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A System of Interaction and Structure V: The Exponentials and Splitting

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Abstract

System NEL is the mixed commutative/non-commutative linear logic BV augmented with linear logic's exponentials, or, equivalently, it is MELL augmented with the non-commutative self-dual connective seq. System NEL is Turing-complete, it is able to directly express process algebra sequential composition and it faithfully models causal quantum evolution. In this paper, we show cut elimination for NEL, based on a property that we call *splitting*. NEL is presented in the calculus of structures, which is a deep-inference formalism, because no Gentzen formalism can express it analytically. The splitting theorem shows how and to what extent we can recover a sequent-like structure in NEL proofs. Together with the decomposition theorem, proved in the previous paper of the series, this immediately leads to a cut-elimination theorem for NEL.

1 Introduction

This is the fifth in a series of papers dedicated to the proof theory of a self-dual non-commutative operator, called *seq*, in the context of linear logic.

Together with the closely related fourth paper *"A System of Interaction and Structure IV: The Exponentials and Decomposition"* [SG09], it studies the normalization theory of the logic of *seq* in the presence of linear logic's exponentials. The overall objective of this series of papers is to establish a proof system, and its normalization theory, for the most expressive logic achievable around seq.

The first paper *"A System of Interaction and Structure"* [Gug07] introduced seq in the context of multiplicative linear logic. The resulting logic is called BV. The proof

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system for BV is presented in the formalism called the *calculus of structures*, which is the simplest formalism in the methodology of *deep inference*.

In the second paper *"A System of Interaction and Structure II: The Need for Deep Inference"* [Tiu06], Alwen Tiu shows that deep inference is necessary to obtain analyticity for BV. In other words, traditional Gentzen proof theory is not sufficient to deal with seq. This is achieved by a (remarkable) construction exploiting seq for nesting logical structures at unbounded depth level, in such a way that any sound non-deep-inference proof system has to be incomplete.

The third paper, currently being elaborated, explores the connection between BV and pomset logic [Ret97]. Pomset logic is a variant of linear logic possessing self-dual non-commutativity, and it has been studied with more conventional methods than deep inference.

Self-Dual Non-Commutativity, Pomset Logic and System BV

This line of research started from the study of pomset logic, which is the first example known to us of a logic with a self-dual non-commutative operator. Pomset logic naturally derives from the study of coherence spaces for multiplicative linear logic (see Girard's [Gir87]), and its self-dual operator has a close correspondence to sequential operators as defined in process algebras. For example, the '.' operator of CCS is self-dual and non-commutative.

Pomset logic is derived from coherence spaces and is presented both in proof nets and in a modified Gentzen sequent calculus. Pomset logic possesses the three logical operators *before* (which is self-dual and non-commutative), and the par and tensor that are familiar from linear logic. No complete analytic system is known for pomset logic, and we strongly suspect that, in order to get one, deep inference is necessary; in other words, Gentzen theory would not be sufficient, for the same reason that it is not sufficient for BV [Tiu06].

The 'before' operator is a natural source of sequentialisation. For example, the cut elimination procedure in the proof nets of pomset logic gets sequentialised by the non-commutative links. This naturally induces a computational model where sequentiality plays a role as important as parallelism, which is interesting in the light of the Curry-Howard correspondence. Non-commutative logics are also important in linguistics, their use dating back to the Lambek calculus [Lam58].

Similarly to pomset logic, system BV contains the logical operators seq (which is self-dual and non-commutative), par and tensor, and it has been derived from semantics. Differently from pomset logic, the semantics adopted are relation webs (instead of coherence spaces) and the syntax is the calculus of structures, *i.e.*, a form of deep inference.

BV and pomset logic share their motivations: essentially, non-commutativity brings sequentiality, and sequentiality is an archetypal algebraic composition both for human and computer languages. As a matter of fact, BV's self-dual non-commutativity captures very precisely the sequentiality notion of CCS [Mil89] (and so of other process algebras), as Bruscoli shows in [Bru02].

We know that system BV is NP-complete [Kah07], and its feasibility for proof search has been studied [Kah04]. Retoré and Straßburger are working at the conjecture that BV and pomset logic are equivalent.

Recently, BV has been employed to axiomatise causal quantum computation better than linear logic does [BPS08], and it gave rise to a new class of categorical models [BPS09].

The Calculus of Structures and Deep Inference

Deep inference was born together with BV, precisely for giving BV a normalisation theory, and directly out of the relation web semantics. The ideology of deep inference is very simple, and it simply generalises Gentzen's methodology: we stipulate that proofs can be composed by the same logical operators as formulae. In Gentzen's proof theory, this composition is bounded in such a way that the shape of a proof is a tree, which, in the analytic case, is determined by the formula structure of the proof conclusion.

The freedom of composition in deep inference leads to a seemingly more complicated normalisation theory, because the principal formula in a deep-inference cut rule instance does not determine the shape of the subproofs above it. On the other hand, the same freedom of composition is what allows us to design an absolutely straightforward analytic and complete system for seq, which is what Tiu, in [Tiu06], showed to be impossible in Gentzen's theory (no matter how complicated a system we might design).

Deep inference is an increasingly influential ideology in proof theory. Several logics which were lacking analytic Gentzen proof systems have been shown to enjoy very simple analytic proof systems in deep inference. This is especially true of modal logics [SS05, Sto07, Brü06], but also of Yetter's non-commutative logic [DG04], and there is work in progress for several intermediate logics.

Deep inference has the peculiar property of allowing proof systems whose rules are *local* (*i.e.*, rules whose complexity is bounded by a constant [BT01, Str02]). Locality is important for normalisation, because it allows unprecedented possibilities of manipulating proofs. Recently, locality led us to develop geometric control structures for normalisation in classical logic, called atomic flows [GG07], which are objects similar to the proof nets in [LS05]. Furthermore, the locality of deep inference has led to proof nets for multiplicative linear logic with units [SL04, LS06], which could not be properly captured by proof nets based on the sequent calculus [Gir96].

Finally, we shall mention that the compositional freedom of deep inference led to the design of proof systems for propositional logic that are as efficient as Frege or Gentzen systems in terms of proof complexity. However, they have the following characteristics:

- Contrary to Gentzen systems, and like Frege systems, they can be extended with Tseitin's extension rule or with Frege's substitution one, and they retain the polynomial equivalence with the corresponding Frege extensions, which are the most powerful known proof systems in terms of proof complexity.

- Contrary to Frege systems, the deep-inference systems have a normalisation theory, *i.e.*, a proof theory.

- Compared to analytic Gentzen systems, deep-inference analytic systems exhibit an exponential speed-up, for example Statman formulae have polynomial proofs in deep inference.

These results can be found in [BG09].

In this series of papers, we adopt the calculus of structures, which is the simplest formalism conceivable in deep inference, and the only one that has been fully developed so far.

The Exponentials and Splitting

This fifth paper, and the fourth paper in the series are devoted to the proof theory of system BV when it is enriched with linear logic's exponentials. We call NEL (non-commutative exponential linear logic) the resulting system. We can also consider NEL as MELL (multiplicative exponential linear logic [Gir87]) plus seq. NEL, which was first presented in [GS02], is conservative over both BV and over MELL augmented by the mix and nullary mix rules [FR94, Ret93, AJ94]. Note that, like BV, NEL cannot be analytically expressed outside of deep inference. System NEL can be immediately understood by anybody acquainted with the sequent calculus, and is aimed at the same range of applications as MELL, but it offers, of course, explicit sequential composition.

NEL is especially interesting because it has been proved to be Turing-complete [Str03c]. The complexity of MELL is currently unknown, but MELL is widely conjectured to be decidable. If that was the case, then the line towards Turing-completeness would clearly be crossed by seq, which, in fact, has been interpreted already as an effective mechanism to structure a Turing machine tape. This is something that MELL, which is fully commutative, apparently cannot do.

Each of the two papers is devoted to a theorem: *splitting* in this paper and *decomposition* in the previous paper. Together, the two theorems immediately yield cut-elimination, which is claimed in this paper.

Because of Tiu's counterexample in [Tiu06], and because NEL is conservative over BV, we know that there can be no complete analytic presentation of NEL outside of deep inference. This is essentially due to the possibility of building, with par and seq, deeply nested structures that are essential to provability in BV and NEL. The typical Gentzen structural induction builds proof trees by proceeding from the outside and going inwards into the formula to be proved. Tiu's counterexample recursively hides pairs of crucial pieces of information deeply inside formulae, so that the Gentzen induction can only reach them when they have been separated into different branches of the proof tree, where they cannot help any more in building a proof.

Splitting (first pioneered in [Gug07]) is, in a sense, a way of rebuilding into deep inference the structure of Gentzen sequent calculus proofs, to the extent possible in the presence of par and seq. The technique consists in blocking the access of inference rules to a part of the formula to be proved, however deep; then, we remove from the context of this blocked part as much 'logical material' as possible. In other words, we prove as much as we can, of a given formula, in the presence of a part of it that has been blocked. The splitting theorem states properties of what is left of the context of the blocked part, in relation to the shape of the blocked part. It turns out that the splitting property is nothing else than a generalization of the shape of Gentzen calculi rules. It precisely coincides with them when we stipulate that the blocked part of a formula is at the shallowest possible level.

Splitting is a hard theorem to prove, but, thanks to the decomposition theorem proved in [SG09], we only need to prove it for a fragment of NEL, and this is what we do in this paper. Once splitting is available, cut elimination follows immediately.

The main results of this paper have already been presented, without proof, in [GS02]. For several years, the proofs of the statements have been available in a manuscript on the web.

2 The System

We define the language for system NEL and its variants, as an extension of the language for BV, defined in [Gug07]. Intuitively, $[S_1 \parr \cdots \parr S_h]$ corresponds to a sequent $\vdash S_1, \dots, S_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity (and associativity). The structure $(S_1 \otimes \cdots \otimes S_h)$ corresponds to the associative and commutative tensor connection of S_1, …, S_h. The structure $\langle S_1 \lhd \cdots \lhd S_h \rangle$ is associative and *non-commutative*: this corresponds to the new logical operator, called *seq*, that we add to those of MELL.[1]

Definition 2.1. There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by a, b, …. *Structures* are denoted by S, P, Q, R, T, U, V, W, X and Z. The structures of the *language* NEL are generated by

$$S ::= a \mid \circ \mid [\underbrace{S \parr \cdots \parr S}_{>0}] \mid (\underbrace{S \otimes \cdots \otimes S}_{>0}) \mid \langle \underbrace{S \lhd \cdots \lhd S}_{>0} \rangle \mid ?S \mid !S \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not an atom and $\bar{S}$ is the *negation* of the structure S. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R \parr T]$ stands for $S\{[R \parr T]\}$.

Structures come with equational theories establishing some basic, decidable algebraic laws by which structures are indistinguishable. These are analogous to the laws of associativity, commutativity, idempotency, and so on, usually imposed on sequents. The difference is that we merge the notions of formula and sequent, and we extend the equations to formulae. The structures of the language NEL are equivalent modulo the relation $=$, defined in Figure 1. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (elements of the sequences are separated by $\parr$, $\lhd$, or $\otimes$, as appropriate in the context).

Definition 2.2. An (*inference*) *rule* is any scheme

$$\rho \frac{T}{R} \quad ,$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*; R or T, but not both, may be missing. A (*proof*) *system*, denoted by $\mathscr{S}$, is a set of rules. A *derivation* in a system $\mathscr{S}$ is a finite chain of instances of rules of $\mathscr{S}$, and is denoted by Δ; a derivation can consist of just one structure. The topmost structure in a derivation is called its *premise*; the bottommost structure is called *conclusion*.

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[1] Please note that we slightly change the syntax with respect to [Gug07, Tiu06]: In these papers commas where used in the places of the connectives $\parr$, $\otimes$, and $\lhd$. Although there is some redundancy in having the connectives and the three different types of brackets, we think, it is easier to parse for the reader.

Associativity

$$\begin{aligned}[\vec{R} \parr [\vec{T}] \parr \vec{U}] &= [\vec{R} \parr \vec{T} \parr \vec{U}] \\ (\vec{R} \otimes (\vec{T}) \otimes \vec{U}) &= (\vec{R} \otimes \vec{T} \otimes \vec{U}) \\ \langle \vec{R} \lhd \langle \vec{T} \rangle \lhd \vec{U} \rangle &= \langle \vec{R} \lhd \vec{T} \lhd \vec{U} \rangle\end{aligned}$$

Commutativity

$$\begin{aligned}[\vec{R} \parr \vec{T}] &= [\vec{T} \parr \vec{R}] \\ (\vec{R} \otimes \vec{T}) &= (\vec{T} \otimes \vec{R})\end{aligned}$$

Unit

$$\begin{aligned}[\circ \parr \vec{R}] &= [\vec{R}] \\ (\circ \otimes \vec{R}) &= (\vec{R}) \\ \langle \circ \lhd \vec{R} \rangle &= \langle \vec{R} \rangle \\ \langle \vec{R} \lhd \circ \rangle &= \langle \vec{R} \rangle\end{aligned}$$

Singleton

$$[R] = (R) = \langle R \rangle = R$$

Negation

$$\begin{aligned}\overline{\circ} &= \circ \\ \overline{[R_1 \parr \cdots \parr R_h]} &= (\bar{R}_1 \otimes \cdots \otimes \bar{R}_h) \\ \overline{(R_1 \otimes \cdots \otimes R_h)} &= [\bar{R}_1 \parr \cdots \parr \bar{R}_h] \\ \overline{\langle R_1 \lhd \cdots \lhd R_h \rangle} &= \langle \bar{R}_1 \lhd \cdots \lhd \bar{R}_h \rangle \\ \overline{?R} &= !\bar{R} \\ \overline{!R} &= ?\bar{R} \\ \bar{\bar{R}} &= R\end{aligned}$$

Contextual Closure

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

Figure 1: Basic equations for the syntactic equivalence =

A derivation Δ whose premise is T, conclusion is R, and whose rules are in $\mathscr{S}$ is denoted by

$$\begin{array}{c} T \\ \mathscr{S} \Big\Vert \Delta \\ R \end{array} \quad .$$

The typical inference rules are of the kind

$$\rho \frac{S\{T\}}{S\{R\}} \quad .$$

This rule scheme ρ specifies that if a structure matches R, in a context $S\{\ \}$, it can be rewritten as specified by T, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). A rule corresponds to implementing in the deductive system *any axiom* $T \Rightarrow R$, where $\Rightarrow$ stands for the implication we model in the system, in our case linear implication. The case where the context is empty corresponds to the sequent calculus. For example, the linear logic sequent calculus rule

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$$

could be simulated easily in the calculus of structures by the rule

$$\otimes' \frac{(\Gamma \otimes [A \parr \Phi] \otimes [B \parr \Psi])}{(\Gamma \otimes [(A \otimes B) \parr \Phi \parr \Psi])} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. The structure Γ stands for the times structure of the other hypotheses in the derivation tree. More precisely, any sequent calculus derivation

$$\begin{array}{c} \underbrace{\vdash \Gamma_1 \;\; \cdots \;\; \vdash \Gamma_{i-1} \quad \otimes\dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad \vdash \Gamma_{i+1} \;\; \cdots \;\; \vdash \Gamma_h}_{\Delta} \\ \vdash \Sigma \end{array}$$

containing the $\otimes$ rule can by simulated by

$$\begin{array}{c} \otimes' \dfrac{(\Gamma'_1 \otimes \ldots \otimes \Gamma'_{i-1} \otimes [A' \parr \Phi'] \otimes [B' \parr \Psi'] \otimes \Gamma'_{i+1} \otimes \ldots \otimes \Gamma'_h)}{(\Gamma'_1 \otimes \ldots \otimes \Gamma'_{i-1} \otimes [(A' \otimes B') \parr \Phi' \parr \Psi'] \otimes \Gamma'_{i+1} \otimes \ldots \otimes \Gamma'_h)} \\ \Big\Vert \, \Delta' \\ \Sigma' \end{array} \quad ,$$

in the calculus of structures, where Γ'_j, A', B', Φ', Ψ', Δ' and Σ' are obtained from their counterparts in the sequent calculus by the obvious translation. This means that by this method every system in the one-sided sequent calculus can be ported trivially to the calculus of structures.

Of course, in the calculus of structures, rules could be used as axioms of a generic Hilbert system, where there is no special, structural relation between T and R: then all the good proof theoretical properties of sequent systems would be lost. We will be careful to design rules in a way that is conservative enough to allow us to prove cut elimination, and such that they possess the subformula property.

In our systems, rules come in pairs,

$$\rho{\downarrow}\,\frac{S\{T\}}{S\{R\}} \text{ (down version)} \qquad \text{and} \qquad \rho{\uparrow}\,\frac{S\{\bar{R}\}}{S\{\bar{T}\}} \text{ (up version)} \quad .$$

Sometimes rules are self-dual, i.e., the up and down versions are identical, in which case we omit the arrows. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\Rightarrow$ is the implication and $\bar{(\cdot)}$ the negation of the logic. In the case of NEL these are linear implication and linear negation. We will be able to get rid of the up rules without affecting provability—after all, $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$ are equivalent statements in many logics. Remarkably, the cut rule reduces into several up rules, and this makes for a modular decomposition of the cut elimination argument because we can eliminate up rules one independently from the other.

Let us now define system NEL by starting from an up-down symmetric variation, that we call SNEL. It is made by two sub-systems that we will call conventionally *interaction* and *structure*. The interaction fragment deals with negation, i.e., duality. It corresponds to identity and cut in the sequent calculus. In our calculus these rules become mutually top-down symmetric and both can be reduced to their atomic counterparts.

The structure fragment corresponds to logical and structural rules in the sequent calculus; it defines the logical operators. Differently from the sequent calculus, the operators need not be defined in isolation, rather complex contexts can be taken into consideration. In the following system we consider *pairs* of logical relations, one inside the other.

$$\mathsf{ai}{\downarrow}\,\frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad\qquad \mathsf{ai}{\uparrow}\,\frac{S(a \otimes \bar{a})}{S\{\circ\}}$$

$$\mathsf{s}\,\frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]}$$

$$\mathsf{q}{\downarrow}\,\frac{S\langle [R \parr U] \lhd [T \parr V] \rangle}{S[\langle R \lhd T \rangle \parr \langle U \lhd V \rangle]} \qquad\qquad \mathsf{q}{\uparrow}\,\frac{S(\langle R \lhd U \rangle \otimes \langle T \lhd V \rangle)}{S\langle (R \otimes T) \lhd (U \otimes V) \rangle}$$

$$\mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad\qquad \mathsf{p}{\uparrow}\,\frac{S(?R \otimes !T)}{S\{?(R \otimes T)\}}$$

$$\mathsf{e}{\downarrow}\,\frac{S\{\circ\}}{S\{!\circ\}} \qquad\qquad \mathsf{e}{\uparrow}\,\frac{S\{?\circ\}}{S\{\circ\}}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\circ\}}{S\{?R\}} \qquad\qquad \mathsf{w}{\uparrow}\,\frac{S\{!R\}}{S\{\circ\}}$$

$$\mathsf{b}{\downarrow}\,\frac{S[?R \parr R]}{S\{?R\}} \qquad\qquad \mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R \otimes R)}$$

$$\mathsf{g}{\downarrow}\,\frac{S\{??R\}}{S\{?R\}} \qquad\qquad \mathsf{g}{\uparrow}\,\frac{S\{!R\}}{S\{!!R\}}$$

Figure 2: System SNEL

Definition 2.3. In Figure 2, *system* SNEL (*symmetric non-commutative exponential linear logic*) is shown. The rules $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, s, $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{e}{\downarrow}$, $\mathsf{e}{\downarrow}$, $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{g}{\downarrow}$, and $\mathsf{g}{\uparrow}$ are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *empty*, *coempty*, *weakening*, *coweakening*, *absorption*, *coabsorption*, *digging*, and *codigging*. The *down fragment* of SNEL is $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{e}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{g}{\downarrow}\}$, the *up fragment* is $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{e}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{g}{\uparrow}\}$.

There is a straightforward two-way correspondence between structures not involving seq and formulae of MELL: for example

$$![(?a \otimes b) \parr \bar{c} \parr !\bar{d}] \quad \text{corresponds to} \quad !((?a \otimes b) \parr c^{\perp} \parr !d^{\perp}) \quad ,$$

and vice versa. Units are mapped into $\circ$ (since $1 \equiv \perp$, when mix and mix0 are added to MELL). System SNEL is just the merging of systems SBV and SELS shown in [Gug07, GS01, Str03b, Str03a]; there one can find details on the correspondence between our systems and linear logic.[2] The rules s, $\mathsf{q}{\downarrow}$ and $\mathsf{q}{\uparrow}$ are the same as in pomset logic viewed as a calculus of cographs [Ret99].

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[2] Note that there is a change of our system with respect to the system SELS in [Str03b] and the version of SNEL presented in the conference version [GS02] of this paper: Here we have added the rules $\mathsf{e}{\downarrow}$, $\mathsf{e}{\uparrow}$, $\mathsf{g}{\downarrow}$, and $\mathsf{g}{\uparrow}$, whereas previously we used the equations $??R = ?R$ and $!!R = !R$, as well as $!\circ = \circ = ?\circ$ in [GS02] and $!1 = 1$ and $?\perp = \perp$ in [Str03b] (see also [Str03a]). From the

All equations are typical of a sequent calculus presentation, save those for units and contextual closure. Contextual closure just corresponds to equivalence being a congruence, it is a necessary ingredient of the calculus of structures. All other equations can be removed and replaced by rules (see, e.g., [Str05]), as in the sequent calculus. This might prove necessary for certain applications. For our purposes, this setting makes for a much more compact presentation, at a more effective abstraction level.

Negation is involutive and can be pushed to atoms; it is convenient always to imagine it directly over atoms. Please note that negation does not swap arguments of seq, as happens in the systems of Lambek [Lam58] and Abrusci-Ruet [AR00]. The unit $\circ$ is self-dual and common to par, times and seq. One may think of it as a convenient way of expressing the empty sequence. Rules become very flexible in the presence of the unit. For example, the following notable derivation is valid:

$$\begin{array}{rc}
 & (a \otimes b) \\ \cline{2-2}
\mathsf{q}\uparrow & \langle a \triangleleft b \rangle \\ \cline{2-2}
\mathsf{q}\downarrow & [a \parr b]
\end{array}
\qquad \equiv \qquad
\begin{array}{rc}
 & (a \otimes b) \\ \cline{2-2}
= & (\langle a \triangleleft \circ \rangle \otimes \langle \circ \triangleleft b \rangle) \\ \cline{2-2}
\mathsf{q}\uparrow & \langle (a \otimes \circ) \triangleleft (\circ \otimes b) \rangle \\ \cline{2-2}
= & \langle a \triangleleft b \rangle \\ \cline{2-2}
= & \langle [a \parr \circ] \triangleleft [\circ \parr b] \rangle \\ \cline{2-2}
\mathsf{q}\downarrow & [\langle a \triangleleft \circ \rangle \parr \langle \circ \triangleleft b \rangle] \\ \cline{2-2}
= & [a \parr b]
\end{array} \quad .$$

The right-hand side above is just a complicated way of writing the left-hand side. Using the “fake inference rule =” sometimes eases the reading of a derivation.

Each inference rule in Figure 2 corresponds to a linear implication that is sound in MELL plus mix and $\mathsf{mix0}$. For example, promotion corresponds to the implication $!(R \parr T) \multimap (!R \parr ?T)$. Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

Definition 2.4. The following rules are called *interaction* and *cut*:

$$\mathsf{i}\!\downarrow \frac{S\{\circ\}}{S[R \parr \bar{R}]} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{S(R \otimes \bar{R})}{S\{\circ\}} \quad ,$$

where R and $\bar{R}$ are called *principal structures.*

The sequent calculus rule

$$\mathsf{cut}\ \frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$

is realized as

$$\begin{array}{rc}
 & ([A \parr \Phi] \otimes [\bar{A} \parr \Psi]) \\ \cline{2-2}
\mathsf{s} & [([A \parr \Phi] \otimes \bar{A}) \parr \Psi] \\ \cline{2-2}
\mathsf{s} & [(A \otimes \bar{A}) \parr \Phi \parr \Psi] \\ \cline{2-2}
\mathsf{i}\!\uparrow & [\Phi \parr \Psi]
\end{array} \quad ,$$

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viewpoint of provability, there is no difference between the two approaches, but certain properties of the system can be demonstrated in a cleaner way. Also from the viewpoint of denotational semantics, our system is now more easily accessible. For example in coherence spaces [Gir87] we do not have an isomorphism between $!R$ and $!!R$.

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. Notice how the tree shape of derivations in the sequent calculus is realized by making use of tensor structures: in the derivation above, the premise corresponds to the two branches of the cut rule. For this reason, in the calculus of structures rules are allowed to access structures deeply nested into contexts.

The cut rule in the calculus of structures can mimic the classical cut rule in the sequent calculus in its realization of transitivity, but it is much more general. We believe a good way of understanding it is thinking of the rule as being about lemmas *in context*. The sequent calculus cut rule generates a lemma which is valid in the most general context; the new cut rule does the same, but the lemma only affects the limited portion of structure that can interact with it.

We easily get the next two propositions, which say: 1) The interaction and cut rules can be reduced into their atomic forms—note that in the sequent calculus it is possible to reduce interaction to atomic form, but not cut. 2) The cut rule is as powerful as the whole up fragment of the system, and vice versa.

Definition 2.5. A rule ρ is *derivable* in the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and

$$\text{for every instance}\quad \rho\,\frac{T}{R} \quad\text{there exists a derivation}\quad \begin{array}{c} T \\ \mathscr{S}\,\Big\Vert\,\Delta \\ R \end{array} \quad .$$

The systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if

$$\text{for every derivation}\quad \begin{array}{c} T \\ \mathscr{S}\,\Big\Vert\,\Delta \\ R \end{array} \quad\text{there exists a derivation}\quad \begin{array}{c} T \\ \mathscr{S}'\,\Big\Vert\,\Delta' \\ R \end{array} \quad ,$$

and vice versa.

Proposition 2.6. *The rule* $\mathsf{i}{\downarrow}$ *is derivable in* $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{e}{\downarrow}\}$*, and, dually, the rule* $\mathsf{i}{\uparrow}$ *is derivable in the system* $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{e}{\uparrow}\}$*.*

Proof. Induction on principal structures. We show the inductive cases for $\mathsf{i}{\uparrow}$:

$$= \dfrac{\mathsf{i}{\uparrow},\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(P \otimes Q \otimes [\bar{P} \bindnasrepma \bar{Q}])}{S(Q \otimes [(P \otimes \bar{P}) \bindnasrepma \bar{Q}])}}{S[(P \otimes \bar{P}) \bindnasrepma (Q \otimes \bar{Q})]}}{S[\circ \bindnasrepma \circ]}}{S\{\circ\}} \qquad = \dfrac{\mathsf{i}{\uparrow},\mathsf{i}{\uparrow}\,\dfrac{\mathsf{q}{\uparrow}\,\dfrac{S(\langle P \lhd Q\rangle \otimes \langle \bar{P} \lhd \bar{Q}\rangle)}{S\langle (P \otimes \bar{P}) \lhd (Q \otimes \bar{Q})\rangle}}{S\langle \circ \lhd \circ\rangle}}{S\{\circ\}} \qquad \mathsf{e}{\uparrow}\,\dfrac{\mathsf{i}{\uparrow}\,\dfrac{\mathsf{p}{\uparrow}\,\dfrac{S(?P \otimes !\bar{P})}{S\{?(P \otimes \bar{P})\}}}{S\{?\circ\}}}{S\{\circ\}} \quad .$$

The cases for $\mathsf{i}{\downarrow}$ are dual. ☐

Note that in the proof above we tacitly used (for the sake of saving paper) another helpful notation: writing $\mathsf{i}{\uparrow},\mathsf{i}{\uparrow}$ just means that two instances of $\mathsf{i}{\uparrow}$ applied one after the other, where the order does not matter.

Proposition 2.7. *Each rule* $\rho{\uparrow}$ *in* SNEL *is derivable in* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\downarrow}\}$*, and, dually, each rule* $\rho{\downarrow}$ *in* SNEL *is derivable in the system* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\uparrow}\}$*.*

$$\circ{\downarrow}\,\frac{}{\circ} \qquad \mathsf{ai}{\downarrow}\,\frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad \mathsf{e}{\downarrow}\,\frac{S\{\circ\}}{S\{!\circ\}}$$

$$\mathsf{s}\,\frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]} \qquad \mathsf{q}{\downarrow}\,\frac{S\langle [R \parr U] \lhd [T \parr V] \rangle}{S[\langle R \lhd T \rangle \parr \langle U \lhd V \rangle]} \qquad \mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R \parr R]}{S\{?R\}} \qquad \mathsf{g}{\downarrow}\,\frac{S\{??R\}}{S\{?R\}}$$

Figure 3: System NEL

Proof. Each instance

$$\rho{\uparrow}\,\frac{S\{T\}}{S\{R\}}$$

can be replaced by

$$\mathsf{i}{\uparrow}\,\cfrac{\rho{\downarrow}\,\cfrac{\mathsf{s}\,\cfrac{\mathsf{i}{\downarrow}\,\cfrac{S\{T\}}{S(T \otimes [R \parr \bar{R}])}}{S[R \parr (T \otimes \bar{R})]}}{S[R \parr (T \otimes \bar{T})]}}{S\{R\}}$$

and dually. □

In the calculus of structures, we call *core* the set of rules that is used to reduce interaction and cut to atomic form. We use the term *hard core* to denote the set of rules in the core other than atomic interaction/cut and empty/coempty. Rules that are not in the core are called *non-core*.

Definition 2.8. The *core* of SNEL is $\{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{e}{\downarrow}, \mathsf{e}{\uparrow}\}$, denoted by SNELc. The *hard core*, denoted by SNELh, is $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}\}$, and the *non-core* is $\{\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}, \mathsf{g}{\downarrow}, \mathsf{g}{\uparrow}\}$.

System SNEL is up-down symmetric, and the properties we saw are also symmetric. Provability is an asymmetric notion: we want to observe the possible conclusions that we can obtain from a unit premise. We now break the up-down symmetry by adding an inference rule with no premise, and we join this logical axiom to the down fragment of SNEL.

Definition 2.9. The following rule is called *unit*:

$$\circ{\downarrow}\,\frac{}{\circ} \quad .$$

System NEL is shown in Figure 3.

As an immediate consequence of Propositions 2.6 and 2.7 we get:

Proposition 2.10. *The systems* $\mathsf{NEL} \cup \{\mathsf{i}{\uparrow}\}$ *and* $\mathsf{SNEL} \cup \{\circ{\downarrow}\}$ *are strongly equivalent.*

Definition 2.11. A derivation with no premise is called a *proof*, denoted by Π. A system $\mathscr{S}$ *proves* R if there is in the system $\mathscr{S}$ a proof Π whose conclusion is R, written

$$\begin{array}{c} \mathscr{S}\,\Big\Vert\,\Pi \\ R \end{array} \quad .$$

We say that a rule ρ is *admissible* for the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and

$$\text{for every proof} \quad \begin{array}{c} \mathscr{S} \cup \{\rho\}\,\Big\Vert\,\Pi \\ R \end{array} \quad \text{there is a proof} \quad \begin{array}{c} \mathscr{S}\,\Big\Vert\,\Pi' \\ R \end{array} \quad .$$

Two systems are *equivalent* if they prove the same structures.

Except for cut and coweakening, all rules in the systems SNEL and NEL enjoy a kind of subformula property (which we treat as an asymmetric property, by going from conclusion to premise): premises are made of substructures of the conclusions.

To get cut elimination, so as to have a system whose rules all enjoy the subformula property, we could just get rid of $\mathsf{ai}\!\uparrow$ and $\mathsf{w}\!\uparrow$, by proving their admissibility for the other rules. But we can do more than that: the whole up fragment of SNEL (except for s which also belongs to the down fragment) is admissible. This entails a *modular* scheme for proving cut elimination. In the remainder of this paper we will give the proof of the cut elimination theorem:

Theorem 2.12. *System* NEL *is equivalent to* $\mathsf{SNEL} \cup \{\circ\!\downarrow\}$.

Corollary 2.13. *The rule* $\mathsf{i}\!\uparrow$ *is admissible for system* NEL.

Our cut elimination proof relies on the following theorem, which is a special case of a more general one, and whose proof can be found in [SG09].

Theorem 2.14 (Decomposition). *For every derivation* $\begin{array}{c} T \\ \Delta\,\Big\Vert\,\mathsf{SNEL} \\ R \end{array}$ *there is a deriva-*

tion of the shape

$$\begin{array}{rc}
 & T \\
\{\mathsf{g}{\uparrow}\} & \Big\Vert \\
 & T_1 \\
\{\mathsf{b}{\uparrow}\} & \Big\Vert \\
 & T_2 \\
\{\mathsf{w}{\uparrow}\} & \Big\Vert \\
 & T_3 \\
\{\mathsf{e}{\downarrow}\} & \Big\Vert \\
 & T_4 \\
\{\mathsf{ai}{\downarrow}\} & \Big\Vert \\
 & T_5 \\
\mathsf{SNELh} & \Big\Vert \\
 & R_5 \\
\{\mathsf{ai}{\uparrow}\} & \Big\Vert \\
 & R_4 \\
\{\mathsf{e}{\uparrow}\} & \Big\Vert \\
 & R_3 \\
\{\mathsf{w}{\downarrow}\} & \Big\Vert \\
 & R_2 \\
\{\mathsf{b}{\downarrow}\} & \Big\Vert \\
 & R_1 \\
\{\mathsf{g}{\downarrow}\} & \Big\Vert \\
 & R
\end{array}$$

with the same premise and conclusion.

Any linear implication $T \multimap R$, i.e., $[\bar{T} \parr R]$, is related to derivability by:

Corollary 2.15. *For any two structures T and R, we have*

$$\begin{array}{rc} & T \\ \mathsf{SNEL} & \Big\Vert \\ & R \end{array} \qquad \text{if and only if} \qquad \begin{array}{rc} \mathsf{NEL} & \overline{\Big\Vert} \\ & [\bar{T} \parr R] \end{array} \quad .$$

Proof. For the first direction, perform the following transformations:

$$\begin{array}{rcl} & T & \\ \mathsf{SNEL} & \Big\Vert & \Delta \\ & R & \end{array} \quad \overset{1}{\leadsto} \quad \begin{array}{rcl} & [\bar{T} \parr T] & \\ \mathsf{SNEL} & \Big\Vert & \Delta' \\ & [\bar{T} \parr R] & \end{array} \quad \overset{2}{\leadsto} \quad \begin{array}{rcl} & \mathsf{i}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{}{\circ}}{[\bar{T} \parr T]} & \\ \mathsf{SNEL} & \Big\Vert & \Delta' \\ & [\bar{T} \parr R] & \end{array} \quad \overset{3}{\leadsto} \quad \begin{array}{rcl} \mathsf{NEL} & \overline{\Big\Vert} & \Pi \\ & [\bar{T} \parr R] & \end{array} \quad .$$

In the first step we replace each structure S occurring inside Δ by $[\bar{T} \parr S]$, or, in other words, the derivation Δ' is obtained by putting Δ into the context $[\bar{T} \parr \{\ \}]$. This is then transformed into a proof by adding an instance of $\mathsf{i}\!\downarrow$ and $\circ\!\downarrow$. Then we apply Proposition 2.6 and cut elimination (Theorem 2.12) to obtain a proof in system NEL. For the other direction, we proceed as follows:

$$\begin{array}{c} \mathsf{NEL}\,\overline{\Big\|}\,\Pi \\ [\bar{T} \parr R] \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \circ \\ \mathsf{NEL}\setminus\{\circ\!\downarrow\}\,\Big\|\,\Delta \\ [\bar{T} \parr R] \end{array} \quad\rightsquigarrow\quad \begin{array}{c} T \\ \mathsf{NEL}\setminus\{\circ\!\downarrow\}\,\Big\|\,\Delta' \\ \mathsf{i}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{(T \otimes [\bar{T} \parr R])}{[(T \otimes \bar{T}) \parr R]}}{R} \end{array} \quad\rightsquigarrow\quad \begin{array}{c} T \\ \mathsf{SNEL}\,\Big\| \\ R \end{array} \quad ,$$

where the first two steps are trivial, and the last one is an application of Proposition 2.6. □

It is easy to prove that system NEL is a conservative extension of BV and of MELL plus mix and $\mathsf{mix0}$ (see [Gug07, Str03a]). The locality properties shown in [GS01, Str03b] still hold in this system, of course. In particular, the promotion rule is local, as opposed to the same rule in the sequent calculus.

3 Cut Elimination

The classical arguments for proving cut elimination in the sequent calculus rely on the following property: when the principal formulae in a cut are active in both branches, they determine which rules are applied immediately above the cut. This is a consequence of the fact that formulae have a root connective, and logical rules only hinge on that, and nowhere else in the formula.

This property does not necessarily hold in the calculus of structures. Further, since rules can be applied anywhere deep inside structures, everything can happen above a cut. This complicates considerably the task of proving cut elimination. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form, which happens simply and independently of cut elimination. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus: complex interactions with their context are possible. As a consequence, our techniques are largely different from the traditional ones.

Three approaches to cut elimination in the calculus of structures have been explored in previous papers: in [GS01, Str03b] we relied on permutations of rules, in [BT01] Brünnler and Tiu relied on semantics, and in [Brü03] Brünnler presents a simple syntactic method that employs the atomicity of cut together with certain proof theoretical properties of classical logic. Neither method can be applied in our case: We know a counterexample (found by Alwen Tiu) that shows that the coseq rule cannot be permuted up by the same technique that has been used in [GS01, Str03b]. And, so far, there is no provability semantics for NEL (in the sense of phase spaces [Oka99] or other model theoretic semantics) that we could use for a completeness argument.

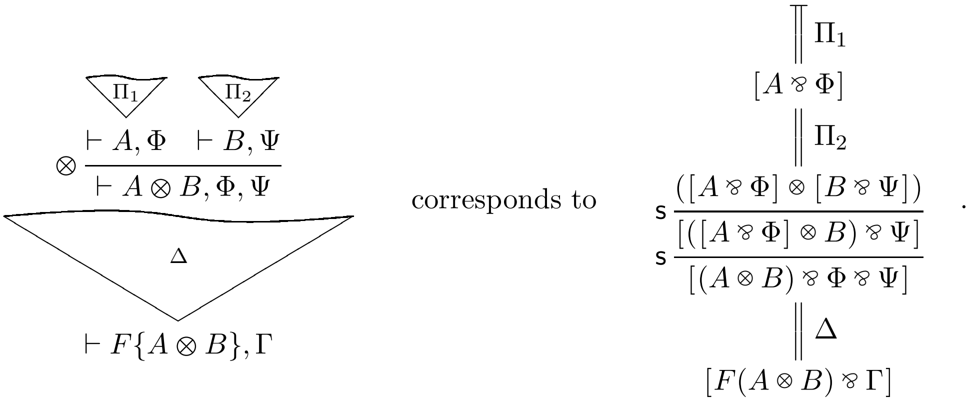

Figure 4: The basic idea behind splitting and context reduction

In this paper we use a fourth technique, called *splitting* [Gug07], which has the advantage of being more uniform than the one based on permutations and which yields a much simpler case analysis. It also establishes a clear connection with the sequent calculus, at least for the fragments of systems that allow for a sequent calculus presentation (in this case, the commutative fragment). Since many systems are expressed in the sequent calculus, our method appears to be entirely general; still, it is independent of the sequent calculus and of completeness with respect to some semantics.

Splitting can be best understood by considering a sequent system with no weakening and absorption (or contraction). Consider, for example, multiplicative linear logic: If we have a proof of the sequent $\vdash F\{A \otimes B\}, \Gamma$, where $F\{A \otimes B\}$ is a formula that contains the subformula $A \otimes B$, we know for sure that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context. This is indicated in Figure 4. We can consider, as shown at the left, the proof for the given sequent as composed of three pieces, Δ, Π_1 and Π_2. In the calculus of structures, many different proofs correspond to the sequent calculus one: they differ for the possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all these proofs, there must also be one that fits the scheme at the right of Figure 4. This precisely illustrates the idea behind the splitting technique.

The derivation Δ in Figure 4 implements a *context reduction* and a proper splitting. We can state, in general, these principles as follows:

1. Context reduction: If $S\{R\}$ is provable, then $S\{\ \}$ can be reduced to the structure $[\{\ \} \parr U]$, such that $[R \parr U]$ is provable. In the example above, $[F\{\ \} \parr \Gamma]$ is reduced to $[\{\ \} \parr \Gamma']$, for some Γ'.

2. Splitting: If $[(R \otimes T) \parr P]$ is provable, then P can be reduced to $[P_1 \parr P_2]$, such that $[R \parr P_1]$ and $[T \parr P_2]$ are provable. In the example above Γ' is reduced to $[\Phi \parr \Psi]$. Splitting holds for all logical operators.

Context reduction is in turn proved by splitting, which is then at the core of the matter. The biggest difficulty resides in proving splitting, and this mainly requires finding the right induction measure.

4 Splitting

In this section we will state and prove splitting, as we will need it for cut elimination. For notational convenience, we define *system* NELc to be the system obtained from NEL by removing the non-core rules:

$$\mathsf{NELc} = \mathsf{NEL} \setminus \{\mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{g}{\downarrow}\} = \{\circ{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{e}{\downarrow}\} = \mathsf{SNELc}{\downarrow} \cup \{\circ{\downarrow}\} \quad .$$

Lemma 4.1 (Splitting). *Let R, T, P be any* NEL *structures.*

(i) *If $[(R \otimes T) \parr P]$ is provable in* NELc*, then there are structures P_R and P_T, such that*

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [R \parr P_R] \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [T \parr P_T] \end{array} \quad .$$

(ii) *If $[\langle R \lhd T \rangle \parr P]$ is provable in* NELc*, then there are structures P_R and P_T, such that*

$$\begin{array}{c} \langle P_R \lhd P_T \rangle \\ \Big\Vert\,\mathsf{NELc} \\ P \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [R \parr P_R] \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [T \parr P_T] \end{array} \quad .$$

Proof. We prove both statements simultaneously by structural induction on the number of atoms in the conclusion and the length (number of rule instances) of the proof, ordered lexicographically. Without loss of generality, assume $R \neq \circ \neq T$ (otherwise both statements are trivially true).

(i) Consider the bottommost rule instance ρ in the proof of $[(R \otimes T) \parr P]$. We can distinguish between three different kinds of cases:

(a) The first kind appears when the redex of ρ is inside R, T or P. Then we have the following situation:

$$\begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ \rho\,\dfrac{[(R' \otimes T) \parr P]}{[(R \otimes T) \parr P]} \end{array}$$

where we can apply the induction hypothesis to Π because it is one rule shorter (if $\rho = \mathsf{ai}{\downarrow}$ also the conclusion is smaller). We get

$$\begin{array}{c} [P_{R'} \parr P_T] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_R \\ [R' \parr P_R] \end{array} \quad \text{and} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_T \\ [T \parr P_T] \end{array}$$

From Π_R, we can get

$$\begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi'_R \\ \rho\,\dfrac{[R' \parr P_R]}{[R \parr P_R]} \end{array}$$

and we are done. If the redex of ρ is inside T or P, the situation is similar.

(b) In the second kind of case the substructure $(R \otimes T)$ is inside the redex of ρ, but is not modified by ρ. These cases can be compared with the "commutative cases" in the usual sequent calculus cut elimination argument. We show only one representative example (a complete case analysis can be found in [Gug07] and [Str03a]): Suppose we have

$$\begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi \\ \mathsf{q}{\downarrow}\,\dfrac{[\langle[(R \otimes T) \parr P_1 \parr P_3] \lhd P_2\rangle \parr P_4]}{[(R \otimes T) \parr \langle P_1 \lhd P_2\rangle \parr P_3 \parr P_4]} \end{array}$$

We can apply the induction hypothesis to Π because it is one rule shorter (the size of the conclusion does not change). This gives us

$$\begin{array}{c} \langle Q_1 \lhd Q_2\rangle \\ \mathsf{NELc}\,\Big\Vert\,\Delta_1 \\ P_4 \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi_1 \\ [(R \otimes T) \parr P_1 \parr P_3 \parr Q_1] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi_2 \\ [P_2 \parr Q_2] \end{array}$$

We can apply the induction hypothesis again to Π_1, because now the the number of atoms in the conclusion is strictly smaller (because we can assume that the instance of $\mathsf{q}{\downarrow}$ is not trivial). We get

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi_R \\ [R \parr P_R] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi_T \\ [T \parr P_T] \end{array}$$

From Δ_1, Δ_2 and Π_2 we can build the following derivation

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \\ \mathsf{NELc}\,\Big\Vert\,\Pi_2 \\ \mathsf{q}{\downarrow}\,\dfrac{[\langle[P_1 \parr Q_1] \lhd [P_2 \parr Q_2]\rangle \parr P_3]}{[\langle P_1 \lhd P_2\rangle \parr P_3 \parr \langle Q_1 \lhd Q_2\rangle]} \\ \mathsf{NELc}\,\Big\Vert\,\Delta_1 \\ [\langle P_1 \lhd P_2\rangle \parr P_3 \parr P_4] \end{array}$$

and we are done. All other cases in this group are similar.

(c) In the last type of case the substructure $(R \otimes T)$ is destroyed by ρ. These cases can be compared to the "key cases" in a standard sequent calculus cut elimination argument. We have only one possibility. The most general situation is as follows:

$$\begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi \\ \mathsf{s}\,\dfrac{[([(R_1 \otimes T_1) \parr P_1] \otimes R_2 \otimes T_2) \parr P_2]}{[(R_1 \otimes R_2 \otimes T_1 \otimes T_2) \parr P_1 \parr P_2]} \end{array}$$

where one of R_1 and R_2 might be $\circ$, but not both of them (similarly for T_1 and T_2). As before, we can apply the induction hypothesis to Π and get

$$\begin{array}{c}[Q_1 \parr Q_2]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_1\\ P_2\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_1\\ [(R_1 \otimes T_1) \parr P_1 \parr Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_2\\ [(R_2 \otimes T_2) \parr Q_2]\end{array}$$

We can apply the induction hypothesis again to Π_1 and Π_2. (Because we assume that the instance of s is not trivial, the conclusions are strictly smaller than the one of the original proof.) We get:

$$\begin{array}{c}[P_{R_1} \parr P_{T_1}]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_3\\ [P_1 \parr Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_{R_1}\\ [R_1 \parr P_{R_1}]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_{T_1}\\ [T_1 \parr P_{T_1}]\end{array}$$

and

$$\begin{array}{c}[P_{R_2} \parr P_{T_2}]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_4\\ Q_2\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_{R_2}\\ [R_2 \parr P_{R_2}]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_{T_2}\\ [T_2 \parr P_{T_2}]\end{array}$$

Now let $P_R = [P_{R_1} \parr P_{R_2}]$ and $P_T = [P_{T_1} \parr P_{T_2}]$. We can build

$$\begin{array}{c}[P_{R_1} \parr P_{R_2} \parr P_{T_1} \parr P_{T_2}]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_4\\ [P_{R_1} \parr P_{T_1} \parr Q_2]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_3\\ [P_1 \parr Q_1 \parr Q_2]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_1\\ [P_1 \parr P_2]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi_{R_1}\\ [R_1 \parr P_{R_1}]\\ \mathsf{NELc}\,\Big\Vert\,\Pi_{R_2}\\ \mathsf{s}\,\dfrac{[(R_1 \otimes [R_2 \parr P_{R_2}]) \parr P_{R_1}]}{[(R_1 \otimes R_2) \parr P_{R_1} \parr P_{R_2}]}\end{array}$$

and a similar proof of $[(T_1 \otimes T_2) \parr P_{T_1} \parr P_{T_2}]$, and we are done.

(ii) The case for $[\langle R \lhd T\rangle \parr P]$ is similar to the one for $[(R \otimes T) \parr P]$, and we leave it to the reader. ☐

Lemma 4.2 (Splitting for Modalities). *Let R and P be any* NEL *structures.*

(i) *If $[!R \parr P]$ is provable in* NELc*, then there are structures $P_1, \dots, P_h$ for some $h \geq 0$, such that*

$$\begin{array}{c}[?P_1 \parr \cdots \parr ?P_h]\\ \mathsf{NELc}\,\Big\Vert\\ P\end{array} \quad\textit{and}\quad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\\ [R \parr P_1 \parr \cdots \parr P_h]\end{array} \quad .$$

(ii) *If* $[?R \parr P]$ *is provable in* NELc, *then there is a structure* P_R, *such that*

$$\begin{array}{c} !P_R \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \qquad \textit{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [R \parr P_R] \end{array} \quad .$$

Proof. The proof is similar to the previous one. We use the same induction measure and the same pattern in the case analysis as before.

(i) We consider again the bottommost rule instance ρ in the proof of $[!R \parr P]$, and we have the same three classes of cases as in the proof of Lemma 4.1.

(a) The redex of ρ is inside R or P. This case is the same as in the proof of Lemma 4.1.

(b) The substructure $!R$ is inside the redex of ρ, but is not changed by ρ. This case is almost literally the same as for Lemma 4.1. We only have to replace $(R \otimes T)$ by $!R$, and

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \qquad \text{by} \qquad \begin{array}{c} [?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array}$$

(As for the previous lemma, the full details can be found in [Str03a].)

(c) The substructure $!R$ is destroyed by ρ. There are two possibilities ($\rho = \mathsf{e}{\downarrow}$ and $\rho = \mathsf{p}{\downarrow}$):

$$\begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ \mathsf{e}{\downarrow}\,\dfrac{[\circ \parr P]}{[!\circ \parr P]} \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ \mathsf{p}{\downarrow}\,\dfrac{[![R \parr P_1] \parr Q_2]}{[!R \parr ?P_1 \parr Q_2]} \end{array}$$

For $\rho = \mathsf{e}{\downarrow}$ we are done immediately by letting $h = 0$. For $\rho = \mathsf{p}{\downarrow}$ we can apply the induction hypothesis to Π and get structures $P_2, \ldots, P_h$ such that

$$\begin{array}{c} [?P_2 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert \\ Q_2 \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [R \parr P_1 \parr P_2 \parr \cdots \parr P_h] \end{array}$$

We immediately get

$$\begin{array}{c} [?P_1 \parr ?P_2 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert \\ [?P_1 \parr Q_2] \end{array} \quad .$$

(ii) As before, consider the bottommost rule instance ρ in the proof of $[?R \parr P]$.

(a) The redex of ρ is inside R or P. This case is the same as before.

(b) The substructure $?R$ is inside the redex of ρ, but is not changed by ρ. As before, this case is almost literally the same as in the proof of Lemma 4.1. This time we have to replace $(R \otimes T)$ by $?R$, and

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \qquad \text{by} \qquad \begin{array}{c} !P_R \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array}$$

(c) The substructure $?R$ is destroyed by ρ. For this case there is only one possibility:

$$\begin{array}{c} \mathsf{NELc} \,\overline{\Big\Vert}\, \Pi \\ \mathsf{p}{\downarrow}\dfrac{[![R \parr P_1] \parr P_2]}{[?R \parr !P_1 \parr P_2]} \end{array}$$

We can apply part (i) of the lemma and get

$$\begin{array}{c} [?Q_1 \parr \cdots \parr ?Q_h] \\ \mathsf{NELc} \Big\Vert \Delta \\ P_2 \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc} \,\overline{\Big\Vert}\, \Pi_R \\ [R \parr P_1 \parr Q_1 \parr \cdots \parr Q_h] \end{array}$$

Now let $P_R = [P_1 \parr Q_1 \parr \ldots \parr Q_h]$. We can build

$$\begin{array}{c} ![P_1 \parr Q_1 \parr \cdots \parr Q_h] \\ \{\mathsf{p}{\downarrow}\} \Big\Vert \\ [!P_1 \parr ?Q_1 \parr \cdots \parr ?Q_h] \\ \mathsf{NELc} \Big\Vert \Delta \\ [!P_1 \parr P_2] \end{array}$$

as desired. □

Lemma 4.3 (Splitting for Atoms). *Let a be any atom and P be any* NEL *structure.*

If there is a proof $\begin{array}{c} \mathsf{NELc} \,\overline{\Big\Vert} \\ [a \parr P] \end{array}$ *then there is a derivation* $\begin{array}{c} \bar{a} \\ \mathsf{NELc} \Big\Vert \\ P \end{array}$.

Proof. After the previous two proofs this is an almost trivial exercise: The case (a) is as before, and for (b), we have to replace $(R \otimes T)$ by a, and

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \qquad \text{by} \qquad \begin{array}{c} \bar{a} \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \quad .$$

For case (c), the only possibility is

$$\begin{array}{c} \mathsf{NELc} \,\overline{\Big\Vert}\, \Pi' \\ \mathsf{ai}{\downarrow}\dfrac{P_1}{[a, \bar{a}, P_1]} \end{array}$$

from which we immediately get

$$\begin{array}{c} \bar{a} \\ \mathsf{NELc}\,\Big\Vert \\ [\bar{a}, P_1] \end{array} .$$

as desired. □

5 Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \} \parr P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

Before giving the statement, we need to define the *modality depth* of a context $S\{\ \}$ to be the number of ! and ? in whose scope the $\{\ \}$ occurs. In the following lemma, the $\{\ \}$ is treated as ordinary atom.

Lemma 5.1 (Context Reduction). *Let R be a* NEL *structure and $S\{\ \}$ be a context. If $S\{R\}$ is provable in* NELc*, then there is a structure P_R, such that*

$$\begin{array}{c} !\cdots![\{\ \} \parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta \\ S\{\ \} \end{array} \qquad \textit{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ [R \parr P_R] \end{array}$$

where the number of ! in front of $[\{\ \} \parr P_R]$ is the modality depth of $S\{\ \}$.

Proof. We proceed by structural induction on the context $S\{\ \}$. The base case when $S\{\ \} = \{\ \}$ is trivial. Now we can distinguish four cases

(a) $S\{\ \} = [(S'\{\ \} \otimes T) \parr P]$ where,without loss of generality, $T \neq \circ$. Note that we do allow $P = \circ$. We can apply splitting (Lemma 4.1) to the proof of $[(S'\{R\} \otimes T) \parr P]$ and get:

$$\begin{array}{c} [P_S \parr P_T] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_S \\ [S'\{R\} \parr P_S] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_T \\ [T \parr P_T] \end{array}$$

Because $T \neq \circ$ we can now apply the induction hypothesis to Π_S and get:

$$\begin{array}{c} !\cdots![\{\ \} \parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \} \parr P_S] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ [R \parr P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \}\parr P_S] \\ \mathsf{NELc}\,\Big\Vert\,\Pi_T \\ \mathsf{s}\,\dfrac{[(S'\{\ \}\otimes[T\parr P_T])\parr P_S]}{[(S'\{\ \}\otimes T)\parr P_S\parr P_T]} \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ [(S'\{\ \}\otimes T)\parr P] \end{array}$$

as desired.

(b) The cases $S\{\ \}=[\langle S'\{\ \}\lhd T\rangle\parr P]$ and $S\{\ \}=[\langle T\lhd S'\{\ \}\rangle\parr P]$ are handled similarly to (a).

(c) If $S\{\ \}=[!S'\{\ \}\parr P]$, then we can apply splitting (Lemma 4.2) to the proof of $[!S'\{R\}\parr P]$ and get:

$$\begin{array}{c} [?P_1\parr\cdots\parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_S \\ [S'\{R\}\parr P_1\parr\cdots\parr P_h] \end{array} \quad .$$

By applying the induction hypothesis to Π_S we get P_R such that

$$\begin{array}{c} !\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \}\parr P_1\parr\cdots\parr P_h] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ [R\parr P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !!\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ ![S'\{\ \}\parr P_1\parr\cdots\parr P_h] \\ \{\mathsf{p}{\downarrow}\}\,\Big\Vert \\ [!S'\{\ \}\parr ?P_1\parr\cdots\parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ [!S'\{\ \}\parr P] \end{array}$$

Note that in this case the number of ! in front of $[\{\ \}\parr P_R]$ increases.

(d) The case where $S\{\ \}=[?S'\{\ \}\parr P]$ is similar to (c). ☐

6 Elimination of the Up Fragment

In this section, we will first show four lemmas, which are all easy consequences of splitting and which say that the core up rules of system SNEL are admissible if they are applied in a shallow context $[\{\ \} \parr P]$. Then we will show how context reduction is used to extend these lemmas to any context. As a result, we get a proof of cut elimination that can be considered modular, in the sense that the four core up rules $\mathsf{ai}\!\uparrow$, $\mathsf{q}\!\uparrow$, $\mathsf{p}\!\uparrow$, and $\mathsf{e}\!\uparrow$ are shown to be admissible, one independently from the other.

Lemma 6.1. *Let P be a structure and let a be an atom. If $[(a \otimes \bar{a}) \parr P]$ is provable in NELc, then P is also provable in NELc.*

Proof. Apply splitting to the proof of $[(a \otimes \bar{a}) \parr P]$. This yields:

$$\begin{array}{c} [P_a \parr P_{\bar{a}}] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [a \parr P_a] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [\bar{a} \parr P_{\bar{a}}] \end{array} .$$

By applying Lemma 4.3, we get a derivation from $\bar{a}$ to P_a and one from a to $P_{\bar{a}}$. From these we can build our proof

$$\begin{array}{c} \mathsf{ai}\!\downarrow \dfrac{\circ\!\downarrow \dfrac{}{\circ}}{[\bar{a} \parr a]} \\ \mathsf{NELc}\,\Big\Vert \\ [P_a \parr P_{\bar{a}}] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array}$$

as desired. □

Lemma 6.2. *Let R, T, U, V and P be any NEL structures. If $[(\langle R \lhd U \rangle \otimes \langle T \lhd V \rangle) \parr P]$ is provable in NELc, then $[\langle (R \otimes T) \lhd (U \otimes V) \rangle \parr P]$ is also provable in NELc.*

Proof. By applying splitting several times to the proof of $[(\langle R \lhd U \rangle \otimes \langle T \lhd V \rangle) \parr P]$, we get structures P_R, P_T, P_U, and P_V such that

$$\begin{array}{c} [\langle P_R \lhd P_U \rangle \parr \langle P_T \lhd P_V \rangle] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [R \parr P_R] \end{array} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [U \parr P_U] \end{array} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [T \parr P_T] \end{array} \quad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [V \parr P_V] \end{array}$$

By putting things together, we can build the proof

$$\begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ \mathsf{q}\!\downarrow,\mathsf{q}\!\downarrow \dfrac{\mathsf{s},\mathsf{s},\mathsf{s},\mathsf{s}\, \dfrac{\langle ([R \parr P_R] \otimes [T \parr P_T]) \lhd ([U \parr P_U] \otimes [V \parr P_V]) \rangle}{\langle [(R \otimes T) \parr P_R \parr P_T] \lhd [(U \otimes V) \parr P_U \parr P_V] \rangle}}{[\langle (R \otimes T) \lhd (U \otimes V) \rangle \parr \langle P_R \lhd P_U \rangle \parr \langle P_T \lhd P_V \rangle]} \\ \mathsf{NELc}\,\Big\Vert \\ [\langle (R \otimes T) \lhd (U \otimes V) \rangle \parr P] \end{array}$$

as desired. ☐

Lemma 6.3. *Let R, T and P be any* NEL *structures. If $[(?R \otimes !T) \parr P]$ is provable in* NELc*, then $[?(R \otimes T) \parr P]$ is also provable in* NELc*.*

Proof. As above, we apply splitting several times to the proof of $[(?R \otimes !T) \parr P]$ and get structures P_R, $P_1, \ldots, P_h$ such that:

$$\begin{array}{c} [!P_R \parr ?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\Vert \\ [R \parr P_R] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\Vert \\ [T \parr P_1 \parr \cdots \parr P_h] \end{array}$$

By putting things together, we can build the proof

$$\begin{array}{c} \mathsf{NELc}\,\Big\Vert \\ \mathsf{s},\mathsf{s}\,\dfrac{!([R \parr P_R] \otimes [T \parr P_1 \parr \cdots \parr P_h])}{![(R \otimes T) \parr P_R \parr P_1 \parr \cdots \parr P_h]} \\ \{\mathsf{p}\!\downarrow\}\,\Big\Vert \\ [?(R \otimes T) \parr !P_R \parr ?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert \\ [?(R \otimes T) \parr P] \end{array}$$

as desired. ☐

Lemma 6.4. *Let P be any* NEL *structure. If $[?\circ \parr P]$ is provable in* NELc*, then $[\circ \parr P]$ is also provable in* NELc*.*

Proof. This is now a trivial exercise, that we leave to the reader. ☐

By the use of context reduction (Lemma 5.1), we can extend the statements of Lemmas 6.1–6.4 from shallow contexts $[\{\ \} \parr P]$ to arbitrary contexts $S\{\ \}$, which is done by the following lemma.

Lemma 6.5. *Let R, T, U and V be any structures, let a be an atom and let $S\{\ \}$ be any context. Then we have the following*

(i) *If $S(a \otimes \bar{a})$ is provable in* NELc*, then so is $S\{\circ\}$.*

(ii) *If $S(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle)$ is provable in* NELc*, then so is $S\langle(R \otimes T) \lhd (U \otimes V)\rangle$.*

(iii) *If $S(?R \otimes !T)$ is provable in* NELc*, then so is $S\{?(R \otimes T)\}$.*

(iv) *If $S\{?\circ\}$ is provable in* NELc*, then so is $S\{\circ\}$.*

Proof. All four statements are proved similarly. We will here show only the third: Let a proof of $S(?R \otimes !T)$ be given and apply context reduction, to get a structure P, such that

$$\begin{array}{c} !\cdots![\{\ \} \parr P] \\ \mathsf{NELc}\,\Big\Vert\,\Delta \\ S\{\ \} \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi \\ [(?R \otimes !T) \parr P] \end{array}$$

By Lemma 6.3 there is a proof Π' of $[?(R \otimes T) \parr P]$. By plugging $?(R \otimes T)$ into the hole of Δ, we can build

$$\begin{array}{c}
\{\circ\downarrow, \mathsf{e}\downarrow\} \,\overset{\top}{\Vert}\\
!\cdots!\circ\\
\mathsf{NELc}\,\Vert\,\Pi'\\
!\cdots![?(R \otimes T) \parr P]\\
\mathsf{NELc}\,\Vert\,\Delta\\
S\{?(R \otimes T)\}
\end{array}$$

It is obvious that the other statements are proved in the same way. ☐

Lemma 6.6. *If a structure R is provable in $\mathsf{NELc} \cup \{\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow\}$ then it is also provable in NELc.*

Proof. The instances of the rules $\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow$ are removed one after the other (starting with the topmost one) via Lemma 6.5. ☐

Now we can very easily give a proof for the cut elimination theorem for the system NEL.

Proof of Theorem 2.12. Cut elimination is obtained in two steps:

$$\begin{array}{c}
\circ\downarrow \dfrac{}{\circ}\\
\mathsf{SNEL}\,\Vert\\
R
\end{array}
\quad \xrightarrow{1} \quad
\begin{array}{c}
\mathsf{NELc} \cup \{\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow\} \,\overset{\top}{\Vert}\\
R'\\
\{\mathsf{w}\downarrow, \mathsf{b}\downarrow, \mathsf{g}\downarrow\}\,\Vert\\
R
\end{array}
\quad \xrightarrow{2} \quad
\begin{array}{c}
\mathsf{NELc} \,\overset{\top}{\Vert}\\
R'\\
\{\mathsf{w}\downarrow, \mathsf{b}\downarrow, \mathsf{g}\downarrow\}\,\Vert\\
R
\end{array}$$

Step 1 is an application of the decomposition (Theorem 2.14). The instances of $\mathsf{g}\uparrow, \mathsf{b}\uparrow, \mathsf{w}\uparrow$ disappear because their premise must be the unit $\circ$, which is impossible. Step 2 is just Lemma 6.6. ☐

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules from the cut. So, our technique is more general than cut elimination in the sequent calculus, for two reasons:

1. it applies to connectives that admit no sequent calculus definition, as seq;
2. it can be used to show admissibility of non-infinitary rules that involve no negation, like $\mathsf{q}\uparrow$ and $\mathsf{p}\uparrow$.

7 Perspectives

We now briefly mention the current developments of this work, and those that we expect.

We think that the techniques developed here for splitting can be exported to the many modal logics already available in deep inference (some of which have no known analytic presentation in Gentzen formalisms). The reason is that linear logic modalities have a similar behavior to those of modal logic. This is particularly obvious if we observe that the promotion rule of NEL is the same as the K rule of all modal logics in deep inference (corresponding to the K axiom of basic modal logic). Of course, contraction in linear logic, and in NEL, is restricted, but the splitting theorems, crucially, do not make any use of it.

We mentioned the applications of BV to process algebras and causal quantum evolution. We expect NEL to find uses in the same directions. In the case of process algebras, this is almost obvious, given that NEL is Turing-complete and that exponentials have been justified since their first introduction as ways of controlling resources (*i.e.*, messages, processes). The logic BV has also been used to define BV-categories [BPS09] for providing an axiomatic description of probabilistic coherence spaces [Gir03].

Apart from its use in getting cut elimination, splitting is a powerful tool for reducing proof search non-determinism in deep inference proof systems. This is explored in the works [Kah06, Kah08].

We are currently investigating, in the context of the INRIA ARC project REDO, the relations between splitting and the focusing technique in linear logic [Mil96], which is at the basis of ludics [Gir01]. It appears that focusing can be justified and greatly generalized by splitting in deep inference. It seems like splitting is a way to explore the duality between any subformula and its context, so unveiling a new logical symmetry.

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