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# Lower Bounds for Pinning Lines by Balls (Extended Abstract)

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## Abstract

It is known that if  $n \geq 2d$  pairwise disjoint balls in  $\mathbb{R}^d$  have a unique line  $\ell$  intersecting them in a given order  $\prec$ , one can always remove a ball so that  $\ell$  remains the only line intersecting the balls in the order induced by  $\prec$ . We show that the constant  $2d$  is best possible, in any dimension, and derive lower bounds on Helly numbers for sets of line transversals to disjoint balls in arbitrary dimension.

*Keywords:* Discrete Geometry, Geometric Transversal, Helly-type Theorem.

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# 1 Introduction

Let  $\mathcal{F}$  be a family of pairwise disjoint compact convex sets in  $\mathbb{R}^d$ . A *line transversal* to  $\mathcal{F}$  is a line that intersects every member of  $\mathcal{F}$ . The set  $\mathcal{T}(\mathcal{F})$  of all line transversals to  $\mathcal{F}$  forms a subspace of the affine Grassmannian, which is called the *space of transversals* to  $\mathcal{F}$ . If a line  $\ell$  is an isolated point of  $\mathcal{T}(\mathcal{F})$ , we say that  $\mathcal{F}$  *pins*  $\ell$ . Each orientation of a line transversal to  $\mathcal{F}$  induces an ordering on  $\mathcal{F}$ , namely the order of the intersections along the line; the pair of reverse orderings on  $\mathcal{F}$  induced by  $\ell$  is called the *geometric permutation* of  $\ell$ . When  $\mathcal{F}$  consists of disjoint balls, geometric permutations characterize the connected components of  $\mathcal{T}(\mathcal{F})$  [1], and in particular  $\mathcal{F}$  pins a line  $\ell$  if and only if  $\mathcal{F}$  admits no other line transversal with the same geometric permutation. If  $\mathcal{F}$  pins  $\ell$  and no proper subset of  $\mathcal{F}$  does, then  $\mathcal{F}$  is a *minimal* pinning of  $\ell$ . A minimal pinning configuration consisting of (pairwise) disjoint balls in  $\mathbb{R}^d$  has size at most  $2d - 1$  [1,2]. In this paper, we show that this constant is best possible in all dimensions:

**Theorem 1.1** *For any  $d \geq 2$ , there exists a minimal pinning by  $2d-1$  disjoint congruent balls in  $\mathbb{R}^d$ .*

This lower bound follows from two new conditions for a configuration of disjoint balls tangent to a line to pin it, a *geometric* necessary condition and a *combinatorial* sufficient condition, which are of independent interest.

The study of how the structure of  $\mathcal{T}(\mathcal{F})$  depends on the geometry of  $\mathcal{F}$  goes back to the 1930's, when Vincensini [8] asked whether Helly's theorem on the intersection of convex sets generalizes to sets of lines (or  $k$ -flat) transversals to convex sets. While the answer is negative in general, Helly-type theorems for sets of transversals do hold when the geometry of  $\mathcal{F}$  is adequately constrained. Danzer [3] showed that this is the case for disjoint unit disks in the plane and conjectured that the same holds in arbitrary dimension, a conjecture recently settled in the positive [2,6]. The corresponding Helly number, for transversals to disjoint unit balls in  $\mathbb{R}^d$ , is known to be between 5 and  $4d - 1$ . Hadwiger [5] showed that if any three in a family  $\mathcal{F}$  of convex sets have a

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line transversal consistent with some global ordering on  $\mathcal{F}$  then  $\mathcal{F}$  has a line transversal. While this statement does not generalize to higher dimension [7], it holds for collections of disjoint balls in  $\mathbb{R}^d$ . In that setting, the “Hadwiger” number is known to be between 3 and  $2d$  [1]. Our proof of Theorem 1.1 shows, more precisely, that there exists a minimal pinning configuration  $(\mathcal{F}, \ell)$  of size  $2d - 1$  where  $\ell$  is the only line transversal to  $\mathcal{F}$ . In particular, the interior of the balls in  $\mathcal{F}$  have no line transversal, whereas the interior of any proper subset of balls in  $\mathcal{F}$  have some line transversals consistent with the geometric permutation of  $\ell$ . Thus, slightly shrinking the balls of  $\mathcal{F}$  yields a lower-bound of  $2d - 1$  for both the Helly and Hadwiger number. In particular, this shows that these numbers must increase with the dimension, which answers another old question of Danzer [3].

The study of isolated transversal relates to other natural questions in discrete and computational geometry. For instance, lines pinned by the unit balls centered in a point set  $P$  arise in geometric optimization (as axis of locally-minimal unit cylinders enclosing  $P$ ) and in robotics (as axis of unit cylinders immobilized – up to translation along and rotations around its axis – by contacts placed at  $P$  on their inside). In geometric transversal theory, the generalization of Hadwiger’s transversal theorem to hyperplane transversals [4] relies on a characterization of pinnings of hyperplanes.

## 2 Outline of the proof

The main ingredient of our proof are *stable pinnings*: families of disjoint balls that pin a line and keep pinning it after arbitrary sufficiently small rigid motions preserving  $\ell$  are applied to each ball independently. We show that stable pinning exist in  $\mathbb{R}^3$  by a simple observation on the combinatorial structure of their projections, and their existence in  $\mathbb{R}^d$  follows by a compactness argument. We then argue that any stable pinning in  $\mathbb{R}^d$  consists of at least  $2d - 1$  objects by analyzing the geometry of the first-order approximations of the solids  $\mathcal{T}(B_i)$  in some adequate parameterization of the space of lines.

A *halfplane pattern* is a sequence  $\mathcal{H} = (H_1, \dots, H_n)$  of halfplanes in  $\mathbb{R}^2$  bounded by lines through the origin. A halfplane pattern can be naturally associated to any pair  $(\mathcal{F}, \ell)$ , where  $\mathcal{F}$  is a family of 3-dimensional balls tangent to a line  $\ell$ : number the balls according to the order of the contact points along  $\ell$ , consider a plane  $\Pi$  orthogonal to  $\ell$  and let  $H_i$  denote the halfplane in  $\Pi$  bounded by a line through  $\ell \cap \Pi$  that contains the orthogonal projection of the  $i^{\text{th}}$  ball on  $\Pi$ . We call a halfplane pattern a *pinning pattern* if no two

halfplanes are bounded by the same line, and if for every directed line  $\Delta$  not meeting the origin and intersecting each halfplane there exist indices  $i < j$  such that  $\Delta$  exits  $H_j$  before entering  $H_i$ . If the halfplane pattern of  $(\mathcal{F}, \ell)$  is a pinning pattern, then  $\mathcal{F}$  has no other line transversal realizing the same ordering as  $\ell$ , and it pins  $\ell$ . In fact, pinning patterns are invariant under small perturbations of the halfplanes (that is, if each halfplane is rotated about the origin by a sufficiently small angle). More precisely, two halfplane patterns are equivalent with respect to the pinning pattern property if the cyclic order of the inward and outward normals of the halfplanes is identical. Thus, if the halfplane pattern of  $(\mathcal{F}, \ell)$  is a pinning pattern, then  $\mathcal{F}$  is not only a pinning of  $\ell$ , but a stable one.

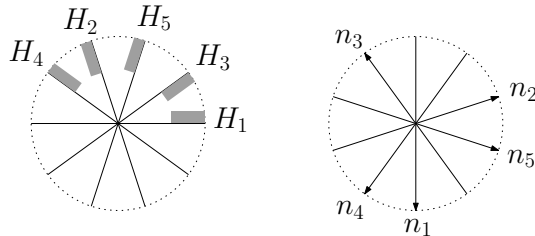


Fig. 1. A  $\sigma_5$ -patterns, as an arrangement of halfplanes through the origin (left) and as a cyclic order of outward and inward normals on  $\mathbb{S}^1$  (right).

Let  $n_i \in \mathbb{S}^1$  denote the outward normal of  $H_i$ . We define a  $\sigma_5$ -pattern as a halfplane pattern where the in- and outward normals appear in the order:  $n_1, -n_3, n_5, n_2, -n_4, -n_1, n_3, -n_5, -n_2, n_4$  (see Figure 1). Any  $\sigma_5$ -pattern is a pinning pattern, and the existence of stable pinnings in  $\mathbb{R}^3$  follows. Since the set of line transversals to disjoint balls realizing the same geometric permutation is connected, a line in  $\mathbb{R}^d$  is pinned if and only if it is pinned in every 3-space that contains it. Let  $\ell$  be a line in  $\mathbb{R}^d$  and pick, for every 3-space containing it, a stable pinning by 5 balls. Any such pinning is stable, so it ensures that  $\ell$  is pinned not only in “its” 3-space, but also in sufficiently close 3-spaces containing  $\ell$ . By compactness, we can find finitely many balls that pin  $\ell$  in every 3-space and retain this property under sufficiently small perturbation; the existence of stable pinnings in  $\mathbb{R}^d$  follows.

It remains to establish a lower bound on the size of stable pinnings. Let  $\ell$  be the  $x_d$ -axis in  $\mathbb{R}^d$ . For  $\lambda \in \mathbb{R}$  and direction vector  $n \in \mathbb{S}^{d-2}$ , consider the set

$$\mathcal{S}(\lambda, n) := \{(x, \lambda) \in \mathbb{R}^d \mid x \in \mathbb{R}^{d-1}, \langle n, x \rangle \leq 0\},$$

where  $(a, b)$  denotes the vector formed by the coordinates  $a$  followed by those

of  $b$ . We call  $\mathcal{S}(\lambda, n)$  a *screen*. The screen  $\mathcal{S}(\lambda, n)$  is a  $(d - 1)$ -dimensional halfspace of a hyperplane orthogonal to  $\ell$ , tangent to  $\ell$  in the point  $(0, \lambda)$ . Identifying the line through  $(u_0, 0)$  and  $(u_1, 1)$  with the point  $(u_0, u_1) \in \mathbb{R}^{2d-2}$  recasts the set of line transversals to a screen as a halfspace in  $\mathbb{R}^{2d-2}$ . We then define the screen of a ball  $B$  tangent to  $\ell$  as  $\mathcal{S}(\lambda, -p/||p||)$ , where  $(p, \lambda)$  denotes the ball's center; the screen of  $B$  meets  $\ell$  in the same point as  $B$  and its boundary is tangent to  $B$  in  $\ell \cap B$ . Now, the key observation is that the relative interiors of the screens of a family of balls pinning  $\ell$  must have no line transversal. In  $\mathbb{R}^{2d-2}$ , this recasts into the corresponding halfspaces intersecting with empty interior, which forces their normals to be linearly dependant. Observing that any configuration of  $k \leq 2d - 2$  screens can be perturbed so that their sets of transversals have linearly independent normals, we get that any stable pinning by balls has size at least  $2d - 1$ .

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