleleft P_T \rangle \\ \parallel \\ P \end{array} \quad \text{and} \quad \text{NELc} \begin{array}{c} \parallel \\ [R \wp P_R] \\ \parallel \end{array} \quad \text{and} \quad \text{NELc} \begin{array}{c} \parallel \\ [T \wp P_T] \\ \parallel \end{array} .$$

*Proof.* We prove both statements simultaneously by structural induction on the number of atoms in the conclusion and the length (number of rule instances) of the proof, ordered lexicographically. Without loss of generality, assume  $R \neq \circ \neq T$  (otherwise both statements are trivially true).

(i) Consider the bottommost rule instance  $\rho$  in the proof of  $[(R \otimes T) \wp P]$ . We can distinguish between three different kinds of cases:

(a) The first kind appears when the redex of  $\rho$  is inside  $R$ ,  $T$  or  $P$ . Then we have the following situation:

$$\text{NELc} \begin{array}{c} \parallel \\ \Pi \\ \parallel \\ [(R' \otimes T) \wp P] \\ \parallel \\ \rho \frac{[(R \otimes T) \wp P]}{[(R \otimes T) \wp P]} \end{array}$$

where we can apply the induction hypothesis to  $\Pi$  because it is one rule shorter (if  $\rho = \text{ai}\downarrow$  also the conclusion is smaller). We get

$$\text{NELc} \begin{array}{c} \parallel \\ [P_{R'} \wp P_T] \\ \parallel \\ \Delta_P \\ \parallel \\ P \end{array} \quad \text{and} \quad \text{NELc} \begin{array}{c} \parallel \\ \Pi_R \\ \parallel \\ [R' \wp P_R] \\ \parallel \end{array} \quad \text{and} \quad \text{NELc} \begin{array}{c} \parallel \\ \Pi_T \\ \parallel \\ [T \wp P_T] \\ \parallel \end{array}$$

From  $\Pi_R$ , we can get

$$\text{NELc} \begin{array}{c} \parallel \\ \Pi'_R \\ \parallel \\ [R' \wp P_R] \\ \parallel \\ \rho \frac{[R' \wp P_R]}{[R \wp P_R]} \end{array}$$

and we are done. If the redex of  $\rho$  is inside  $T$  or  $P$ , the situation is similar.

(b) In the second kind of case the substructure  $(R \otimes T)$  is inside the redex of  $\rho$ , but is not modified by  $\rho$ . These cases can be compared with the ‘commutative cases’ in the usual sequent calculus cut elimination argument. We show only one representative example (a complete case analysis can be found in (Gug07) and (Str03a)): Suppose we have

$$\text{NELc} \begin{array}{c} \parallel \\ \Pi \\ \parallel \\ [(\langle (R \otimes T) \wp P_1 \wp P_3 \rangle \triangleleft P_2) \wp P_4] \\ \parallel \\ \text{q}\downarrow \frac{[(\langle (R \otimes T) \wp P_1 \wp P_3 \rangle \triangleleft P_2) \wp P_4]}{[(R \otimes T) \wp \langle P_1 \triangleleft P_2 \rangle \wp P_3 \wp P_4]} \end{array}$$

We can apply the induction hypothesis to  $\Pi$  because it is one rule shorter (the

size of the conclusion does not change). This gives us

$$\begin{array}{c} \langle Q_1 \triangleleft Q_2 \rangle \\ \text{NELc} \parallel \Delta_1 \\ P_4 \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_1 \\ [(R \otimes T) \wp P_1 \wp P_3 \wp Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_2 \\ [P_2 \wp Q_2] \end{array}$$

We can apply the induction hypothesis again to  $\Pi_1$ , because now the number of atoms in the conclusion is strictly smaller (because we can assume that the instance of  $\text{q}\downarrow$  is not trivial). We get

$$\begin{array}{c} [P_R \wp P_T] \\ \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_R \\ [R \wp P_R] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_T \\ [T \wp P_T] \end{array}$$

From  $\Delta_1$ ,  $\Delta_2$  and  $\Pi_2$  we can build the following derivation

$$\begin{array}{c} [P_R \wp P_T] \\ \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] \\ \text{NELc} \parallel \Pi_2 \\ \text{q}\downarrow \frac{[\langle [P_1 \wp Q_1] \triangleleft [P_2 \wp Q_2] \rangle \wp P_3]}{[\langle P_1 \triangleleft P_2 \rangle \wp P_3 \wp \langle Q_1 \triangleleft Q_2 \rangle]} \\ \text{NELc} \parallel \Delta_1 \\ [\langle P_1 \triangleleft P_2 \rangle \wp P_3 \wp P_4] \end{array}$$

and we are done. All other cases in this group are similar.

- (c) In the last type of case the substructure  $(R \otimes T)$  is destroyed by  $\rho$ . These cases can be compared to the ‘key cases’ in a standard sequent calculus cut elimination argument. We have only one possibility. The most general situation is as follows:

$$\begin{array}{c} \text{NELc} \parallel \Pi \\ \text{s} \frac{[\langle [(R_1 \otimes T_1) \wp P_1] \otimes R_2 \otimes T_2 \rangle \wp P_2]}{[(R_1 \otimes R_2 \otimes T_1 \otimes T_2) \wp P_1 \wp P_2]} \end{array}$$

where one of  $R_1$  and  $R_2$  might be  $\circ$ , but not both of them (similarly for  $T_1$  and  $T_2$ ). As before, we can apply the induction hypothesis to  $\Pi$  and get

$$\begin{array}{c} [Q_1 \wp Q_2] \\ \text{NELc} \parallel \Delta_1 \\ P_2 \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_1 \\ [(R_1 \otimes T_1) \wp P_1 \wp Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_2 \\ [(R_2 \otimes T_2) \wp Q_2] \end{array}$$

We can apply the induction hypothesis again to  $\Pi_1$  and  $\Pi_2$ . (Because we assume that the instance of  $\text{s}$  is not trivial, the conclusions are strictly smaller than the

one of the original proof.) We get:

$$\begin{array}{c} [P_{R_1} \wp P_{T_1}] \\ \text{NELc} \parallel \Delta_3 \\ [P_1 \wp Q_1] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_{R_1} \\ [R_1 \wp P_{R_1}] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_{T_1} \\ [T_1 \wp P_{T_1}] \end{array}$$

and

$$\begin{array}{c} [P_{R_2} \wp P_{T_2}] \\ \text{NELc} \parallel \Delta_4 \\ Q_2 \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_{R_2} \\ [R_2 \wp P_{R_2}] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_{T_2} \\ [T_2 \wp P_{T_2}] \end{array}$$

Now let  $P_R = [P_{R_1} \wp P_{R_2}]$  and  $P_T = [P_{T_1} \wp P_{T_2}]$ . We can build

$$\begin{array}{c} [P_{R_1} \wp P_{R_2} \wp P_{T_1} \wp P_{T_2}] \\ \text{NELc} \parallel \Delta_4 \\ [P_{R_1} \wp P_{T_1} \wp Q_2] \\ \text{NELc} \parallel \Delta_3 \\ [P_1 \wp Q_1 \wp Q_2] \\ \text{NELc} \parallel \Delta_1 \\ [P_1 \wp P_2] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_{R_1} \\ [R_1 \wp P_{R_1}] \\ \text{NELc} \parallel \Pi_{R_2} \\ \text{s} \frac{[(R_1 \otimes [R_2 \wp P_{R_2}]) \wp P_{R_1}]}{[(R_1 \otimes R_2) \wp P_{R_1} \wp P_{R_2}]} \end{array}$$

and a similar proof of  $[(T_1 \otimes T_2) \wp P_{T_1} \wp P_{T_2}]$ , and we are done.

- (ii) The case for  $[(R \triangleleft T) \wp P]$  is similar to the one for  $[(R \otimes T) \wp P]$ , and we leave it to the reader.  $\square$

We can now tackle modalities, for which we can exhibit, too, a splitting lemma.

**Lemma 3.3 (Splitting for Modalities).** Let  $R$  and  $P$  be any NEL structures.

- (i) If  $[!R \wp P]$  is provable in NELc, then there are structures  $P_1, \dots, P_h$  for some  $h \geq 0$ , such that

$$\begin{array}{c} [?P_1 \wp \dots \wp ?P_h] \\ \text{NELc} \parallel \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [R \wp P_1 \wp \dots \wp P_h] \end{array} .$$

- (ii) If  $[?R \wp P]$  is provable in NELc, then there is a structure  $P_R$ , such that

$$\begin{array}{c} !P_R \\ \text{NELc} \parallel \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [R \wp P_R] \end{array} .$$

*Proof.* The proof is similar to the previous one. We use the same induction measure and the same pattern in the case analysis as before.

- (i) We consider again the bottommost rule instance  $\rho$  in the proof of  $[!R \wp P]$ , and we have the same three classes of cases as in the proof of Lemma 3.2.

- (a) The redex of  $\rho$  is inside  $R$  or  $P$ . This case is the same as in the proof of Lemma 3.2.
- (b) The substructure  $!R$  is inside the redex of  $\rho$ , but is not changed by  $\rho$ . This case is almost literally the same as for Lemma 3.2. We only have to replace  $(R \otimes T)$  by  $!R$ , and

$$\begin{array}{ccc} [P_R \wp P_T] & & [?P_1 \wp \dots \wp ?P_h] \\ \text{NELc} \parallel \Delta_2 & \text{by} & \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] & & [P_1 \wp P_3 \wp Q_1] \end{array}$$

(As for the previous lemma, the full details can be found in (Str03a).)

- (c) The substructure  $!R$  is destroyed by  $\rho$ . There are two possibilities ( $\rho = e\downarrow$  and  $\rho = p\downarrow$ ):

$$\begin{array}{ccc} \text{NELc} \parallel & & \text{NELc} \parallel \Pi \\ e\downarrow \frac{[\circ \wp P]}{[! \circ \wp P]} & \text{and} & p\downarrow \frac{[![R \wp P_1] \wp Q_2]}{[!R \wp ?P_1 \wp Q_2]} \end{array}$$

For  $\rho = e\downarrow$  we are done immediately by letting  $h = 0$ . For  $\rho = p\downarrow$  we can apply the induction hypothesis to  $\Pi$  and get structures  $P_2, \dots, P_h$  such that

$$\begin{array}{ccc} [?P_2 \wp \dots \wp ?P_h] & & \\ \text{NELc} \parallel & \text{and} & \text{NELc} \parallel \\ Q_2 & & [R \wp P_1 \wp P_2 \wp \dots \wp P_h] \end{array}$$

We immediately get

$$\begin{array}{c} [?P_1 \wp ?P_2 \wp \dots \wp ?P_h] \\ \text{NELc} \parallel \\ [?P_1 \wp Q_2] \end{array} .$$

- (ii) As before, consider the bottommost rule instance  $\rho$  in the proof of  $[?R \wp P]$ .

- (a) The redex of  $\rho$  is inside  $R$  or  $P$ . This case is the same as before.
- (b) The substructure  $?R$  is inside the redex of  $\rho$ , but is not changed by  $\rho$ . As before, this case is almost literally the same as in the proof of Lemma 3.2. This time we have to replace  $(R \otimes T)$  by  $?R$ , and

$$\begin{array}{ccc} [P_R \wp P_T] & & !P_R \\ \text{NELc} \parallel \Delta_2 & \text{by} & \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] & & [P_1 \wp P_3 \wp Q_1] \end{array}$$

- (c) The substructure  $?R$  is destroyed by  $\rho$ . For this case there is only one possibility:

$$\begin{array}{c} \text{NELc} \parallel \Pi \\ p\downarrow \frac{[![R \wp P_1] \wp P_2]}{[?R \wp !P_1 \wp P_2]} \end{array}$$

We can apply part (i) of the lemma and get

$$\begin{array}{c} [?Q_1 \wp \dots \wp ?Q_h] \\ \text{NELc} \parallel \Delta \\ P_2 \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_R \\ [R \wp P_1 \wp Q_1 \wp \dots \wp Q_h] \end{array}$$

Now let  $P_R = [P_1 \wp Q_1 \wp \dots \wp Q_h]$ . We can build

$$\begin{array}{c} ![P_1 \wp Q_1 \wp \dots \wp Q_h] \\ \{p\downarrow\} \parallel \\ [!P_1 \wp ?Q_1 \wp \dots \wp ?Q_h] \\ \text{NELc} \parallel \Delta \\ [!P_1 \wp P_2] \end{array}$$

as desired.  $\square$

**Lemma 3.4 (Splitting for Atoms).** Let  $a$  be any atom and  $P$  be any NEL structure.

$$\text{If there is a proof } \begin{array}{c} \text{NELc} \parallel \\ [a \wp P] \end{array} \text{ then there is a derivation } \begin{array}{c} \bar{a} \\ \text{NELc} \parallel \\ P \end{array} .$$

*Proof.* After the previous two proofs this is an almost trivial exercise: The case (a) is as before, and for (b), we have to replace  $(R \otimes T)$  by  $a$ , and

$$\begin{array}{c} [P_R \wp P_T] \\ \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] \end{array} \quad \text{by} \quad \begin{array}{c} \bar{a} \\ \text{NELc} \parallel \Delta_2 \\ [P_1 \wp P_3 \wp Q_1] \end{array} .$$

For case (c), the only possibility is

$$\text{ai}\downarrow \frac{\text{NELc} \parallel \Pi'}{[a, \bar{a}, P_1]}$$

from which we immediately get

$$\begin{array}{c} \bar{a} \\ \text{NELc} \parallel \\ [\bar{a}, P_1] \end{array} .$$

as desired.  $\square$

#### 4. Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context  $S\{ \}$  to a problem that concerns only a shallow context  $[\{ \} \wp P]$ . In the case of cut elimination, for example, we will then be able to apply splitting.

Before giving the statement, we need to define the *modality depth* of a context  $S\{ \}$  to be the number of ! and ? in whose scope the  $\{ \}$  occurs. In the following lemma, the  $\{ \}$  is treated as ordinary atom.

**Lemma 4.1 (Context Reduction).** Let  $R$  be a NEL structure and  $S\{ \}$  be a context. If  $S\{R\}$  is provable in NELc, then there is a structure  $P_R$ , such that

$$\begin{array}{c} ! \cdots ! [\{ \} \wp P_R] \\ \text{NELc} \parallel \Delta \\ S\{ \} \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi \\ [R \wp P_R] \end{array}$$

where the number of ! in front of  $[\{ \} \wp P_R]$  is the modality depth of  $S\{ \}$ .

*Proof.* We proceed by structural induction on the context  $S\{ \}$ . The base case when  $S\{ \} = \{ \}$  is trivial. Now we can distinguish four cases

- (a)  $S\{ \} = [(S'\{ \} \otimes T) \wp P]$  where, without loss of generality,  $T \neq \circ$ . Note that we do allow  $P = \circ$ . We can apply splitting (Lemma 3.2) to the proof of  $[(S'\{R\} \otimes T) \wp P]$  and get:

$$\begin{array}{c} [P_S \wp P_T] \\ \text{NELc} \parallel \Delta_P \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_S \\ [S'\{R\} \wp P_S] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_T \\ [T \wp P_T] \end{array}$$

Because  $T \neq \circ$  we can now apply the induction hypothesis to  $\Pi_S$  and get:

$$\begin{array}{c} ! \cdots ! [\{ \} \wp P_R] \\ \text{NELc} \parallel \Delta' \\ [S'\{ \} \wp P_S] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi \\ [R \wp P_R] \end{array}$$

From this we can build

$$\begin{array}{c} ! \cdots ! [\{ \} \wp P_R] \\ \text{NELc} \parallel \Delta' \\ [S'\{ \} \wp P_S] \\ \text{NELc} \parallel \Pi_T \\ \frac{[(S'\{ \} \otimes [T \wp P_T]) \wp P_S]}{[(S'\{ \} \otimes T) \wp P_S \wp P_T]} \\ \text{NELc} \parallel \Delta_P \\ [(S'\{ \} \otimes T) \wp P] \end{array}$$

as desired.



- (b) The cases  $S\{ \} = [\langle S'\{ \} \triangleleft T \rangle \wp P]$  and  $S\{ \} = [\langle T \triangleleft S'\{ \} \rangle \wp P]$  are handled similarly to (a).
- (c) If  $S\{ \} = [!S'\{ \} \wp P]$ , then we can apply splitting (Lemma 3.3) to the proof of  $[!S'\{R\} \wp P]$  and get:

$$\begin{array}{c} [?P_1 \wp \dots \wp ?P_h] \\ \text{NELc} \parallel \Delta_P \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi_S \\ [S'\{R\} \wp P_1 \wp \dots \wp P_h] \end{array} .$$

By applying the induction hypothesis to  $\Pi_S$  we get  $P_R$  such that

$$\begin{array}{c} !\dots![\{ \} \wp P_R] \\ \text{NELc} \parallel \Delta' \\ [S'\{ \} \wp P_1 \wp \dots \wp P_h] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \Pi \\ [R \wp P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !!\dots![\{ \} \wp P_R] \\ \text{NELc} \parallel \Delta' \\ ![S'\{ \} \wp P_1 \wp \dots \wp P_h] \\ \{p\downarrow\} \\ [!S'\{ \} \wp ?P_1 \wp \dots \wp ?P_h] \\ \text{NELc} \parallel \Delta_P \\ [!S'\{ \} \wp P] \end{array}$$

Note that in this case the number of  $!$  in front of  $[\{ \} \wp P_R]$  increases.

- (d) The case where  $S\{ \} = [?S'\{ \} \wp P]$  is similar to (c).  $\square$

## 5. Elimination of the Up Fragment

In this section, we will first show four lemmas, which are all easy consequences of splitting and which say that the core up rules of system SNEl are admissible if they are applied in a shallow context  $[\{ \} \wp P]$ . Then we will show how context reduction is used to extend these lemmas to any context. As a result, we get a proof of cut elimination that can be considered modular, in the sense that the four core up rules  $\text{ai}\uparrow$ ,  $\text{q}\uparrow$ ,  $\text{p}\uparrow$ , and  $\text{e}\uparrow$  are shown to be admissible, one independently from the other.

**Lemma 5.1.** Let  $P$  be a structure and let  $a$  be an atom. If  $[(a \otimes \bar{a}) \wp P]$  is provable in NELc, then  $P$  is also provable in NELc.

*Proof.* Apply splitting to the proof of  $[(a \otimes \bar{a}) \wp P]$ . This yields:

$$\begin{array}{c} [P_a \wp P_{\bar{a}}] \\ \text{NELc} \parallel P \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [a \wp P_a] \end{array} \quad \text{and} \quad \begin{array}{c} \text{NELc} \parallel \\ [\bar{a} \wp P_{\bar{a}}] \end{array} .$$

By applying Lemma 3.4, we get a derivation from  $\bar{a}$  to  $P_a$  and one from  $a$  to  $P_{\bar{a}}$ . From these we can build our proof

$$\begin{array}{c}
 \circ\downarrow \frac{-}{\circ} \\
 \text{ai}\downarrow \frac{-}{[\bar{a} \wp a]} \\
 \text{NELc} \parallel \\
 [P_a \wp P_{\bar{a}}] \\
 \text{NELc} \parallel \\
 P
 \end{array}$$

as desired.  $\square$

**Lemma 5.2.** Let  $R, T, U, V$  and  $P$  be any NEL structures. If  $[(\langle R \triangleleft U \rangle \otimes \langle T \triangleleft V \rangle) \wp P]$  is provable in NELc, then  $[(\langle R \otimes T \rangle \triangleleft (U \otimes V)) \wp P]$  is also provable in NELc.

*Proof.* By applying splitting several times to the proof of  $[(\langle R \triangleleft U \rangle \otimes \langle T \triangleleft V \rangle) \wp P]$ , we get structures  $P_R, P_T, P_U,$  and  $P_V$  such that

$$\begin{array}{c}
 [ \langle P_R \triangleleft P_U \rangle \wp \langle P_T \triangleleft P_V \rangle ] \\
 \text{NELc} \parallel \\
 P
 \end{array}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [R \wp P_R]
 \end{array}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [U \wp P_U]
 \end{array}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [T \wp P_T]
 \end{array}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [V \wp P_V]
 \end{array}$$

By putting things together, we can build the proof

$$\begin{array}{c}
 \text{NELc} \parallel \\
 \frac{\text{s, s, s, s} \quad \langle [R \wp P_R] \otimes [T \wp P_T] \rangle \triangleleft ([U \wp P_U] \otimes [V \wp P_V])}{\langle [R \otimes T] \wp P_R \wp P_T \rangle \triangleleft [(U \otimes V) \wp P_U \wp P_V]} \\
 \text{q}\downarrow, \text{q}\downarrow \frac{-}{[(\langle R \otimes T \rangle \triangleleft (U \otimes V)) \wp \langle P_R \triangleleft P_U \rangle \wp \langle P_T \triangleleft P_V \rangle]} \\
 \text{NELc} \parallel \\
 [(\langle R \otimes T \rangle \triangleleft (U \otimes V)) \wp P]
 \end{array}$$

as desired.  $\square$

**Lemma 5.3.** Let  $R, T$  and  $P$  be any NEL structures. If  $[(?R \otimes !T) \wp P]$  is provable in NELc, then  $[?(R \otimes T) \wp P]$  is also provable in NELc.

*Proof.* As above, we apply splitting several times to the proof of  $[(?R \otimes !T) \wp P]$  and get structures  $P_R, P_1, \dots, P_h$  such that:

$$\begin{array}{c}
 [!P_R \wp ?P_1 \wp \dots \wp ?P_h] \\
 \text{NELc} \parallel \\
 P
 \end{array}
 \quad
 \text{and}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [R \wp P_R]
 \end{array}
 \quad
 \text{and}
 \quad
 \begin{array}{c}
 \text{NELc} \parallel \\
 [T \wp P_1 \wp \dots \wp P_h]
 \end{array}$$

By putting things together, we can build the proof

$$\begin{array}{c}
\text{NELc} \parallel \\
\frac{!([R \wp P_R] \otimes [T \wp P_1 \wp \dots \wp P_h])}{![(R \otimes T) \wp P_R \wp P_1 \wp \dots \wp P_h]} \\
\text{s, s} \\
\{p\downarrow\} \parallel \\
[?(R \otimes T) \wp !P_R \wp ?P_1 \wp \dots \wp ?P_h] \\
\text{NELc} \parallel \\
[?(R \otimes T) \wp P]
\end{array}$$

as desired.  $\square$

**Lemma 5.4.** Let  $P$  be any NEL structure. If  $[?\circ \wp P]$  is provable in NELc, then  $[\circ \wp P]$  is also provable in NELc.

*Proof.* This is now a trivial exercise, that we leave to the reader.  $\square$

By the use of context reduction (Lemma 4.1), we can extend the statements of Lemmas 5.1–5.4 from shallow contexts  $[\{ \} \wp P]$  to arbitrary contexts  $S\{ \}$ , which is done by the following lemma.

**Lemma 5.5.** Let  $R, T, U$  and  $V$  be any structures, let  $a$  be an atom and let  $S\{ \}$  be any context. Then we have the following

- (i) If  $S(a \otimes \bar{a})$  is provable in NELc, then so is  $S\{\circ\}$ .
- (ii) If  $S(\langle R \triangleleft U \rangle \otimes \langle T \triangleleft V \rangle)$  is provable in NELc, then so is  $S(\langle (R \otimes T) \triangleleft (U \otimes V) \rangle)$ .
- (iii) If  $S(?R \otimes !T)$  is provable in NELc, then so is  $S\{?(R \otimes T)\}$ .
- (iv) If  $S\{?\circ\}$  is provable in NELc, then so is  $S\{\circ\}$ .

*Proof.* All four statements are proved similarly. We will here show only the third: Let a proof of  $S(?R \otimes !T)$  be given and apply context reduction, to get a structure  $P$ , such that

$$\begin{array}{ccc}
! \dots ![\{ \} \wp P] & & \\
\text{NELc} \parallel \Delta & \text{and} & \text{NELc} \parallel \Pi \\
S\{ \} & & [?(R \otimes !T) \wp P]
\end{array}$$

By Lemma 5.3 there is a proof  $\Pi'$  of  $[?(R \otimes T) \wp P]$ . By plugging  $?(R \otimes T)$  into the hole of  $\Delta$ , we can build

$$\begin{array}{c} \{\circ\downarrow, e\downarrow\} \parallel \\ \dots \circ \\ \text{NELc} \parallel \Pi' \\ \dots ![?(R \otimes T) \wp P] \\ \text{NELc} \parallel \Delta \\ S\{?(R \otimes T)\} \end{array}$$

It is obvious that the other statements are proved in the same way.  $\square$

**Lemma 5.6.** If a structure  $R$  is provable in  $\text{NELc} \cup \{\text{ai}\uparrow, \text{q}\uparrow, \text{p}\uparrow, \text{e}\uparrow\}$  then it is also provable in  $\text{NELc}$ .

*Proof.* The instances of the rules  $\text{ai}\uparrow, \text{q}\uparrow, \text{p}\uparrow, \text{e}\uparrow$  are removed one after the other (starting with the topmost one) via Lemma 5.5.  $\square$

Now we can very easily give a proof for the cut elimination theorem for the system  $\text{NEL}$ .

*Proof of Theorem 2.8* Cut elimination is obtained in two steps:

$$\begin{array}{ccc} \begin{array}{c} \circ\downarrow \text{---} \\ \circ \\ \text{SNEL} \parallel \\ R \end{array} & \xrightarrow{1} & \begin{array}{c} \text{NELc} \cup \{\text{ai}\uparrow, \text{q}\uparrow, \text{p}\uparrow, \text{e}\uparrow\} \parallel \\ R' \\ \{\text{w}\downarrow, \text{b}\downarrow, \text{g}\downarrow\} \parallel \\ R \end{array} & \xrightarrow{2} & \begin{array}{c} \text{NELc} \parallel \\ R' \\ \{\text{w}\downarrow, \text{b}\downarrow, \text{g}\downarrow\} \parallel \\ R \end{array} \end{array}$$

Step 1 is an application of the decomposition (Theorem 2.11). The instances of  $\text{g}\uparrow, \text{b}\uparrow, \text{w}\uparrow$  disappear because their premise must be the unit  $\circ$ , which is impossible. Step 2 is just Lemma 5.6.  $\square$

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules from the cut. So, our technique is more general than cut elimination in the sequent calculus, for two reasons:

- 1 it applies to connectives that admit no sequent calculus definition, as  $\text{seq}$ ;
- 2 it can be used to show admissibility of non-infinitary rules that involve no negation, like  $\text{q}\uparrow$  and  $\text{p}\uparrow$ .

## 6. Perspectives

We think that the techniques developed here for splitting can be exported to the many modal logics already available in deep inference (some of which have no known cut-free presentation in Gentzen formalisms). The reason is that linear logic modalities have a similar behaviour to those of modal logic. This is particularly obvious if we observe that

the promotion rule of NEL is the same as the K rule of all modal logics in deep inference (corresponding to the K axiom of basic modal logic). Of course, contraction in linear logic, and in NEL, is restricted, but the splitting theorems, crucially, do not make any use of it.

Apart from its use in getting cut elimination, splitting is a powerful tool for reducing proof search non-determinism in deep inference proof systems. This is explored in the works (Kah06; Kah08).

We are currently investigating, in the context of the INRIA ARC project REDO, the relations between splitting and the focusing technique in linear logic (And92; Mil96), which is at the basis of ludics (Gir01). It appears that focusing can be justified and greatly generalized by splitting in deep inference. It seems like splitting is a way to explore the duality between any subformula and its context, so unveiling a new logical symmetry.

## Acknowledgments

We thank Dale Miller, as well as David Baelde, Nicolas Guenot, and Vivek Nigam for fruitful discussions on focusing and splitting.

## References

- Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic*, 59(2):543–574, 1994.
- Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
- Richard F. Blute, Alessio Guglielmi, Ivan T. Ivanov, Prakash Panangaden, and Lutz Straßburger. A logical basis for quantum evolution and entanglement. Submitted, 2010.
- Richard Blute, Prakash Panangaden, and Lutz Straßburger. The logic BV and quantum causality. In *Trends in Logic VI: Logic and the foundations of physics: space, time and quanta*, 2008.
- Richard Blute, Prakash Panangaden, and Sergey Slavnov. Deep inference and probabilistic coherence spaces. Submitted, 2009.
- Paola Bruscoli. A purely logical account of sequentiality in proof search. In Peter J. Stuckey, editor, *Logic Programming, 18th International Conference*, volume 2401 of *Lecture Notes in Artificial Intelligence*, pages 302–316. Springer-Verlag, 2002.
- Kai Brünnler. Atomic cut elimination for classical logic. In M. Baaz and J. A. Makowsky, editors, *CSL 2003*, volume 2803 of *Lecture Notes in Computer Science*, pages 86–97. Springer-Verlag, 2003.
- Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *LNAI*, pages 347–361. Springer, 2001.
- Arnaud Fleury and Christian Retoré. The mix rule. *Math. Structures in Comp. Science*, 4(2):273–285, 1994.
- Alessio Guglielmi and Tom Gundersen. Normalisation control in deep inference via atomic flows. *Logical Methods in Computer Science*, 4(1:9):1–36, 2008.
- Alessio Guglielmi, Tom Gundersen, and Michel Parigot. A proof calculus which reduces syntactic bureaucracy. In Christopher Lynch, editor, *21st International Conference on Rewriting*

- Techniques and Applications*, volume 6 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 135–150. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2010.
- Alessio Guglielmi, Tom Gundersen, and Lutz Straßburger. Breaking paths in atomic flows for classical logic. In *LICS 2010*, 2010.
- Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
- Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Math. Structures in Comp. Science*, 11(3):301–506, 2001.
- Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.
- Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.
- Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1), 2007.
- Ozan Kahramanoğulları. System BV without the equalities for unit. In C. Aykanat, T. Dayer, and I. Körpeoğlu, editors, *19th International Symposium on Computer and Information Sciences, ISCIS 2004*, volume 3280 of *Lecture Notes in Computer Science*, pages 986–995. Springer-Verlag, 2004.
- Ozan Kahramanoğulları. Reducing nondeterminism in the calculus of structures. In M. Hermann and A. Voronkov, editors, *LPAR 2006*, volume 4246 of *Lecture Notes in Artificial Intelligence*, pages 272–286. Springer-Verlag, 2006.
- Ozan Kahramanoğulları. System BV is NP-complete. *Annals of Pure and Applied Logic*, 2007. In press.
- Ozan Kahramanoğulları. Interaction and depth against nondeterminism in proof search. Submitted, 2008.
- Dale Miller. Forum: A multiple-conclusion specification logic. *Theoretical Computer Science*, 165:201–232, 1996.
- Mitsuhiro Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, pages 333–396, 1999.
- Christian Retoré. *Réseaux et Séquents Ordonnés*. PhD thesis, Université Paris VII, 1993.
- Christian Retoré. Pomset logic: A non-commutative extension of classical linear logic. In Ph. de Groote and J. R. Hindley, editors, *Typed Lambda Calculus and Applications, TLCA'97*, volume 1210 of *LNCS*, pages 300–318. Springer, 1997.
- Lutz Straßburger and Alessio Guglielmi. A system of interaction and structure IV: The exponentials and decomposition, 2009. To appear in ACM ToCL.
- Lutz Straßburger. *Linear Logic and Noncommutativity in the Calculus of Structures*. PhD thesis, Technische Universität Dresden, 2003.
- Lutz Straßburger. MELL in the Calculus of Structures. *Theoretical Computer Science*, 309(1–3):213–285, 2003.
- Lutz Straßburger. System NEL is undecidable. In Ruy De Queiroz, Elaine Pimentel, and Lucília Figueiredo, editors, *10th Workshop on Logic, Language, Information and Computation (WoLLIC)*, volume 84 of *Electronic Notes in Theoretical Computer Science*, 2003.
- Alwen Fernando Tiu. A system of interaction and structure II: The need for deep inference. *Logical Methods in Computer Science*, 2(2):1–24, 2006.