Universal Sets of n Points for One-bend Drawings of Planar Graphs with n Vertices

Hazel Everett, Sylvain Lazard, Giuseppe Liotta, Steve Wismath

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Hazel Everett † Sylvain Lazard† Giuseppe Liotta ‡
Stephen Wismath §

Abstract

This paper shows that any planar graph with n vertices can be point-set embedded with at most one bend per edge on a universal set of n points in the plane. An implication of this result is that any number of planar graphs admit a simultaneous embedding without mapping with at most one bend per edge.

1 Introduction

Let S be a set of m distinct points in the plane and let G be a planar graph with n vertices ($n \leqslant m$). A *point-set embedding* of G on S is a planar drawing of G such that each vertex is drawn as a point of S and the edges are drawn as polylines. The problem of computing point-set embeddings of planar graphs has a long tradition both in the graph drawing and in the computational geometry literature (see, e.g., [7, 8, 10]). Considerable attention has been devoted to the study of universal sets of points. We say that a set S of m points is *h-bend universal* for a family of planar graphs with n vertices ($n \leqslant m$) if each graph in the family admits a point-set embedding onto S that has at most h bends along each edge.

Gritzman, Mohar, Pach and Pollack [7] proved that every set of n distinct points in general position in the plane is 0-bend universal for the class of outerplanar graphs with n vertices. De Fraysseix, Pach, and Pollack [4] and independently Schnyder [11] proved that a grid with $O(n^2)$ points is 0-bend universal for all planar graphs with n vertices. De Fraysseix et al. [4] also showed that a 0-bend universal set of points for all planar graphs having n vertices cannot have $n+o(\sqrt{n})$ points. This last lower bound was improved by Chrobak and Karloff [3] and later by Kurowski [9] who showed that linearly many extra

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†INRIA Nancy Grand Est - Université Nancy 2, LORIA, Nancy, France. Firstname.Name@loria.fr

‡Dip. di Ingegneria Elettronica e dell'Informazione, Università degli Studi di Perugia. liotta@diei.unipg.it

§Department of Mathematics and Computer Science, University of Lethbridge, Alberta, Canada. wismath@uleth.ca

points are necessary for a 0-bend universal set of points for all planar graphs having n vertices. On the other hand, if two bends along each edge are allowed, a tight bound on the size of the point-set is known: Kaufmann and Wiese [8] proved that every set of n distinct points in the plane is 2-bend universal for planar graphs with n vertices. Finally, if one bend along each edge is allowed, Di Giacomo et al. [5] proved a related result that every planar graph can be drawn with its vertices on any given convex curve; however, the positions of the points on the curve depend on the planar graph.

In this paper we study the cardinality of a universal set of points for all planar graphs with n vertices under the assumption that at most one bend per edge is allowed in the point-set embedding. We prove the following theorem.

Theorem 1 *Let $\mathcal{F}_n$ be the family of all planar graphs with n vertices. There exists a set of n distinct points in the plane that is* 1*-bend universal for $\mathcal{F}_n$.*

The proof is constructive; an example is shown in Figure 1. We define a set S of n points (Figure 1(a)) and show how to compute in $O(n)$ time an embedding of any planar graph with n vertices on S such that the resulting drawing has at most one bend per edge. The drawing procedure starts by computing a special type of book embedding defined in Section 2 (see Figure 1(c)) and then uses this book embedding to construct the point-set embedding with the algorithm described in Section 3 (see Figure 1(d)).

Our universal set of n points can be defined either (i) on an integer grid of size $n2^n$ by n or (ii) with algebraic coordinates such that they are the vertices of a convex chain with unit-length edges. In the former case, the graphs can be drawn with all bend-points also on the grid points of the $n2^n$ by n grid. In the latter case, all planar graphs of $\mathcal{F}_n$ can be drawn with all bend-points and vertices in a square of size n by n at distance at least $\frac{1}{2(d+1)}$ apart, where d is the maximum degree of the graph. Furthermore, relaxing the unit edge length constraint, a universal set of points can be chosen on any given convex curve.

We conclude this introduction by noting a result that is immediately implied by Theorem 1. Two planar graphs G_1 and G_2 with the same set of vertices are said to admit a *simultaneous embedding without mapping* if there exists a set of points in the plane that supports a point-set embedding of both G_1 and G_2 [2].[1] It is not known whether any two planar graphs admit a simultaneous embedding without mapping such that all edges are straight-line segments. However, Brass et al. [2] showed that any planar graph has a straight-line simultaneous embedding without mapping with any number of outerplanar graphs. Kaufmann and Wiese [8] also proved that any two planar graphs have a simultaneous embedding without mapping such that each edge is drawn with at most two bends. In this context, Theorem 1 fills the gap for the one-bend case and implies the following.

Corollary 2 *Any number of planar graphs with n vertices admit a simultaneous embedding without mapping with at most one bend per edge.*

The paper is organized as follows. In Section 2, we first introduce some notation and review preliminary results about topological book embeddings. We then prove Theorem 1 in Section 3; we first present, in Section 3.1, a construction of sets of n points in convex

[1]A simultaneous embedding is said to be *with mapping* if the same vertex is mapped to the same point in both the drawings of G_1 and of G_2 [2].

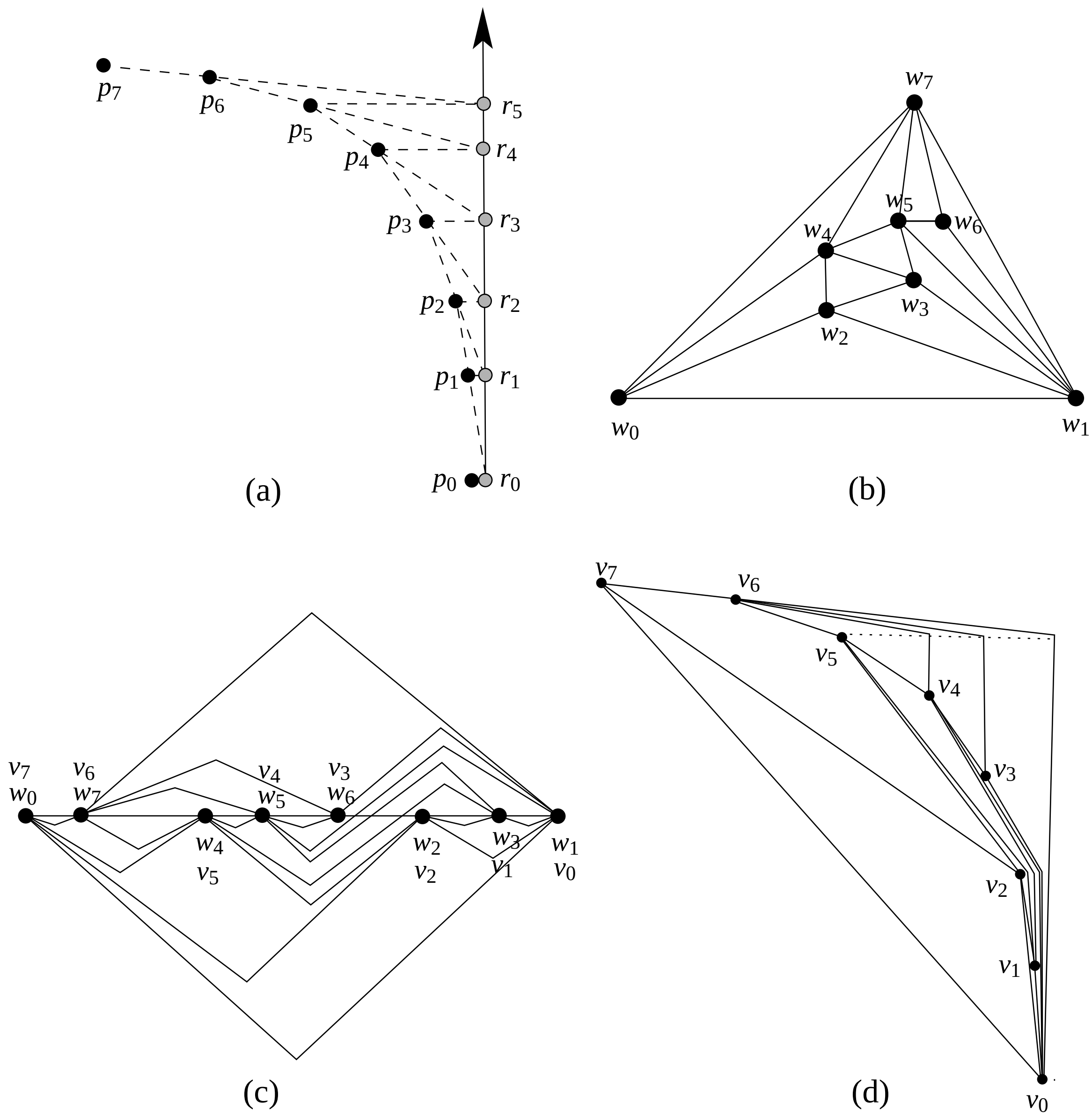

Figure 1: (a) A universal set, S, of 8 points, $p_0, \ldots, p_7$. (b) A graph G with 8 vertices. (c) A proper monotone topological book embedding, Γ, of G. The vertices are labeled both by their labels in graph G and drawing $\hat{\Gamma}$. (d) The drawing, $\hat{\Gamma}$, of G on the universal point set S.

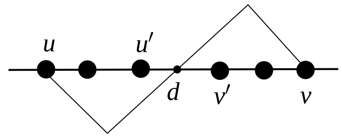

Figure 2: Edge (u,v) with spine crossing d and bounding vertices u' and v'.

position, called necklaces and describe, in Section 3.2, an algorithm that computes a 1-bend point-set embedding of any planar graph with n vertices on a necklace of n points. We then prove the correctness of the algorithm in Section 3.3 before concluding in Section 4.

2 Topological Book Embeddings

In this section, we review the notions of monotone, proper, and augmented topological book embeddings (see *e.g.,* [5, 6]). We start with preliminary notation.

Consider the Cartesian coordinate system (O,x,y) and let p,q be two points in the plane. We say that p is *left of* q and we denote it as $p < q$ if the x-coordinate of p is less than the x-coordinate of q; we shall also use the notation $p \leqslant q$ to mean that either p is left of q or p coincides with q; we define similarly $p > q$ and $p \geqslant q$. Let $\overline{pq}$ denote the line segment from p to q.

A *spine* is a horizontal line. Let ℓ be a spine and let p,q be two points on ℓ. Let $p < q$ and let b be a point on the perpendicular bisector of $\overline{pq}$ and not on ℓ. An *arc* (p,q) with *bend-point b* consists of segments $\overline{pb}$ and $\overline{bq}$; points p and q are respectively the *left* and *right endpoint* of (p,q). Arc (p,q) can be either in the half-plane above the spine or in the half-plane below the spine (such half-planes are assumed to be closed sets); in the first case we say that the arc is in the *top page* of ℓ, otherwise it is in the *bottom page* of ℓ.

Let $G = (V,E)$ be a planar graph. A *monotone topological book embedding* of G [5], denoted Γ, is a planar drawing such that all vertices of G are represented as points of a spine ℓ and each edge is either represented as an arc in the bottom page, or as an arc in the top page, or as a polyline that crosses the spine and consists of two consecutive arcs. More precisely, let (u,v) be an edge of a monotone topological book embedding that crosses the spine at a point d; assuming that u is left of v along the spine, (u,v) is such that (see Figure 2): (i) $u < d < v$, (ii) arc (u,d) is in the bottom page, and (iii) arc (d,v) is in the top page. Point d is called the *spine crossing* of (u,v) and is also denoted as $d(u,v)$.

Di Giacomo et al. [5] presented a constructive proof of the existence of monotone topological book embeddings. They also presented an algorithm to prove that any planar graph admits a one-bend drawing with all vertices on any given convex curve. Their algorithm can be viewed as follows. First, a monotone topological book embedding is constructed. Second, the spine is deformed into a convex curve while moving the vertices on the curve so that the bends in the bottom page straighten, resulting in a one-bend drawing. More precisely, the vertices and incident edges are added, one by one, to the drawing according to a canonical ordering; the position of a vertex thus depends on the position of the vertices and edges drawn in previous steps. In our approach, we carefully choose the position of the vertices on a convex curve and prove that the edges of any planar

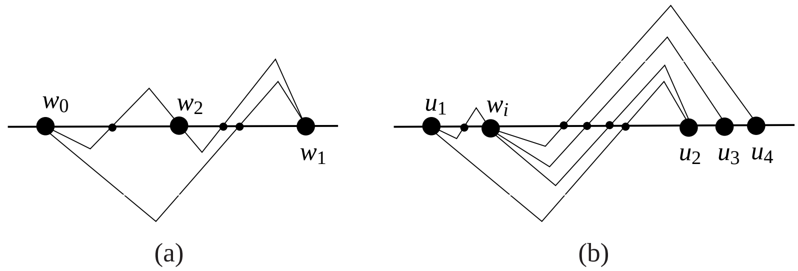

Figure 3: Construction of a monotone topological book embedding.

graph can then be drawn by a technique similar to that of Di Giacomo et al. [5] on that point set. Since monotone topological book embeddings are crucial in our construction, we sketch here, for completeness, the constructive proof of their existence.

Theorem 3 ([5]) *Every planar graph admits a monotone topological book embedding which can be computed in linear time in the size of the graph.*

Sketch of Proof. We describe an algorithm that takes a planar embedding of a maximal planar graph (every planar graph can be augmented to become maximally planar) and produces a linear ordering of its vertices and spine crossings for its edges. This linear ordering defines a monotone topological book embedding of the input graph.

Let G be a maximal embedded planar graph with n vertices and let w_0, w_1, w_{n-1} denote the vertices of its outer face. A monotone topological book embedding of G is constructed by adding a single vertex per step. The algorithm chooses the next vertex to be added by following a *canonical ordering* [4] of G. A canonical ordering of G with respect to edge (w_0, w_1) is an ordering of the vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ of G such that for every integer k ($3 \leqslant k < n$) the following properties hold (see Figure 1(b)):

- The embedded subgraph $G_{k-1} \subseteq G$ induced by $w_0, w_1, \ldots, w_{k-1}$ is biconnected and the external boundary C_{k-1} of G_{k-1} contains edge (w_0, w_1);
- w_k is a vertex (of $G \setminus G_{k-1}$) in the external face of G_{k-1}, and its neighbors in G_{k-1} form a subpath of the path $C_{k-1} \setminus \{(w_0, w_1)\}$.

Now, let $w_0, w_1, w_2, \ldots, w_{n-1}$ be a canonical ordering of G. At Step i, $0 \leqslant i < n$, the algorithm processes vertex w_i and all edges from w_i to some $w_{j<i}$. All edges are drawn with a spine crossing (that is, as a concatenation of an arc in the bottom page and one in the top page). Also, the monotone topological book embedding Γ_i computed at the end of Step i ($2 \leqslant i < n$) satisfies the following invariants:

1. Γ_i is a monotone topological book embedding of G_i that preserves the embedding of G_i.
2. For every pair of vertices u and v that are consecutive on the path $C_i \setminus \{(w_0, w_1)\}$ ($2 \leqslant i < n$) there is no vertex or spine crossing between u and $d(u, v)$ in the linear ordering of Γ_i.

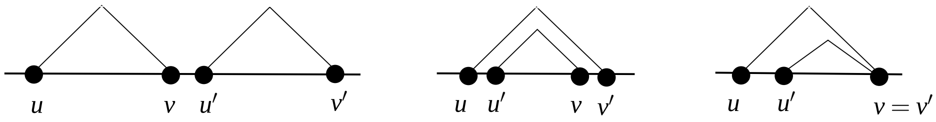

Figure 4: Illustration for Property 1.

In the first three steps of the algorithm, vertices w_0, w_1, w_2, and the division vertices of the edges between them are placed so that the linear ordering is the following: $w_0, d(w_0,w_2), w_2, d(w_2,w_1), d(w_0,w_1), w_1$ (see Figure 3(a)). Notice that the two invariants hold at the end of third step. The vertex to be placed at Step i is w_i ($3 \leqslant i < n$); let $u_1, u_2, \ldots, u_h$ be the neighbors of w_i in G_{i-1}. By Invariant 2, there is no vertex or spine crossing between u_1 and $d(u_1,u_2)$. Vertex w_i and all the division vertices $d(w_i,u_j)$ ($1 \leqslant j \leqslant h$) are placed between u_1 and $d(u_1,u_2)$ in such a way that their order is u_1, $d(u_1,w_i)$, w_i, $d(w_i,u_h)$, $d(w_i,u_{h-1})$, $\ldots$, $d(w_i,u_2)$, $d(u_1,u_2)$ (see Figure 3(b)). At the end of Step $n-1$, the computed linear ordering defines a monotone topological book embedding of G. That the running time of the algorithm is linear in the size of the graph is straightforward. □

Let $G = (V,E)$ be a planar graph, let Γ be a monotone topological book embedding of G computed according to Theorem 3, let (u,v) be an edge of Γ, and let d be its spine crossing. Also (see Figure 2), let u' be the rightmost vertex along the spine of Γ such that $u' < d$ and let v' be the leftmost vertex of the spine of Γ such that $d < v'$. We say that u' and v' are the two *bounding vertices* of d. We say that d is a *proper spine crossing* if its bounding vertices u' and v' are such that $u < u' < d < v' < v$. A monotone topological book embedding is *proper* if all of its spine crossings are proper. Since an edge that crosses the spine with a non-proper spine crossing can be replaced by a single arc (in the top or in the bottom page) without introducing an intersection, we directly obtain the following lemma from Theorem 3.

Lemma 4 *Every planar graph has a proper monotone topological book embedding which can be computed in linear time in the size of the graph.*

Let now Γ be a proper monotone topological book embedding of a planar graph G. If we insert a dummy vertex for each spine crossing of Γ, we obtain a new proper topological book embedding Γ' such that Γ' represents a planar subdivision G' of G obtained by splitting with a vertex some of the edges of G. We call the graph G' an *augmented form* of G and the drawing Γ' an *augmented proper topological book embedding* of G.

A vertex of G' that is also a vertex of G is called a *real vertex* of Γ'; a vertex of G' that corresponds to a spine crossing of Γ is called a *division vertex* of Γ'. Note that every division vertex of Γ' has degree two and that every edge of Γ' is either an arc in the top page or an arc in the bottom page. The *bounding vertices of a division vertex d* of Γ' are the two real vertices that form the bounding vertices of the spine crossing corresponding to d in Γ. The following property is a consequence of the planarity of Γ' (see Figure 4).

Property 1 *Let (u,v) and (u',v') be two distinct arcs of Γ' that are in the same page and such that $u < u'$. Then, (i) $u < v \leqslant u' < v'$ or (ii) $u < u' < v' \leqslant v$.*

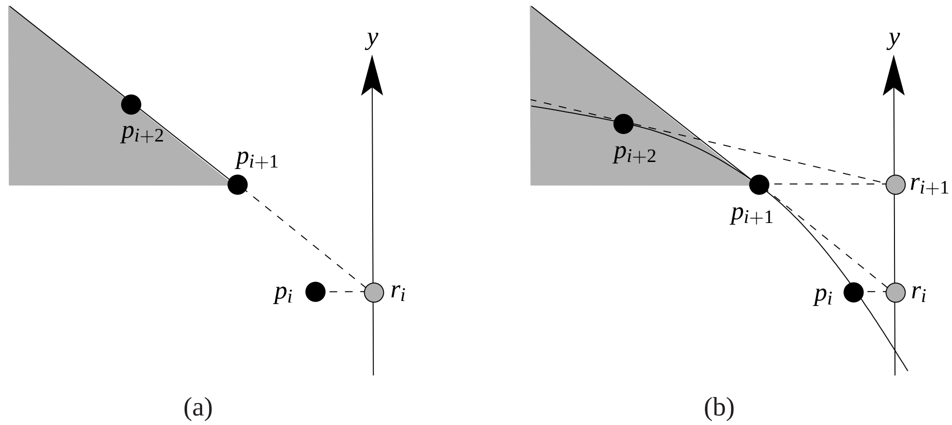

Figure 5: Three consecutive points of a necklace: p_{i+2} lies in the grey region defined by p_i and p_{i+1}. In (b), given p_{i+2} and p_{i+1} on a convex curve, p_i is chosen on the curve such that the grey region it defines (with p_{i+1}) contains p_{i+2}.

3 Proof of Theorem 1

We prove Theorem 1 by first defining a family of sets of n points in convex position, called necklaces (Section 3.1), and then by describing an algorithm that computes a 1-bend point-set embedding of any planar graph with n vertices on a necklace of n points (Section 3.2). We prove the correctness of the algorithm, and thus Theorem 1, in Section 3.3.

3.1 Necklaces

Refer to Figure 5(a) and to Figure 1(a) for a complete example. Let p_0 be any point on the x-axis strictly left of the vertical y-axis and p_1 be any point strictly in the top-left quadrant of p_0. We construct p_{i+2}, for $0 \leqslant i \leqslant n-2$, from p_i and p_{i+1} as follows. Let r_i be the projection of p_i on the vertical y-axis. Point p_{i+2} is chosen anywhere on or below the line through r_i and p_{i+1} and strictly above the horizontal line through p_{i+1}. Let S be any set of n points defined by the above procedure; we call S a *necklace* of n points.

We observe that, if we allow the points to have real (algebraic) coordinates, we can choose p_{i+2} on the line through r_i and p_{i+1} such that p_{i+1} is equidistant to p_i and p_{i+2}. Then it is straightforward that if the distance between p_i and p_{i+1} is 1 for all i, then all points lie in a square of size n by n. Notice also that we can construct a necklace, with algebraic coordinates, on any convex curve. Indeed, given any convex arc, we can choose the y-axis and the points $p_{n-1}, \ldots, p_0$, in that order, such that p_i is the point on the curve whose projection on the y-axis lies on the tangent to the curve at p_{i+1} (see Figure 5(b)).[2]

We can alternatively place the vertices on grid points as follows. Consider p_0 and p_1 at $(-1,0)$ and $(-2,1)$, respectively, and let p_{i+2} be chosen as the symmetric point of r_i with respect to p_{i+1}. Then, it is straightforward that p_i has coordinates $(-2^i, i)$, for $i \geqslant 0$.

[2]In other words, $p_{n-1}, \ldots, p_0$ is the sequence of points obtained by applying Newton's root-finding method starting from p_{n-1} (the y-axis plays here the role of the x-axis in Newton's method).

3.2 Computing 1-bend point-set embeddings

We describe a drawing algorithm, called `1-Bend Universal Drawer` or 1-BUD for short, that takes as input a planar graph G with n vertices and a necklace S of n points and returns a point-set embedding of G on S such that every edge of G is drawn with at most one bend.

Before describing our algorithm, we introduce the notion of *bend-lines* and *vertical strips* (see Figure 6(b)). The *bend-line* of p_i ($i > 1$) is the horizontal segment from p_{i-1} to r_{i-1}. The *vertical strip* of p_i ($i < n-1$), denoted $V(p_i)$, is the strip bounded by the vertical lines through p_i and p_{i+1}; the strip is considered open on the left side and closed on the right side. Denote also by $CH(S)$ and $\partial CH(S)$ the convex hull of S and its boundary.

The algorithm consists of the following steps.

Step 1: Draw the vertices of G. Compute a proper monotone topological book embedding Γ of G and the corresponding augmented proper topological book embedding Γ' (see Section 2). Let ℓ be the spine of Γ'. Label the real vertices of Γ' (that is, the vertices of Γ) on ℓ by $v_{n-1}, \ldots, v_0$ in that order from left to right, *i.e.*, $v_i < v_{i-1}$ (see Figure 1(c)). Modify Γ' so that every edge connecting two *consecutive* real vertices is drawn as an arc in the bottom page. Map every real vertex v_i to point p_i of the necklace (see Figure 1(d)).

Step 2: Draw the bends of the arcs of the top page of Γ'. Refer to Figure 6(a). For every real vertex v_i of Γ', let $a_{i0}, a_{i1}, \cdots, a_{ik}$ be the sequence of arcs in the top page of Γ' whose right endpoint is v_i; assume that $a_{i0}, a_{i1}, \cdots, a_{ik}$ are encountered in this order when going counterclockwise around v_i by starting the tour from a point on ℓ slightly to the right of v_i. Let w_{ij} denote the left (real or division) vertex of a_{ij} in Γ' ($0 \leqslant j \leqslant k$). If w_{ij} is a real vertex of Γ' (such as w_{i0} in Figure 6(a)), it is also labeled as vertex $v_{h_{ij}}$ for some $0 \leqslant h_{ij} \leqslant n-1$. Otherwise, w_{ij} is a division vertex of Γ' (such as w_{i1} in Figure 6(a)) and we define h_{ij} such that $v_{h_{ij}}$ and $v_{h_{ij}-1}$ are the two bounding real vertices of w_{ij} in Γ'.

Refer now to Figure 6(b). For each arc a_{ij}, we draw the bend of a_{ij} at a point q_{ij} on the bend-line of $p_{h_{ij}}$ (*i.e.*, through $p_{h_{ij}-1}$) such that (i) q_{i0} lies on the vertical ray above p_i and, for $j > 0$, (ii) q_{ij} lies strictly to the right of the vertical line through p_{i+1}, and strictly to the left of the line through p_i and $q_{i(j-1)}$.

Step 3: Draw the division vertices of Γ'. With the notation of Step 2, for each division vertex w_{ij} of Γ', let $(v_{k_{ij}}, w_{ij})$ be the arc Γ' whose right endpoint is w_{ij} (see Figure 6(a)). Draw w_{ij} at the intersection point, $p_{w_{ij}}$, between segment $\overline{p_{k_{ij}} q_{ij}}$ and $\partial CH(S)$ (see Figure 6(b)). (Note that, as will show in Lemma 6, $p_{w_{ij}}$ is not a point of the necklace $\{p_0, \ldots, p_{n-1}\}$.)

Step 4: Draw the arcs of Γ'. For each arc (u,v) of Γ', let p_u, p_v be the points representing u and v along $\partial CH(S)$. If (u,v) is an arc in the bottom page, draw it as the chord $\overline{p_u p_v}$. For instance, arc $(v_{k_{i1}}, w_{i1})$ in Figure 6(a) is drawn as chord $\overline{p_{k_{i1}} p_{w_{i1}}}$ in Figure 6(b).

If (u,v) is an arc in the top page of Γ', let q be the point computed at Step 2 that represents the bend-point of (u,v). Draw (u,v) as the polyline consisting of segments $\overline{p_u q}$ and $\overline{q p_v}$. For instance, arc $a_{i0} = (w_{i0}, v_i)$ in Figure 6(a) is drawn as the polyline consisting of $\overline{p_{h_{i0}} q_{i0}}$ and $\overline{q_{i0} p_i}$ in Figure 6(b) and arc $a_{i1} = (w_{i1}, v_i)$ is drawn as $\overline{p_{w_{i1}} q_{i1}}$ and $\overline{q_{i1} p_i}$.

Step 5. Let $\hat{\Gamma}$ be the drawing computed in Steps 1 to 4. The final drawing of G is obtained by ignoring the division vertices in $\hat{\Gamma}$.

Note that, in Step 2, we can trivially draw all the bends on integer grid points if there are n grid points between p_i and p_{i+1} on any horizontal line. Hence, all bends can be

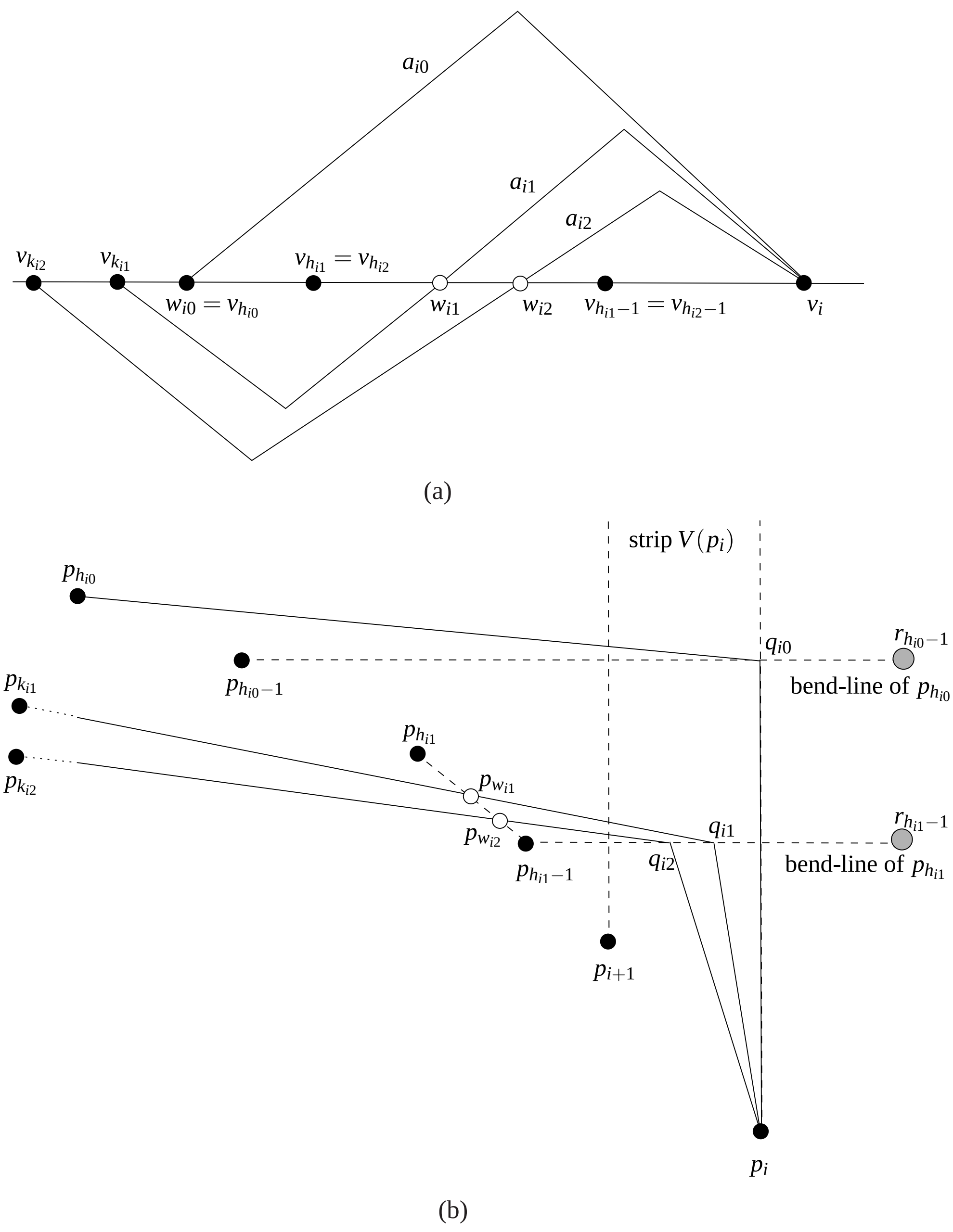

Figure 6: Illustration for Algorithm `1-Bend Universal Drawer`. (a) Part of an augmented proper monotone topological book embedding. (b) Sketch of the corresponding drawing on the necklace. Points $p_{k_{i1}}$ and $p_{k_{i2}}$ are much farther to the left than shown.

drawn on integer grid points if the necklace consists of point p_0 with coordinates $(-1,0)$ and points p_i with coordinates $(-n2^{i-1}, i)$, for $i > 0$. Similarly, if a necklace consists of the vertices (with algebraic coordinates) of a convex chain with unit-length edges, the bend-points can be drawn in Step 2 such that the horizontal distance between every consecutive pair of points of a sequence $p_i, q_{i0}, \ldots, q_{ik}, p_{i+1}$ is at least the horizontal distance between p_i and p_{i+1} times $\frac{1}{d+1}$ where where d is the maximum degree of the graph.[3] Furthermore, if the necklace is constructed such that the horizontal distance between p_0 and p_1 (and thus between any p_i and p_{i+1}) is at least $\frac{1}{2}$, then all the bend-points and vertices lie in a square of size n by n at distance at least $\frac{1}{2(d+1)}$ apart.

Proving Theorem 1 amounts to showing that the above algorithm correctly computes a point-set embedding of G on S such that each edge has at most one bend. The idea is to prove that the drawing computed at the end of Step 5 maintains the topology of Γ and that the geometric properties of the proper monotone topological book embedding and of the necklace implies a point-set embedding of the graph without edge-crossings and with at most one bend per edge. This proof is the subject of the next section.

3.3 Proof of correctness

We use here the same notation as in the previous section. We prove in this section the following main lemma which directly yields the correctness of the algorithm (Theorem 5).

Main Lemma. $\hat{\Gamma}$ is a planar drawing.

Based on this lemma we prove the correctness of the algorithm.

Theorem 5 *Algorithm* 1-BUD *computes a 1-bend point-set embedding of a planar graph with n vertices on a necklace with n points.*

Proof. Observe that every real vertex of Γ' is drawn as a point of S in $\hat{\Gamma}$. Since $\hat{\Gamma}$ does not have edge crossings (Main Lemma), removing the division vertices (which are of degree two) from $\hat{\Gamma}$ gives a point-set embedding of G on S. Also, by construction, the two edges incident on a division vertex of $\hat{\Gamma}$ form a flat angle, and thus removing the division vertices from $\hat{\Gamma}$ does not increase the number of bends. It follows that the drawing computed by Algorithm 1-BUD is a point-set embedding of G on S such that each edge has at most one bend. □

We now prove the Main Lemma. We start with three preliminary lemmas. We then prove, in different cases depending on which page two arcs of Γ' lie, that their drawings in $\hat{\Gamma}$ do not cross.

Let $a = (u,v)$ and $a' = (u',v')$ be two arcs of Γ' such that $u \leqslant u'$ and let p_u, p_v, $p_{u'}$, and $p_{v'}$ be the points of $\hat{\Gamma}$ representing u, v, u', and v', respectively. Recall that if u is a real vertex of Γ' then p_u is a point of the necklace $S = \{p_0, \ldots, p_{n-1}\}$, otherwise, u is a division vertex of Γ' and p_u is not *a priori* a point of the necklace; similarly for v, u' and v'.

Lemma 6 *If u is a division vertex of Γ' whose bounding vertices are v_t and v_{t-1}, then $p_t < p_u < p_{t-1}$.*

[3] Note that q_{i0} should then be drawn strictly inside the strip $V(p_i)$ instead of vertically above p_i.

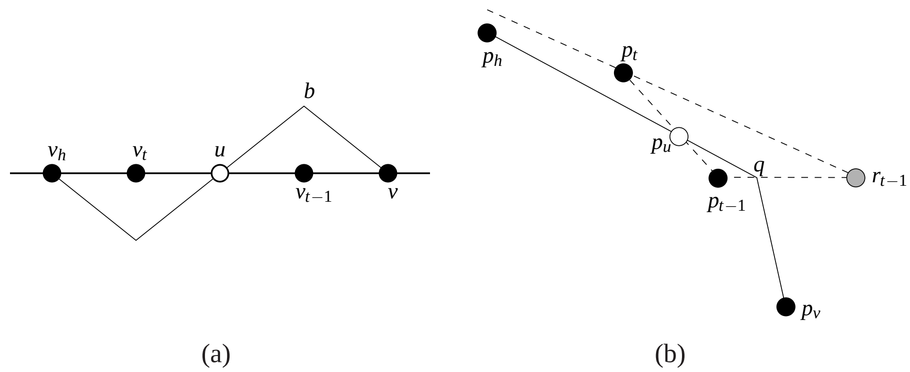

Figure 7: Illustration for Lemmas 6, 7 and 8 for the case where u is a division vertex (otherwise, $p_u = p_t$ and p_h is not relevant).

Proof. Refer to Figure 7. Let (v_h, u) and (u, v) be the two arcs of Γ' incident to u that lie, respectively, in the bottom page and top page of Γ'. Since Γ' is a proper monotone topological book embedding, we have that $v_h < v_t < u < v_{t-1} < v$. Both $v_h < v_t$ are real vertices of Γ' and they are thus mapped to $p_h < p_t$, respectively, in Step 1 of the algorithm. In Step 2, the bend of arc (u, v) is drawn as a point of the bend-line of p_t, denoted as q. The division vertex u is drawn in Step 3, as the intersection point, p_u, between segment $\overline{p_h q}$ and $\partial CH(S)$. Since q lies on the relatively-open segment $\overline{p_{t-1} r_{t-1}}$ and p_h lies (i) strictly below the line through p_t and r_{t-1}, (ii) strictly above the line through p_{t-1} and r_{t-1}, and (iii) strictly to the left of the line through p_t and p_{t-1}, the point p_u lies on the relative interior of segment $\overline{p_t p_{t-1}}$. Hence, $p_t < p_u < p_{t-1}$. □

Lemma 7 *If arc $a = (u, v)$ is in the top page of Γ' such that its bend is drawn in $\hat{\Gamma}$ on the bend-line of p_t, then $v_t \leqslant u < v_{t-1} \leqslant v$ and $p_t \leqslant p_u < p_{t-1} \leqslant p_v$.*

Proof. First, note that v is a real vertex of Γ' since a is in the top page. It directly follows from Step 2 of the algorithm that u is either equal to v_t or is a division vertex of Γ' whose bounding vertices are v_t and v_{t-1}. In the former case, $u = v_t < v_{t-1} \leqslant v$ and since they are all real vertices, Step 1 implies $p_u = p_t < p_{t-1} \leqslant p_v$. In the latter case, we have $v_t < u < v_{t-1} < v$ (since Γ' is an augmented proper monotone topological book embedding) and, by Lemma 6, $p_t < p_u < p_{t-1}$; furthermore, since $v_{t-1} < v$ are both real vertices, we also have $p_{t-1} < p_v$. □

Lemma 8 *If arc $a = (u, v)$ is in the top page of Γ', it is drawn in $\hat{\Gamma}$ as a polyline joining p_u to p_v with one bend-point q such that q (and thus the whole polyline) lies below the horizontal line through p_u and above the horizontal line through p_v.*

Proof. The bend of a is drawn on the bend-line of, say, p_t (that is on the horizontal line through p_{t-1}). By Lemma 7, $p_u < p_{t-1} \leqslant p_v$ and thus p_{t-1} is below the horizontal line through p_u and above the horizontal line through p_v, hence the result. □

Lemma 9 *If $a = (u, v)$ and $a' = (u', v')$ are in the top page of Γ' such that $u < u'$ and their bends are drawn in $\hat{\Gamma}$ on the same bend-line, then their drawings in $\hat{\Gamma}$ do not intersect, except possibly at a common endpoint. Furthermore, $p_u < p_{u'}$.*

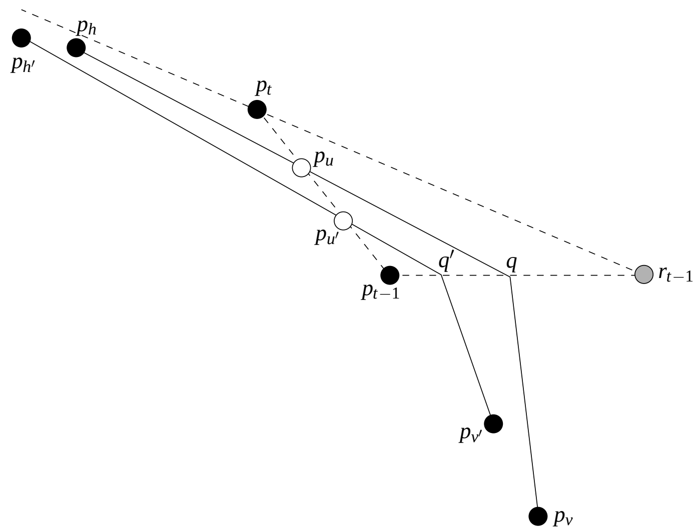

Figure 8: Illustration for Lemma 9.

Proof. Refer to Figure 8. Let q and q' be the bends of a and a' in $\hat{\Gamma}$, respectively, and assume that q and q' are drawn on the bend-line of p_t. By Lemma 7, we have $u' < v_{t-1}$ and $v_{t-1} \leqslant v$, thus $u' < v$. Property 1 thus implies that (ii) $u < u' < v' \leqslant v$ (see Figure 4(b-c)).

We first show that q' is to the left of q, *i.e.* $q' < q$. Points q and q' are in the vertical strips $V(p_v)$ and $V(p_{v'})$, respectively (see Step 2). Recall that $v' \leqslant v$ in Γ'. If $v' < v$ in Γ', we have that $p_{v'} < p_v$ since both v and v' are real vertices. Then, since any point in $V(p_{v'})$ is to the left of any point in $V(p_v)$, point q' is to the left of q. Otherwise, $v = v'$ and both q and q' are inside $V(p_v)$. In this case, consider the arcs incident to vertex $v = v'$ of Γ' and visit them by going counterclockwise around v (start the clockwise tour at a point on the spine of Γ' slightly to the right of v). Since $u < u'$ and Γ' is planar, we have that arc (u', v') is encountered after arc (u, v) (see Figure 4(c)). Hence, Step 2 implies that q' is to the left of q on the bend-line of p_t.

Segments $\overline{p_u q}$ and $\overline{p_{u'} q'}$ do not intersect each other. By Lemma 7, $v_t \leqslant u < v_{t-1}$ and $v_t \leqslant u' < v_{t-1}$. Thus, if u (resp. u') is a real vertex, it is equal to v_t. Note also that at most one of u and u' can be equal to v_t (since $u \neq u'$). Thus, if u' is a real vertex, $u' = v_t < u < v_{t-1}$, contradicting the fact that $u < u'$. Otherwise, if u is a real vertex, $u = v_t < u' < v_{t-1}$ and thus $p_u = p_t \neq p_{u'}$; Lemma 7 then implies that $p_u = p_t < p_{u'} < p_{t-1}$. Thus, points p_u, $p_{u'}$, q', and q appear in that order on the polyline formed by segments $\overline{p_t p_{t-1}}$ and $\overline{p_{t-1} r_{t-1}}$ (as in Figure 8 except that, here, $p_u = p_t$). Therefore, segments $\overline{p_u q}$ and $\overline{p_{u'} q'}$ do not intersect each other. Note that we have also proved that $p_u < p_{u'}$.

We now consider the case where u and u' are both division vertices of Γ'. In this case, we have $v_t < u < u' < v_{t-1}$. Let (v_h, u) and $(v_{h'}, u')$ be the two arcs in the bottom page of Γ' incident to u and u'. Since Γ' is an augmented proper monotone topological book embedding, $v_h < v_t$ and $v_{h'} < v_t$. Furthermore, since Γ' is a planar drawing, we have $v_{h'} \leqslant v_h < v_t < u < u'$. Finally, since $v_{h'}$, v_h, and v_t are real vertices of Γ', we have $p_{h'} \leqslant p_h < p_t$ (by Step 1).

Since $q' < q$ both lie on the bend-line of p_t and $p_{h'} \leqslant p_h < p_t$, the segments $\overline{p_h q}$ and $\overline{p_{h'} q'}$ do not intersect each other, except possibly at $p_h = p_{h'}$ (see Figure 8). Since p_u (resp.

$p_{u'}$) lies by construction (see Step 3) on segment $\overline{p_h q}$ (resp. $\overline{p_{h'} q'}$), segments $\overline{p_u q}$ and $\overline{p_{u'} q'}$ do not intersect each other. Furthermore, since p_u and $p_{u'}$ also lie on segment $\overline{p_t p_{t-1}}$ (by Lemma 7) and $q' < q$, points p_u, $p_{u'}$, q', and q appear in that order on the polyline formed by segments $\overline{p_t p_{t-1}}$ and $\overline{p_{t-1} r_{t-1}}$, we have $p_u < p_{u'}$.

The drawings of a and a' in $\hat{\Gamma}$ do not intersect, except possibly at $p_v = p_{v'}$. If $v \neq v'$, the vertical strips $V(p_v)$ and $V(p_{v'})$ are disjoint and, since they contain q and q', respectively, segments $\overline{q p_v}$ and $\overline{q' p_{v'}}$ do not intersect. On the other hand, if $v = v'$, segments $\overline{q p_v}$ and $\overline{q' p_{v'}}$ intersect only at their common endpoint $p_v = p_{v'}$ since they are not collinear by construction (see Step 2).

Finally, segment $\overline{p_u q}$ (resp. $\overline{p_{u'} q'}$) does not intersect segments $\overline{q' p_{v'}}$ (resp. $\overline{q p_v}$) because they are separated by the bend-line containing q and q' (by Lemma 8) and $q \neq q'$. Therefore, the drawings of arcs a and a' in $\hat{\Gamma}$ do not intersect each other except possibly at $p_v = p_{v'}$. □

Before proving that the drawings of a and a' in $\hat{\Gamma}$ do not cross each other in the other cases, we prove that the left-to-right order along the spine of Γ is preserved along $\partial CH(S)$.

Lemma 10 *If $u < u'$ in Γ', then $p_u < p_{u'}$.*

Proof. We consider different cases depending on whether u and u' are real or division vertices.

Case (i): both u and u' are real vertices. Step 1 of the algorithm assigns to u a label with index larger than that of u' and thus it maps u to a point of the necklace with index larger than that of u'. Hence, by construction of the necklace, the point p_u representing u in $\hat{\Gamma}$ is to the left of the point $p_{u'}$ representing u' in $\hat{\Gamma}$.

Case (ii): u is a division vertex and u' is a real vertex. Let v_t and v_{t-1} be the two bounding vertices of u in Γ'; we have $v_t < u < v_{t-1}$. Since u' is a real vertex such that $u < u'$, we have $v_{t-1} \leqslant u'$. It thus follows from Step 1 that $p_{t-1} \leqslant p_{u'}$ and Lemma 6 implies $p_u < p_{u'}$.

Case (iii): u is a real vertex and u' is a division vertex. A symmetric argument as above gives the result.

Case (iv): both u and u' are division vertices. Let $v_t < v_{t-1}$ be the bounding vertices of u and $v_{t'} < v_{t'-1}$ the bounding vertices of u' in Γ'. If these pairs of bounding vertices do not coincide then, since $u < u'$, we have $v_t < u < v_{t-1} \leqslant v_{t'} < u' < v_{t'-1}$. Step 1 then implies that $p_t < p_{t-1} \leqslant p_{t'} < p_{t'-1}$. Also, Lemma 6 implies that $p_t < p_u < p_{t-1}$ and $p_{t'} < p_{u'} < p_{t'-1}$, thus $p_u < p_{u'}$. On the other hand, if the pairs of bounding vertices coincide (*i.e.*, $t = t'$), then the bends of two arcs (u,v) and (u',v') in the top page of Γ' are both drawn on the bend-line of p_t and Lemma 9 implies that $p_u < p_{u'}$. □

We now resume proving that the drawings of a and a' do not cross each other.

Lemma 11 *If $a = (u,v)$ and $a' = (u',v')$ are in the top page of Γ' such that $u < u'$ and their bends are drawn in $\hat{\Gamma}$ on distinct bend-lines, then their drawings in $\hat{\Gamma}$ do not intersect, except possibly at a common endpoint.*

Proof. Let q and q' be the bends of a and a' in $\hat{\Gamma}$ and assume that they lie on the bend-lines of p_t and $p_{t'}$, respectively, with $t \neq t'$. Property 1 implies that (i) $u < v \leqslant u' < v'$ or (ii) $u < u' < v' \leqslant v$. We consider the two cases separately.

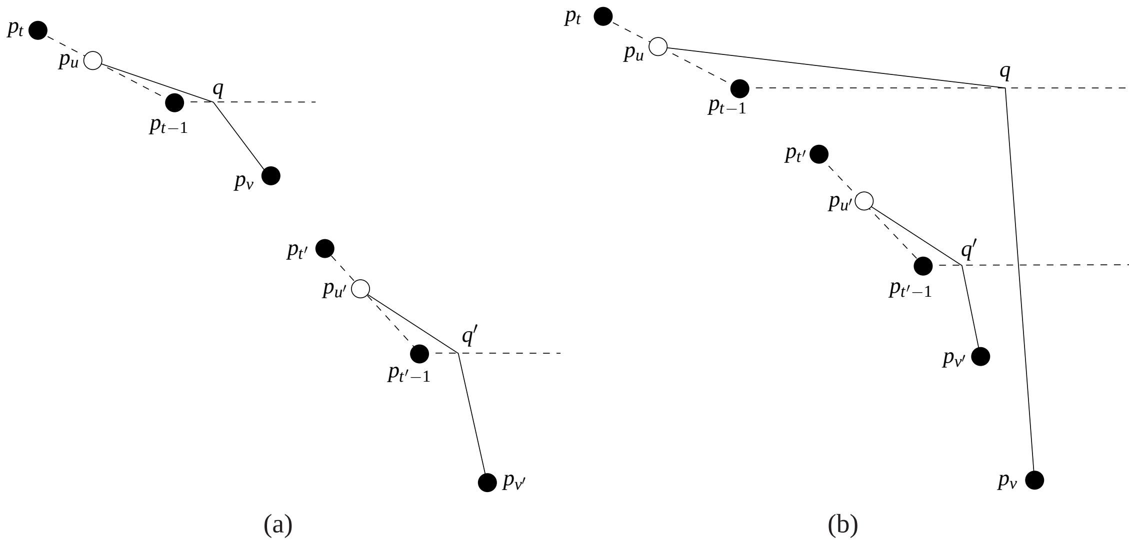

Figure 9: Illustration for Lemma 11. (a) Case (i), $u < v \leqslant u'$. (b) Case (ii), $u < u' < v' \leqslant v$.

Case (i) $u < v \leqslant u'$. Refer to Figure 9(a). By Lemma 8, arc a is drawn above the horizontal line through p_v and arc a' is drawn below the horizontal line through $p_{t'}$. Moreover, by Lemma 7, $v_t \leqslant u < v_{t-1}$ and $v_{t'} \leqslant u' < v_{t'-1}$. Since v_{t-1} is a bounding vertex of u such that $u < v_{t-1}$ and v is a real vertex such that $u < v$, we have $v_{t-1} \leqslant v$. Similarly, since $v_{t'}$ is a bounding vertex of u' such that $v_{t'} \leqslant u'$ and v is a real vertex such that $v \leqslant u'$, we have $v \leqslant v_{t'}$. Thus, $v_{t-1} \leqslant v \leqslant v_{t'}$ and, by Lemma 10, $p_v \leqslant p_{t'}$. Thus p_v is above, or equal to, $p_{t'}$ and the drawings of a and a' in $\hat{\Gamma}$ do not intersect, except possibly at a common endpoint.

Case (ii) $u < u' < v' \leqslant v$. Refer to Figure 9(b). By Lemma 7, $v_t \leqslant u < v_{t-1}$ and $v_{t'} \leqslant u' < v_{t'-1} \leqslant v'$ and, since $u < u'$ and $t \neq t'$, we have $v_t \leqslant u < v_{t-1} \leqslant v_{t'} \leqslant u' < v_{t'-1} \leqslant v' \leqslant v$. Thus, by Lemma 10,

$$p_t \leqslant p_u < p_{t-1} \leqslant p_{t'} \leqslant p_{u'} < p_{t'-1} \leqslant p_{v'} \leqslant p_v. \qquad (1)$$

In particular, we have $p_{t-1} \leqslant p_{u'}$ and thus (by Lemma 8) segment $\overline{p_u q}$, which is above the bend-line (of p_t) through p_{t-1}, is above $p_{u'}$, which is also above the drawing of a' in $\hat{\Gamma}$ (by Lemma 8). Hence $\overline{p_u q}$ does not intersect the drawing of a', except possibly at $q = p_{u'}$. If $q = p_{u'}$, then q lies on $\partial CH(S)$ and thus $q = p_{t-1}$. In Step 2 of the algorithm, q is drawn at p_{t-1} only in the case when $p_{t-1} = p_v$, which contradicts (1). Hence, segment $\overline{p_u q}$ does not intersect the drawing of a' in $\hat{\Gamma}$.

As in the proof of Lemma 9, the two segments $\overline{q p_v}$ and $\overline{q' p_{v'}}$ do not intersect except at $p_v = p_{v'}$ when $v = v'$. Thus the drawings of a and a' in $\hat{\Gamma}$ can only intersect at $p_v = p_{v'}$ or at an intersection point between segments $\overline{q p_v}$ and $\overline{p_{u'} q'}$.

If $v \neq v'$, then (1) implies $p_u < p_{u'} < p_{v'} < p_v$. The drawings of a and a' then have distinct endpoints and thus intersect an even number of times (since $q \neq q'$ as they lie on distinct bend-lines). Since there is at most one possible intersection point (between the two segments $\overline{q p_v}$ and $\overline{p_{u'} q'}$), the drawings of a and a' in $\hat{\Gamma}$ do not intersect.

If $v = v'$, then $u < u' < v = v'$ and the planarity of Γ' imply that the arc a' is encountered (in Γ') before a when going clockwise around $v = v'$ by starting the tour from a point on the spine ℓ slightly to the left of $v = v'$ (see Figure 4(c)). Thus, by Step 2, q' is drawn on a ray (from $p_v = p_{v'}$) that is to the left of the ray supporting q. Moreover, since segment $\overline{p_{u'}q'}$ is drawn to the left of the ray supporting q', $\overline{p_{u'}q'}$ does not intersect segment $\overline{qp_v}$. Hence the drawings of a and a' intersect only at their common endpoint $p_v = p_{v'}$. □

Lemma 12 *If $a = (u,v)$ and $a' = (u',v')$ are in the top page of Γ' such that $u = u'$, then their drawings in $\hat{\Gamma}$ only intersect at their common endpoint $p_u = p_{u'}$.*

Proof. The bends of arcs a and a' are drawn at two distinct points q and q' that lie on the same bend-line and that lie in the vertical strips $V(p_v)$ and $V(p_{v'})$, respectively (see Step 2). The arcs a and a' are drawn as polylines that consist of segments $\overline{p_uq}$, $\overline{qp_v}$ and $\overline{p_{u'}q'}$, $\overline{q'p_{v'}}$. The two segments $\overline{p_uq}$ and $\overline{p_{u'}q'}$ intersect only at their common endpoint $p_u = p_{u'}$ and they do not properly intersect with the two other segments since they are separated by the bend-line containing q and q' (by Lemma 8). Also the two segments $\overline{qp_v}$ and $\overline{q'p_{v'}}$ do not intersect since they lie, respectively, in the two vertical strips $V(p_v)$ and $V(p_{v'})$ which are distinct (indeed $v \neq v'$ since $u = u'$) and do not intersect by construction. Therefore, the drawings of arcs a and a' intersect only at $p_u = p_{u'}$. □

Lemma 13 *If a and a' are in the bottom page of Γ', their drawings in $\hat{\Gamma}$ do not intersect, except possibly at a common endpoint.*

Proof. Arc a is drawn as segment $\overline{p_up_v}$ and a' as segment $\overline{p_{u'}p_{v'}}$ (see Step 4). By Property 1, either (i) $u < v \leqslant u' < v'$, or (ii) $u < u' < v' \leqslant v$, or (iii) $u = u'$. In all cases, Lemma 10 and the convexity of S imply that $\overline{p_up_v}$ and $\overline{p_{u'}p_{v'}}$ do not intersect each other, except possibly at a common endpoint. □

Lemma 14 *If a and a' are in opposite pages of Γ', their drawings in $\hat{\Gamma}$ do not intersect, except possibly at a common endpoint.*

Proof. Assume that $a = (u,v)$ is in the bottom page of Γ' and that $a' = (u',v')$ is in the top page.

The arc a is drawn as a chord in $CH(S)$. If v is a division vertex of Γ' with bounding vertices $v_t < v_{t-1}$, then $u < v_t < v$ (since Γ' is an augmented proper monotone topological book embedding); thus $p_u < p_t < p_v$ (by Lemma 10) and a is drawn as a chord strictly inside $CH(S)$ (except for its endpoints). Otherwise, v is a real vertex of Γ' and thus, both p_u and p_v are vertices of $CH(S)$. Hence, a is drawn as a chord strictly inside $CH(S)$ (except for its endpoints) or it is drawn as an edge of $CH(S)$.

On the other hand, arc a' is drawn as a polyline joining $p_{u'}$ to $p_{v'}$ with bend-point q'. Since $p_{u'}$ and $p_{v'}$ lie on the boundary of $CH(S)$ and q' lies strictly outside $CH(S)$ (see Step 2), the arc a' is drawn, in $\hat{\Gamma}$, strictly outside $CH(S)$, except for its endpoints. Thus, if a is drawn strictly inside $CH(S)$ (except for its endpoints), the drawings of a and a' do not intersect, except possibly at a common endpoint.

Suppose now that a is drawn as an edge of $CH(S)$. Then the drawings of a and a' do not intersect, except possibly at a common endpoint, unless $p_{u'}$ lies on the relative interior of the edge $\overline{p_up_v}$ of $CH(S)$. In that case, u' is a division vertex of Γ' whose bounding vertices are u and v (by Lemma 7); thus (u,v) cannot be an arc of Γ' (since Γ' is an augmented proper monotone topological book embedding). Hence, the drawings of a and a' do not intersect, except possibly at a common endpoint. □

Lemmas 9, 11, 12, 13, and 14 imply that no two edges of $\hat{\Gamma}$ intersect, except possibly at common endpoints. This concludes the proof of the Main Lemma that the drawing $\hat{\Gamma}$ is planar because, by Lemma 10, all the vertices of Γ' are mapped to distinct vertices on the boundary of $CH(S)$.

4 Conclusion

This paper shows that there exists a set S of n distinct points in the plane such that every planar graph with n vertices admits a point-set embedding onto S where every edge is drawn as a polyline having at most one bend. Moreover, such a universal set S can be constructed on any convex curve. An application of this result to the simultaneous embeddability problem without mapping is also described.

We remark that our result closes a gap about universal sets of points for planar graphs. Indeed, it was already known that a universal set of (exactly) n points supporting straight-line drawings of planar graphs does not exist [3, 4, 9], while any set of n points can be universal if two bends per edge are allowed [8]. Also, notice that not all sets of points can be 1-bend universal: for example, if the points of S are collinear, exactly the family of sub-hamiltonian planar graphs has a 1-bend point-set embedding on S [1]. However, it is an open problem to determine whether any strictly convex point set is 1-bend universal.

When the vertices (and possibly the bend points) are required to lie on a regular grid, our construction of universal point-sets for one-bend drawing of planar graphs uses grids of exponential size (in the size of the graph). Moreover, if both the vertices and the bend points are to be drawn on a regular grid, the angle between adjacent incident edges may be exponentially small. We leave as an open problem to find a universal point-set for one-bend drawing of planar graphs in a polynomial-size regular grid and such that the minimum angle between adjacent incident edges of the drawings is at least (inversely) polynomial in the size of the graph.

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