Inverse EEG (electroencephalography) Problem:

From measurements by electrodes of the electric potential \( u \) on the scalp, recover a distribution of \( m \) pointwise dipolar current sources \( C_k \) with moments \( p_k \) located in the brain (modeling the presence of epileptic foci).

Model: The head \( \Omega \) is modeled as a set of 3 spherical or ellipsoidal nested regions \( \Omega_k \subset \mathbb{R}^3, i = 0, 1, 2 \) (brain, skull, scalp), separated by interfaces \( S_i \) (with \( S_2 = \partial \Omega \)) and with piecewise constant conductivity \( \sigma_i = \sigma_i > 0 \).

\[
\begin{align*}
\text{Macroscopic model} &: \quad \text{quasi-static approximation of Maxwell equations} \rightarrow \text{Spatial behavior of } u \text{ in } \Omega \text{ [1]: (P)} \quad \begin{cases} 
\text{div} (\sigma \nabla u) - \sum_{k=1}^{m} p_k V \nabla C_k & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}
\end{align*}
\]

where \( g \) and \( \phi \) denote the given potential and current flux on the scalp (or approximate interpolation of these quantities).

The resolution of this inverse problem can be divided into 3 main steps:

1. Data propagation (Cortical mapping step):

\[ u \text{ at electrodes } \implies \text{Cortical problem} \implies \text{Brain mapping} \implies \text{Scalp reconstruction}. \]

Since \( C_k \in \Omega_k \), the function \( u \) is harmonic in the outer layers \( \Omega_2 \) and \( \Omega_3 \) where boundary conditions are given by the continuity relations

\[ u_k = [\sigma_k \partial_{n_k} u] = 0 \text{ on } S_i, \quad i = 1, 2, \quad \text{denotes the jump across the surface } S_i. \]

Based on these boundary conditions, data propagation can be achieved by using boundary element methods [2], or by using robust harmonic approximation techniques and expansions on appropriate basis [3].

2. Anti-harmonic projection (Signal space separation):

\[ \begin{cases} 
\Delta u = \sum_{k=1}^{m} p_k \nabla C_k & \text{in } \Omega_k \\
u = g & \text{on } S_1
\end{cases} \]

assumes the form: \( u(x) = f(x) + \sum_{i=1}^{m} \phi_i(x) - \sum_{i=1}^{m} \chi_i(x) \), \( \phi_i \) a harmonic function in \( \Omega_k \).

In order to recover the \( C_k \) inside \( \Omega_k \), the knowledge of the singular function \( u_k \) is required on \( S_0 \). This can be deduced from available boundary data by expanding \( u \) on bases of spherical or ellipsoidal harmonics [1, 4].

3. Best rational approximation on planar sections (Source localization):

\[ \text{(pointwise values of } u_k \text{ on } S_1 \implies \text{2D best approximation schemes in planar sections of the boundary} \implies \text{localization of sources } C_k \). \]

Illustrations (Spherical model):

- Slice \( \Omega_2 \) along a family of planes \( \Pi_0, \Pi_1 \cap \Omega_2 = \Gamma_0 \) (circles or ellipses).
- From pointwise values of the singular part \( u_k \) on \( \Gamma_1 \), approximate \( f_1 = (P, u_0) \) the square of the anti-analytic projection on \( \Gamma_1 \) [3, 4].
- Ellipses may then be preliminarily mapped by a conformal rational transformation onto circles.
- \( f_1 \) is a meromorphic function, analytic outside \( \partial \Omega_2 \), which is strongly and explicitly linked with the sources \( C_k \).
- Approximate the \( Q_1, \Gamma_2 \) by \( \bar{Q}_1, \bar{\Gamma}_2 \) the poles of the best \( L^2 \) or \( L^\infty \) rational approximation to \( f_1 \) on \( \Gamma_1 \) (degree \( \geq m \) for a sphere, \( 2m \) for an ellipsoid), the poles \( Q_1, \Gamma_2 \) accumulate to the singularities \( Q_m, \Gamma_m \) [5].
- Varying \( p \), this allows us to approximately locate the \( m \) sources \( C_k \) in \( \Omega_2 \).

The error estimate is around \( 10^{-2} \).

Numerical results:

Spherical model:

- In this case the relation between \( Q_{1, \mu} \) and \( C_k \) when the slice \( p \) varies is the following:
  \( Q_{1, \mu} \) are aligned together and also with the complex coordinates \( C_k \) of \( C_k \).
  \( Q_{1, \mu} \) is minimum at \( \mu = Q_k \) in \( \mu = Q_k \) (the \( k \)-th source section).

The error estimate is around \( 10^{-4} \) for the 1 source case and around \( 10^{-2} \) for the 3 sources case.

Perspectives:

- Recover the moments \( p_k \), improve sources estimation (slicing along different directions).
- Software development (FindSourcesID).
- More realistic geometries.
- The inverse MEG source problem.

The error estimate is around \( 10^{-5} \) for the 1 source case and around \( 10^{-2} \) for the 2 sources case.

Ellipsoidal model:

- In each slice \( p \) we construct a real valued polynomial \( \phi, \mu(\zeta) \) whose coefficients are related to the conformal rational transformation from ellipse to circle. \( \phi, \mu(\zeta) \) is minimum for \( \zeta = Q_k \).

These numerical results represent exact sources \( \phi \) and approximated ones \( \phi \).

The error estimate is around \( 10^{-4} \) for the 1 source case and around \( 10^{-2} \) for the 2 sources case.

FindSourcesID

(view of \( u \) on \( S_2 \) then of \( C_k \) into \( \Omega_2 \).)

References: