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# THE HILBERT SCHEME OF POINTS AND ITS LINK WITH BORDER BASIS

M.E. ALONSO<sup>(1)</sup>, J. BRACHAT, AND B. MOURRAIN

ABSTRACT. In this paper, we give new explicit representations of the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  as a projective subvariety of a Grassmannian variety. This new explicit description of the Hilbert scheme is simpler than the previous ones and global. It involves equations of degree 2. We show how these equations are deduced from the commutation relations characterizing border bases. Next, we consider infinitesimal perturbations of an input system of equations on this Hilbert scheme and describe its tangent space. We propose an effective criterion to test if it is a flat deformation, that is if the perturbed system remains on the Hilbert scheme of the initial equations. This criterion involves in particular formal reduction with respect to border bases.

## 1. INTRODUCTION

In the study of solutions of polynomial equations, the following question naturally arises:

*How can the families of ideals which defines a fixed number  $\mu$  of points counted with multiplicities, be characterized ?*

It is motivated by practical issues related to the solution of polynomial systems, given with approximate coefficients. Understanding the allowed perturbations of a zero-dimensional algebra, which keep the number of solutions constant, is an actual challenge, in the quest for efficient and stable numerical polynomial solvers. From a theoretical point of view, this question is closely linked with the study of the Hilbert Scheme of  $\mu$  points, which is an active area of investigation in Algebraic Geometry.

The Hilbert scheme of  $\mu$  points in the projective space  $\mathbb{P}_{\mathbb{K}}^n$  is the set of homogeneous saturated ideals of  $\mathbb{K}[x_0, \dots, x_n]$ , which define  $\mu$  projective points (counted with multiplicity). It was introduced by [8], following a schematic description: it is defined as a scheme representing a contravariant functor from the category of schemes to the one of sets. This functor associates to any scheme  $S$  the set of flat families  $\chi \subset \mathbb{P}^r \times S$  of closed subschemes of  $\mathbb{P}^r$  parametrized by  $S$ , whose fibers have Hilbert polynomial  $\mu$ . Many works were developed to analyze its geometric properties (see eg. [16]), which are still not completely understood. Among them, it is known to be reducible for  $n > 2$  [15], but the components are not known for  $\mu \geq 8$  [3]. Its connectivity firstly proved by Hartshorne (1965), is studied in [24] with a more constructive approach. Further investigations related to its defining equations were also developed for instance in [2], [7], [14], or on toric varieties in [10]. These equations are obtained from rank conditions in the vector space of polynomials in two successive “degrees”. The stratification of the Hilbert scheme by

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subsets of ideals with a fixed initial ideal for a given term ordering have been studied in several works, starting from [5], including more recent one like [18]. These strata can be embedded into affine open subsets of the variety corresponding to ideals associated to quotient algebras with a given monomial basis; They define an open covering charts of the Hilbert scheme and are studied in [9] (for  $n = 2$ ), [12], [13], or [26].

In this paper, we concentrate on punctual Hilbert schemes. We first recall an explicit description of these open covering charts, involving border relations. The main ingredients used in this description are the commutation relations characterizing a border basis, developed in [20] in a completely different setting. This link between border bases and equations on open subsets of the Hilbert scheme is also studied in [25], [17].

We extend this local description and give new explicit global equations for the Hilbert scheme  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  of  $\mu$  points in  $\mathbb{P}^n$ . We show how the commutation relations characterizing border basis can be exploited further to yield an explicit complete set of defining equations for  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  as a projective variety. Following a dual point of view, we obtain new equations which are of degree two in the Plücker coordinates of the ideal in degree  $\mu$  associated to an element of the Hilbert scheme. This new description of the Hilbert scheme is much more explicit than the one introduced in [8] and less complex than those in [2], [14], providing Plücker-like relations defining globally  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ .

The study of the Hilbert scheme of points leads to the analysis of the neighborhood around a point of this variety. From a practical point of view, we are interested in characterizing perturbations which preserve the multiplicity of the ideal. Such a deformation is called a *flat deformation*. A general setting for this problem consists in considering a system of equations depending on a family of parameters (or a scheme  $X$  which maps onto a scheme  $S$  of parameters), such that when the parameters are specialized at 0, we have a given variety (or scheme)  $X_0$ . The problem consists in finding a subvariety of the parameter space  $S$  so that the fibers above this subvariety are flat deformations of  $X_0$ . Such a problem goes back to works like [11]. An effective construction was proposed for instance in [6] based on a standard (or Grobner) basis approach. In this paper, we consider only the case where  $X_0$  is a set of  $\mu$  points counted with multiplicity. We provide an effective tests to check locally if a given perturbation of the input equations is a flat deformation, that is, if the perturbed saturated ideal still defines  $\mu$  points counted with multiplicity. This test can then be used to define on which subvariety of the parameter space, we have a flat deformation.

To handle this effective flatness test problem, we exploit the properties of commutation characterizing border bases. We first analyze locally these quadratic equations which define the Hilbert scheme of  $\mu$  points and describe its tangent space at a given point in terms of border bases. We consider an infinitesimal perturbation of the input equations and propose an effective criterion to test if the perturbed system remains on the Hilbert scheme of the initial equations, that is if we have a flat deformation of the zero-dimensional quotient algebra. This explicit effective criteria for a flat deformation, is given in terms of the input equations and the border basis of these input equations. This criterion involves in particular formal reduction with respect to border bases. It extends the work in [23] and ideas developed in [1].

**1.1. Notations.** In the following,  $R = \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[\underline{\mathbf{x}}]$  is the set of polynomials in the variables  $x_1, \dots, x_n$  and coefficients in the field  $\mathbb{K}$ . We also denote by  $S = \mathbb{K}[x_0, \dots, x_n] = \mathbb{K}[\mathbf{x}]$  the polynomial ring in  $x_0, \dots, x_n$  for a new variable  $x_0$  "of homogenization". For any  $\alpha \in \mathbb{N}^{n+1}$ , we set  $\mathbf{x}^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$  and  $\underline{\mathbf{x}}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . The canonical basis of  $\mathbb{N}^{n+1}$  is denoted by  $(e_i)_{i=0, \dots, n}$ , so that  $\mathbf{x}^{\alpha+e_i} = \mathbf{x}^\alpha x_i$  ( $\alpha \in \mathbb{N}^{n+1}, i = 0, \dots, n$ ). We identify  $\mathbb{N}^n$  with the set of exponents  $\alpha \in \mathbb{N}^{n+1}$  whose first coordinate  $\alpha_0$  is 0. For  $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ , we denote by  $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ .

We fix a set  $B = \{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_D}\}$  of monomials in  $x_1, \dots, x_n$ . Hereafter,  $B$  will be identified with its set of exponents. For a set of exponents  $B \subset \mathbb{N}^{n+1}$ , we will also denote by  $\mathbf{x}^B$  the set of monomials with exponents in  $B$ . Hereafter, we will also identify  $B$  and  $\mathbf{x}^B$ .

For  $B \subset \mathbb{N}^n$ , we say that  $B$  is connected to 1 if  $\mathbf{0} \in B$  (or  $1 \in \underline{\mathbf{x}}^B$ ) and for all  $\beta \in B - \{\mathbf{0}\}$ , there exist  $\beta' \in B$  and  $i \in 1 \dots n$  such that  $\beta = \beta' + e_i$  (or  $\underline{\mathbf{x}}^\beta = \underline{\mathbf{x}}^{\beta'} x_i$ ).

Given an ideal  $I$  of  $S$ , we denote by  $I_d$  the vector space of homogeneous polynomials of degree  $d$  which belong to  $I$ . We also note  $s_d$  the dimension of  $S_d$ .

For  $B \subset R$ , we denote by  $B^+ = x_1 B \cup \dots \cup x_n B \cup B$  and  $\partial B = B^+ - B$ . For  $B \subset S$ ,  $\partial_i B = \{\mathbf{x}^{\beta - e_i + e_j}, j = 0, \dots, n, \beta \in B \text{ with } \beta_i > 0\} - \mathbf{x}^B$ .

A *rewriting family* associated to a set  $B \subset R$  of monomials is a set of polynomials of the form  $(h_\alpha)_{\alpha \in \partial B}$  with:

$$h_\alpha(\underline{\mathbf{x}}) = \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha, \beta} \underline{\mathbf{x}}^\beta$$

with  $z_{\alpha, \beta} \in \mathbb{K}$  for all  $\alpha \in \partial B$ . We call it a *border basis* of  $B$  if moreover  $B$  is a basis of  $\mathcal{A} = R/(h_\alpha(\underline{\mathbf{x}}))$ .

If  $B = (\beta_1, \dots, \beta_m) \subset \mathbb{N}^{n+1}$  and  $\beta \in \mathbb{N}^{n+1}$ ,  $B^{\beta_i|\beta}$  is the sequence  $(\beta_1, \dots, \beta_{i-1}, \beta, \beta_{i+1}, \dots, \beta_m)$  obtained from  $B$ , by replacing  $\beta_i$  by  $\beta$ . Finally we note  $\langle B \rangle$  the vector space generated by  $B$ .

We are going to study the set of  $\mathbb{K}$ -algebras  $\mathcal{A}$  generated by  $x_1, \dots, x_n$ , which admit  $B$  as a monomial basis. For any  $a \in \mathcal{A}$ , we consider the operator  $M_a$  of multiplication by  $a$  in  $\mathcal{A}$ :

$$\begin{aligned} M_a &: \mathcal{A} \rightarrow \mathcal{A} \\ &b \mapsto ab \end{aligned}$$

As  $\mathcal{A}$  is a commutative algebra, the operators of multiplication by the variables  $x_i$  commute. Thus for any  $p \in R$ , we can define the operator  $p(M_{x_1}, \dots, M_{x_n})$  obtained by substitution of the variable  $x_i$  by  $M_{x_i}$  ( $i = 1, \dots, n$ ).

We define  $\mathfrak{J}(\mathcal{A}) := \{p \in R; p(M_{x_1}, \dots, M_{x_n}) = 0\}$  and call it the ideal associated to  $\mathcal{A}$ .  $\mathfrak{J}$  is a bijection between ideals defining  $\mu$  points counted with multiplicity and quotient algebras of dimension  $\mu$ .

Finally, let the "affinization by  $x_0$ " be the application from  $S$  to  $R$  that takes a polynomial  $p \in S$  and gives  $p(1, x_1, \dots, x_n) \in R$ . For any subset  $I \subset S$ , we will note  $\underline{I}$  its image by the affinization.

**1.2. The Hilbert scheme.** Let  $\mu \in \mathbb{N}$  and  $S = k[x_0, \dots, x_n]$  be the polynomial ring in  $n + 1$  variables over the field  $k$ . Let  $S_k \subset S$  be the subvector space of  $S$  generated by the homogeneous polynomials of degree  $k$ . The *Hilbert function* of  $I$

is the dimension of  $I_k$  as a function of  $k \in \mathbb{N}$ . When  $k$  large enough, it coincides with a polynomial called the *Hilbert polynomial* of  $I$ .

**Definition 1.1.** Let  $I$  be a graded ideal of  $S$ .  $I$  is said to be saturated if for all integers  $k \leq d$ ,  $I_d : S_k = I_{d-k}$ .

**Definition 1.2.** The Hilbert scheme of  $\mu$  points, denoted  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ , is the set of all saturated graded ideals of  $S$  whose Hilbert polynomial is the constant polynomial  $\mu$ .

For example, let  $a$  be any point  $\mathbb{P}^n$  different for the origin. Let  $m_a$  be the graded ideal generated by all the homogeneous polynomials  $P$  such that  $P(a) = 0$ . The prime ideal  $m_a$  is clearly saturated. The dimension of  $S_d/m_{a,d}$  is equal to 1 for  $d > 0$  because it is isomorphic to the field  $k$  by the evaluation in  $a$ . Thus  $m_a$  is a point of the Hilbert scheme of 1 point.

Similarly, for any integer  $k$ ,  $m_a^k$  is a point of the Hilbert scheme of  $\mu$  points (where  $\mu$  is actually the number of monomials of degree  $\leq k-1$ ).

**Definition 1.3.** Let  $\mathbf{u}$  be a linear form, then  $U_{\mathbf{u}}^\mu$  is the subset of  $\mathbf{Hilb}^\mu$  defined by:

$$U_{\mathbf{u}}^\mu := \{I \in \mathbf{Hilb}^\mu \mid (I : u) = I\}$$

**Proposition 1.4.** The family  $(U_{\mathbf{u}}^\mu)_{\mathbf{u} \in S_1}$  consists of a covering of  $\mathbf{Hilb}^\mu$ .

*Proof.* Let  $I \in \mathbf{Hilb}^\mu$ , its primary decomposition is of the form:

$$I = \bigcap_{0 \leq i \leq n} Q_{a_i}$$

where  $a_i$  are points in  $\mathbb{P}^n$  and  $Q_{a_i}$  are  $m_{a_i}$ -primary ideals. Because  $I$  is saturated, none of these points are equal to the origin. Thus, there exists a linear form  $u$  such that  $u(a_i) \neq 0$  for all  $i$ . By the primary decomposition of  $I$ , we get that:

$$(I : u) = I.$$

□

From now on we will assume  $\mathbf{u} = x_0$ . We will then dehomogenize by setting  $x_0 = 1$  in  $I$  and get an ideal  $\underline{I}$  of  $R = \mathbb{K}[x_1, \dots, x_n]$ .

**Proposition 1.5.**  $U_{x_0}^\mu$  is in bijection with the set :

$$\underline{U}^\mu = \{\tilde{I} \subset R \mid \mathcal{A} = R/\tilde{I} \text{ is of dimension } \mu\}$$

*Proof.* The bijection consists of the affinization:

$$\begin{aligned} \text{Aff} & : U_{x_0}^\mu \rightarrow \underline{U}^\mu \\ & I \mapsto \underline{I} \end{aligned}$$

From [14], we get that  $I \in U_{x_0}^\mu$  satisfies: dimension of  $S_d/I_d = \mu$  for all  $d \geq \mu$ . Moreover the affinization induces a surjective linear application between  $R_{\leq d}/\underline{I}_{\leq d}$  and  $S_d/I_d$  which is actually injective because  $(I : x_0 = I)$ . Thus  $R_{\leq d}/\underline{I}_{\leq d}$  is of dimension  $\mu$  for all  $d \geq \mu$ . The filtration:

$$R_{\leq d}/\underline{I}_{\leq d} \subset R_{\leq d+1}/\underline{I}_{\leq d+1} \subset \dots \subset R/\underline{I}$$

gives us (using dimension) that  $\mathcal{A} = R/\underline{I}$  is equal to  $R_{\leq d}/\underline{I}_{\leq d}$  for all  $d \geq \mu$  and thus is of dimension  $\mu$ .

To prove this application is a bijection we construct its inverse. Consider the map that takes an ideal  $J$  of  $R$  and associates to it the ideal  $I$  of  $S$  defined by:

$$I = \{p \in S \mid p(1, x_1, \dots, x_r) \in J\}$$

It is easy to check that this application is the inverse of  $\text{Aff}$ , since  $(I : x_0^*) = I$  implies that  $I \in U_{x_0}^\mu$ .  $\square$

Notice that a quotient algebra  $\mathcal{A} = R/I$  of dimension  $\mu$ , has always a monomial basis  $B$  connected to 1 and thus its elements are of degree  $< \mu$ . One can take for instance the monomial basis which is the complementary of the initial of  $I$ , for a give monomial ordering in  $R = \mathbb{K}[\underline{x}]$  [[chap. 15].

## 2. HILBERT SCHEME REPRESENTATION

In this section, we give a new description of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ , starting with a local one involving border bases and producing a global one based on duality and explicit homogeneous polynomials defining  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  as a projective subvariety of the Grassmannian  $\mathbf{Gr}^\mu(S_d^*)$ .

**2.1. Border basis representation.** Suppose that  $\mathcal{A}$  is a quotient algebra with a monomial basis  $B$  of cardinal  $\mu$ , connected to 1. Then for any  $\alpha \in \partial B$ , the monomial  $\underline{x}^\alpha$  is a linear combination in  $\mathcal{A}$  of the monomials of  $B$ : For any  $\alpha \in \partial B$ , there exists  $z_{\alpha, \beta} \in \mathbb{K}$  ( $\beta \in B$ ) such that  $h_\alpha^z(\underline{x}) := \underline{x}^\alpha - \sum_{\beta \in B} z_{\alpha, \beta} \underline{x}^\beta \equiv 0$  in  $\mathcal{A}$ . The equations  $h_\alpha^z(\underline{x})$  will be called, hereafter, the *border relations* of  $B$  in  $\mathcal{A}$ .

Given these border relations, we define a normal form  $N^z : \langle B^+ \rangle \rightarrow \langle B^+ \rangle$  by:

- $N^z(\underline{x}^\beta) = \underline{x}^\beta$  if  $\beta \in B$ ,
- $N^z(\underline{x}^\alpha) = \underline{x}^\alpha - h_\alpha^z(\underline{x}) = \sum_{\beta \in B} z_{\alpha, \beta} \underline{x}^\beta$  if  $\alpha \in \partial B$ .

This construction is extended by linearity to  $\langle B^+ \rangle$ .

Similarly, the tables of multiplication  $M_{x_i}^z : \langle B \rangle \rightarrow \langle B \rangle$  are constructed as  $M_{x_i}^z(\underline{x}^\beta) = N^z(x_i \underline{x}^\beta)$  for  $\beta \in B$ . Notice that the coefficients of the matrix of  $M_{x_i}^z$  in the basis  $B$  are linear in the parameters  $\mathbf{z}$ .

More generally, a monomial  $m$  can be reduced modulo the polynomials  $(h_\alpha^z(\underline{x}))_{\alpha \in \partial B}$  to a linear combination of monomials in  $B$ , as follows: decompose  $m = x_{i_1} \cdots x_{i_l}$  and compute  $N^z(m) = M_{x_{i_1}}^z \circ \cdots \circ M_{x_{i_l}}^z(1)$ . We easily check that  $m - N^z(m) \in (h_\alpha^z(\underline{x}))_{\alpha \in \partial B}$ .

From the knowledge of the multiplicative structure of  $\mathcal{A}$  (its tables of multiplication), we can deduce the roots of  $\mathcal{J}(\mathcal{A})$  with their multiplicities (see eg. [4]). Thus knowing these border relations yields all the information we need on the ideal  $\mathcal{J}(\mathcal{A})$  and its roots.

Given a quotient algebra with basis  $B$  of cardinal  $\mu$ , we have seen that here exists  $(z_{\alpha, \beta})_{\alpha \in \partial B, \beta \in B}$  which describe completely an ideal defining  $\mu$  points with multiplicity. Conversely, we are interested in characterizing the parameters  $\mathbf{z} := (z_{\alpha, \beta})_{\alpha \in \partial B, \beta \in B}$  such that the polynomials  $(h_\alpha^z(\underline{x}))_{\alpha \in \partial B}$  are the border relations of some quotient algebra  $\mathcal{A}^z$  with basis  $B$ .

The following result [20] also used in [25, 17] for special cases of basis  $B$  answers the question:

**Theorem 2.1.** *Let  $B$  be a set of  $\mu$  monomials connected to 1. The polynomials  $h_\alpha^z(\underline{\mathbf{x}})$  are the border relations of some quotient algebra  $\mathcal{A}^z$  of basis  $B$  iff*

$$(1) \quad M_{x_i}^z \circ M_{x_j}^z - M_{x_j}^z \circ M_{x_i}^z = 0 \quad \text{for } 1 \leq i < j \leq n.$$

*Proof.* As  $B$  is connected to 1, by [20],  $\mathbf{h}^z := (h_\alpha^z(\underline{\mathbf{x}}))$  is the border basis of a quotient algebra iff the matrices of multiplication  $M_{x_i}^z$  commute.  $\square$

Notice that the equations (1) are of degree  $\leq 2$  in  $\mathbf{z}$ .

**Proposition 2.2.** *Let  $B \subset S_d$  such that  $\underline{B}$  (its affinization by  $x_0$ ) is connected to 1, then*

$$H_{x_0}^B = \{I \subset S \mid I \text{ is saturated and } \mathcal{A} = R/I \text{ has basis } \underline{B}\}$$

*is a variety of  $\mathbb{K}^{\mu \times N}$  defined by  $\mathfrak{H}_{\underline{B}} := \{\mathbf{z} \in \mathbb{K}^{\mu \times N}; M_{x_i}^z \circ M_{x_j}^z - M_{x_j}^z \circ M_{x_i}^z = 0, 1 \leq i < j \leq n\}$ . We call it the variety of quotient algebras with basis  $B$ .*

These varieties depending on monomial sets  $B$  are used in [13] to define the global punctual Hilbert scheme, via a glueing construction.

**Proposition 2.3.** *The family  $H_{x_0}^B$  for all set  $B$  of monomials of size  $\mu$  connected to 1 is a covering of  $U_{x_0}^\mu$ .*

*Proof.* It comes from the previous bijection between  $U_{x_0}^\mu$  and the set  $\underline{U}^\mu$  which is covered by the family  $H_{x_0}^B$  for  $B$  a monomial set connected to one, of degree  $< \mu$ .  $\square$

**Proposition 2.4.**  *$H_{x_0}^B$  is an affine chart of the Hilbert Scheme of  $\mu$  points (for a set  $B$  of monomials of size  $\mu$  connected to 1).*

*In particular,  $H_{x_0}^B$  is isomorphic to the set of all  $(z_{\alpha,\beta})_{x^\alpha \in \partial B, x^\beta \in B}$  such that the operators  $M_{x_i}$  of multiplication by  $x_i$  in  $\langle B \rangle$ , given by the rewriting rules:*

$$f_\alpha = x^\alpha - \sum_{\beta \in B} z_{\alpha,\beta} x^\beta$$

*commute.*

*Proof.*  $M_{x_i}$  is the linear operator in  $\langle B \rangle$  that maps  $\underline{\mathbf{x}}^\gamma$  to  $x_i \underline{\mathbf{x}}^\gamma$  if  $x_i \underline{\mathbf{x}}^\gamma \in B$  and  $\underline{\mathbf{x}}^\gamma$  to  $\sum_{\beta \in B} z_{\alpha,\beta} \underline{\mathbf{x}}^\beta$  if  $\underline{\mathbf{x}}^\alpha = x_i \underline{\mathbf{x}}^\gamma$  is in  $\partial B$ . Thus the matrices of these operators in the basis  $B$  of  $\langle B \rangle$  depend linearly on  $(z_{\alpha,\beta})$ . Then, the proposition comes from the previous Theorem 2.1.  $\square$

**2.2. Grassmannian representation.** The usual description of the Hilbert Scheme introduced by [8] is based on the property that for  $d$  high enough and for any ideal  $\mathcal{J}$  defining  $\mu$  points with multiplicity, the vector space  $\mathcal{J}_d \subset S_d$  of polynomials of  $\mathcal{J}$  is of dimension  $\mu$  and defines a point in a certain Grassmannian variety of  $\mu$  hyperplanes. Based on the result of [7], one can take  $d \geq \mu$  and make this construction more precise.

Recall that  $S = \mathbb{K}[x_0, \dots, x_n]$  and that  $S_d$  is the vector space of homogeneous polynomials of degree  $d$ . We denote by  $s_d$  its dimension.

Let  $\mathbf{Gr}^n(E)$ , where  $E$  is any vector space and  $n$  an integer, be the set of  $n$ -dimensional vector subspace of  $E$ .

Using [14] and the persistence theorem from [7] gives us a necessary and sufficient rank condition on  $I_d$ , to have a saturated homogeneous ideal  $I$  in the Hilbert scheme of  $\mu$  points:

$$\dim(S_1 \cdot I_d) = s_{d+1} - \mu.$$

**Proposition 2.5.** *Given an integer  $d$  such that  $d \geq \mu$ , the Hilbert scheme of  $\mu$  points is in bijection with the subset  $W$  of  $\mathbf{Gr}^{s_d - \mu}(S_d) \times \mathbf{Gr}^{s_{d+1} - \mu}(S_{d+1})$  defined by*

$$\{(I_d, I_{d+1}) \in \mathbf{Gr}^{s_d - \mu}(S_d) \times \mathbf{Gr}^{s_{d+1} - \mu}(S_{d+1}) \mid S_1 \cdot I_d \subset I_{d+1}\}.$$

The Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  is the projection of this variety of  $\mathbf{Gr}^{s_d - \mu}(S_d) \times \mathbf{Gr}^{s_{d+1} - \mu}(S_{d+1})$  on the first factor. This description of the Hilbert scheme of  $\mu$  points extends to Hilbert scheme of ideals with a given Hilbert polynomial. It can be restated in terms of rank conditions:

**Proposition 2.6.** *Given an integer  $d$  such that  $d \geq \mu$ , the Hilbert scheme of  $\mu$  points is in bijection with the subset  $G$  of  $\mathbf{Gr}(s_d - \mu, S_d)$  given by*

$$\{I_d \in \mathbf{Gr}(s_d - \mu, S_d) \mid \dim_k S_1 \cdot I_d = s_{d+1} - \mu\}.$$

From the previous proposition 2.6, we can get local equations of the Hilbert scheme on affine coordinate charts of the grassmanian  $\mathbf{Gr}(s_d - \mu, S_d)$ :

**Definition 2.7.** *Let  $K$  be any subset of monomials of size  $\mu$  in  $S_d$ . Then  $\mathbf{A}_K$  is the subset of  $\mathbf{Gr}(s_d - \mu, S_d)$  of  $(s_d - \mu)$ -dimensional subspaces  $V \subset S_d$  complementary to  $\langle K \rangle$ .*

It is well known that  $\mathbf{A}_K$  is isomorphic to  $k^{\mu(s_d - \mu)}$ . The next proposition is a direct consequence of proposition 2.6 (cf. [11]).

**Proposition 2.8.** *In the affine coordinate chart  $\mathbf{A}_K$  of the grassmanian  $\mathbf{Gr}^{s_d - \mu}(S_d)$ , the Hilbert scheme  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  is defined by the ideal of all the  $(s_{d+1} - \mu + 1) \times (s_{d+1} - \mu + 1)$  minors of the  $s_{d+1} \times (n + 1) \cdot s_d$  matrix whose columns are  $x_i$  times the basis elements  $\in S_{d+1}$  of  $V \in \mathbf{A}_K$  for all  $0 \leq i \leq r$  and all  $1 \leq j \leq s_d - \mu$ .*

This description is given by rank conditions on  $S_1 \cdot W_d$  and yields determinantal equations of order  $s_{d+1} - \mu + 1$  on matrices of size  $(n + 1) s_d \times s_{d+1}$ . For more details, see [2], [7], [14].

**2.3. Dual and global description.** We are going to relate these affine varieties to an open subset of the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  and to give a global definition of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  as a projective subvariety of another Grassmannian defined on the dual. We will connect it with the border basis description of Section 2.1. An element of this projective variety corresponds to an ideal of  $S$ , which defines  $\mu$  points counted with multiplicity.

Instead of working with the elements of an ideal  $I$  in degree  $d$ , we consider the orthogonal  $I_d^\perp$  which is a subvector space of dimension  $\mu$  (if  $d \geq \mu$ ) in the dual space  $S_d^* := \text{Hom}(S_d, \mathbb{K})$ . It defines an element  $\Delta$  of the Grassmannian  $\mathbf{Gr}^\mu(S_d^*)$  of vector spaces of dimension  $\mu$  in  $S_d^*$ . We denote by  $(\mathbf{d}^\alpha)_{|\alpha|=d}$  the basis of  $S_d^*$ , dual to the monomial basis  $(\mathbf{x}^\alpha)_{|\alpha|=d}$  of  $S_d$ . The Grassmannian  $\mathbf{Gr}^\mu(S_d^*)$  is embedded in projective space of the exterior product  $\wedge^\mu S_d^*$ . A basis of  $\wedge^\mu S_d^*$  is  $(\mathbf{d}^{\alpha_1} \wedge \cdots \wedge \mathbf{d}^{\alpha_\mu})_{|\alpha_i|=d, \alpha_1 < \cdots < \alpha_\mu}$  (for some monomial ordering  $<$  in  $S$ ). The corresponding coordinates of an element  $\Delta \in \mathbb{P}(\wedge^\mu S_d^*)$  are denoted by  $\Delta_{\alpha_1, \dots, \alpha_\mu}$ . The Grassmannian  $\mathbf{Gr}^\mu(S_d^*)$  is the set of elements  $\Delta \in \wedge^\mu S_d^*$  whose coordinates



$(\Delta_{\alpha_1, \dots, \alpha_\mu})$  satisfy the well-known Plücker relations. For  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$ , we define  $\ker \Delta = \{p \in S_d, \text{ st. } \forall \Lambda \in \Delta, \Lambda(p) = 0\}$ .

Given a saturated homogeneous ideal  $I \subset S$ , defining  $\mu$  points with multiplicity, and described by a basis  $\delta_1, \dots, \delta_\mu$  of its orthogonal  $I^\perp$ , its Plücker coordinates in  $\wedge^\mu S_d^*$  are

$$\Delta_{\beta_1, \dots, \beta_\mu} = \begin{vmatrix} \delta_1(\mathbf{x}^{\beta_1}) & \cdots & \delta_1(\mathbf{x}^{\beta_\mu}) \\ \vdots & & \vdots \\ \delta_\mu(\mathbf{x}^{\beta_1}) & \cdots & \delta_\mu(\mathbf{x}^{\beta_\mu}) \end{vmatrix}$$

for  $|\beta_i| = d, \beta_1 < \cdots < \beta_\mu \in S_d$ .

We are going to describe the border relations with respect to a basis  $B \subset S_d$  of  $S_d/I_d$  in terms of these Plücker coordinates.

**Lemma 2.9.** *Let  $B = (b_1, \dots, b_\mu)$  be a set of polynomials of degree  $d$ , such that  $B$  is a basis of  $S_d/I_d$ . Then  $\Delta_B \neq 0$  and  $\mathcal{I}_d$  is generated by the relations*

$$a - \sum_{i=1}^{\mu} \frac{\Delta_{B^{[b_i|a]}}}{\Delta_B} b_i, \text{ for } a \in S_d$$

where  $B^{[b_i|a]} = (b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_\mu)$ .

*Proof.* As  $B$  is a basis of  $S_d/\mathcal{I}_d$ ,  $\mathcal{I}_d$  is generated by relations of the form

$$a \equiv \sum_{i=1}^{\mu} z_{a, b_i} b_i \text{ modulo } \mathcal{I}_d,$$

for all  $a \in S_d$ . This implies that  $\mathbf{z} = [z_{a, b_i}]_{i=1, \dots, \mu}$  is solution of the system:

$$\begin{bmatrix} \delta_1(a) \\ \vdots \\ \delta_\mu(a) \end{bmatrix} = \begin{bmatrix} \delta_1(b_1) & \cdots & \delta_1(b_\mu) \\ \vdots & & \vdots \\ \delta_\mu(b_1) & \cdots & \delta_\mu(b_\mu) \end{bmatrix} \mathbf{z}.$$

As  $B$  is a basis of  $S_d/\mathcal{I}_d$ ,  $\delta_B = \det(\delta_i(\beta_j))_{1 \leq i, j \leq \mu} \neq 0$  and the solution of this system is unique. By Cramer's rule, it is given by

$$z_{a, b_i} = \frac{\Delta_{B^{[b_i|a]}}}{\Delta_B}.$$

□

**Theorem 2.10.** *The Hilbert scheme of  $\mu$  points in  $\mathbb{P}^n$  is the projection on  $\mathbf{Gr}^\mu(S_d^*)$  of the variety of  $\mathbf{Gr}^\mu(S_d^*) \times \mathbf{Gr}^\mu(S_{d+1}^*)$  defined by the equations*

$$(2) \quad \Delta_B \Delta'_{B', x_k a} - \sum_{b \in B} \Delta_{B^{[b|a]}} \Delta'_{B', x_k b} = 0,$$

for all subset  $B$  (resp.  $B'$ ) of  $\mu$  (resp.  $\mu - 1$ ) monomials of degree  $d$  (resp.  $d + 1$ ), all monomial  $a \in S_d$  and for every  $k$ .

*Proof.* First of all, let  $(\Delta, \Delta')$  be an element of  $\mathbf{Gr}^\mu(S_d^*) \times \mathbf{Gr}^\mu(S_{d+1}^*)$  satisfying the equations (2). We need to prove that  $\ker \Delta$  is a point the Hilbert scheme of  $\mu$  points. By [7] we just need to prove that  $S_1 \cdot \ker \Delta \subset \ker \Delta'$ . Let  $B$  be a basis of  $S_d/\ker \Delta$ , and let  $p$  be an element of  $\ker \Delta$ . Equations (2) imply that  $\Delta'_{B', x_k p}$  is equal to zero for all  $k = 1, \dots, n$  and all subset  $B'$  of  $\mu - 1$  monomials of degree  $d + 1$ . Thus,  $x_k \cdot p$  belongs in  $\ker \Delta'$  for all  $k = 1, \dots, n$  and  $S_1 \cdot \ker \Delta \subset \ker \Delta'$ .

Conversely, let  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$  be a point of the Hilbert scheme associated to the ideal  $I$  so that  $I_d = \ker \Delta$ . Let  $\Delta' = I_{d+1}^\perp$  be the component of degree  $d+1$  associated to  $I$ . We are going to prove that the equations (2) are satisfied for  $(\Delta, \Delta')$ .

Suppose first that  $B$  is a basis of  $S_d/I_d = S_d/\ker \Delta$ , (2) is just a consequence of  $S_1 \cdot \ker \Delta \subset \ker \Delta'$ . In fact,

$$a = \sum_{b \in B} z_{a,b} b \pmod{(\ker \Delta)}.$$

Thus,

$$a x_k = \sum_{b \in B} z_{a,b} b x_k \pmod{(\ker \Delta')}$$

because  $S_1 \cdot \ker \Delta \subset \ker \Delta'$ . By linearity, we get

$$\sum_{b \in B} \Delta_{B^{[b|a]}} \Delta'_{B', x_k b} = \sum_{b \in B} \Delta_B \Delta'_{B', x_k z_{a,b} b} = \Delta_B \Delta'_{B', x_k a}$$

which is (2).

Suppose that  $B$  is no longer a basis of  $S_d/I_d$ . If it is of rank less than  $\mu - 2$  then (2) is obvious. We can assume that it is of rank  $\mu - 1$  and that

$$\sum_{i=1 \dots \mu-1} \lambda_i b_i = b_\mu$$

in  $S_d/\ker \Delta$ . Thus,

$$\Delta_{B^{[b_i|a]}} = -\lambda_i \Delta_{B^{[b_\mu|a]}}$$

and by linearity

$$\sum_i \Delta_{B^{[b_i|a]}} \Delta'_{B', x_k b_i} = -\sum_i \Delta_{B^{[b_\mu|a]}} \Delta'_{B', x_k b_i} \lambda_i$$

where  $\lambda_\mu = -1$ . But since we have

$$\sum_{i=1 \dots \mu-1} \lambda_i b_i = b_\mu$$

in  $S_d/\ker \Delta$  and as  $S_1 \cdot \ker \Delta \subset \ker \Delta'$  we get

$$\sum_{i=1 \dots \mu-1} \lambda_i b_i x_k - b_\mu x_k = 0 = \sum_{i=1 \dots \mu} \lambda_i b_i x_k$$

and

$$\Delta_{B^{[b_i|a]}} \Delta'_{B', x_k b_i} = -\sum_i \Delta_{B^{[b_\mu|a]}} \Delta'_{B', x_k b_i} \lambda_i = 0$$

which is (2) in the case where  $B$  is not a basis of  $S_d/I_d$ .  $\square$

Hereafter, we describe the equations of the projection on the first factor  $\mathbf{Gr}^\mu(S_d^*)$ , of the variety defined by the equations (2). Let us first introduce a generic linear form  $\mathbf{u} = u_0 x_0 + \dots + u_n x_n$  where  $u_i$  are parameters.

For any sequence  $B = \{b_1, \dots, b_\mu\} \subset S_{d-1}$ , we define

$$\Delta_{\mathbf{u}, B} = \det(\delta_i(\mathbf{u} \cdot b_j)) = \sum_{I \in \{0, \dots, n\}^\mu} \mathbf{u}^{(I)} \Delta_{I, B},$$

where  $\Delta_{I, B} = \Delta_{x_{I_1} b_1, \dots, x_{I_\mu} b_\mu}$  and  $(I)$  is the element of  $\mathbb{N}^{n+1}$  such that  $\mathbf{u}^{(I)} = u_{I_1} \dots u_{I_\mu}$  for all  $I \in \{0, \dots, n\}^\mu$ . Hereafter, we will identify  $I \in \{0, \dots, n\}^\mu$  with

an element of  $\{0, \dots, n\}^B$  so that for  $b \in B$ ,  $I_b$  is the coordinate of  $I$  indexed by  $b \in B$ .

For two sequences of monomials  $B, B' \subset S_{d-1}$ , we have

$$\Delta_{\mathbf{u} \cdot B} \Delta_{\mathbf{u} \cdot B'} = \sum_{K \in \mathbb{N}^{n+1}, |K|=2\mu} \mathbf{u}^K \sum_{I, J \in \{0, \dots, n\}^\mu, (I)+(J)=K} \Delta_{I \cdot B} \Delta_{J \cdot B'}.$$

**Proposition 2.11.** *Let  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$  be an element in the Hilbert scheme  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  of  $\mu$  points in  $\mathbb{P}^n$ . Then for all subset  $B$  of  $\mu$  monomials of degree  $d-1 = \mu$ , for all  $K \in \mathbb{N}^{n+1}$  with  $|K| = 2\mu$ , for all  $b, b'' \in B$  and all  $1 \leq i < j \leq n$ , we have*

$$(3) \quad \sum_{(I)+(J)=K} \sum_{b' \in B} (\Delta_{I \cdot B^{[x_{I_{b'} b'] | x_i b]}} \Delta_{J \cdot B^{[x_{J_{b''} b''} | x_j b']}} - \Delta_{I \cdot B^{[x_{I_{b'} b'] | x_j b]}} \Delta_{J \cdot B^{[x_{J_{b''} b''} | x_i b']}}) = 0.$$

*Proof.* Notice that the relations (3) are obtained as the coefficients in  $\mathbf{u}$  of the relations

$$(4) \quad \sum_{b' \in B} (\Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b' | x_i b]}} \Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b'' | x_j b']}} - \Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b' | x_j b]}} \Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b'' | x_i b']}}) = 0.$$

Proving (3) is thus equivalent to proving (4) for generic values of  $\mathbf{u}$ .

Let  $\Delta = \delta_1 \wedge \dots \wedge \delta_\mu \in \mathbf{Gr}^\mu(S_d^*)$  be an element of the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  associated to the ideal  $\mathcal{I} \subset S$ , and let  $B = \{b_1, \dots, b_\mu\} \subset S_{d-1}$ .

Let us assume first that  $\Delta_{\mathbf{u} \cdot B} \neq 0$  so that  $\mathbf{u} \cdot B \subset S_d$  is a basis of  $S_d/\mathcal{I}_d$ . As  $\mathbf{u}$  is generic, we can assume that  $\mathbf{u}$  is not a zero divisor in  $S/\mathcal{I}$ . By a change of variables which transforms  $\mathbf{u}$  in  $x_0$  and after dehomogenization (setting  $x_0 = 1$ ), we obtain a basis  $\underline{B} = (\underline{b}_1, \dots, \underline{b}_\mu)$  of  $\mathcal{A} = R/\underline{\mathcal{I}}$ . By Lemma 2.9, the operators of multiplication by the variables  $x_i$  ( $i = 1, \dots, n$ ) in the basis  $\underline{B}$  are given by the matrices

$$M_i = (z_{x_i b_k, x_0 b_j})_{1 \leq j, k \leq \mu} = \frac{1}{\Delta_{x_0 B}} (\Delta_{x_0 B^{[x_0 b_j | x_i b_k]}})_{1 \leq j, k \leq \mu}.$$

As these operators commute, we have for any  $\underline{b} \in \underline{B}$  and  $i, j \in \{1, \dots, n\}$ , we have

$$\begin{aligned} M_j \circ M_i(\underline{b}) &= M_j \left( \sum_{b' \in B} z_{x_i b, x_0 b'} \underline{b}' \right), \\ &= \sum_{b' \in B} \sum_{b'' \in B} z_{x_i b, x_0 b'} z_{x_j b', x_0 b''} \underline{b}'' \\ M_i \circ M_j(\underline{b}) &= \sum_{b' \in B} \sum_{b'' \in B} z_{x_j b, x_0 b'} z_{x_i b', x_0 b''} \underline{b}'' . \end{aligned}$$

By identification of the coefficients in the basis  $\underline{B}$ , we deduce that for all  $b'' \in B$ ,

$$\sum_{b' \in B} (z_{x_i b, x_0 b'} z_{x_j b', x_0 b''} - z_{x_j b, x_0 b'} z_{x_i b', x_0 b''}) = 0$$

After simplification by  $\Delta_B^2$ , we obtain

$$\sum_{b' \in B} (\Delta_{x_0 B^{[x_0 b' | x_i b]}} \Delta_{x_0 B^{[x_0 b'' | x_j b']}} - \Delta_{x_0 B^{[x_0 b' | x_j b]}} \Delta_{x_0 B^{[x_0 b'' | x_i b']}}) = 0.$$

These are the relations (4) for  $x_0 = \mathbf{u}$ .

Let us prove that the relations (4) are also satisfied when  $\Delta_{\mathbf{u} \cdot B} = 0$ , that is when  $B = (\mathbf{u} \cdot b_1, \dots, \mathbf{u} \cdot b_\mu)$  is not a basis of  $S_d/\mathcal{I}_d$ . We denote by  $\delta_1, \dots, \delta_\mu$  a basis of  $\mathcal{I}_d^\top$ .

In this case, either  $B$  is a family of rank  $< \mu - 1$  in  $S_d/\mathcal{I}_d$  and for any  $k = 1, \dots, \mu$  and  $b \in S_{d-1}$ , we have

$$\begin{aligned} & \Delta_{B[\mathbf{u} \cdot b_k | \mathbf{u} \cdot b]} \\ &= \begin{vmatrix} \delta_1(\mathbf{u} \cdot b_1) & \cdots & \delta_1(\mathbf{u} \cdot b_{k-1}) & \delta_1(\mathbf{u} \cdot b) & \delta_1(\mathbf{u} \cdot b_{k+1}) & \cdots & \delta_1(\mathbf{u} \cdot b_\mu) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \delta_\mu(\mathbf{u} \cdot b_1) & \cdots & \delta_\mu(\mathbf{u} \cdot b_{k-1}) & \delta_\mu(\mathbf{u} \cdot b) & \delta_\mu(\mathbf{u} \cdot b_{k+1}) & \cdots & \delta_\mu(\mathbf{u} \cdot b_\mu) \end{vmatrix} \\ &= 0, \end{aligned}$$

which implies the relations (3). Or  $\mathbf{u} \cdot B$  is a family of rank  $\mu - 1$  in  $S_d/\mathcal{I}_d$ . By invariance of the relations (3) by permutation of the elements in  $B$ , we can assume that  $\mathbf{u} \cdot b_1, \dots, \mathbf{u} \cdot b_{\mu-1}$  are linearly independent and that  $\mathbf{u} \cdot b_\mu \equiv \sum_{l=1}^{\mu-1} \lambda_l \mathbf{u} \cdot b_l$  in  $S_d/\mathcal{I}_d$ , for some  $\lambda_l \in \mathbb{K}$ . Equivalently,  $\sum_{l=1}^{\mu-1} \lambda_l \mathbf{u} \cdot b_l \equiv 0$  with  $\lambda_\mu = -1$ . Let  $B' = \{b_1, \dots, b_{\mu-1}\}$ . Then by multilinearity and antisymmetry of the determinant,

$$\Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b_l | b^*]} = -\lambda_l \Delta_{\mathbf{u} \cdot B', b^*},$$

for every  $b^* \in S_d$ , and  $l = 1, \dots, \mu$ . We deduce that

$$\begin{aligned} \sum_{l=1}^{\mu} \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b_l | x_i b]} \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b'' | x_j b_l]} &= \sum_{l=1}^{\mu} -\lambda_l \Delta_{\mathbf{u} \cdot B', x_i b} \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b'' | x_j b_l]} \\ &= -\Delta_{\mathbf{u} \cdot B', x_i b} \begin{vmatrix} \cdots & \delta_1(\sum_{l=1}^{\mu} \lambda_l x_j b_l) & \cdots \\ \vdots & \vdots & \\ \cdots & \delta_\mu(\sum_{l=1}^{\mu} \lambda_l x_j b_l) & \cdots \end{vmatrix}, \end{aligned}$$

by linearity of the determinant with respect to the column indexed by  $b''$ . As we have  $\mathbf{u} \cdot (\sum_{l=1}^{\mu-1} \lambda_l b_l) \equiv 0$  modulo  $\mathcal{I}_d$  and as  $\mathbf{u}$  is a generic linear form, thus a non-zero divisor in  $S/\mathcal{I}$ , we also have  $x_j (\sum_{l=1}^{\mu-1} \lambda_l b_l) \equiv 0$  in  $S_d/\mathcal{I}_d$ . This implies that the determinant above vanishes. Exchanging the role of  $i$  and  $j$ , we deduce the relations (4) when  $\Delta_{\mathbf{u} \cdot B} = 0$ .  $\square$

**Proposition 2.12.** *Let  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$  be an element in the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$ . Then for all subset  $B$  of  $\mu$  monomials of degree  $d - 1 = \mu$ , for all  $K \in \mathbb{N}^{n+1}$  with  $|K| = 2\mu$ , for all monomial  $a \in S_{d-1}$ , for all  $b \in B$  and for all  $k = 0, \dots, n$ , we have*

$$(5) \quad \sum_{(I)+(J)=K} \left( \Delta_{I \cdot B[x_{I_b} b | x_k a]} \Delta_{J \cdot B} - \sum_{b' \in B} \Delta_{I \cdot B[x_{I_b} b | x_k b']} \Delta_{J \cdot B[x_{J_{b'}} b' | x_{J_{b'}} a]} \right) = 0.$$

*Proof.* Here also the relations (5) are obtained as the coefficients in  $\mathbf{u}$  of the relations

$$(6) \quad \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b | x_k a]} \Delta_{\mathbf{u} \cdot B} - \sum_{b' \in B} \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b | x_k b']} \Delta_{\mathbf{u} \cdot B[\mathbf{u} \cdot b' | x_k a]} = 0.$$

Proving (5) is thus equivalent to proving (6) for generic values of  $\mathbf{u}$ .

Let  $\Delta = \delta_1 \wedge \cdots \wedge \delta_\mu \in \mathbf{Gr}^\mu(S_d^*)$  be an element of the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  associated to the saturated ideal  $\mathcal{I} \subset S$ , and let  $B = \{b_1, \dots, b_\mu\} \subset S_{d-1}$ .

We consider first the case where  $\Delta_{\mathbf{u} \cdot B} \neq 0$  so that  $\mathbf{u} \cdot B$  is a basis of  $S_d/\mathcal{I}_d$ . As  $\mathbf{u}$  is generic, we can assume that  $\mathbf{u}$  is not a zero divisor in  $S/\mathcal{I}$ . As before, by a change of variables which transforms  $\mathbf{u}$  in  $x_0$  and after dehomogenization (setting

$x_0 = 1$ ), we obtain a basis  $\underline{B} = (b_1, \dots, b_\mu)$  of  $\mathcal{A} = R/\underline{I}$ . In this basis, the matrices of multiplication by the variables  $x_k$  are given by the matrices

$$M_k = \frac{1}{\Delta_{x_0 B}} (\Delta_{x_0 B^{[x_0 b_i | x_k b_j]}})_{1 \leq i, j \leq \mu}.$$

We denoted by  $\delta_1, \dots, \delta_\mu$  elements of  $S_d^*$  such that  $\delta_1 \wedge \dots \wedge \delta_\mu = \Delta$ . By a linear combination of the elements  $\delta_i$ , we can assume that  $(\delta_1, \dots, \delta_\mu)$  is the dual basis of  $(x_0 b_1, \dots, x_0 b_\mu)$  in  $S_d/\mathcal{I}_d$ . Then, for any monomial  $a \in S_{d-1}$ , we have

$$\begin{bmatrix} \delta_1(x_k a) \\ \vdots \\ \delta_\mu(x_k a) \end{bmatrix} = M_k \cdot \begin{bmatrix} \delta_1(x_0 a) \\ \vdots \\ \delta_\mu(x_0 a) \end{bmatrix}.$$

By Lemma 2.9, for any  $a \in S_{d-1}$  and  $i = 1, \dots, \mu$ , we have

$$\delta_i(x_0 a) = \frac{\Delta_{x_0 B^{[x_0 b_j | x_0 a]}}}{\Delta_{x_0 B}}, \delta_i(x_k a) = \frac{\Delta_{x_0 B^{[x_0 b_j | x_k a]}}}{\Delta_{x_0 B}},$$

which yields the relations (6) for  $x_0 = \mathbf{u}$ , after multiplication by  $\Delta_{x_0 B}^2$ .

Suppose now that  $\Delta_{\mathbf{u} \cdot B} = 0$ . If  $B$  is of rank  $< \mu - 1$  in  $S_d/\mathcal{I}_d$  then for  $c \in S_d$ ,  $\Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b | c]}} = 0$  and the relations (5) are trivially satisfied. If  $B$  is of rank  $\mu - 1$  in  $S_d/\mathcal{I}_d$ , we may suppose as above that  $\mathbf{u} \cdot b_1, \dots, \mathbf{u} \cdot b_{\mu-1}$  are linearly independent so that  $\Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b_j | b]}} = -\lambda_j \Delta_{\mathbf{u} \cdot B', b}$  where  $\mathbf{u} \cdot b_\mu \equiv \sum_{j=1}^{\mu-1} \lambda_j \mathbf{u} \cdot b_j$  and  $\lambda_\mu = -1$ . Then, as in the previous proof, we have

$$\sum_{j=1}^{\mu} \Delta_{\mathbf{u} \cdot B^{[\mathbf{u} \cdot b_i | x_k b_j]}} \Delta_{B^{[\mathbf{u} \cdot b_j | x_0 a]}} = - \sum_{j=1}^{\mu} \lambda_j \Delta_{B^{[\mathbf{u} \cdot b_i | x_k b_j]}} \Delta_{\mathbf{u} \cdot B', x_0 a} = 0,$$

which yields the relations (6).  $\square$

**Theorem 2.13.** *An element  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$  is in the Hilbert scheme of  $\mu$  points in  $\mathbb{P}^r$  iff it satisfies the relations (3) and (5).*

*Proof.* From the previous propositions 2.11 and 2.12, if  $\Delta$  is in the Hilbert scheme of  $\mu$  points then (3) and (5) are satisfied. Conversely, we need to prove that if equations (3) and (5) are satisfied, then  $\Delta \in \mathbf{Gr}^\mu(S_d^*)$  is in the Hilbert scheme of  $\mu$  points. We also know that relations (3) and (5) are equivalent to generic relations (4) and (6). Let us prove that we can find a linear form  $\mathbf{u} = u_0 x_0 + \dots + u_n x_n$  and a subset  $B' \subset S_{d-1}$  of  $\mu$  monomials such that  $B = \mathbf{u} \cdot B'$  is a basis of  $S_d/\ker \Delta$ . In fact, if there are no such  $\mathbf{u}$  and  $B' = (b_1, \dots, b_\mu) \subset S_{d-1}$ , this means that for all  $\mathbf{u}$  and  $B'$ :

$$\Delta_{\mathbf{u} \cdot B'} = \sum_{i_1, \dots, i_\mu \in [0, n]^\mu} u_{i_1} \dots u_{i_\mu} \Delta_{x_{i_1} b_1, \dots, x_{i_\mu} b_\mu} = 0.$$

Thus for all  $i_1, \dots, i_\mu \in 0, \dots, n^\mu$  and all  $B' = (b_1, \dots, b_\mu) \subset S_{d-1}$ :  $\Delta_{x_{i_1} b_1, \dots, x_{i_\mu} b_\mu} = 0$ . It implies that  $\Delta_B = 0$  for all subset  $B \subset S_d$  of  $\mu$  monomials, which is impossible since  $\Delta \in \mathbf{Gr}^\mu(S_d^*) \neq 0$ . Thus we can find a linear form  $\mathbf{u} = u_0 x_0 + \dots + u_n x_n$  and a subset  $B' \subset S_{d-1}$  of  $\mu$  monomials such that  $B = \mathbf{u} \cdot B'$  is a basis of  $S_d/\ker \Delta$ . By a linear change of coordinates, we can assume that there exists a subset  $B' \subset S_{d-1}$  of  $\mu$  monomials such that  $B = x_0 B'$  is a basis of  $S_d/\ker \Delta$ .

Let  $I_d = \ker \Delta \subset S_d$  and  $\underline{I}_d$  its affinization (the subvector space of  $R_{\leq d}$  defined by putting  $x_0 = 1$  in  $I_d$ ). Let  $\pi : S_d/I_d \rightarrow \langle B \rangle$  be the natural isomorphism of vector space between  $S_d/I_d$  and  $\langle B \rangle$ . Moreover the affinization is an isomorphism

between  $R_{\leq d}/\underline{I}_d$  and  $S_d/I_d$ . Thus  $R_{\leq d}/\underline{I}_d$  is naturally isomorphic to  $\langle B \rangle$ . Let  $\psi$  be this isomorphism. As in proposition 2.11, we introduce the linear operators  $(M_i)_{i=1,\dots,n}$  operating on  $\langle B \rangle$ . Relations (4) and (6) for  $\mathbf{u} = x_0$  give us that the operators  $(M_i)_{i=1,\dots,n}$  commute and that, for every  $\alpha \in \mathbb{N}^{n+1}$  (resp.  $\alpha \in \mathbb{N}^n$ ) such that  $|\alpha| = d-1$  (resp.  $|\alpha| \leq d-1$ ), and every  $i = 1, \dots, n$ ,  $\pi(\underline{\mathbf{x}}^\alpha x_i) = M_i(\pi(\underline{\mathbf{x}}^\alpha x_0))$  (resp.  $\psi(\underline{\mathbf{x}}^\alpha x_i) = M_i(\psi(\underline{\mathbf{x}}^\alpha))$ ).

We define  $\sigma$ , an application from  $R$  to  $\langle B \rangle$ , as follows:  $\forall p \in R, \sigma(p) = p(M)(\pi(x_0^d))$  (or  $p(M)(\psi(1))$ ) where  $\underline{\mathbf{x}}^\alpha(M) = M_1^{\alpha_1} \dots M_n^{\alpha_n}$ . As the operators  $(M_i)_{i=1,\dots,n}$  commute,  $p(M)$  is well defined and  $\mathfrak{J} = \ker \sigma$  is an ideal of  $R$ . Let us prove by induction on the degree of  $p$  that  $\sigma(p) = \psi(p)$  for polynomials  $p$  in  $R_{\leq d}$ . In fact,  $\sigma(x_i x^\alpha) = M_i(\sigma(x^\alpha))$  where  $|\alpha| = k \leq d-1$ . By induction, we have  $\sigma(x_i x^\alpha) = M_i(\psi(x^\alpha)) = M_i(\pi(x^\alpha x_0^{d-k}))$ . From the end of the previous paragraph, this is equal to  $\psi(x^\alpha x_i)$ .

Then it follows that  $\sigma$  is surjective because for all  $b \in B, \sigma(b) = \pi(b) = b$ . Thus we get that  $R/\mathfrak{J}$  is of dimension  $\mu = |B|$ . Moreover, since  $\sigma$  and  $\psi$  coincide on  $R_{\leq d}$ ,  $\mathfrak{J}_{\leq d} = \underline{I}_d$ . Let  $J$  be the homogenization of  $\mathfrak{J}$ , then  $J^h$  defines  $\mu$  points with multiplicity and  $J_d = I_d$  (because  $\mathfrak{J}_{\leq d} = \underline{I}_d$ ). This proves that if the relations (3) and (5) are satisfied then  $\Delta$  is the dual in degree  $d$  of the ideal  $J \subset S$  defining  $\mu$  points in  $\mathbb{P}^n$ .  $\square$

### 3. TANGENT SPACE

Our objective in this section is to determine the tangent space at a point  $I_0$  of the variety  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  defined by the equations (3) and (5).

To compute the tangent space at a point  $I_0$ , we consider an open affine subset of the Hilbert scheme  $U_{\mathbf{u}}^\mu$  containing  $I_0$ . After a change of coordinates ( $\mathbf{u} = x_0$ ) we can assume this open subset is of the form  $U_{\mathbf{x}_0}^\mu$ . Using proposition 2.2, we can assume  $I_0$  is in the affine open subset  $H_{x_0}^B$  with  $\underline{B}$  connected to 1. From section 2.3,  $H_{\mathbf{x}_0}^B$  is an affine variety whose system of coordinates are the parameters  $(z_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B}$  such that

$$h_\alpha^0 = \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} z_{\alpha,\beta} \underline{\mathbf{x}}^\beta$$

is a border basis of  $\underline{I}_0$  for  $B$  where  $(z_{\alpha,\beta})_{\alpha \in \partial B, \beta \in B} = \left( \frac{\Delta_{x_0 B^{[x_0 b_\beta | b_\alpha]}}}{\Delta_{x_0 B}} \right)_{\alpha \in \partial B, \beta \in B}$  (the rest of the coordinates  $\Delta_{B'}$  for  $B'$  not of the form  $x_0 B^{[x_0 b_\beta | b_\alpha]}$ , can be expressed as polynomial functions of  $\left( \frac{\Delta_{x_0 B^{[x_0 b_\beta | b_\alpha]}}}{\Delta_{x_0 B}} \right)_{\alpha \in \partial B, \beta \in B}$ ).

Thus we can compute the tangent space of the Hilbert scheme using the previous system of coordinates. From section 2.3, the equations of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  in this system of coordinates reduce to the commutation relations:

$$M_i^0(\mathbf{z}) M_j^0(\mathbf{z}) - M_j^0(\mathbf{z}) M_i^0(\mathbf{z}) = 0,$$

where  $M_i^0(\mathbf{z})$  is the operator of multiplication by  $x_i$  in the basis  $B$  modulo the affine ideal  $\underline{I}_0$ .

By definition, the tangent space of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  at  $I_0$  is the set of vectors  $\mathbf{h}^1 = (h_{\alpha,\beta}^1)$  such that line  $h_\alpha^\varepsilon(\mathbf{x}) = h_\alpha^0(\mathbf{x}) + \varepsilon h_\alpha^1(\mathbf{x})$  intersects  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  with multiplicity  $\geq 2$ , where  $h_\alpha^1(\mathbf{x}) := \sum_{\beta \in B} h_{\alpha,\beta}^1 \underline{\mathbf{x}}^\beta$ .

Substituting in the commutation relations, we obtain

$$\begin{aligned}
& M_{x_i}^\varepsilon \circ M_{x_j}^\varepsilon - M_{x_j}^\varepsilon \circ M_{x_i}^\varepsilon \\
&= (M_{x_i}^0 \circ M_{x_j}^0 - M_{x_j}^0 \circ M_{x_i}^0) \\
&\quad + \varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon(M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

where  $M_{x_i}^\varepsilon = M_{x_i}^0 + \varepsilon M_{x_i}^1$ ,  $M_{x_i}^0$  is the operator of multiplication by  $x_i$  in  $\mathcal{A}^0$  and  $M_{x_i}^1$  is linear in  $\mathbf{h}^1$ . We deduce the linear equations in  $\mathbf{h}^1$  defining the tangent space of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  at  $I_0$ :

$$(7) \quad M_{x_i}^1 \circ M_{x_j}^0 + M_{x_i}^0 \circ M_{x_j}^1 - M_{x_j}^1 \circ M_{x_i}^0 - M_{x_j}^0 \circ M_{x_i}^1 = 0 \quad (1 \leq i < j \leq n).$$

**Definition 3.1.** Let  $\mathbf{h}^0 := (h_\alpha^0)_{\alpha \in \partial B}$  be a border basis for  $B$ . We denote by  $T_{\mathbf{h}^0}$ , the set of  $\mathbf{h}^1 = (h_\alpha^1)_{\alpha \in \partial B}$  with  $h_\alpha^1 \in \langle B \rangle$ , which satisfies the linear equations (7).

In the following, we will also denote by  $H^0 : \langle B^+ \rangle \rightarrow \langle B^+ \rangle$  the linear map such that for  $\beta \in B$ ,  $H^0(\mathbf{x}^\beta) = 0$  and for  $\alpha \in \partial B$ ,  $H^0(\mathbf{x}^\alpha) = h_\alpha^0$ . We also denote by  $N_0 : \langle B^+ \rangle \rightarrow \langle B \rangle$  the normal form modulo  $\mathfrak{J}_0$ , that is the projection of  $\langle B^+ \rangle$  on  $\langle B \rangle$  along  $\langle h_\alpha^0 \rangle$ . By construction,  $p = \sum_{\alpha \in B^+} \lambda_\alpha \mathbf{x}^\alpha \in \langle B^+ \rangle$ ,  $H^0(p) = \sum_{\alpha \in \partial B} \lambda_\alpha h_\alpha^0$  and we have  $N^0 + H^0 = \text{Id}_{\langle B^+ \rangle}$ . Similarly, we also denote by  $H^1 : \langle B^+ \rangle \rightarrow \langle B \rangle$  the map defined by  $H^1(\mathbf{x}^\beta) = 0$  if  $\mathbf{x}^\beta \in B$ ,  $H^1(\mathbf{x}^\alpha) = h_\alpha^1$  if  $\alpha \in \partial B$ . By construction, for all  $m \in B$ ,  $M_i^1(m) = H^1(x_i m)$ .

**Theorem 3.2.** Let  $I_0 \in H_{x_0}^B$  be an ideal, with the border relations  $\mathbf{h}^0 := (h_\alpha^0)_{\alpha \in \partial B}$  for the basis  $B$  of  $\mathcal{A}^0 = R/\mathfrak{J}_0$ , where  $\mathfrak{J}_0 = \underline{I}_0$ . Then

$$\begin{aligned}
\phi : T_{\mathbf{h}^0} &\rightarrow \text{Hom}_R(\mathfrak{J}_0, R/\mathfrak{J}_0) \\
\mathbf{h}^1 &\rightarrow \phi(\mathbf{h}^1) : h_\alpha^0 \mapsto h_\alpha^1
\end{aligned}$$

is an isomorphism of  $\mathbb{K}$ -vector spaces.

*Proof.* We first prove that  $\phi(\mathbf{h}^1)$  is well-defined, ie. if  $g = \sum_i u_\alpha h_\alpha^0 = \sum_\alpha u'_\alpha h_\alpha^0 \in \mathfrak{J}_0$  with  $u_\alpha, u'_\alpha \in R$ , then  $\sum_\alpha u_\alpha h_\alpha^1 = \sum_\alpha u'_\alpha h_\alpha^1$  in  $R/\mathfrak{J}_0$ . In other words, if  $\sum_\alpha v_\alpha h_\alpha^0 = 0$  in  $R$ , then  $\sum_\alpha v_\alpha h_\alpha^1 \equiv 0$  in  $R/\mathfrak{J}_0$ . By [23], the syzygies of the border basis elements  $\mathbf{h}^0 := (h_\alpha^0)$  are generated by the commutation polynomials:

$$x_i H^0(x_{i'} m) - x_{i'} H^0(x_i m) + H^0(x_i N^0(x_{i'} m)) - H^0(x_{i'} N^0(x_i m)),$$

for all  $m \in B$ ,  $1 \leq i < i' \leq n$ . Let us prove that these relations are also satisfied modulo  $\mathfrak{J}_0$ , if we replace  $H^0$  by  $H^1$ . As  $\mathbf{h}^1 = (h_\alpha^1) \in T_{\mathbf{h}^0}$ , we have

$$\begin{aligned}
0 &= M_{x_i}^0 \circ M_{x_{i'}}^1(m) - M_{x_{i'}}^0 \circ M_{x_i}^1(m) + M_{x_i}^1 \circ M_{x_{i'}}^0(m) - M_{x_{i'}}^1 \circ M_{x_i}^0(m) \\
&= N^0(x_i H^1(x_{i'} m)) - N^0(x_{i'} H^1(x_i m)) + H^1(x_i N^0(x_{i'} m)) - H^1(x_{i'} N^0(x_i m)) \\
&= x_i H^1(x_{i'} m) - H^0(x_i H^1(x_{i'} m)) \\
&\quad - x_{i'} H^1(x_i m) + H^0(x_{i'} H^1(x_i m)) \\
&\quad + H^1(x_i N^0(x_{i'} m)) - H^1(x_{i'} N^0(x_i m)) \\
&\equiv x_i H^1(x_{i'} m) - x_{i'} H^1(x_i m) + H^1(x_i N^0(x_{i'} m)) - H^1(x_{i'} N^0(x_i m)) \text{ modulo } \mathfrak{J}_0.
\end{aligned}$$

This proves that the generating syzygies are mapped by  $\phi(\mathbf{h}^1)$  to 0 in  $R/\mathfrak{J}_0$  and thus the image by  $\phi(\mathbf{h}^1)$  of any syzygy of  $\mathbf{h}^0$  is 0, that is,  $\phi(\mathbf{h}^1)$  is a well-defined element of  $\text{Hom}_R(\mathfrak{J}_0, R/\mathfrak{J}_0)$ .

Conversely, let us prove that if  $\psi_0 \in \text{Hom}_R(\mathfrak{J}_0, R/\mathfrak{J}_0)$ , then  $\mathbf{h}^1 := (\psi_0(h_a^0)) \in T_{\mathbf{h}^0}$ . As  $\psi_0 \in \text{Hom}_R(\mathfrak{J}_0, R/\mathfrak{J}_0)$ , the syzygies of  $\mathbf{h}^0$  are mapped by  $\psi_0$  to 0. Thus, for all  $m \in B$ ,  $1 \leq i < i' \leq n$ ,

$$\begin{aligned} 0 &\equiv \psi_0(x_i H^0(x_{i'} m) - x_{i'} H^0(x_i m) + H^0(x_i N^0(x_{i'} m)) - H^0(x_{i'} N^0(x_i m))) \\ &\equiv x_i H^1(x_{i'} m) - x_{i'} H^1(x_i m) + H^1(x_i N^0(x_{i'} m)) - H^1(x_{i'} N^0(x_i m)) \text{ modulo } \mathfrak{J}_0. \end{aligned}$$

As  $H^1(p) \in \langle B \rangle$  and  $N^0(H^1(p)) = H^1(p)$  for all  $p \in \langle B^+ \rangle$ , we have

$$\begin{aligned} 0 &= N^0(x_i H^1(x_{i'} m)) - N^0(x_{i'} H^1(x_i m)) + H^1(x_i N^0(x_{i'} m)) - H^1(x_{i'} N^0(x_i m)) \\ &= M_i^0 \circ M_{i'}^1(m) - M_{i'}^0 \circ M_i^1(m) + M_i^1 \circ M_{i'}^0(m) - M_{i'}^1 \circ M_i^0(m), \end{aligned}$$

which proves that  $\mathbf{h}^1 \in T_{\mathbf{h}^0}$ .  $\square$

We can notice that the tangent space of the variety  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  locally defined by the equations (7) is also isomorphic to  $\text{Hom}_R(\mathfrak{J}_0/\mathfrak{J}_0^2, R/\mathfrak{J}_0)$ , which is a well known result that we can find for example in [27][p. 217].

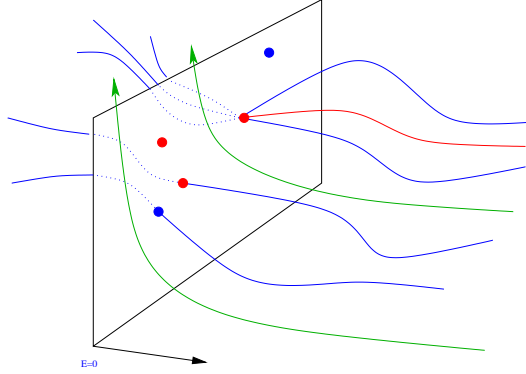
#### 4. DEFORMATION ON THE HILBERT SCHEME

In this section, we consider again an ideal  $I^0$  of  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ . We assume that the point  $I^0$  is in  $U_{x_0}^\mu$  and denote by  $\mathfrak{J}^0$  the corresponding affine ideal of  $R$ . We assume  $B \subset R$ , connected to 1, is a basis of the associated affine quotient  $\mathcal{A}^0 = R/\mathfrak{J}^0$ . We denote by  $f_0^0, \dots, f_m^0 \in R$  a generating set of  $\mathfrak{J}^0$ , which we are going to “deform”.

Let  $R^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}]$  be the ring of polynomials with formal power series in  $\varepsilon$  as coefficients. We consider a small perturbation of the equations of the form  $f_k^\varepsilon = f_k^0 + \varepsilon f_k^1 + \dots$  ( $k = 1 \dots m$ ) as elements of  $R^\varepsilon$ . Let  $\mathfrak{J}^\varepsilon$  be the ideal generated by  $(f_1^\varepsilon, \dots, f_m^\varepsilon)$  in  $R^\varepsilon$ , and the corresponding quotient algebra  $\mathcal{A}^\varepsilon = R^\varepsilon/\mathfrak{J}^\varepsilon$ .

The primary decomposition of  $\mathfrak{J}^\varepsilon$  in  $R^\varepsilon$  is of the form  $\mathfrak{J}^\varepsilon = \mathfrak{J}_\infty^\varepsilon \cap \mathfrak{J}_a^\varepsilon$  where

- $\mathfrak{J}_a^\varepsilon$  is the intersection of the primary components of  $\mathfrak{J}^\varepsilon$ , which intersect the hyperplane  $\varepsilon = 0$ .
- $\mathfrak{J}_\infty^\varepsilon$  is the intersection of the primary components of  $\mathfrak{J}^\varepsilon$ , which do not intersect the hyperplane  $\varepsilon = 0$ .



As illustrated above, the ideal  $\mathfrak{J}_a^\varepsilon$  corresponds to the roots of  $\mathfrak{J}^\varepsilon$  with coordinates in the quotient field of Puiseux algebraic power series over the algebraic closure  $\overline{\mathbb{K}}$ , denoted by  $\overline{\mathbb{K}}((\varepsilon^*))$  which have a limit in  $\overline{\mathbb{K}}^n$  when  $\varepsilon$  tends to 0. Similarly  $\mathfrak{J}_\infty^\varepsilon$  corresponds to the roots  $\in \overline{\mathbb{K}}((\varepsilon^*))^n$  which “go to infinity” when  $\varepsilon$  tends to 0.

More generally, for any ideal  $\mathfrak{J} \subset R^\varepsilon$ , we define similarly  $\mathfrak{J}_a$  (resp.  $\mathfrak{J}_\infty$ ) as the intersection of the primary components of  $\mathfrak{J}$ , which (resp. do not) intersect the hyperplane  $\varepsilon = 0$ . Let  $G = \{g(\varepsilon, \mathbf{x}) \in R^\varepsilon; g(0, \mathbf{x}) = 1\}$ . It is a multiplicative subset of  $R^\varepsilon$ . Its elements are of the form  $g(\varepsilon, \mathbf{x}) = 1 + \varepsilon g'(\varepsilon, \mathbf{x})$  for  $g' \in R^\varepsilon$ .



**Lemma 4.1.** *For any ideal  $\mathfrak{J} \subset R^\varepsilon$ ,  $G^{-1}\mathfrak{J} = G^{-1}\mathfrak{J}_a$  in  $G^{-1}R^\varepsilon$ .*

*Proof.* By definition of  $\mathfrak{J}_\infty$  and by Hilbert Nullstellensatz, each primary component of  $\mathfrak{J}_\infty$  contains an element of the form  $1 + \varepsilon g(\varepsilon, \underline{\mathbf{x}}) \in G$ . Therefore, we have  $G^{-1}\mathfrak{J} = G^{-1}\mathfrak{J}_a$ .  $\square$

We are interested in characterizing when the zero-set of  $\mathfrak{J}_a^\varepsilon$  is a flat deformation of the initial zero-set of  $\mathfrak{J}^0$ . That is, from an algebraic point of view, when the structure of the algebra  $G^{-1}\mathcal{A}^\varepsilon$  as  $\mathbb{K}[[\varepsilon]]$ -algebra, gives “in the limit” the  $\mathbb{K}$ -algebra  $\mathcal{A}^0$  and when it has the same dimension as  $\mathcal{A}^0$ . Notice that, by the previous lemma, we have  $G^{-1}\mathcal{A}^\varepsilon = G^{-1}\mathcal{A}_a^\varepsilon = G^{-1}R^\varepsilon/\mathfrak{J}_a^\varepsilon$ , so that  $G^{-1}\mathcal{A}^\varepsilon$  describes locally the roots in with  $\overline{\mathbb{K}}((\varepsilon^*))$  with a finite limit.

We will clarify what it means that the structure of the algebra  $G^{-1}\mathcal{A}^\varepsilon$  gives in the limit  $\mathcal{A}^0$ , by considering border basis representation with respect to a fixed basis  $B$ . The following lemma shows that if we have an analytic perturbation of a border basis for a basis  $B$ , then there is no root which “goes to infinity”.

**Lemma 4.2.** *Let  $h_\alpha^\varepsilon = \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} u_{\alpha,\beta}(\varepsilon) \underline{\mathbf{x}}^\beta$  with  $u_{\alpha,\beta}(\varepsilon) \in \mathbb{K}[[\varepsilon]]$  for  $\alpha \in \partial B$ ,  $\beta \in B$ . Let  $\mathcal{H}^\varepsilon = (h_\alpha^\varepsilon) \subset R^\varepsilon$  be the ideal generated by the  $h_\alpha^\varepsilon$  for  $\alpha \in \partial B$ . Suppose that  $h_\alpha^0 = \underline{\mathbf{x}}^\alpha - \sum_{\beta \in B} u_{\alpha,\beta}(0) \underline{\mathbf{x}}^\beta$ ,  $\alpha \in \partial B$  is a border basis of  $B$  for  $\mathcal{H}^0$ . Then  $\mathcal{H}^\varepsilon = \mathcal{H}_a^\varepsilon$ .*

*Proof.* Let us prove that any element of  $G$  of the form  $1 + \varepsilon g(\varepsilon, \underline{\mathbf{x}})$  is invertible modulo  $\mathcal{H}^\varepsilon$  in  $R^\varepsilon$ . We search an inverse of the form  $d(\varepsilon, \underline{\mathbf{x}}) = \sum_{\beta \in B} d_\beta \underline{\mathbf{x}}^\beta$ , where the  $\mathbf{d} = (d_\beta)_{\beta \in B}$  are considered as variables.

By reduction by the polynomials  $(h_\alpha^\varepsilon)_{\alpha \in \partial B}$ , we reduce the polynomial  $(1 + \varepsilon g(\varepsilon, \underline{\mathbf{x}})) d(\varepsilon, \underline{\mathbf{x}})$  to a linear combinations of the form

$$\sum_{\beta \in B} l_\beta(\varepsilon, \mathbf{d}) \underline{\mathbf{x}}^\beta$$

where  $l_\beta$  is linear in the variables  $\mathbf{d} = (d_\beta)$ . Moreover, we have  $l_\beta(0, \mathbf{d}) = d_\beta$ , which shows that the system

$$(8) \quad l_1(\varepsilon, \mathbf{d}) = 1, l_\beta(\varepsilon, \mathbf{d}) = 0 \quad \text{for } \beta \in B - \{1\}.$$

has a unique solution for  $\varepsilon = 0$ . By the Implicit Function Theorem, the system (8) has a (unique) analytic solution in  $\mathbf{d}$ , so that there exists  $d(\varepsilon, \underline{\mathbf{x}}) \in R^\varepsilon$  such that  $(1 + \varepsilon g(\varepsilon, \underline{\mathbf{x}})) d(\varepsilon, \underline{\mathbf{x}}) = 1$  modulo  $\mathcal{H}^\varepsilon$ . Therefore any element of  $G$  is invertible modulo  $\mathcal{H}^\varepsilon$ ,  $\mathcal{H}_\infty^\varepsilon \equiv R^\varepsilon$  modulo  $\mathcal{H}^\varepsilon$  and all primary components of  $\mathcal{H}^\varepsilon$  intersects  $\varepsilon = 0$ . We deduce that  $\mathcal{H}^\varepsilon = \mathcal{H}_a^\varepsilon$ .  $\square$

Let  $B$  is a (monomial) basis of  $\mathcal{A}^0 = R/\mathfrak{J}^0$  and  $(h_\alpha^0)_{\alpha \in \partial B}$  a border basis of  $B$  for  $\mathfrak{J}^0$ . Let us call *formal border basis*, the family of polynomials  $h_\alpha^{\varepsilon, \mathbf{v}} = h_\alpha^0 + \varepsilon \sum_{\beta \in B} v_{\alpha,\beta} \underline{\mathbf{x}}^\beta$ , ( $\alpha \in \partial B$ ), where  $v_{\alpha,\beta}$  are unknowns. We denote by  $\mathbf{v}_\alpha(\underline{\mathbf{x}}) := \sum_{\beta \in B} v_{\alpha,\beta} \underline{\mathbf{x}}^\beta$  so that  $h_\alpha^{\varepsilon, \mathbf{v}} = h_\alpha^0 + \varepsilon \mathbf{v}_\alpha(\underline{\mathbf{x}})$ .

From the *formal border basis*, we can write *formal multiplication matrices*  $M_{x_j}^{\mathbf{v}}$ , whose entries are polynomials in the variables  $\mathbf{v}$ .

As  $\{f_1^0, \dots, f_m^0\}$  and  $\{h_\alpha^0\}_{\alpha \in \partial B}$  generate the same ideal  $\mathfrak{J}^0$  of  $R$ , for suitable polynomials  $p_{\alpha,j}^0 \in R$ , we have  $h_\alpha^0 = \sum_j p_{\alpha,j}^0 f_j^0$ . The perturbation relations  $f_j^\varepsilon$  define a *lifting of the border relations*, namely for every  $\alpha \in \partial B$ ,

$$\tilde{h}_\alpha^\varepsilon := \sum_j p_{\alpha,j}^0 f_j^\varepsilon = \sum_j p_{\alpha,j}^0 (f_j^0 + \varepsilon f_j^1 + \dots) = h_\alpha^0 + \varepsilon \sum_j p_{\alpha,j}^0 f_j^1 + \dots$$

Let  $\tilde{\mathcal{H}}^\varepsilon = (\tilde{h}_\alpha^\varepsilon)_{\alpha \in \partial B} \subset R^\varepsilon$  be the “lifted” ideal generated by  $\tilde{h}_\alpha^\varepsilon$  for  $\alpha \in \partial B$ . We are going to construct deformed analytic border relations which define locally the same ideal as the lifted border relations.

**Proposition 4.3.** *Let  $B$  be a basis of  $\mathcal{A}^0 = R/\mathfrak{I}^0$  and  $(h_\alpha^0)_{\alpha \in \partial B}$  a border basis of  $\mathfrak{I}^0$  with respect to  $B$ . Then,  $B$  is a generating set as  $\mathbb{K}[[\varepsilon]]$ -module of the quotient  $R^\varepsilon/\mathfrak{I}_a^\varepsilon$ . Moreover, there exists a rewriting family  $\mathbf{h}^\varepsilon := (h_\alpha^\varepsilon)_{\alpha \in \partial B} \subset \mathfrak{I}_a^\varepsilon$  for  $B$  of the form  $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon \sum_{\beta \in B} u_{\alpha,\beta}(\varepsilon) \mathbf{x}^\beta$  with  $u_{\alpha,\beta}(\varepsilon) \in \mathbb{K}[[\varepsilon]]$  analytic in  $\varepsilon$ .*

*Proof.* We are going to make the obvious reduction of the polynomial  $\tilde{h}_\alpha^\varepsilon$  (as polynomial in the  $\mathbf{x}$ 's), with respect to the formal border basis  $h_\alpha^{\varepsilon,\mathbf{v}}$ . In this reduction, we start with the monomials the further to the boundary  $\partial B$  and iterate the reduction by  $h_\alpha^\varepsilon$  until all monomials of the reduced polynomial are in  $B$ .

We have that

$$\tilde{h}_\alpha^\varepsilon - h_\alpha^{\varepsilon,\mathbf{v}} = -\varepsilon \mathbf{v}_\alpha(\mathbf{x}^\beta) + \varepsilon \sum_j p_{\alpha,j}^0 f_j^1 + \dots$$

Notice that  $-\varepsilon \mathbf{v}_\alpha(\mathbf{x}^\beta) = -\varepsilon \sum_{\beta \in B} v_{\alpha,\beta} \mathbf{x}^\beta$  has its support in  $B$  and will not be reduced by the border relations  $\{h_\alpha^{\varepsilon,\mathbf{v}}\}_{\alpha \in \partial B}$ . Through this reduction, the monomials of the support of  $\sum_j p_{\alpha,j}^0 f_j^1 + \dots$  which are not in  $B$  are replaced recursively by sum of terms which coefficients are polynomial of the same degree in  $\varepsilon$  and  $\mathbf{v}$ . At the end of the reduction, we obtain a polynomial with support in  $B$ . Hence when the reduction is finished, for every  $\alpha$ , we have:

$$\tilde{h}_\alpha^\varepsilon - h_\alpha^{\varepsilon,\mathbf{v}} = -\varepsilon \mathbf{v}_\alpha(\mathbf{x}^\beta) + \sum_{\alpha' \in \partial B} Q_{\alpha,\alpha'}(\varepsilon, \mathbf{v}, \mathbf{x}) h_{\alpha'}^\varepsilon + \varepsilon \sum_{\beta \in B} R_{\alpha,\beta}(\varepsilon, \mathbf{v}) \mathbf{x}^\beta$$

where  $Q_{\alpha,\alpha'}$  and  $R_{\alpha,\beta}$  are polynomial functions, with terms of the same degree in  $\varepsilon$  and  $\mathbf{v}$ . This can be rewritten as

$$\tilde{h}_\alpha^\varepsilon = \sum_{\alpha' \in \partial B} (1_{\alpha=\alpha'} + Q_{\alpha,\alpha'}(\varepsilon, \mathbf{v}, \mathbf{x})) h_{\alpha'}^\varepsilon + \varepsilon \sum_{\beta \in B} (R_{\alpha,\beta}(\varepsilon, \mathbf{v}) - v_{\alpha,\beta}) \mathbf{x}^\beta$$

where  $1_{\alpha=\alpha'}$  is 1 iff  $\alpha = \alpha'$  and 0 otherwise.

Let  $\mathbf{Q}(\varepsilon, \mathbf{v}, \mathbf{x})$  be the matrix of  $Q_{\alpha,\alpha'}(\varepsilon, \mathbf{v}, \mathbf{x})$  for  $\alpha, \alpha' \in \partial B$ . Each coefficient  $Q_{\alpha,\alpha'}(\varepsilon, \mathbf{v}, \mathbf{x})$ , obtained by reduction of  $\varepsilon \sum_j p_{\alpha,j}^0 f_j^1 + \dots$  by the formal border basis  $h_\alpha^{\varepsilon,\mathbf{v}}$ , is divisible by  $\varepsilon$ . Hence, we have  $\det(\text{Id} + \mathbf{Q}(\varepsilon, \mathbf{v}, \mathbf{x})) = 1 + \varepsilon D(\varepsilon, \mathbf{v}, \mathbf{x})$  for some suitable polynomial  $D(\varepsilon, \mathbf{v}, \mathbf{x})$ .

Moreover, by the Implicit Function Theorem, there is an analytic solution  $\mathbf{u}$ , to the equation in  $\mathbf{v}$ :

$$(9) \quad \Sigma_{\alpha,\beta}(\varepsilon, \mathbf{v}) := v_{\alpha,\beta} - R_{\alpha,\beta}(\varepsilon, \mathbf{v}) = 0, \alpha \in \partial B, \beta \in B.$$

That is, there exist  $u_{\alpha,\beta} \in \mathbb{K}[[\varepsilon]]$ , analytic in  $\varepsilon$ , such that calling  $h_\alpha^\varepsilon := h_\alpha^0 + \varepsilon \sum_{\beta \in B} u_{\alpha,\beta} \mathbf{x}^\beta$ , ( $\alpha \in \partial B$ ) reduce  $\tilde{h}_\alpha^\varepsilon$  to 0.

Let  $\mathcal{H}^\varepsilon = (h_\alpha^\varepsilon)_{\alpha \in B} \subset R^\varepsilon$ . By Lemma 4.2, we have  $\mathcal{H}^\varepsilon = \mathcal{H}_a^\varepsilon$ .

By construction,  $\tilde{\mathcal{H}}^\varepsilon = (\tilde{h}_\alpha^\varepsilon) \subset \mathcal{H}^\varepsilon$ . As  $\det(\text{Id} + \mathbf{Q}(\varepsilon, \mathbf{v}, \mathbf{x})) \in G$ , we also have  $G^{-1}\tilde{\mathcal{H}}^\varepsilon = G^{-1}\mathcal{H}^\varepsilon$ . By construction, we also have  $\tilde{\mathcal{H}}^\varepsilon \subset \mathfrak{I}^\varepsilon$ . Therefore we deduce that

$$G^{-1}\tilde{\mathcal{H}}^\varepsilon = G^{-1}\mathcal{H}^\varepsilon = G^{-1}\mathcal{H}_a^\varepsilon \subset G^{-1}\mathfrak{I}_a^\varepsilon,$$

which implies that  $\mathcal{H}^\varepsilon = \mathcal{H}_a^\varepsilon \subset \mathfrak{I}_a^\varepsilon$ . Consequently, as any monomial not in  $B$ , can be reduced modulo  $\mathcal{H}^\varepsilon$  to a linear combination of monomials in  $B$  with coefficients in  $\mathbb{K}[[\varepsilon]]$ ,  $B$  is a generating set of  $R^\varepsilon/\mathfrak{I}_a^\varepsilon$ , which concludes the proof.  $\square$

**Example 4.4.** We consider the perturbation

- $f_1^\varepsilon = x^2 - \varepsilon x$ ,
- $f_2^\varepsilon = xy - \varepsilon x$ ,
- $f_3^\varepsilon = xy - \varepsilon y$ ,
- $f_4^\varepsilon = y^2 - \varepsilon y$ ,
- $f_5^\varepsilon = \varepsilon x - \varepsilon^2$ ,
- $f_6^\varepsilon = \varepsilon y - \varepsilon^2$ .

We have  $\mathfrak{J}^0 = (x^2, xy, y^2)$  and  $\mathfrak{J}^\varepsilon = (f_1^\varepsilon, \dots, f_6^\varepsilon) = (x, y, \varepsilon) \cdot (x - \varepsilon, y - \varepsilon) = \mathfrak{J}_a^\varepsilon$ .

The set  $B = \{1, x, y\}$  is a basis of  $R/\mathfrak{J}^0$  and the border relations are  $h_{x^2}^0 = x^2$ ,  $h_{xy}^0 = xy$ ,  $h_{y^2}^0 = y^2$ . As  $h_{x^2}^0 = f_1^0$ ,  $h_{xy}^0 = f_2^0$ ,  $h_{y^2}^0 = f_4^0$ , these border relations lift in

- $\tilde{h}_{x^2}^\varepsilon = f_1^\varepsilon = x^2 - \varepsilon x$ ,
- $\tilde{h}_{xy}^\varepsilon = f_2^\varepsilon = xy - \varepsilon x$ ,
- $\tilde{h}_{y^2}^\varepsilon = f_4^\varepsilon = y^2 - \varepsilon y$ .

After reduction by the formal border relations and resolution of the corresponding (linear) system (9), we have  $h_{x^2}^\varepsilon = \tilde{h}_{x^2}^\varepsilon$ ,  $h_{xy}^\varepsilon = \tilde{h}_{xy}^\varepsilon$ ,  $h_{y^2}^\varepsilon = \tilde{h}_{y^2}^\varepsilon$ .

We check that the multiplication operators by  $x$  and  $y$  commute, so that the polynomials  $h_{x^2}^\varepsilon, h_{xy}^\varepsilon, h_{y^2}^\varepsilon$  are border relations for  $B$ . This implies that  $R^\varepsilon/\mathcal{H}^\varepsilon$  is a free  $\mathbb{K}[[\varepsilon]]$ -module with basis  $B$ . In this example, we have  $\mathcal{H}^\varepsilon \subsetneq \mathfrak{J}^\varepsilon$  but  $\mathcal{H}^0 = \mathfrak{J}^0$ .

Notice that there is not a unique way to construct the relations  $\{h_\alpha^\varepsilon\}_{\alpha \in \partial B}$  and that these relations are not necessarily border relations as it is illustrated in the next example. This is the case iff the commutation relations (1) are satisfied.

**Example 4.5.** We consider the same deformation as in example 4.4, but we rewrite  $x^2 = f_1^0 + f_3^0 - f_2^0$  and  $y^2 = f_4^0 + f_2^0 - f_3^0$  which lift in  $\tilde{h}_{x^2}^\varepsilon = f_1^\varepsilon + f_3^\varepsilon - f_2^\varepsilon$  and  $\tilde{h}_{y^2}^\varepsilon = f_4^\varepsilon + f_2^\varepsilon - f_3^\varepsilon$ . The other relation  $h_{xy}^0$  is lift as in the previous example. After reduction by the formal border relations and resolution of the corresponding (linear) system, we obtain the following rewriting relations:

- $h_{x^2}^\varepsilon = x^2 - \varepsilon y$ ,
- $h_{xy}^\varepsilon = xy - \varepsilon x$ ,
- $h_{y^2}^\varepsilon = y^2 - \varepsilon x$ .

We easily check that the matrix of multiplication  $M_x, M_y$  are not commuting, since  $M_x \circ M_y(x) = \varepsilon^2 y$  and  $M_y \circ M_x(x) = \varepsilon^2 x$ . The polynomials  $h_{x^2}^\varepsilon, h_{xy}^\varepsilon, h_{y^2}^\varepsilon$  are not a border basis for  $B$ .

As above, we can reduce the polynomials  $f_i^\varepsilon$  by the formal border basis  $h_\alpha^{\varepsilon, \mathbf{v}}$  in order to obtain the relations

$$(10) \quad f_i^\varepsilon = \sum_{\beta \in B} \rho_{i, \beta}(\mathbf{v}) \mathbf{x}^\beta + (h_\alpha^{\varepsilon, \mathbf{v}}).$$

**Theorem 4.6.** Let  $B$  be a monomial basis connected to 1 of  $\mathcal{A}^0 = R/\mathfrak{J}^0$  where  $\mathfrak{J}^0 = (f_k^0)$ . Let  $(h_\alpha^0)_{\alpha \in \partial B}$  be the border basis of  $(\mathfrak{J}^0)$  for  $B$ . Let  $f_k^\varepsilon \in \mathbb{K}[\mathbf{x}, \varepsilon]$  be a deformation of  $f_k^0$ . We denote by  $\Sigma_{\alpha, \beta}(\varepsilon, \mathbf{v})$  ( $\alpha \in \partial B, \beta \in B$ ) the polynomials (9), obtained by lifting  $h_\alpha^0$  and reduction. Then  $\mathcal{A}_a^\varepsilon = R^\varepsilon/\mathfrak{J}_a^\varepsilon$  is a flat  $\mathbb{K}[[\varepsilon]]$ -module with basis  $B$  iff

- the coefficients of the commutation expression with the formal matrices:  $M_{x_i}^{\mathbf{v}} \circ M_{x_j}^{\mathbf{v}} - M_{x_j}^{\mathbf{v}} \circ M_{x_i}^{\mathbf{v}}$ ,  $1 \leq i < j \leq n$ , and
- the polynomials  $\rho_{i, \beta}(\mathbf{v})$ ,  $\beta \in B, i = 1, \dots, k$  in (10),

are reduced to 0 by the relations  $\Sigma_{\alpha,\beta}(\varepsilon, \mathbf{v})$  in the local ring  $\mathbb{K}[\varepsilon, \mathbf{v}]_{(\varepsilon, \mathbf{v})}$ .

*Proof.* Let us call  $u \in \mathbb{K}[[\varepsilon]]$  the unique solution to the equations (9), and the corresponding  $\{h_\alpha^\varepsilon\}_{\alpha \in \partial B}$ . The formal commutation relations and the polynomials  $\rho_{i,\beta}(\mathbf{v})$  reduce to 0 by the relations (9) in the local ring  $\mathbb{K}[\varepsilon, \mathbf{v}]_{(\varepsilon, \mathbf{v})}$  iff the matrices  $M_{x_i}^{\mathbf{u}}$  (that is, the formal ones making the substitution  $\mathbf{v} \rightarrow \mathbf{u}$ ) verify the commutation identities in the ring  $\mathbb{K}[[\varepsilon]][\mathbf{x}]$  and  $f_i^\varepsilon \in \mathcal{H}^\varepsilon$ .

If the relations (10) are satisfied, we have  $\mathcal{J}^\varepsilon \subset \mathcal{H}^\varepsilon$ . By proposition 4.3, we know that  $\mathcal{H}^\varepsilon \subset \mathcal{J}_a^\varepsilon$ . This shows that  $\mathcal{J}^\varepsilon \subset \mathcal{H}^\varepsilon \subset \mathcal{J}_a^\varepsilon$  and that  $\mathcal{H}_a^\varepsilon = \mathcal{J}_a^\varepsilon$ .

We have to prove that the set of monomials  $B$  is a basis of  $\mathcal{A}_a^\varepsilon = R^\varepsilon/\mathcal{J}_a^\varepsilon$  as  $\mathbb{K}[[\varepsilon]]$ -module, which is equivalent to prove that

$$\mathcal{A}_a^\varepsilon \otimes_{\mathbb{K}[[\varepsilon]]} \mathbb{K}((\varepsilon)) = \mathbb{K}((\varepsilon))[\mathbf{x}]/\mathcal{J}_a^\varepsilon$$

is a  $\mathbb{K}((\varepsilon))$ -algebra having as basis the monomial set  $B$ . As the multiplication matrices  $M_{x_i}^{\mathbf{u}}$  w.r.t.  $B$  in this  $\mathbb{K}((\varepsilon))$ -algebra commute, by [20, 22],  $B$  is a basis of  $\mathbb{K}((\varepsilon))[\mathbf{x}]/\mathcal{J}_a^\varepsilon$ . Thus  $B$  is free in  $\mathcal{A}_a^\varepsilon = \mathbb{K}[[\varepsilon]][\mathbf{x}]/\mathcal{J}_a^\varepsilon$ . By proposition 4.3,  $B$  is also a generating set of  $\mathcal{A}_a^\varepsilon$  as  $\mathbb{K}[[\varepsilon]]$ -module. Thus  $B$  is a basis of this quotient algebra:  $\mathcal{A}_a^\varepsilon$  is a flat  $\mathbb{K}[[\varepsilon]]$ -module with basis  $B$ .

Conversely if  $\mathcal{A}_a^\varepsilon = R^\varepsilon/\mathcal{J}_a^\varepsilon$  is a flat  $\mathbb{K}[[\varepsilon]]$ -module of basis  $B$ , for any monomial  $\mathbf{x}^\alpha$  in  $\partial B$ , there exist  $u_{\alpha,\beta} \in \mathbb{K}[[\varepsilon]]$  such that

$$\mathbf{x}^\alpha - \sum_{\beta \in \partial B} u_{\alpha,\beta} \mathbf{x}^\beta \equiv 0 \text{ modulo } \mathcal{J}_a^\varepsilon.$$

This defines the border relations  $\mathcal{H}^\varepsilon = (h_\alpha^\varepsilon)$  of  $\mathcal{J}_a^\varepsilon$  for  $B$  which correspond to the unique solution  $\mathbf{u}$  of relations (9). Moreover as  $\mathcal{J}^\varepsilon \subset \mathcal{J}_a^\varepsilon = \mathcal{H}^\varepsilon$ , the polynomials  $f_i^\varepsilon \in \mathcal{J}^\varepsilon$  reduce to 0 modulo  $\mathcal{H}^\varepsilon$ , so that  $\rho_{i,\beta}(\mathbf{u}) = 0$ .  $\square$

**Remark 4.7.** *Both constraints (1) and (10) are necessary:*

- In the example 4.4, the commutation relations (1) are satisfied but not the relations (10). The polynomials  $f_3^\varepsilon, f_5^\varepsilon, f_6^\varepsilon$  do not reduce to 0 by the border basis  $(h_{x_2}^\varepsilon, h_{xy}^\varepsilon, h_{y^2}^\varepsilon)$ .
- In the example 4.5, the matrices  $M_x^{\mathbf{u}}, M_y^{\mathbf{u}}$  are not commuting.

These examples correspond to deformations  $\mathcal{J}^\varepsilon$  of  $\mathcal{J}^0$ , which are not flat.

This reduction steps can be interpreted as an equality test to 0 in the ring  $\mathbb{K}[\varepsilon, \mathbf{v}]_{(\varepsilon, \mathbf{v})}/(\Sigma_{\alpha,\beta})$ . In this local ring one can solve the membership problem using Mora's NORMALFORM algorithm [19]. Indeed, notice also that the equations  $\Sigma_{\alpha,\beta}$ 's are a standard basis for any local term ordering degree-anticompatible, since their initial are linear and they are linearly independent.

If instead of a parameter  $\varepsilon$  for the deformation, we consider a deformations depending on a family of parameters  $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_s)$ . The same construction applies and yields relations of the form

$$\Sigma_{\alpha,\beta}(\underline{\varepsilon}, \mathbf{v}) := v_{\alpha,\beta} - R_{\alpha,\beta}(\underline{\varepsilon}, \mathbf{v})$$

with  $R_{\alpha,\beta}(\underline{\varepsilon}, \mathbf{v}) \in \mathbb{K}[\mathbf{x}, \underline{\varepsilon}]$  for  $\alpha \in \partial B, \beta \in B$ . The condition to have a flat deformation is that the relations (1) and (10) reduces to 0, by  $\Sigma_{\alpha,\beta}(\underline{\varepsilon}, \mathbf{v})$ . Performing this reduction yields polynomials in  $\mathbf{v}$  with coefficients in  $\underline{\varepsilon}$ . The variety in the parameter space defined by the vanishing of these coefficients is the set above which the deformation is flat.

5. PATH FOLLOWING ON  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ 

In this section, we assume we have an analytic flat deformation  $f_0^\varepsilon, \dots, f_m^\varepsilon$  of  $\mathfrak{J}^0$ . In other words,  $\mathfrak{J}^\varepsilon$  stays on  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ . The objective is to describe a Newton-type method which follow this path. This problem can be connected to homotopy methods based on the extension to multivariate systems, of Weierstrass method as described in [21]. For this purpose, we are going to express the tangent vector to the path in  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$  with respect to the first terms of the deformation of  $f_i^\varepsilon$  and the reduction by the border basis at  $\mathfrak{J}^0$ .

Consider the reduction of  $f_0^\varepsilon, \dots, f_m^\varepsilon$  by  $h_\alpha^\varepsilon$ , which yields the following equality:

$$f_k^0 + \varepsilon f_k^1 + \mathcal{O}(\varepsilon^2) = \sum_{\alpha \in \partial B} q_{k,\alpha}^\varepsilon h_\alpha^\varepsilon = \sum_{\alpha \in \partial B} q_{k,\alpha}^0 h_\alpha^0 + \varepsilon \left( \sum_{\alpha \in \partial B} q_{k,\alpha}^1 h_\alpha^0 + q_{k,\alpha}^0 h_\alpha^1 \right) + \mathcal{O}(\varepsilon^2).$$

Taking  $(q_\alpha^0)_\alpha$  such that  $f_k^0 = \sum_{\alpha \in \partial B} q_{k,\alpha}^0 h_\alpha^0$ , we deduce the relations

$$(11) \quad N^0(f_k^1) - \sum_{\alpha \in \partial B} N^0(q_{k,\alpha}^0 h_\alpha^1) = 0,$$

where  $N^0$  is the normal form modulo  $(h_\alpha^0)$  (ie. the projection on  $\langle B \rangle$  along  $(h_\alpha^0)$ ).

The equations (7) and (11) are linear in  $\mathbf{h}^1 = (h_{\alpha,\beta}^1)_{\alpha \in \partial B, \beta \in B}$ . We show now that they uniquely define the tangent vector to the deformation path at  $\mathfrak{J}^0$ .

**Theorem 5.1.** *Suppose that  $\mathcal{A}^0 = R/\mathfrak{J}^0$  admits  $B$  connected to 1 as a basis. Let  $f_0^0, \dots, f_m^0$  be generators of  $\mathfrak{J}^0$  and let  $f_k^\varepsilon = f_k^0 + \varepsilon f_k^1 + \dots$  ( $k = 1 \dots m$ ) define a flat deformation  $\mathcal{A}^\varepsilon = R^\varepsilon/\mathfrak{J}^\varepsilon$  of  $\mathcal{A}^0$ , where  $\mathfrak{J}^\varepsilon := (f_k^\varepsilon)$ . Then the border relations of  $\mathcal{A}^\varepsilon$  are of the form  $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon h_\alpha^1 \sum_{\beta \in B} h_{\alpha,\beta}^1 \mathbf{x}^\beta + \mathcal{O}(\varepsilon^2)$  where  $\mathbf{h}^0 = (h_\alpha^0)$  is the border basis of  $\mathcal{A}^0$  for  $B$ ,  $h_\alpha^1 := \sum_{\beta \in B} h_{\alpha,\beta}^1 \mathbf{x}^\beta$  and  $\mathbf{h}^1 = (h_{\alpha,\beta}^1)_{\alpha \in \partial B, \beta \in B}$  is the unique solution of the linear equations (7) and (11).*

*Proof.* By proposition 4.3, there exists a border basis  $\mathbf{h}^\varepsilon = (h_\alpha^\varepsilon)$  for  $B$  analytic in  $\varepsilon$ , and such that  $(f_k^\varepsilon) \subset (h_\alpha^\varepsilon)$ . Then as described above, the linear terms  $h_\alpha^1$  of  $h_\alpha^\varepsilon = h_\alpha^0 + \varepsilon h_\alpha^1 + \dots$  satisfy the linear equations (7) and (11).

To prove the uniqueness of a solution of (7) and (11), we decompose  $h_\alpha^0 = \sum_k p_{\alpha,k}^0 f_k^0$ . As  $f_k^0 = \sum_{\alpha \in \partial B} q_{k,\alpha}^0 h_\alpha^0$ , we deduce that

$$\sum_{\alpha \in \partial B} \left( \sum_k p_{\alpha',k}^0 q_{k,\alpha}^0 - 1_{\alpha'=\alpha} \right) h_\alpha^0 = 0$$

where  $1_{\alpha=\alpha'} = 1$  iff  $\alpha = \alpha'$  and 0 otherwise. As  $\mathbf{h}^1 \in T_{\mathbf{h}^0}$ , it implies that

$$\sum_{\alpha \in \partial B} \left( \sum_k p_{\alpha',k}^0 q_{k,\alpha}^0 - 1_{\alpha'=\alpha} \right) h_\alpha^1 \equiv 0 \quad \text{in } R/\mathfrak{J}_0.$$

This proves in particular that  $N^0 \left( \sum_{\alpha \in \partial B} \sum_k p_{\alpha',k}^0 q_{k,\alpha}^0 h_\alpha^1 \right) = h_{\alpha'}^1$ . Therefore, we deduce by linear combination of equations (11) that for all  $\alpha' \in \partial B$ ,

$$\begin{aligned} N^0 \left( \sum_k p_{\alpha',k}^0 f_k^1 \right) &= N^0 \left( \sum_k \sum_{\alpha \in \partial B} p_{\alpha',k}^0 q_{k,\alpha}^0 h_\alpha^1 \right) \\ &= N^0(h_{\alpha'}^1) = h_{\alpha'}^1. \end{aligned}$$

Consequently, the linear systems formed by the equation (7) and (11) has a unique solution  $h_{\alpha'}^1 = N^0 \left( \sum_k p_{\alpha',k}^0 f_k^1 \right)$ , for  $\alpha' \in \partial B$ .  $\square$

This leads to the following predictor-corrector procedure for following a path  $f_0^t, \dots, f_k^t \in R[t]$  such that  $\mathcal{J}^t := (\mathbf{f}^t) = (f_0^t, \dots, f_k^t)$  stays on  $\mathbf{Hilb}^\mu(\mathbb{P}^n)$ . The predictor step, which move in the tangent direction, is as follows:

- Compute the border basis  $\mathbf{h}^0$  of  $\mathbf{f}^{t_0}$  for a set  $B$  connected to 1 and let  $N^0$  be the projection on  $\langle B \rangle$  along  $\mathcal{J}^{t_0}$ ;
- Compute  $f_i^1 := \frac{\partial f_i^t}{\partial t}(t_0)$ ,  $i = 1, \dots, k$ ;
- Take variables  $\mathbf{h}^1 = (h_{\alpha, \beta}^1)_{\alpha \in \partial B, \beta \in B}$  and define the formal polynomials  $h_\alpha^1(\mathbf{x}) := \sum_{\beta \in B} h_{\alpha, \beta}^1 \mathbf{x}^\beta$  for  $\alpha \in \partial B$  and  $N^1 : \langle B^+ \rangle \rightarrow \langle B \rangle$  such that  $N^1(\mathbf{x}^\alpha) = h_\alpha^1(\mathbf{x})$  for  $\alpha \in \partial B$  and  $N^1(\mathbf{x}^\beta) = 0$  for  $\beta \in B$ .
- Compute the linear equations in  $\mathbf{h}^1$ :

$$N^0(f_k^1) - \sum_{\alpha \in \partial B} N^0(q_{k, \alpha}^0 h_\alpha^1) = 0,$$

obtained by reduction with respect to the border basis  $\mathbf{h}^0$ ;

- For each  $\beta \in B, 1 \leq i < j \leq n$ , compute the linear equations in  $\mathbf{h}^1$

$$N^1(x_i N^0(x_j \mathbf{x}^\beta)) + N^0(x_i N^1(x_j \mathbf{x}^\beta)) - N^1(x_j N^0(x_i \mathbf{x}^\beta)) + N^0(x_j N^1(x_i \mathbf{x}^\beta)) = 0$$

- Solve all these linear equations in  $\mathbf{h}^1$  and replace  $\mathbf{h}^0$  by  $\tilde{\mathbf{h}}^0 := \mathbf{h}^0 + s \mathbf{h}^1$  where  $s$  is the step size from  $t_0$  to  $t_1$ .

The corrector step, which projects back onto the variety  $H_{x_0}^B$ , is as follows:

- Compute the projection operator  $\tilde{N}^0 : \langle B^+ \rangle \rightarrow \langle B \rangle$  associated to the rewriting family  $\tilde{\mathbf{h}}^0$  such that  $\tilde{N}^0(\mathbf{x}^\alpha) = -h_\alpha(\mathbf{x})$  for  $\alpha \in \partial B$  and  $\tilde{N}^0(\mathbf{x}^\beta) = \mathbf{x}^\beta$  for  $\beta \in B$ .
- Take variables  $\mathbf{h}^1 = (h_{\alpha, \beta}^1)_{\alpha \in \partial B, \beta \in B}$  and define the formal polynomials  $h_\alpha^1(\mathbf{x}) := \sum_{\beta \in B} h_{\alpha, \beta}^1 \mathbf{x}^\beta$  for  $\alpha \in \partial B$  and  $N^1 : \langle B^+ \rangle \rightarrow \langle B \rangle$  such that  $N^1(\mathbf{x}^\alpha) = h_\alpha^1(\mathbf{x})$  for  $\alpha \in \partial B$  and  $N^1(\mathbf{x}^\beta) = 0$  for  $\beta \in B$ .
- For each  $\beta \in B, 1 \leq i < j \leq n$ , compute the linear equations in  $\mathbf{h}^1$

$$\begin{aligned} N^1(x_i \tilde{N}^0(x_j \mathbf{x}^\beta)) + \tilde{N}^0(x_i N^1(x_j \mathbf{x}^\beta)) - N^1(x_j \tilde{N}^0(x_i \mathbf{x}^\beta)) + \tilde{N}^0(x_j N^1(x_i \mathbf{x}^\beta)) \\ = -\tilde{N}^0(x_i \tilde{N}^0(x_j \mathbf{x}^\beta)) + \tilde{N}^0(x_j \tilde{N}^0(x_i \mathbf{x}^\beta)) \end{aligned}$$

- Solve this system in  $\mathbf{h}^1$ , replace  $\tilde{\mathbf{h}}^0$  by  $\tilde{\mathbf{h}}^0 + \mathbf{h}^1$  and iterate this loop until the norm of  $\mathbf{h}^1$  is small enough.

The corrector step consists simply in applying Newton iteration on the equations (1) and in iterating the correction until the error is small enough.

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