INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Asymptotic expansion for the solution of a penalized control constrained semilinear elliptic problems

J. Frédéric Bonnans — Francisco J. Silva

N° 7126

Novembre 2009

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7126--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

INRIA

centre de recherche
SACLAY – ÎLE-DE-FRANCE

Asymptotic expansion for the solution of a penalized control constrained semilinear elliptic problems

J. Frédéric Bonnans* , Francisco J. Silva†

Thème NUM — Systèmes numériques
Équipes-Projets Commands

Rapport de recherche n° 7126 — Novembre 2009 — 25 pages

Abstract: In this work we consider the optimal control problem of a semilinear elliptic PDE with a Dirichlet boundary condition, where the control variable is distributed over the domain and is constrained to be nonnegative. The approach is to consider an associated family of penalized problems, parametrized by $\varepsilon > 0$, whose solutions define a central path converging to the solution of the original problem. Our aim is to obtain an asymptotic expansion for the solutions of the penalized problems around the solution of the original problem. This approach allows us to obtain some specific error bounds in various norms and for a general class of barrier functions. In this manner, we generalize the results of [2] which were obtained in the ODE framework.

Key-words: Optimal control of PDE, interior-point algorithms, control constraints, expansion of solutions.

* INRIA-Saclay and CMAP, École Polytechnique, 91128 Palaiseau, France (Frederic.Bonnans@inria.fr)

† INRIA-Saclay and CMAP, École Polytechnique, 91128 Palaiseau, France (silva@cmap.polytechnique.fr)

Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université
4, rue Jacques Monod, 91893 ORSAY Cedex
Téléphone : +33 1 72 92 59 00

Développement asymptotique de la solution d'un problème de commande optimale semi linéaire elliptique pénalisé

Résumé : Dans ce travail nous considérons le problème de commande optimale d'une équation semi linéaire elliptique avec conditions de Dirichlet homogène au bord, la commande étant distribuée sur le domaine et positive. L'approche est de considérer une famille de problèmes pénalisés par $\varepsilon > 0$, dont la solution définit une trajectoire centrale qui converge vers la solution du problème original. Notre but est d'obtenir un développement asymptotique de la solution du problème pénalisé au voisinege de la solution du problème original. Notre approche nous permet d'obtenir des estimations d'erreur dans différentes normes et pour une classe générale de fonctions barrière. Ceci étend les résultats de [2], obtenus dans un cadre de commande optimale d'équations différentielles.

Mots-clés : Commande optimale des EDP, algorithmes de points intérieurs, contraintes sur la commande, d'eveloppement des solutions.

1 Introduction

Optimal control of control constrained PDEs is a very rich subject from the theoretical and applied point of view. For an overview of the theory we refer the reader to the classic book [21] and the more recent monographs [15, 20, 19, 25]. Sensitivity analysis as well as second order conditions have been established in [7, 12, 28].

Numerical methods for these types of problems have been an very active subject of research and we can distinguish two main approaches that are usually referred as direct and indirect methods. Direct methods are those based on the *discretize and then optimize* approach, which means that the infinite dimensional problem is transformed into a finite dimensional one with a very large dimension. Then standard methods of nonlinear programming optimization are used to solve the discretized problem, see for example [3, 4, 11, 13, 23, 22]. In contrast, indirect methods are based on the *optimize and then discretize* approach where optimality conditions are obtained for the infinite dimensional problem and the resulting variational inequalities are discretized, see for example [18, 30, 31].

Interior point methods are among the most popular methods in the indirect approach. They have been investigated, even in the state constraint case [26], extensively in [5, 6, 27, 32, 33]. Specifically, in [27], for box constraints over the control, the optimal solution u_0, with associated state y_0, can be expressed pointwisely as a projection of a linear function of the adjoint state p_0. This enables to avoid the explicit discretization of the control and leads to a very efficient implementation of the method. From the theoretical point of view, the method consists in introducing a family of penalized problems parametrized by $\varepsilon > 0$ whose solution u_ε are *strictly* feasible and studying the convergence of the central path defined by $(y_\varepsilon, p_\varepsilon)$, the state and adjoint state associated with u_ε, towards (y_0, p_0).

Motivated by these works, we consider the optimal control of a semilinear PDE where the control is distributed over the domain Ω and is constrained to be nonnegative. Associated with any isolated solution u_0 we consider a family of localized penalized problems parametrized by $\varepsilon > 0$. We study in detail the relationship between the solution u_ε of the penalized problem and u_0. Our approach is the same that in [2], which was studied in the ODE framework, and consists in obtaining an asymptotic expansion for state y_ε and the adjoint state p_ε, which are associated to u_ε, around the state y_0 and adjoint state p_0, which are associated to u_0. In this sense, our approach is complementary to that in [27] where the slope of the central path, defined by $(y_\varepsilon, p_\varepsilon)$, is integrated in order to obtain error bounds. Under very general hypothesis we can show that $(y_\varepsilon, p_\varepsilon)$ can be expressed as (y_0, p_0) plus a *principal* term which is characterized as being the state and adjoint state associated to the solution of a *tangent* optimization problem. This fact enable us to obtain, as a corollary, precise error bounds for the central path in various Sobolev norms and for a rather general class of penalty functions.

The paper is organized as follows: In Section 2, after introducing the necessary notations, we state the problem as well as its penalized versions. Regularity results are specified and convergence of the central path is obtained, which allows us to write the solution of the penalized problem in term of its associated adjoint state. This fact will be crucial for Section 3, since the optimality system for the penalized problem can be written in terms of $(y_\varepsilon, p_\varepsilon)$ only. Then we show, by means of a Restoration theorem as in [2] and under very general conditions, that is possible to obtain the desired asymptotic expansion of the central path around (y_0, p_0). We finalize Section 3 by obtaining that error bounds for the infinite dimensional problem, in various norms, can be obtained from its finite dimensional counterparts, generalizing the result of [2]. In particular, for the logarithmic penalty, we recover in Section 4 an error for the control of $O(\sqrt{\varepsilon})$ in the L^∞ norm and under more restrictive hypothesis we improve this bound in the L^2 norm to $O(\varepsilon^{3/4})$. Similar results are obtained for the error of the central path $(y_\varepsilon, p_\varepsilon)$ in the H^2 norm.

2 Problem statement and preliminary results

Consider the following semilinear elliptic equation

$$\left\{ \begin{array}{rcll} -\Delta y(x) + \phi(y(x)) & = & g(x) & \text{for} \quad x \in \Omega, \\ y(x) & = & 0 & \text{for} \quad x \in \partial\Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with C^2 boundary, $g \in L^2(\Omega)$ and ϕ is a nondecreasing real valued function over $\mathbb{R}$, Lipschitz with associated constant L_ϕ and continuously differentiable. Given $s \in [2, \infty]$, denote by $||\cdot||_s$ the standard norm in $L^s(\Omega)$. For $m \in \mathbb{N}$ set

$$W^{m,s}(\Omega) := \{y \in L^s(\Omega) \ ; \ D^\alpha y \in L^s(\Omega) \text{ for } \alpha \text{ such that } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + ... + \alpha_n$ and

$$D^\alpha := \frac{\partial^{\alpha_1 + ... + \alpha_n}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

represents a derivative operator in the distribution sense. As usual, for $s = 2$ we will write $H^m(\Omega) := W^{m,2}(\Omega)$. It is well know that $W^{m,s}(\Omega)$ endowed with the norm

$$||y||_{m,s} := \sum_{0 \leq |\alpha| \leq m} ||D^\alpha y||_s$$

is a Banach space and $H^m(\Omega)$ endowed with the norm

$$||y||_{m,2} := \left(\sum_{0 \leq |\alpha| \leq m} ||D^\alpha y||_2^2 \right)^{1/2}$$

is a Hilbert space. We also denote $W_0^{m,s}(\Omega)$, which will be written as $H_0^m(\Omega)$ when $s = 2$, the space defined as the closure of $\mathcal{D}(\Omega)$ in $W^{m,s}(\Omega)$, where $\mathcal{D}(\Omega)$ denotes the set of C^∞ functions with compact support in Ω. For the reader convenience we recall the following Sobolev embeddings (cf. [1], [14], [16])

$$W^{m,s}(\Omega) \subseteq \left\{ \begin{array}{lll} L^{q_1}(\Omega) & \text{with } \frac{1}{q_1} = \frac{1}{s} - \frac{m}{n} & \text{if } s < \frac{n}{m} \\ L^q(\Omega) & \text{with } q \in [1, +\infty) & \text{if } s = \frac{n}{m} \\ C^{m - [\frac{n}{s}] - 1, \gamma(n,s)}(\Omega) & & \text{if } s > \frac{n}{m} \end{array} \right. \tag{2}$$

where $\gamma(n,s)$ is defined as

$$\gamma(n,s) = \begin{cases} [\frac{n}{s}] + 1 - \frac{n}{s}, & \text{if } \frac{n}{s} \notin \mathbb{Z} \\ \text{any positive number} < 1 & \text{if } \frac{n}{s} \in \mathbb{Z} \end{cases} \tag{3}$$

and $C^{m-[\frac{n}{s}]-1,\gamma(n,s)}(\Omega)$ denotes the Holder space with exponents $m-[\frac{n}{s}]-1$ and $\gamma(n,s)$ (for the definition see [14] p. 240). In this work we will use repeatedly the fact that $W^{2,s}(\Omega) \subseteq C(\Omega)$ when $s > n/2$ ($s = 2$ if $n \leq 3$). This is equivalent to the existence of a constant c_s such that

$$||y||_\infty \leq c_s ||y||_{2,s}. \tag{4}$$

An space that will play an important role is $\mathcal{Y}^s := W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ which endowed with the norm $|| \cdot ||_{2,s}$ is a Banach space.

In the following $s \in [2, \infty)$ will be fixed and we will assume, without loss of generality, that $\phi(0) = 0$. We collect in the next proposition some properties of the PDE (1) (see for example [7, 9]).

Proposition 1. *If $g \in L^s(\Omega)$ the following holds:*
(i) *The semilinear equation (1) has a unique solution $y_g \in \mathcal{Y}^s$ and there exists a constant $\bar{c}_s > 0$ such that*

$$||y_g||_{2,s} \leq \bar{c}_s ||g||_s. \tag{5}$$

(ii) *The mapping $g \to y_g$ is continuous from $L^s(\Omega)$ into $\mathcal{Y}^s$, both spaces endowed with the weak topology.*

Proof. (i) Equation (1) can be interpreted as the optimality system, in the weak sense, of the variational problem

$$\operatorname*{Min}_{y} \int_\Omega \left\{ \tfrac{1}{2}|\nabla y(x)|^2 + \Phi(y(x)) - g(x)y(x) \right\} \mathrm{d}x \quad \text{subject to } y \in H_0^1(\Omega), \tag{6}$$

where $\Phi : [0, +\infty) \to \mathbb{R}$ is defined by $\Phi(t) := \int_0^t \phi(t)$. Since $|\Phi(t)| \leq \frac{1}{2} L_\phi t^2$, the convex mapping $y \in H_0^1(\Omega) \to \int_\Omega \Phi(y(x))\mathrm{d}x \in \mathbb{R}$ is bounded over the bounded sets and whence is continuous. In addition, the cost function is strongly convex and continuous and thus problem (6) has a unique solution $y_g \in H_0^1(\Omega)$. Multiplying equation (1) by y_g and using Green's formula yields

$$\int_\Omega \left\{ |\nabla y_g(x)|^2 + \phi(y_g(x))y_g(x) \right\} \mathrm{d}x = \int_\Omega g y_g(x)\mathrm{d}x.$$

Since $\phi(y_g)y_g \geq 0$, by the Cauchy-Schwarz and Poincaré inequalities we obtain that

$$||y_g||_{1,2} \leq ||g||_2. \tag{7}$$

On the other hand, since $\phi(0) = 0$, it holds that $||\phi(y_g)||_r \leq L_\phi ||y_g||_r$ for all $r \in [1, +\infty)$. Hence, in view of (7), an standard boostraping argument yields the existence of $a_s > 0$ such that $||y_g||_s \leq a_s ||g||_s$. Thus $||\Delta y_g||_s \leq (L_\phi a_s + 1)||g||_s$, from which (5) follows.
(ii) Let $(g_k)_{k\in\mathbb{N}}$ converge weakly to $\bar{g}$. Then the sequence g_k is bounded in $L^s(\Omega)$ and consequently, by (7), the associated states $y_k := y_{g_k}$ are bounded in $\mathcal{Y}^s$. Thus, extracting a subsequence if necessary, y_k converges weakly in $\mathcal{Y}^s$ to some $\bar{y}$ and hence strongly in $L^s(\Omega)$. This implies, since ϕ is Lipschitz, that $\phi(y_k) \to \phi(\bar{y})$ strongly in $L^s(\Omega)$. Passing to the weak limit in $L^s(\Omega)$ in equation (1) yields that $\bar{y} = y_g$ from which the conclusion follows. □

Denote respectively by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the subsets of nonnegative and positive real numbers. Also, set $\mathcal{U}_+^s := L^s(\Omega; \mathbb{R}_+)$.

Suppose that $g = f + u$, where $f \in L^s(\Omega)$ and $u \in L^2(\Omega)$. By proposition 1 we have that $u \in L^2(\Omega) \to y_{f+u} \in \mathcal{Y}^2$ is well defined. In the following f will be a fixed function and, in order to simplify the notation, we will write y_u for the unique solution in $\mathcal{Y}^2$ of

$$\left\{ \begin{array}{rcllr} -\Delta y(x) + \phi(y(x)) & = & f(x) + u(x) & \text{for} & x \in \Omega, \\ y(x) & = & 0 & \text{for} & x \in \partial\Omega. \end{array} \right. \tag{8}$$

Let us define the *cost function* $J_0 : L^2(\Omega) \to \mathbb{R}_+$ by

$$J_0(u) := \tfrac{1}{2} \int_\Omega (y_u(x) - \bar{y}(x))^2 dx + \tfrac{1}{2} N \int_\Omega u(x)^2 \mathrm{d}x, \tag{9}$$

where $N > 0$ and $\bar{y} \in L^\infty(\Omega)$ is a reference state function. It holds that:

Corollary 2. *The function $J_0 : L^2(\Omega) \to \mathbb{R}$ is w.l.s.c. (weakly lower semicontinuous).*

Proof. Since the $u \in L^2(\Omega) \to ||u||_2^2$ is w.l.s.c. and $J_0(\cdot) = \frac{1}{2}||\cdot||_2^2 + \frac{1}{2}N||y_{(\cdot)} - \bar{y}||_2^2$, the result follows by proposition 1(ii). $\square$

Now, consider the following optimal control problem

$$\text{Min } J_0(u) \quad \text{subject to } u \in \mathcal{U}_+^s. \tag{$\mathcal{CP}_0^s$}$$

By constrast to the case when (8) is linear in y (for example when $\phi \equiv 0$), problem $(\mathcal{CP}_0^2)$ is not necessarily convex. Thus, the classical argument to show the existence and uniqueness of a solution of $(\mathcal{CP}_0^2)$ does not apply. Instead, we have the following existence result.

Proposition 3. *Problem $(\mathcal{CP}_0^2)$ has (at least) one solution.*

Proof. Any minimizing sequence u_k for $(\mathcal{CP}_0^2)$ is bounded in $L^2(\Omega)$. Therefore, extracting a subsequence if necessary, we may suppose that it weakly converges to some $u_0 \in L^2(\Omega)$. Since $\mathcal{U}_+^2$ is weakly closed, we have that $u_0 \in \mathcal{U}_+^2$ and, in view of corollary 2, it is a solution of $(\mathcal{CP}_0^2)$. $\square$

As usual in optimal control theory, it will be convenient to write the derivative of J_0 in terms of an adjoint state. For every $u \in L^2(\Omega)$ the adjoint equation associated with u is defined by

$$\left\{ \begin{array}{rcllr} -\Delta p(x) + \phi'(y_u(x))p(x) & = & y_u(x) - \bar{y}(x) & \text{for} & x \in \Omega, \\ p(x) & = & 0 & \text{for} & x \in \partial\Omega. \end{array} \right. \tag{10}$$

It holds that (see [8] lemma 6.18):

Lemma 4. *Let $u \in L^2(\Omega)$. Then the adjoint equation has a unique solution $p_u \in H_0^1(\Omega)$, called the adjoint state associated with u. In addition, the function J_0 is of class C^1 and*

$$DJ_0(u) = p_u + Nu. \tag{11}$$

Remark. Note that equation (10) and the Sobolev embeddings (2) imply that $p_u \in \mathcal{Y}^q$ where

$$q = \begin{cases} \frac{2n}{n-4} & \text{if } n > 4, \\ \text{any real number in } [2, \infty) & \text{if } n \leq 4. \end{cases}$$

Now, let u_0 be a solution of $(\mathcal{CP}_0^2)$. In what follows we will write $y_0 := y_{u_0}$ and $p_0 := p_{u_0}$. The first order condition for the optimality of u_0 is given by

$$DJ_0(u_0)(v - u_0) \geq 0, \quad \text{for all } v \in \mathcal{U}_+^2. \tag{12}$$

Expressions (11) and (12) easily yield that

$$u_0 = P_{\mathcal{U}_+^2}(-N^{-1}p_0), \tag{13}$$

where $P_{\mathcal{U}_+^s}$ denotes the orthogonal projection in $L^2(\Omega)$ onto $\mathcal{U}_+^2$. This in turn implies that the following *punctual* relation holds

$$u_0(x) = \pi_0(-N^{-1}p_0(x)) \quad \text{for a.a. } \ x \in \Omega, \tag{14}$$

where for $a \in \mathbb{R}$ we denote $\pi_0(a) := \max\{0, a\}$.

Expression (14) allows us, by a bootstrapping argument and using the Sobolev embeddings, to specify the regularity of (y_0, p_0). In fact, proposition 3 implies the following corollary:

Corollary 5. *Problem $(\mathcal{CP}_0^s)$ has (at least) one solution and it holds that:*

$$\begin{array}{l} y_0 \in \begin{cases} L^{q_1}(\Omega) & \textit{with } q_1 = \frac{ns}{n-2s} & \textit{if } s < \frac{n}{2}, \\ L^q(\Omega) & \textit{with } q \in [1, +\infty) & \textit{if } s = \frac{n}{2}, \\ C^{1-[\frac{n}{s}],\gamma(n,s)}(\Omega) & & \textit{if } s > \frac{n}{2}. \end{cases} \\ p_0 \in \begin{cases} L^{q_2}(\Omega) & \textit{with } q_2 = \frac{ns}{n-4s} & \textit{if } s < \frac{n}{4}, \\ L^q(\Omega) & \textit{with } q \in [1, +\infty) & \textit{if } s = \frac{n}{4}, \\ C^{1-[\frac{n-2s}{s}],\gamma(n,q_1)}(\Omega) & & \textit{if } s > \frac{n}{4}. \end{cases} \end{array} \tag{15}$$

Proof. Let u_0 be a solution of $(\mathcal{CP}_0^2)$. Replacing expression (14) into equations (8) and (10) yields that y_0 and p_0 satisfy

$$\begin{cases} -\Delta y(x) + \phi(y(x)) &= f(x) + \pi_0(-N^{-1}p_0(x)) & \text{for} \quad x \in \Omega. \\ -\Delta p(x) + \phi'(y_u(x))p(x) &= y_u(x) - \bar{y}(x) & \text{for} \quad x \in \Omega \\ y(x) = p(x) &= 0 & \text{for} \quad x \in \partial\Omega \end{cases} \tag{16}$$

An standard boostraping argument in equations (16) implies that $p_0 \in L^{q_2}(\Omega)$ where $q_2 = \frac{ns}{n-4s}$. Since $q_2 > s$, expression (14) yields that $u_0 \in L^s(\Omega)$ and therefore solves $(\mathcal{CP}_0^s)$. Regularity results (15) follow by (2), since using that $f + u_0 \in L^s(\Omega)$. □

Next we consider a *localized* penalized version of $(\mathcal{CP}_0^s)$. Since we could have several solutions of $(\mathcal{CP}_0^s)$, the idea is to localize the problem around an strict solution (if there is any). Let ℓ be a convex function with domain either $\mathbb{R}_+$ or $\mathbb{R}_{++}$, which is C^2 on the interior of its domain, and satisfies:

$$\begin{array}{l} \text{(i) } \lim_{t\downarrow 0} \ell'(t) = -\infty; \quad \text{(ii) } \lim_{t\downarrow 0} \dfrac{\ell''(t)}{\ell'(t)} = -\infty; \\ \text{(iii) There exist } \alpha \geq 0 \text{ such that } |\ell'(t)| \leq \alpha t \quad \forall\, t \geq 1. \end{array} \tag{17}$$

Remark. Standard examples of functions satisfying these properties are:

$$\ell(t) = -\log t \quad ; \quad \ell(t) = t^{-p}, \quad p > 0 \quad ; \quad \ell(t) = -t^p, \quad p \in (0,1) \quad ; \quad \ell(t) = t \log t.$$

Let u_0 be a solution of $(\mathcal{CP}_0^s)$. For $b, \varepsilon > 0$ the localized *penalized* problem is defined as

$$\text{Min } J_\varepsilon(u) := J_0(u) + \varepsilon \int_\Omega \ell(u(x))\mathrm{d}x \quad \text{subject to } u \in \mathcal{U}_+^s \cap \bar{B}_s(u_0, b) \quad (\mathcal{CP}_\varepsilon^{b,s}),$$

where $\bar{B}_s(u_0, b)$ denotes the closed ball in $L^s(\Omega)$ centered at u_0 of radius b. Note that ℓ, being a convex function, is bounded by below by some affine function and thus J_ε takes values in $\mathbb{R} \cup \{+\infty\}$.

Lemma 6. *The function $J_\varepsilon : L^s(\Omega) \to \mathbb{R}$ is w.l.s.c. and problem $(\mathcal{CP}_\varepsilon^{b,s})$ has (at least) one solution.*

Proof. First note that since $(L^s(\Omega))^* \subseteq L^2(\Omega)$ we have that weak continuity from $L^2(\Omega)$ to $\mathbb{R}$ implies weak continuity from $L^s(\Omega)$ to $\mathbb{R}$. Thus, it is enough to show the property for $s = 2$. By corollary 2, the function J_0 is w.l.s.c. hence, adapting the argument of proposition 1 in [2] (which is based in Fatou's lemma), we obtain that $u \in L^2(\Omega) \to \int_\Omega \ell(u(x))\mathrm{d}x$ is convex l.s.c. and hence convex w.l.s.c. which yields the first assertion. The second assertion follows directly by taking a minimizing sequence and using that J_ε is w.l.s.c. □

We give here an elementary argument, for the semilinear case, to prove a well known contraction principle which is a corollary of Stampacchia's results (see [29]).

Lemma 7. *There exists a constant $C_1 > 0$ such that for every $u_1, u_2 \in L^s(\Omega)$ we have*

$$||y_{u_1} - y_{u_2}||_1 \leq C_1 ||u_1 - u_2||_1. \tag{18}$$

Proof. Set $z = y_{u_1} - y_{u_2}$ and $h = u_1 - u_2$. Clearly z satisfies

$$\begin{cases} -\Delta z(x) + \psi_{u_1,u_2}(x) z(x) = h(x) & \text{for} \quad x \in \Omega, \\ z(x) = 0 & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{19}$$

where

$$\psi_{u_1,u_2}(x) := \begin{cases} \dfrac{\phi(y_{u_2}(x)) - \phi(y_{u_1}(x))}{(y_{u_2} - y_{u_1})(x)}, & \text{if } y_{u_2}(x) \neq y_{u_1}(x), \\ \phi'(y_{u_1})(x), & \text{otherwise}. \end{cases} \tag{20}$$

Evidently $0 \leq \psi_{u_1,u_2}(x) \leq L_\phi$ for all $x \in \Omega$. Now, let v_z be the unique solution of

$$\begin{cases} -\Delta v_z(x) + \psi_{u_1,u_2}(x) v_z(x) = \operatorname{sgn}(z(x)) & \text{for} \quad x \in \Omega \\ v_z(x) = 0 & \text{for} \quad x \in \partial\Omega \end{cases} \tag{21}$$

Multiplying by v_z the first equation in (19) and using Green's formula yields that

$$\int_\Omega |z(x)|\mathrm{d}x = \int_\Omega h(x) v_z(x)\mathrm{d}x. \tag{22}$$

On the other hand, by the maximum principle for elliptic equations (see for example [10, proposition IX.29]) it holds that $-v_1 \leq v_z \leq v_1$ where $v_1 \geq 0$ solves

$$\begin{cases} -\Delta v_1(x) + \psi_{u_1,u_2}(x) v_1(x) &= 1 \quad \text{for} \quad x \in \Omega \\ v_1(x) &= 0 \quad \text{for} \quad x \in \partial\Omega. \end{cases} \tag{23}$$

Using that $\psi \geq 0$ and the maximum principle again, we see that $v_1 \leq v_2$ where v_2 solves

$$\begin{cases} -\Delta v_2(x) &= 1 \quad \text{for} \quad x \in \Omega \\ v_2(x) &= 0 \quad \text{for} \quad x \in \partial\Omega. \end{cases} \tag{24}$$

Since v_2 is bounded in $L^\infty(\Omega)$ the result follows from (22). □

The following result yields that the solutions of the penalized problem are bounded in $L^\infty(\Omega)$ by a constant which is independent of ε.

Proposition 8. *Suppose that $s > n/2$ ($s = 2$ if $n \leq 3$) and let u_ε be a solution of $(\mathcal{CP}_\varepsilon^{b,s})$. If ε is small enough, there exists a constant K_ℓ (independent of ε) such that*

$$u_\varepsilon(x) \leq K_\ell \quad \textit{for a.a. } x \in \Omega. \tag{25}$$

Proof. For $K > 2||u_0||_\infty$ set

$$\overline{\Omega}_K := \{x \in \Omega;\ u_\varepsilon(x) \geq K\}$$

and

$$u_\varepsilon^K(x) := \begin{cases} K/2 & \text{if } x \in \overline{\Omega}_K \\ u_\varepsilon(x) & \text{otherwise} \end{cases} \quad ; \quad y_\varepsilon^K(x) := y_{u_\varepsilon^K}(x) \quad \text{for a.a. } x \in \Omega. \tag{26}$$

Note that u_ε^K is feasible. For all $u \in L^s(\Omega)$ we have (omitting the function arguments in the integral)

$$J_0(u) - J_0(u_\varepsilon) = \tfrac{1}{2} \int_\Omega \{(u + u_\varepsilon)(u - u_\varepsilon) + (y_u + y_\varepsilon - 2\bar{y})(y_u - y_\varepsilon)\}\, \mathrm{d}x. \tag{27}$$

Taking $u = u_\varepsilon^K$ in (27) we see that, since $s > n/2$ ($s = 2$ if $n \leq 3$) and $u_\varepsilon \in \bar{B}_s(u_0, b)$, proposition 1(i) implies that $y_\varepsilon^K + y_\varepsilon - 2\bar{y}$ is uniformly bounded by a constant independent of ε and K. In addition, by the very definition of $\overline{\Omega}_K$ and u_ε^K, it holds that

$$\left(u_\varepsilon + u_\varepsilon^K\right)\left(u_\varepsilon - u_\varepsilon^K\right) \geq \frac{3}{2} K (u_\varepsilon - u_\varepsilon^K) \mathbf{1}_{\overline{\Omega}_K} \geq 0$$

where $\mathbf{1}_{\overline{\Omega}_K}$ is the indicator function of $\overline{\Omega}_K$. Therefore, in view of lemma 7, we have the existence of $K_2 > 0$ such that

$$J_0(u_\varepsilon) - J_0(u_\varepsilon^K) \geq \left(\frac{3}{4} K + K_2\right) K\, meas(\overline{\Omega}_K). \tag{28}$$

Using the convexity of ℓ, we obtain that

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_\varepsilon^K) \geq K\, meas(\overline{\Omega}_K) \left(\frac{3}{4} K + K_2 + \tfrac{1}{2}\varepsilon \ell'(\tfrac{1}{2}K)\right). \tag{29}$$

On the other hand, hypothesis (17)(iii) implies, for ε small enough, the existence of K_ℓ (independent of ε) such that $\frac{3}{4}K_\ell + K_2 + \frac{1}{2}\varepsilon\ell'(\frac{1}{2}K_\ell) > 0$. Therefore $meas(\overline{\Omega}_{K_\ell}) = 0$ from which the conclusion follows. □

Let us give an elementary lemma that will be useful in the convergence proof of the central path to the optimal solution (proposition 10). First, define $\bar{F} : \mathcal{Y}^s \times \mathcal{Y}^s \to L^s(\Omega)$ by

$$\bar{F}(y,p) := -\Delta p + \phi'(y)p - y + \bar{y} \tag{30}$$

and for every $y \in \mathcal{Y}^s$ denote by $p[y]$ the unique solution of $\bar{F}(y,p) = 0$. It holds that:

Lemma 9. *Suppose that ϕ is C^2 and that $s > n/2$ ($s = 2$ if $n \leq 3$). Then*
(i) *The function $\bar{F}$ is C^1.*
(ii) *The mapping $y \in \mathcal{Y}^s \to p[y] \in \mathcal{Y}^s$ is C^1.*
(iii) *The mapping $u \in L^s(\Omega) \to y_u \in \mathcal{Y}^s$ is C^2.*

Proof. In order to prove (i) it is enough to note that $\phi'(y)p$ is C^1 since ϕ is C^2 and $s > n/2$ ($s = 2$ if $n \leq 3$). Assertions (ii) and (iii) follow directly by the implicit function theorem. □

For the solutions u_ε of the penalized problems we will write $y_\varepsilon := y_{u_\varepsilon}$ for the state functions and $p_\varepsilon := p_{u_\varepsilon}$ for the adjoint state functions. Now we can state the convergence result.

Proposition 10. *Assume that $s > n/2$ ($s = 2$ if $n \leq 3$) and suppose that there exists $b_0 > 0$ such that u_0 is the unique minimum of $(\mathcal{CP}_0^s)$ in $\bar{B}_s(u_0, b_0)$. Then*
(i) *The controls u_ε, solutions of $(\mathcal{CP}_\varepsilon^{b_0,s})$, strongly converge to u_0 in $L^s(\Omega)$ as $\varepsilon \downarrow 0$.*
(ii) *It holds that $J_\varepsilon(u_\varepsilon) \to J_0(u_0)$ and that $J_0(u_\varepsilon) \downarrow J_0(u_0)$.*
(iii) *The states y_ε converge to y_0 in $\mathcal{Y}^s$ and the adjoint states p_ε converge to p_0 in $\mathcal{Y}^s$.*

Proof. Since u_ε is bounded in $L^2(\Omega)$, extracting a subsequence if necessary, it converges weakly to some $\bar{u}$. Similary, since $J_0(u_\varepsilon)$ is bounded in $\mathbb{R}$, we can assume, extracting a subsequence again, that there exists $\bar{J} \geq 0$ such that $J_0(u_\varepsilon)$ converges to $\bar{J}$.

In view of the optimality of u_ε, for every $\eta > 0$ such that $u_0 + \eta$ is feasible for $(\mathcal{CP}_\varepsilon^{b_0,s})$, we have that

$$J_\varepsilon(u_\varepsilon) \leq J_0(u_0 + \eta) + \varepsilon \int_\Omega \ell(u_0(x) + \eta)\mathrm{d}x.$$

Letting first $\varepsilon \downarrow 0$ and then $\eta \downarrow 0$ yields

$$\overline{\lim}_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon) \leq J_0(u_0). \tag{31}$$

On the other hand, because of the convexity of ℓ, there exist some β_1 and β_2 such $\ell(x) \geq \beta_1 x + \beta_2$ for all $x \in \mathbb{R}_+$. Thus

$$J_\varepsilon(u_\varepsilon) \geq J_0(u_\varepsilon) + \varepsilon \int_\Omega (\beta_1 u_\varepsilon(x) + \beta_2)\, \mathrm{d}x. \tag{32}$$

Using (31), (32) and the fact that J_0 is w.l.s.c. yields that

$$J_0(u_0) \geq \overline{\lim}_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon) \geq \underline{\lim}_{\varepsilon \downarrow 0} J_\varepsilon(u_\varepsilon) \geq \bar{J} \geq J_0(\bar{u}). \tag{33}$$

Since u_0 is the unique minimum of $(\mathcal{CP}_0^s)$ in $\bar{B}_s(u_0, b_0)$, it holds that $\bar{u} = u_0$ and hence (ii) is established.

In order to prove (i) it suffices to note that thanks to proposition 1 (ii) the states y_ε converge strongly in $L^2(\Omega)$ to y_0. Therefore, since $J_0(u_\varepsilon) \to J_0(u_0)$ we have that $||u_\varepsilon||_2 \to ||u_0||_2$. Together with the weak convergence in $L^2(\Omega)$ of u_ε to u_0, we obtain the strong convergence in $L^2(\Omega)$. The convergence in $L^s(\Omega)$ follows directly from the convergence in $L^2(\Omega)$ and the fact that u_ε is uniformly bounded in $L^\infty(\Omega)$ by proposition 8. Finally (iii) is a direct consequence of lemma 9. □

Remark. Note that, under the hypothesis of the theorem above, the convergence in $L^s(\Omega)$ of u_ε to u_0 implies that for ε small enough the constraint $u_\varepsilon \in \bar{B}_s(u_0, b)$ is inactive.

Now we obtain lower bounds for u_ε.

Proposition 11. *Under the hypothesis of proposition 10 there exists a constant $K_1 > 0$ such that for $\varepsilon > 0$ small enough*

$$\ell'(2u_\varepsilon(x)) \geq -\frac{2K_1}{\varepsilon} \quad \textit{for a.a. } x \in \Omega. \tag{34}$$

Proof. By (17)(i) there exists $\zeta > 0$ such that ℓ is decreasing on $(0, \zeta]$. Set

$$\underline{\Omega}^\zeta := \{x \in \Omega;\ u_\varepsilon(x) \leq \zeta/2\}$$

and

$$u_\varepsilon^\zeta(x) := \begin{cases} \zeta & \text{if } x \in \underline{\Omega}^\zeta \\ u_\varepsilon(x) & \text{otherwise} \end{cases} \quad ; \quad y_\varepsilon^\zeta(x) := y_{u_\varepsilon^\zeta}(x) \quad \text{for a.a. } x \in \Omega. \tag{35}$$

Note that, by remark 2, u_ε^ζ is feasible for ζ small enough. In addition,

$$0 \leq \left(u_\varepsilon^\zeta + u_\varepsilon\right)\left(u_\varepsilon^\zeta - u_\varepsilon\right) \leq \frac{3}{2}\zeta(u_\varepsilon^\zeta - u_\varepsilon)\mathbf{1}_{\underline{\Omega}^\zeta}.$$

Thus, taking $u = u_\varepsilon^\zeta$ in (27) and reasoning as in the proof of proposition 8, we obtain the existence of $K_1' > 0$ such that

$$J_\varepsilon(u_\varepsilon^\zeta) - J_\varepsilon(u_\varepsilon) \leq K_1'\zeta \operatorname{meas}(\underline{\Omega}^\zeta) + \varepsilon \int_{\underline{\Omega}^\zeta} \left(\ell(u_\varepsilon^\zeta(x)) - \ell(u_\varepsilon(x))\right) \mathrm{d}x.$$

By the mean value theorem and the convexity of ℓ, which implies that ℓ' is increasing, we find that

$$\ell(u_\varepsilon^\zeta(x)) - \ell(u_\varepsilon(x)) \leq \tfrac{1}{2}\ell'(\zeta)\zeta$$

for a.a. $x \in \underline{\Omega}^\zeta$. This in turn implies that

$$J_\varepsilon(u_\varepsilon^\zeta) - J_\varepsilon(u_\varepsilon) \leq \zeta \operatorname{meas}(\underline{\Omega}^\zeta)\left(K_1' + \tfrac{1}{2}\varepsilon\ell'(\zeta)\right). \tag{36}$$

Therefore, by the optimality of u_ε, if $\operatorname{meas}(\underline{\Omega}^\zeta) > 0$ we have that $K_1' \geq -\frac{1}{2}\varepsilon\ell'(\zeta)$. By choosing ζ' such that $K_1' < -\frac{1}{2}\varepsilon\ell'(\zeta')$ we obtain that for a.a. $x \in \underline{\Omega}^\zeta$

$$\ell'(2u_\varepsilon(x)) \geq \ell'(\zeta').$$

Relation (34) follows by letting $\ell'(\zeta') \uparrow -2K_1\varepsilon$. □

Remark. For the examples given in remark 2 inequality (34) yields
(i) If $\ell(t) = -\log t$ then there exists $C_1 > 0$ such that $u_\varepsilon(x) \geq C_1\varepsilon$ for a.a. $x \in \Omega$.
(ii) If $\ell(t) = t\log t$ then there exists $C_2, C_3 > 0$ such that $u_\varepsilon(x) \geq C_2\exp(-C_3/\varepsilon)$ for a.a. $x \in \Omega$.
(iii) If $\ell(t) = t^{-p}$ with $p > 0$ then there exists $C_4 > 0$ such that $u_\varepsilon(x) \geq C_4\varepsilon^{1/(p+1)}$ for a.a. $x \in \Omega$.
(iv) If $\ell(t) = -t^p$ with $p \in (0,1)$ then there exists $C_5 > 0$ such that $u_\varepsilon(x) \geq C_5\varepsilon^{1/(1-p)}$ for a.a. $x \in \Omega$.

Note that $u \in L^s(\Omega) \to \int_\Omega \ell(u(x))\mathrm{d}x$ is, in general, not continuous and whence not differentiable. This implies that we cannot write directly the first order condition for the optimality of u_ε. However, we can avoid this difficulty by noting that, in view of propositions 8 and 11, $u \in L^\infty(\Omega) \to \int_\Omega \ell(u(x))\mathrm{d}x$ is differentiable at any solution of $(\mathcal{CP}_\varepsilon^{b_0,s})$.

Proposition 12. *Under the hypothesis of proposition 10, for $\varepsilon > 0$ small enough it holds that*

$$u_\varepsilon(x) = \pi_\varepsilon(-N^{-1}p_\varepsilon(x)) \quad \textit{for a.a. } x \in \Omega, \tag{37}$$

where for every $z \in \mathbb{R}$, $\pi_\varepsilon(z)$ is the unique solution of

$$\text{Min } \tfrac{1}{2}(x-z)^2 + \varepsilon\ell(x), \quad \textit{s.t.} \quad x \in \mathbb{R}_{++}. \tag{$\mathcal{P}_{\varepsilon,z}$}$$

Proof. By proposition 8 it holds that $u_\varepsilon \in L^\infty(\Omega)$. Hence, it is a local solution of

$$\text{Min } \ J_\varepsilon(u) \quad \text{subject to } u \in \mathcal{U}_+^s \cap \bar{B}_s(u_0, b_0) \cap L^\infty(\Omega).$$

Proposition 11 implies that $J_\varepsilon : L^\infty(\Omega) \to \mathbb{R}$ is differentiable. Therefore, writing the first order condition for the above problem and noting remark 2, we have

$$DJ_0(u_\varepsilon)h + \varepsilon\int_\Omega \ell'(u_\varepsilon(x))h(x)\mathrm{d}x \quad = \quad 0 \quad \text{for all } h \in L^\infty(\Omega),$$

which implies that

$$Nu_\varepsilon(x) + p_\varepsilon(x) + \varepsilon\ell'(u_\varepsilon) \quad = \quad 0 \quad \text{for a.a. } x \in \Omega. \tag{38}$$

The conclusion follows noting that for $x \in \Omega$, equation (38) is the first order optimality condition of $(\mathcal{P}_{\varepsilon,z})$ with $z = -N^{-1}p_\varepsilon(x)$. □

Remark. *Note that for every $z \in \mathbb{R}$ the function $\pi_\varepsilon(z)$ corresponds to the interior penalty approximation of $\pi_0(z)$.*

We collect in the following lemma, some useful properties of the family $\{\pi_\varepsilon\}_{\varepsilon\geq 0}$ whose proof can be found in [2] Section 3 for a more general case.

Lemma 13. *The family of functions $\{\pi_\varepsilon\}_{\varepsilon\geq 0}$ satisfies*
(i) *There exist c_π, independent of 'ε', such that for all $z_1, z_2 \in \mathbb{R}$,*

$$|\pi_\varepsilon(z_1) - \pi_\varepsilon(z_2)| \leq c_\pi|z_1 - z_2|. \tag{39}$$

(ii) *As* $\varepsilon \downarrow 0$ *the sequence* π_ε *converges to* π_0 *uniformly on each compact set of* $\mathbb{R}$.
(iii) *The function* $(\varepsilon, z) \to D_z\pi_\varepsilon(z)$ *is continuous in* $(\bar{\varepsilon}, \bar{z})$ *for every* $\bar{\varepsilon} \geq 0$ *and* $\bar{z} \neq 0$.
(iv) *The continuous function* $\pi_\varepsilon - \pi_0$ *is increasing in* $(-\infty, 0)$ *and decreasing in* $(0, \infty)$. *Henceforth,*

$$\sup_{z\in\mathbb{R}} |\pi_\varepsilon(z) - \pi_0(z)| = |\pi_\varepsilon(0) - \pi_0(0)| = |\pi_\varepsilon(0)|.$$

(v) *For each compact set* $K \subseteq \mathbb{R}$ *not containing* 0, *it holds that:*

$$\sup_{z\in K} |\pi_\varepsilon(z) - \pi_0(z)| = O(\varepsilon).$$

Remark. Hypothesis (ii) in (17) is used to prove (iii) in the lemma above.

3 Main results

As before, we consider $f \in L^s(\Omega)$ and for the rest of the article we assume that $s > \frac{1}{2}n$ ($s = 2$ if $n \leq 3$). Let u_0 be a solution of $(\mathcal{CP}_0^s)$ and y_0, p_0 its associated state and costate, respectively. Analogously, for $\varepsilon > 0$, $b > 0$ let u_ε be a solution of $(\mathcal{CP}_\varepsilon^{b,s})$ and denote, as in the previous section, by y_ε and p_ε its associated state and costate, respectively. Consider the mapping $F : \mathcal{Y}^s \times \mathcal{Y}^s \times \mathbb{R}_+ \to L^s(\Omega) \times L^s(\Omega)$ defined by

$$F(y, p, \varepsilon) := \begin{pmatrix} \Delta y + \Pi_\varepsilon(-N^{-1}p) + f - \phi(y) \\ \Delta p + y - \bar{y} - \phi'(y)p \end{pmatrix}. \tag{40}$$

In view of (14), proposition 10 and (37) we see that if u_0 is a local strict solution of $(\mathcal{CP}_0^s)$ then for b and $\varepsilon \geq 0$ small enough

$$F(y_\varepsilon, p_\varepsilon, \varepsilon) = 0.$$

Motivated by this fact, our objective is to obtain an "asymptotic expansion" for $(y_\varepsilon, p_\varepsilon)$ around (y_0, p_0). As in the ODE case (see [2]), the mapping F is, in general, not differentiable at $(y_0, p_0, 0)$. In fact, it can be easily seen that $D_\varepsilon F(y_0, p_0, 0)$ does not always exists. Therefore, we cannot apply the standard implicit function theorem in order to obtain such expansion. We will overcome this difficulty in the same way as in [2], i.e. by using the following restoration theorem, whose proof can be found in the Appendix of [2].

Theorem 14. (Restoration theorem) *Let* X *and* Y *be Banach spaces,* E *a metric space and* $F : U \subset X \times E \to Y$ *a continuous mapping on an open set* U. *Let* $(\hat{x}, \varepsilon_0) \in U$ *be such that* $F(\hat{x}, \varepsilon_0) = 0$. *Assume that there exists a surjective linear continuous mapping* $A : X \to Y$ *and a function* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $c(\beta) \downarrow 0$ *when* $\beta \downarrow 0$ *such that, if* $x \in \overline{B}(\hat{x}, \beta)$, $x' \in \overline{B}(\hat{x}, \beta)$ *and* $\varepsilon \in B(\varepsilon_0, \beta)$, *then*

$$\|F(x', \varepsilon) - F(x, \varepsilon) - A(x' - x)\| \leq c(\beta)\|x' - x\|. \tag{41}$$

Then, denoting by B *a bounded right inverse of* A, *for* ε *close to* ε_0, $F(\cdot, \varepsilon)$ *has, in a neighborhood of* $\hat{x}$, *a zero denoted by* x_ε *such that the following expansion holds*

$$x_\varepsilon = \hat{x} - BF(\hat{x}, \varepsilon) + r(\varepsilon) \quad \text{with } ||r(\varepsilon)|| = o\left(\|F(\hat{x}, \varepsilon)\|\right). \tag{42}$$

Remark. *Note that hypothesis* (41) *implies that if A is invertible and β is such that $c(\beta)||A^{-1}||_{Y\to X} < 1$ (where $||\cdot||_{Y\to X}$ denotes the standard norm for the space of bounded linear functionals from Y to X) then for all $\varepsilon \in B(\varepsilon_0, \beta)$ the mapping $F(\cdot, \varepsilon)$ is injective in $\bar{B}(\hat{x}, \beta)$. In particular, for $\varepsilon \in B(\varepsilon_0, \beta)$ there exists a unique $x_\varepsilon \in \bar{B}(\hat{x}, \beta)$ such that $F(x_\varepsilon, \varepsilon) = 0$.*

In order to verify that F, defined in (40), satisfies the hypothesis of theorem 14 we need the following lemmas.

Lemma 15. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and denote by $\mathcal{A}(f)$ the set of points were f is not differentiable. For $s \in [1, \infty)$ set $\bar{f} : L^\infty(\Omega) \to L^s(\Omega)$ defined by*

$$\bar{f}[w](x) := f(w(x)). \tag{43}$$

Then $\bar{f}$ is Fréchet differentiable at every $\bar{w} \in L^\infty(\Omega)$ satisfying that

$$meas\,\{x \in \Omega\ ;\ \bar{w}(x) \in \mathcal{A}(f)\} = 0 \tag{44}$$

and $\left(D\bar{f}[\bar{w}]h\right)(x) = f'(\bar{w}(x))h(x)$ for all $h \in L^\infty(\Omega)$.

Proof. Let $\theta : L^\infty(\Omega) \to \mathbb{R}_+$ defined by

$$\theta(h) := \frac{||\bar{f}(\bar{w}+h) - \bar{f}(\bar{w}) - f'(\bar{w}(\cdot))h||_s^s}{||h||_\infty^s}. \tag{45}$$

We have to show that $\theta(h) \to 0$ as $h \to 0$. In fact we have

$$0 \le \theta(h) \le \int_\Omega \frac{|f(\bar{w}(x)+h(x)) - f(\bar{w}(x)) - f'(\bar{w}(x))h(x)|^s}{|h(x)|^s} \mathrm{d}x \tag{46}$$

and the result follows by the dominated convergence theorem using the fact that f is Lipschitz. □

For $w \in \mathcal{Y}^s$ set

$$Sing(w) := \{x \in \bar{\Omega}\ ;\ w(x) = 0\} \tag{47}$$

and for every $\varepsilon \ge 0$ define $\Pi_\varepsilon : \mathcal{Y}^s \to L^s(\Omega)$ by $(\Pi_\varepsilon(w))(x) := \pi_\varepsilon(w(x))$ for a.a. $x \in \Omega$. Lemmas 13 and 15 allows us to prove the following result.

Lemma 16. *Let $\widehat{w} \in \mathcal{Y}^s$ and suppose $meas(Sing(\widehat{w})) = 0$. Then*
(i) *For every $\varepsilon > 0$, $w \in \mathcal{Y}^s$, the function Π_ε is differentiable at w and for every $h \in \mathcal{Y}^s$*

$$\left(D\Pi_\varepsilon(w)h\right)(x) = \pi'_\varepsilon(w(x))h(x), \quad \textit{for a.a. } x \in \Omega.$$

(ii) *The function Π_0 is differentiable at $\widehat{w}$ and for every $h \in \mathcal{Y}^s$*

$$\left(D\Pi_0(\widehat{w})h\right)(x) = \pi'_0(\widehat{w}(x))h(x), \quad \textit{for a.a. } x \in \Omega.$$

(iii) *There exist a nondecreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\beta\downarrow 0} c(\beta) = 0$ such that for any $w', w \in \mathcal{Y}^s$ with $||w' - \widehat{w}||_{2,s} \le \beta$, $||w - \widehat{w}||_{2,s} \le \beta$ and $\varepsilon \in [0, \beta]$ we have*

$$||\Pi_\varepsilon(w') - \Pi_\varepsilon(w) - D\Pi_0(\widehat{w})(w'-w)||_s \le c(\beta)||w'-w||_{2,s}. \tag{48}$$

Proof. (i) Since, for $\varepsilon > 0$, π_ε is $\mathcal{C}^1$ it holds that Π_ε, viewed as mapping from $L^\infty(\Omega)$ into $L^\infty(\Omega)$, is also $\mathcal{C}^1$. Therefore, noting that $s > n/2$ ($s = 2$ if $n \leq 3$), the result easily follows.
(ii) Consequence of lemma 15 using that $\mathcal{Y}^s \subseteq L^\infty(\Omega)$.
(iii) Note that

$$||\Pi_\varepsilon(w') - \Pi_\varepsilon(w) - D\Pi_0(\widehat{w})(w' - w)||_s =$$
$$\left|\left|\left(\int_0^1 \{D\Pi_\varepsilon(w + s(w' - w)) - D\Pi_0(\widehat{w})\} \, \mathrm{d}s\right)(w' - w)\right|\right|_s$$
$$\leq \sup_{z \in B_{2,s}(\widehat{w},\beta)} ||D\Pi_\varepsilon(z) - D\Pi_0(\widehat{w})||_{\mathcal{Y}^s \to L^s(\Omega)} \, ||w' - w||_{2,s} \, .$$

where $B_{2,s}(\widehat{w}, \beta)$ denotes the ball in $W^{2,s}(\Omega)$ of center $\widehat{w}$ and radius β and $||\cdot||_{\mathcal{Y}^s \to L^s(\Omega)}$ denotes the standard norm for the space of linear bounded functions from $\mathcal{Y}^s$ to $L^s(\Omega)$. Let $h \in \mathcal{Y}^s$ with $||h||_{2,s} \leq 1$. Since $s > n/2$ ($s = 2$ if $n \leq 3$), we have

$$||D\Pi_\varepsilon(z)h - D\Pi_0(\widehat{w})h||_s^s \leq c_s^s \left(\int_\Omega |\pi'_\varepsilon(z(x)) - \pi'_0(\widehat{w}(x))|^s \mathrm{d}x\right)$$

with c_s being defined in (4). Thus,

$$||\Pi_\varepsilon(w') - \Pi_\varepsilon(w) - D\Pi_0(\widehat{w})(w' - w)||_s \leq c(\beta) \, ||w' - w||_{2,s}$$

where $c(\beta)$ is the nondecreasing function defined by

$$c(\beta) := c_s \left(\int_\Omega \sup_{\varepsilon \in [0,\beta]} \sup_{z \in B(\widehat{w}(x),\beta)} |\pi'_\varepsilon(z(x)) - \pi'_0(\widehat{w}(x))|^s \mathrm{d}x\right)^{\frac{1}{s}} .$$

Since $meas(Sing(\widehat{w})) = 0$, lemma 13 (i) and (iii) yields that $c(\beta) \downarrow 0$ as $\beta \downarrow 0$ by the dominated convergence theorem. □

In order to establish our main result we will have to impose a second order sufficient condition at any solution of $(\mathcal{CP}_0^s)$. First let us study the following abstract setting:

Consider a nonempty closed and convex set $K \subseteq L^2(\Omega)$ and define $K_s := K \cap L^s(\Omega)$. We will establish some second order sufficient conditions for the problem

$$\text{Min } J_0(u) \text{ subject to } u \in K_s \, . \tag{$\mathcal{AP}$}$$

Let $\bar{u} \in K$. The radial, tangent, normal cones to K at $\bar{u}$ and the critical cone in $L^2(\Omega)$ at $\bar{u}$ are defined respectively by

$$\begin{array}{rcl} \mathcal{R}_K(\bar{u}) & := & \{h \in L^2(\Omega) \; ; \; \exists \; \sigma > 0; \; \bar{u} + \sigma h \in K\}, \\ T_K(\bar{u}) & := & \{h \in L^2(\Omega) \; ; \; \exists \; u(\sigma) = \bar{u} + \sigma h + o_2(\sigma) \in K, \; \sigma \geq 0, \; \left|\left|\frac{o_2(\sigma)}{\sigma}\right|\right|_2 \to 0\}, \\ N_K(\bar{u}) & := & \{h^* \in L^2(\Omega) \; ; \; \langle h^*, u - \bar{u}\rangle \leq 0, \; \forall \; u \in K\}, \\ C(\bar{u}) & := & \{h \in T_K(u) \text{ and } DJ_0(\bar{u})h \leq 0\}. \end{array} \tag{49}$$

If $\bar{u} \in K_s$ we define analogously the radial, tangent and normal cones to K_s at $\bar{u}$ and the critical cone in $L^s(\Omega)$ at $\bar{u}$ by replacing $L^2(\Omega)$ by $L^s(\Omega)$ and K by K_s in (49). We denote them by $\mathcal{R}_{K_s}, T_{K_s}(\bar{u}), N_{K_s}(\bar{u})$ and $C_s(\bar{u})$ respectively.

We say that J_0 satisfies the local quadratic growth condition at $\bar{u}$ if there exists $\alpha > 0$ and a neighborhood $\mathcal{V}_s$ of $\bar{u}$ in $L^s(\Omega)$ such that

$$J_0(u) \geq J_0(\bar{u}) + \alpha ||u - \bar{u}||_2^2 + o(||u - \bar{u}||_2^2) \quad \text{for all } u \in K_s \cap \mathcal{V}_s. \tag{50}$$

The following notion of polyhedricity will be required (see [17, 24]). The set K_s is said to be *polyhedric* in $L^s(\Omega)$ at $u \in K_s$ if for all $u^* \in N_{K_s}(u)$ (sets of normal of K_s at u), the set $\mathcal{R}_{K_s}(u) \cap (u^*)^\perp$ is dense in $T_{K_s}(u) \cap (u^*)^\perp$ with respect to the $L^s(\Omega)$ norm. If K_s is polyhedric in $L^s(\Omega)$ at each $u \in K_s$ we say that K_s is *s-polyhedric.*

For various types of optimization problems (see [8]), positivity of the second derivative of the cost function over the critical cone at a point u can be related to the quadratic growth condition at u. This is usually referred as a no gap second order sufficient condition which under some hypothesis will be satisfied in our problem.

If ϕ is C^2 then, since $s > n/2$ ($s = 2$ if $n \leq 3$), the function $J_0 : L^s(\Omega) \to \mathbb{R}$ is C^2 (see [8, lemma 6.27]) and for all $u, v \in L^s(\Omega)$ we have

$$D^2 J_0(u)(v, v) = \int_\Omega \left\{ N v(x)^2 + (1 - p_u(x)\phi''(y_u(x)))\, z_v(x)^2 \right\} \mathrm{d}x, \tag{51}$$

where z_v is the unique solution of the linearized state equation

$$\left\{ \begin{array}{rcll} -\Delta z(x) + \phi'(y_u(x)) z(x) & = & v(x) & \text{for} \quad x \in \Omega, \\ z(x) & = & 0 & \text{for} \quad x \in \partial\Omega. \end{array} \right. \tag{52}$$

In addition, it is proved that the quadratic form $D^2 J_0(u)$ has a unique continuous extension over $L^2(\Omega) \times L^2(\Omega)$ and this extension is a Legendre form, which means that it is sequentially w.l.s.c. and that if h_k converges weakly to h in $L^2(\Omega)$ and $D^2 J_0(u)(h_k, h_k) \to D^2 J_0(u)(h, h)$ then h_k converges strongly to h in $L^2(\Omega)$.

The theorem below, which concerns to second order sufficient conditions for $(\mathcal{AP})$, is proved in [8, theorem 6.31].

Theorem 17. *Consider problem* $(\mathcal{AP})$ *and let* $\bar{u} \in K_s$*. If* K_s *is s-polyhedric and* $C_s(\bar{u})$ *is dense in* $C(\bar{u})$*, then the quadratic growth condition* (50)*, the second order condition*

$$\exists\ \alpha > 0, \textit{such that } D^2 J_0(\bar{u})(h, h) \geq \alpha ||h||_2^2 \quad \textit{for all } h \in C(\bar{u}) \tag{53}$$

and the punctual relation

$$D^2 J_0(\bar{u})(h, h) > 0 \quad \textit{for all } h \in C(\bar{u}) \setminus \{0\} \tag{54}$$

are equivalent.

When $K = \mathcal{U}_+^2$ and $u \in K$ it is easy to verify that

$$\begin{array}{rcl} T_K(u) & := & \{ v \in L^2(\Omega)\ ;\ v(x) \geq 0 \ \text{ if } u(x) = 0 \text{ for a.a. } x \in \Omega \} \\ N_K(u) & := & \{ v \in (L^2(\Omega))^*\ ;\ v(x) \leq 0 \text{ and } v(x) = 0 \text{ if } u(x) > 0 \text{ for a.a. } x \in \Omega \}. \end{array} \tag{55}$$

If $u \in K_s$ the correspondig expressions for $T_{K_s}(u)$ and $N_{K_s}(u)$ are obtained by replacing $L^2(\Omega)$ by $L^s(\Omega)$ in (55). If u_0 is a local solution of $(\mathcal{CP}_0^s)$ and $p_0(x) \neq 0$ for almost all $x \in \Omega$, expression (11) yields that

$$C_s(u_0) \quad := \quad \{v \in L^s(\Omega) \ ; \ v(x) = 0 \ \text{ if } u_0(x) = 0 \text{ for a.a. } x \in \Omega\}. \qquad (56)$$

Analogously, if u_0 is a solution of $(\mathcal{CP}_0^2)$, the corresponding expression for $C(u_0)$ is obtained by replacing $L^s(\Omega)$ by $L^2(\Omega)$ in (56).

Now we give a simple proof of the following well known result (see for example [8, proposition 6.33]) which shows that theorem 17 can be applied in our case $(K_s = \mathcal{U}_+^s)$.

Lemma 18. *Suppose that $K_s = \mathcal{U}_+^s$, then*
(i) The set K_s is s-polyhedric.
(ii) If u_0 is a local solution of $(\mathcal{CP}_0^s)$, then $C_s(u_0)$ is dense in $C(u_0)$.

Proof. (i) Let $u \in \mathcal{U}_+^s$ and $u^* \in N_{\mathcal{U}_+^s}(u)$. For $h \in T_{\mathcal{U}_+^s}(u) \cap (u^*)^\perp$ and $k \in \mathbb{N}$ let $h_k \in L^\infty(\Omega)$ be defined as

$$h_k(x) := \begin{cases} 0 & \text{if } 0 < u(x) \leq 1/k \\ \max\{-k, \min\{h(x), k\}\} & \text{otherwise.} \end{cases} \qquad (57)$$

It is easy to check that $h_k \in \mathcal{R}_{\mathcal{U}_+^s} \cap (u^*)^\perp$ and $h_k \to h$ in $L^s(\Omega)$ by the dominated convergence theorem.
(ii) Given $h \in C(u_0)$ the sequence h_k defined in (57) belongs to $C_s(u_0)$ and converges in $L^2(\Omega)$ to h by the dominated convergence theorem. □

To obtain our main result we will assume two hypothesis. The first one allows to ensure that hypothesis (41) holds at $(y_0, p_0, 0)$ for the mapping F defined in (40). The second one will imply that the set of solutions of $(\mathcal{CP}_0^s)$ is isolated and that $D_{(y,p)}F(y_0, p_0, 0)$ is an isomorphism (see lemma 19). We consider the following hypothesis:

(H1) For the adjoint state p_0, associated to any local solution u_0 of $(\mathcal{CP}_0^s)$, it holds that

$$meas(Sing(p_0)) = 0.$$

(H2) At any local solution u_0 of $(\mathcal{CP}_0^s)$, condition (53) holds.

Remark. *Suppose that **(H1)** does not hold. Then, the $W^{2,s}$ regularity of p_0 implies that $-\Delta p_0 = 0$ in $Sing(p_0)$ (see [10] page 195). Therefore, by equations* (8) *and* (10),

$$-\Delta \bar{y}(x) + \phi(\bar{y}(x)) \quad = \quad f(x) \quad for \quad x \in Sing(p_0)$$

which yields a compatibility condition between the data $\bar{y}$ and f.

Lemma 19. *Let u_0 be a solution of $(\mathcal{CP}_0^s)$, suppose that ϕ is C^2 and that **(H1)**, **(H2)** hold. Then F (defined in* (40)*) is differentiable with respect to (y,p) at $(y_0, p_0, 0)$ and the linear mapping $D_{(y,p)}F(y_0, p_0, 0)$ is an isomorphism.*

In addition, for every $(\delta_1, \delta_2) \in L^s(\Omega) \times L^s(\Omega)$, we have that

$$D_{(y,p)}F(y_0, p_0, 0)^{-1}(\delta_1, \delta_2)$$

is the unique solution of the reduced optimality system of

$$Min \left\{ \int_\Omega \left[\tfrac{1}{2} N v^2 + \tfrac{1}{2}\left(1 - p_0 \phi''(y_0)\right) z_{v+\delta_1}^2 + \delta_2 z_{v+\delta_1} \right] \mathrm{d}x \ ; v \in C(u_0) \right\} \qquad (\mathcal{QP}_{\delta_1,\delta_2})$$

where z_v *is defined in* (52).

Proof. In view of assumption **(H1)** and lemma 16, the mapping F is differentiable with respect to (y,p) at $(y_0,p_0,0)$ and

$$D_{(y,p)}F(y_0,p_0,0)(z,q) = \begin{pmatrix} \Delta z - \Pi_0'(-N^{-1}p_0)N^{-1}q - \phi'(y_0)z \\ \Delta q + z - \phi''(y_0)p_0 z - \phi'(y_0)q \end{pmatrix}.$$

Let $\delta_1, \delta_2 \in L^s(\Omega)$, to find $(z,q) \in \mathcal{Y}^s$ such that $D_{(y,p)}F(y_0,p_0,0)(z,q) = (\delta_1,\delta_2)$ is equivalent to solve in $\mathcal{Y}^s \times \mathcal{Y}^s$ the following system of PDE's

$$\begin{aligned} -\Delta z(x) + \phi'(y_0(x))z(x) &= \delta_1(x) - \frac{\Pi_0'(-N^{-1}p_0(x))q(x)}{N} \\ -\Delta q(x) + \phi''(y_0(x))p_0(x)z(x) + \phi'(y_0(x))q(x) &= \delta_2(x) + z(x) \end{aligned}$$

for all $x \in \Omega$. But these equations are exactly the reduced optimality system for problem $(\mathcal{QP}_{\delta_1,\delta_2})$ which can be written, denoting by $\langle \cdot,\cdot \rangle_{L^2}$ the standard duality product in $L^2(\Omega)$, as

$$\text{Min } \tfrac{1}{2} D^2 J_0(u_0)(v,v) + \langle \gamma^*_{\delta_1,\delta_2}, v \rangle_{L^2} + \beta^*_{\delta_1,\delta_2} \quad \text{subject to } v \in C(u_0)$$

for some $\gamma^*_{\delta_1,\delta_2} \in L^2(\Omega)$ and

$$\beta^*_{\delta_1,\delta_2} := \int_\Omega \left[\tfrac{1}{2}\left(1 - p_0\phi''(y_0)\right) z_{\delta_1}^2 + \delta_2 z_{\delta_1} \right] \mathrm{d}x.$$

In fact, since $z_{v+\delta_1} = z_v + z_{\delta_1}$, the cost function of $(\mathcal{QP}_{\delta_1,\delta_2})$ is given by

$$\tfrac{1}{2} D^2 J_0(u_0)(v,v) + \int_\Omega \left[\left(1 - p_0\phi''(y_0)\right) z_v z_{\delta_1} + \delta_2 z_v \right] \mathrm{d}x + \beta^*_{\delta_1,\delta_2}.$$

Since the above integral is a linear form, as a function of v, the existence of $\gamma^*_{\delta_1,\delta_2}$ follows by the Riesz's theorem.

By (H2) this cost function is strongly convex over the closed subspace $C(u_0)$ and therefore has a unique minimum. The $W^{2,s}$ regularity for its associated state and adjoint state follows readily by a boostrapping argument. □

For every $\varepsilon \geq 0$ let us define $q_\varepsilon := -p_\varepsilon / N$. Now we can state our main result.

Theorem 20. *Let* u_0 *be a solution of* $(\mathcal{CP}_0^s)$, *suppose that* ϕ *is* C^2 *and that* ***(H1)***, ***(H2)*** *hold. Denote respectevely by* y_0 *and* p_0 *the state and adjoint state associated to* u_0. *Then there are* $\bar{b} > 0$ *and* $\bar{\varepsilon} > 0$ *such that for* $\varepsilon \in [0,\bar{\varepsilon}]$ *problem* $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ *has a unique solution* u_ε. *In addition, denoting by* y_ε *and* p_ε *the associated state and adjoint state for* u_ε, *the following expansion around* (y_0,p_0) *holds*

$$\begin{pmatrix} y_\varepsilon \\ p_\varepsilon \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix} + D_{(y,p)}F(y_0,p_0,0)^{-1}F(y_0,p_0,\varepsilon) + r(\varepsilon), \qquad (58)$$

where $r(\varepsilon) = o(||F(y_0, p_0, \varepsilon)||_s)$. *Moreover,* $D_{(y,p)}F(y_0, p_0, 0)^{-1}F(y_0, p_0, \varepsilon)$ *is characterized as being the unique solution of* $(\mathcal{QP}_{\delta\Pi(\varepsilon),0})$ *where*

$$\delta\Pi(\varepsilon) := \Pi_\varepsilon(q_0) - \Pi_0(q_0).$$

Proof. Lemma 16 (ii) implies that hypothesis (41) of theorem 14 is satisfied with $A = D_{(y,p)}F(y_0, p_0, 0)$. Lemma 19 yields that A is invertible, whence the first assertion follows from the convergence of $(y_\varepsilon, p_\varepsilon)$ to (y_0, p_0) in $\mathcal{Y}^s \times \mathcal{Y}^s$, established in proposition 10, and remark 3.

Noting that $F(y_0, p_0, \varepsilon) = F(y_0, p_0, \varepsilon) - F(y_0, p_0, 0) = (\delta\Pi(\varepsilon), 0)$, the second assertion follows by theorem 14 and lemma 19 with $\delta_1 = \delta\Pi(\varepsilon)$ and $\delta_2 = 0$. □

Theorem 20 yields, in particular, the following error bounds.

Corollary 21 (Error bounds)**.** *Under the assumptions of theorem 20 we have*
(i) *The error estimates for* $u_\varepsilon, y_\varepsilon$ *and* p_ε *are given by*

$$||u_\varepsilon - u_0||_s + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(||\delta\Pi(\varepsilon)||_s\right). \tag{59}$$

(ii) *The error bound for the control in the infinity norm is given by*

$$||u_\varepsilon - u_0||_\infty = O\left(||\delta\Pi(\varepsilon)||_\infty\right) = O(\pi_\varepsilon(0)). \tag{60}$$

(iii) *The error estimate for the cost is given by*

$$|J_0(u_\varepsilon) - J_0(u_0)| = O\left(||\delta\Pi(\varepsilon)||_s\right). \tag{61}$$

Proof. (i) Theorem 14 yields that

$$||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O(||F(y_0, p_0, \varepsilon)||_s) = O(||\delta\Pi(\varepsilon)||_s). \tag{62}$$

Therefore, using proposition 13 (i) we obtain that

$$||u_\varepsilon - u_0||_s = ||\Pi_\varepsilon(q_\varepsilon) - \Pi_0(q_0)||_s = O(||q_\varepsilon - q_0||_s) + O(||\delta\Pi(\varepsilon)||_s), \tag{63}$$

which combined with (62) yields (59).
(ii) Clearly, as in (i)

$$||u_\varepsilon - u_0||_\infty = O(||q_\varepsilon - q_0||_\infty) + O(||\delta\Pi(\varepsilon)||_\infty), \tag{64}$$

and thus, using that $s > n/2$ ($s = 2$ if $n \leq 3$),

$$||u_\varepsilon - u_0||_\infty = O(||q_\varepsilon - q_0||_{2,s}) + O(||\delta\Pi(\varepsilon)||_\infty).$$

Hence, using the estimation given in (i),

$$||u_\varepsilon - u_0||_\infty = O(||\delta\Pi(\varepsilon)||_s) + O(||\delta\Pi(\varepsilon)||_\infty) = O(||\delta\Pi(\varepsilon)||_\infty),$$

and the result follows from lemma 13(iv).
(iii) We have

$$J_0(u_\varepsilon) - J_0(u_0) = \tfrac{1}{2}\int_\Omega \{(u_\varepsilon + u_0)(u_\varepsilon - u_0) + (y_\varepsilon + y_0 - 2\bar{y})(y_\varepsilon - y_0)\}\,\mathrm{d}x. \tag{65}$$

Since $s > n/2$ ($s = 2$ if $n \leq 3$), proposition 11 and lemma 1 (i) imply that $u_\varepsilon + u_0$ and $y_\varepsilon + y_0 - 2\bar{y}$ are uniformly bounded in $L^\infty(\Omega)$. Henceforth lemma 7 implies that

$$J_0(u_\varepsilon) - J_0(u_0) = O(||u_\varepsilon - u_0||_1) = O(||u_\varepsilon - u_0||_s)$$

and the result follows by (i). □

4 Examples

In this section the results of section 3 are applied to the examples given in remark 2. In subsection 4.1 we obtain precise error bounds for the central path. We pay particular attention to the logarithmic barrier in view of its well known properties as a penalty function. In section 4.2 we study the error for the cost function. in what follows we will assume that ϕ is C^2.

4.1 Error estimates for the central path

First, note that combining (i) and (ii) of corollary 21 yields

$$||u_\varepsilon - u_0||_\infty + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(\pi_\varepsilon(0)\right). \tag{66}$$

First order condition for $(\mathcal{P}_{\varepsilon,0})$ implies that $\pi_\varepsilon(0)$ is the unique solution of

$$t + \varepsilon\ell'(t) = 0. \tag{67}$$

Thus, particularizing ℓ and using (67) will give precise error bounds for the central path.

4.1.1 Negative power penalty

If $\ell(t) = \ell_1(t) := t^{-p}$ with $p > 0$, then (67) yields that $\pi_\varepsilon(0) = O\left(\varepsilon^{1/(2+p)}\right)$ and thus

$$||u_\varepsilon - u_0||_\infty + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(\varepsilon^{1/(2+p)}\right). \tag{68}$$

Expression (68) implies that for every $p > 0$ the error is worst than $O(\sqrt{\varepsilon})$.

4.1.2 Power penalty

When $\ell(t) = \ell_2(t) := -t^p$ with $p \in (0, 1)$, equation (67) yields that $\pi_\varepsilon(0) = O(\varepsilon^{1/(2-p)})$ and thus

$$||u_\varepsilon - u_0||_\infty + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(\varepsilon^{r(p)}\right). \tag{69}$$

where $r(p) := 1/(2 - p) < 1$. Note that $r(p) \uparrow 1$ as $p \uparrow 1$.

4.1.3 Entropy penalty

The case $\ell(t) = \ell_3(t) := t \log t$ will be the one with the smallest error bound. In fact, equation (67) implies that $\pi_\varepsilon(0)$ is the unique solution of

$$t + \varepsilon(\log t + 1) = 0. \tag{70}$$

Even if we do not have an explicit solution for this equation, the monotony of left hand side of (70) can be used in order to obtain a precise estimate for $\pi_\varepsilon(0)$. Indeed, it can be easily seen that for every $k \geq 1$, denoting by

$$\log^k(\cdot) := \log ... \log(\cdot)$$

(there are k logarithms), we have that $\pi_\varepsilon(0) = O(\psi(\varepsilon))$ where

$$\varepsilon \log^k |\log \varepsilon| \leq \psi(\varepsilon) \leq \varepsilon |\log \varepsilon| \quad \text{for } \varepsilon \text{ small enough.}$$

Thus

$$||u_\varepsilon - u_0||_\infty + ||y_\varepsilon - y_0||_{2,s} + ||p_\varepsilon - p_0||_{2,s} = O\left(\psi(\varepsilon)\right). \tag{71}$$

4.1.4 Logarithmic penalty

It is well known that the case $\ell(t) = \ell_4(t) := -\log t$ is particularly important. Fortunately, $\pi_\varepsilon(z)$ can be computed explicitly for all $z \in \mathbb{R}$. Indeed, first order condition for $(\mathcal{P}_{\varepsilon,z})$ implies that $\pi_\varepsilon(z)$ is the unique solution of

$$t - z - \varepsilon/z = 0. \tag{72}$$

Henceforth, $\pi_\varepsilon(z)$ is given by

$$\pi_\varepsilon(z) = \tfrac{1}{2}\left(x + \sqrt{x^2 + 4\varepsilon}\right). \tag{73}$$

If $n \leq 3$ (hence $s = 2$) expression (73) will allow us, using corollary 21(i), to compute the error for the control in the L^2 norm (see (77)).

Theorem 22. *Suppose that the assumptions of theorem 20 hold. Let $\bar{b} > 0$ be such that $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ has a unique solution u_ε for $\varepsilon > 0$ small enough. Then:*
(i) *We have*

$$||u_\varepsilon - u_0||_\infty + ||p_\varepsilon - p_0||_{2,s} + ||y_\varepsilon - y_0||_{2,s} \quad = \quad O(\sqrt{\varepsilon}). \tag{74}$$

(ii) *If in addition $n \leq 3$ (hence $s = 2$), there exist $m \in \mathbb{N}$, positive real numbers $\alpha > 0$, $0 < \bar{\delta} < 1$ and a finite collection of closed C^2 curves $(C_i)_{1\leq i\leq m}$ such that:*

- *The singular set $Sing(p_0)$ can be expressed as*

$$Sing(p_0) = \bigcup_{i=1}^{m} C_i. \tag{75}$$

- *For all $i \in \{1, ..., m\}$, defining $C_i^{\bar{\delta}} := \{x \in \Omega;\ dist(x, C_i) \leq \bar{\delta}\}$, it holds that:*

$$|p_0(x)| \geq \alpha\, dist(x, C_i) \quad \text{for all } x \in C_i^{\bar{\delta}}. \tag{76}$$

Then

$$||u_\varepsilon - u_0||_2 + ||p_\varepsilon - p_0||_{2,2} + ||y_\varepsilon - y_0||_{2,2} = O(\varepsilon^{\frac{3}{4}}). \tag{77}$$

Proof. (i) Follows directly from (66) since (73) implies that $\pi_\varepsilon(0) = 0$.
(ii) In view of corollary 21(i), with $s = 2$, we will estimate the right hand side of (59). For simplicity we assume that $Sing(p_0) = \partial\Omega$ and that $p_0 < 0$ in Ω. We will use an argument based on local mappings. Set

$$Q := \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\ |x'| < 1, |x_n| < 1\right\}.$$

Since $\partial\Omega$ is C^2 there exists $I \in \mathbb{N}$ and $\{(\omega_i, \phi_i)\}_{0\leq i\leq I}$ such that for every $i \in \{1, ..., I\}$ we have that ω_i is an open set and $\phi_i : \omega_i \to Q$ is a C^2 mapping with a C^2 inverse satisfying that $\overline{\omega_0} \subsetneq \Omega$, $\bar{\Omega} \subseteq \cup_{i=0}^I \omega_i$, $\partial\Omega \subseteq \cup_{i=i}^I \omega_i$ and

$$\begin{array}{rcl} \phi_i(\omega_i \cap \Omega) & = & Q \cap \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_n > 0\right\} =: Q^+ \\ \phi_i(\omega_i \cap \partial\Omega) & = & Q \cap \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_n = 0\right\} =: Q^0. \end{array}$$

Clearly $||\Pi_\varepsilon(q_0) - \Pi_0(q_0)||_2^2 \leq \sum_{i=0}^I I_i$ where for every $i \in \{1, ..., I\}$

$$I_i := \int_{\Omega\cap\ \omega_i} |\pi_\varepsilon(q_0(x)) - \pi_0(q_0(x))|^2 \mathrm{d}x.$$

Since $\overline{\omega_0} \subsetneq \Omega$, lemma 13 (iv) yields that $I_0 = O(\varepsilon^2)$. Let us now fix $i \in \{1,...,I\}$ and set $\tau = q_0 \circ \phi_i^{-1}$. By a change of variable we obtain the existence of K_i such that

$$I_i \leq K_i \int_{B_{n-1}} \int_0^1 |\pi_\varepsilon(\tau(x', x_n)) - \pi_0(\tau(x', x_n))|^2 \, \mathrm{d}x_n \mathrm{d}x',$$

where B_{n-1} denotes the unit ball in $\mathbb{R}^{n-1}$. Hypothesis (76) implies the existence of $\bar{\alpha} > 0$ such that

$$\tau(x', x_n) \geq \bar{\alpha} x_n \quad \text{for all } x_n \in [0, \bar{\delta}]. \tag{78}$$

Therefore, using the uniformity with respect to $x' \in B_{n-1}$ in (78), we have that

$$\sum_{i=1}^{I} I_i = O\left(\int_0^1 |\pi_\varepsilon(\alpha x_n) - \pi_0(\alpha x_n)|^2 \, \mathrm{d}x_n \right).$$

Expression (73) yields that

$$\begin{aligned} \textstyle\int_0^1 |\pi_\varepsilon(\alpha x_n) - \pi_0(\alpha x_n)|^2 \, \mathrm{d}x_n &= \textstyle\int_0^1 \left(x^2 + 2\varepsilon - x\sqrt{x^2 + 4\varepsilon} \right) \mathrm{d}x \\ &= \tfrac{1}{3} + 2\varepsilon - \tfrac{1}{3}(1 + 4\varepsilon)^{3/2} + \tfrac{1}{3}(4\varepsilon)^{3/2} \end{aligned}$$

and noting that $(1 + 4\varepsilon)^{3/2} = 1 + 6\varepsilon + O(\varepsilon^2)$, we obtain the desired result. $\square$

4.2 Error estimate for the cost function

Note that by corollary 21(iii) we have directly that

$$J_0(u_\varepsilon) - J_0(u_0) = O(||u_\varepsilon - u_0||_\infty) \tag{79}$$

which is bigger than $O(\varepsilon)$ for the four examples studied in subsection 4.1. Now we improve estimate (79) for $\ell = \ell_2, \ell_3$ and ℓ_4 by generalizing an argument suggested by Anton Schiela, in a personal communication, for the convex case (for example, when $\phi \equiv 0$) and for the logarithmic barrier.

Theorem 23. *Let $\ell = \ell_2, \ell_3, \ell_4$ (defined in subsection 4.1) and suppose that the assumptions of theorem 20 hold. Let $\bar{b} > 0$ be such that $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ has a unique solution for $\varepsilon > 0$ small enough. Then*

$$J_0(u_\varepsilon) - J_0(u_0) = O(\varepsilon) \tag{80}$$

Proof. Since J_0 is of class C^2 we have that

$$J_0(u_0) \geq J_0(u_\varepsilon) + DJ_0(u_\varepsilon)(u_\varepsilon - u_0) - O\left(\sup_{z \in [u_\varepsilon, u_0]} ||D^2 J_0(z)||_{\mathcal{L}(\mathcal{Y}^s, \mathcal{Y}^s)} ||u_\varepsilon - u_0||_\infty^2 \right) \tag{81}$$

where $\mathcal{L}(\mathcal{Y}^s, \mathcal{Y}^s)$ denotes the space of continuous bilinear forms over $\mathcal{Y}^s \times \mathcal{Y}^s$. Expression (51) yields that $\sup_{z \in [u_\varepsilon, u_0]} ||D^2 J_0(z)||_{\mathcal{L}(\mathcal{Y}^s, \mathcal{Y}^s)}$ is uniformly bounded in ε. Therefore by (69), (71) and (74),

$$\sup_{z \in [u_\varepsilon, u_0]} ||D^2 J_0(z)||_{\mathcal{L}(\mathcal{Y}^s, \mathcal{Y}^s)} ||u_\varepsilon - u_0||_\infty^2 = O(||u_\varepsilon - u_0||_\infty^2) = O(\varepsilon). \tag{82}$$

On the other hand, optimality conditions for $(\mathcal{CP}_\varepsilon^{\bar{b},s})$ yield that

$$DJ_0(u_\varepsilon) = -\varepsilon \ell'(u_\varepsilon), \tag{83}$$

hence, using (81) and (82), we have that

$$J_0(u_\varepsilon) - J_0(u_0) \leq -\varepsilon \int_\Omega \ell'(u_\varepsilon(x))(u_\varepsilon(x) - u_0(x))\mathrm{d}x + O(\varepsilon). \tag{84}$$

Since for $\ell_2(t)$ and $\ell_4(t)$ it holds that $\ell_2', \ell_4' \leq 0$, we obtain that

$$J_0(u_\varepsilon) - J_0(u_0) \leq -\varepsilon \int_\Omega \ell'(u_\varepsilon(x))u_\varepsilon(x)\mathrm{d}x + O(\varepsilon). \tag{85}$$

For ℓ_2 inequality (85) yields

$$J_0(u_\varepsilon) - J_0(u_0) \leq \varepsilon p \int_\Omega u_\varepsilon(x)^p \mathrm{d}x + O(\varepsilon) = O(\varepsilon),$$

by (25). For ℓ_4 inequality (85)

$$J_0(u_\varepsilon) - J_0(u_0) \leq -\varepsilon\, meas(\Omega) + O(\varepsilon) = O(\varepsilon).$$

Finally, for ℓ_3 inequality (84) implies that $J_0(u_\varepsilon) - J_0(u_0) \leq I_1 + I_2 + O(\varepsilon)$, where

$$\begin{array}{lll} I_1 & := & -\varepsilon \int_{\{u_\varepsilon(x) \leq e^{-1}\}} \ell'(u_\varepsilon(x))(u_\varepsilon(x) - u_0(x))\mathrm{d}x \quad \text{and} \\ I_2 & := & -\varepsilon \int_{\{u_\varepsilon(x) \geq e^{-1}\}} \ell'(u_\varepsilon(x))(u_\varepsilon(x) - u_0(x))\mathrm{d}x. \end{array}$$

Since $u_\varepsilon \log u_\varepsilon$ is bounded uniformly in ε, we have that

$$I_1 \leq -\varepsilon \int_{\{u_\varepsilon(x) \leq e^{-1}\}} (1 + \log u_\varepsilon(x))\, u_\varepsilon(x)\mathrm{d}x = O(\varepsilon)$$

and

$$I_2 = -\varepsilon \int_{\{u_\varepsilon(x) \geq e^{-1}\}} (1 + \log u_\varepsilon(x))\,(u_\varepsilon(x) - u_0(x))\,\mathrm{d}x = O(\varepsilon)$$

by (25). □

References

[1] R.A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[2] F. Alvarez, J. Bolte, J.F. Bonnans, and F. Silva. Asymptotic expansions for interior penalty solutions of control constrained linear-quadratic problems. *INRIA Report RR-6863*, 2009.

[3] T. Appel, A. Rösch, and G. Winkler. Optimal control in non-convex domains: A priori discretization error estimates. *Calcolo*, 44:137–158, 2007.

[4] N. Arada, E. Casas, and F. Tröltzsch. Error estimates for the numerical approximation of a semilinear elliptic control problem. *Comp. Optim. Appls.*, 23(2):201–229, 2002.

[5] M. Bergounioux, M. Haddou, M. Hintermüller, and K. Kunisch. A comparison of a Moreau-Yosida-based active set strategy and interior point methods for constrained optimal control problems. *SIAM Journal on Optimization*, 11:495–521 (electronic), 2000.

[6] J.T. Betts, S.K. Eldersveld, P.D. Frank, and J.G. Lewis. An interior-point algorithm for large scale optimization. In *Large-scale PDE-constrained optimization (Santa Fe, NM, 2001)*, volume 30 of *Lect. Notes Comput. Sci. Eng.*, pages 184–198. Springer, Berlin, 2003.

[7] J.F. Bonnans. Second order analysis for control constrained optimal control problems of semilinear elliptic systems. *Applied Math. Optimization*, 38-3:303–325, 1998.

[8] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems*. Springer-Verlag, New York, 2000.

[9] H. Brézis. Problèmes unilatéraux. *J. Mathématiques pures et appliquées*, 51:1–168, 1972.

[10] H. Brézis. *Analyse Fonctionnelle. Théorie et Applications.* Collection Mathématiques Appliquées pour la Maîtrise. Paris: Masson, 1983.

[11] E. Casas. Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems. *Adv. Comp. Math.*, 26:137–153, 2007.

[12] E. Casas, F. Tröltzsch, and A. Unger. Second order sufficient optimality conditions for a nonlinear elliptic boundary control problem. *Zeitschrift für Analysis und ihre Anwendungen*, 15:687–707, 1996.

[13] K. Deckelnick and M. Hinze. Convergence of a finite element approximation to a state constrainted elliptic control problem. *SIAM J. Numer. Anal.*, 45:1937–1953, 2007.

[14] L.C. Evans. *Partial differential equations.* Amer. Math Soc., Providence, RI, 1998. Graduate Studies in Mathematics 19.

[15] H.O. Fattorini. *Infinite dimensional optimization and control theory.* Cambridge University Press, New York, 1998.

[16] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer Verlag, Berlin, 1983.

[17] A. Haraux. How to differentiate the projection on a convex set in hilbert space. some applications to variational inequalities. *J. Mathematical Society of Japan*, 29:615–631, 1977.

[18] M. Hintermüller and K. Kunisch. Path-following methods for a class of constrained minimization problems in function space. *SIAM J. on Optimization*, 17:159–187, 2006.

[19] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE constraints.* Springer, New York, 2008.

[20] X. Li and J. Yong. *Optimal control theory for infinite dimensional systems.* Birkhäuser, Boston, 1995.

[21] J.-L. Lions. *Contrôle optimal de systèmes gouvernés par des equations aux dérivées partielles.* Dunod, Paris, 1968.

[22] H. Maurer. Optimization techniques for solving elliptic control problems with control and state constraint. part 2: Distributed control. *Comp. Optim. Applic.*, 18:141–160, 2001.

[23] H. Maurer and H. D. Mittelmann. Optimization techniques for solving elliptic control problems with control and state constraint. part 1: Boundary control. *Comp. Optim. Applic.*, 16:29–55, 2000.

[24] F. Mignot. Contrôle dans les inéquations variationnelles. *J. Functional Analysis*, 22:25–39, 1976.

[25] P. Neittaanmaki, J. Sprekels, and D. Tiba. *Optimization of elliptic systems.* Springer, New York, 2006.

[26] A. Schiela. Barrier methods for optimal control problems with state constraints. *SIAM J. on Optimization.*, 20(2):1002–1031, 2009.

[27] A. Schiela and M.Weiser. Superlinear convergence of the control reduced interior point method for PDE constrained optimization. *Comp.Opt.Appl.*, 39 (3):369–393, 2008.

[28] J. Sokolowski. Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. *SIAM Journal of Control and Optimization*, 25:1542–1556, 1987.

[29] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15:189–258, 1965.

[30] M. Ulbrich and S. Ulbrich. Superlinear convergence of affine-scaling interior-point Newton methods for infinite-dimensional nonlinear problems with pointwise bounds. *SIAM J. Control Optim.*, 38:1938–1984, 2000.

[31] M. Ulbrich and S. Ulbrich. Primal-Dual Interior point methods for PDE-constrained optimization. *Math. Program.*, 117:435–485, 2009.

[32] M. Weiser, T. Gänzler, and A. Schiela. A control reduced primal interior point method for a class of control constrained optimal control problems. *Comp. Opt. Appl.*, 41 (1):127–145, 2008.

[33] M. Weiser and A. Schiela. Function space interior point methods for PDE constrained optimization. *PAMM*, 4 (1):43–46, 2004.

Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 Orsay Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399