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# On DRC-Covering of $K_n$ by Cycles

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## Abstract

This paper considers the cycle covering of complete graphs motivated by the design of survivable WDM networks, where the requests are routed on sub-networks which are protected independently from each other. The problem can be stated as follows : for a given graph  $G$ , find a cycle covering of the edge set of  $K_n$ , where  $V(K_n) = V(G)$ , such that each cycle in the covering satisfies the disjoint routing constraint (DRC), relatively to  $G$ , which can be stated as follows : to each edge of  $K_n$  we associate in  $G$  a path and all the paths associated to the edges of a cycle of the covering must be vertex disjoint. Here we consider the case where  $G = C_n$ , a ring of size  $n$  and we want to minimize the number of cycles in the covering. We give optimal solutions for the problem as well as for variations of the problem, namely, its directed version and the case when the cycle length is fixed to 4.

**Keywords:** Cycle covering, Survivability, Protection, WDM networks

## 1 Introduction

The problem of covering the edges of  $K_n$  by complete graphs  $K_k$ 's has been studied by many people. This problem is known as the covering design problem [8, 10]. Moreover, the problem of finding a perfect covering of the edges of  $K_n$  is the same as a partitioning problem and this is related to the existence of an  $(n, k, 1)$ -design. Partitioning the edges of  $K_n$  into isomorphic graphs in particular cycles  $C_k$ 's has also been well studied [6, 7]. But there seems to be less known results for the problem of covering the edges of  $K_n$  by  $C_k$ 's, where  $k \geq 4$ . In [2], the answer was given for  $k = 4$ .

Here we consider a covering problem arising from the design of a survivable WDM network, where the communication requests are routed on sub-networks which are protected independently from each other. We model the WDM network by a graph, called the physical graph and denoted

by  $G$ . The vertices of the graph represent the optical switches and the edges the fiber-optics links. In fact,  $G$  is an oriented symmetric multigraph ; indeed each time there is a fiber optic from a node  $x$  to a node  $y$  there is also the opposite one. We will consider either the symmetric digraph or the underlying undirected graph

Usually the physical graph has no regularity properties, just enough connectivity in order to ensure a good routing for the demands in case of failures. However many designers of an optical network build it with loops interconnected between themselves. Thus, the first case to consider is when the physical graph is a loop (or ring), which means that the graph  $G$  is either an undirected cycle of length  $n$ , denoted  $C_n$  or the symmetric directed cycle  $C_n^*$ .

Routing a request over  $G$  consists in finding a path over  $G$  between the pair of nodes communicating in the request. The protection problem we consider consists in covering the family of requests by subgraphs  $I_k$ . In general, we wish the  $I_k$  to have a simple structure and a small number of vertices. Therefore the interesting case is when  $I_k$  is a small cycle. Indeed, on the cycle we use half of the capacity for the demands, and in case of failure we reroute the traffic through the failed link via the remaining part of the cycle using the other half of the capacity. It will be interesting to get very small cycles as subnetworks as they are easier to manage and less costly to reroute.

But there is another constraint due to the fact that the requests have to be routed on the physical network  $G$ , which can be modeled as follows (one can think that to each subnetwork, here cycle, we associate a wavelength) : each cycle formed by some requests must be routed vertex disjointly over  $G$ , or this is the same as saying that we can find a set of vertex disjoint paths corresponding to the set of requests over a cycle. We call this property the disjoint routing constraint (DRC). This property can be extended to the directed version of the problem in which the paths in the routing will be directed paths and the cycles in the covering will be directed cycles.

It is clear that not every cycle satisfies DRC. As an illustration, the cycle  $(1, 2, 3, 4, 1)$  can be routed over  $C_4$  and it satisfies DRC. However the cycle  $(1, 3, 4, 2, 1)$  does not satisfy DRC as it is impossible to associate the requests  $(1, 3)$  and  $(2, 4)$  to vertex disjoint paths in  $C_4$ (see Figure 1).

We can model all the requests as the edge set of a logical graph  $I$  undirected or not. The vertices represent the nodes of the physical graph and the edges correspond to the requests between these nodes. In general, it is a digraph ; but in telecommunication networks used for applications like telephone the requests are symmetric (i.e. for each communication request from node  $A$  to node  $B$  there is a similar request from node  $B$  to node  $A$ ) and furthermore, the routing of symmetric requests is done by a symmetric routing. There is no constraint forcing that, but it is the way requests are actually managed. So the symmetry of the routing explains why we can also consider undirected graphs for the physical one. Finally the instance of communication called *total exchange* (or All-to-All), where each node wants to communicate with all the others simultaneously, is important. In such a case, the logical graph  $I$  will be the complete graph  $K_n$  (or the symmetric complete digraph  $K_n^*$ ).

Finding a cycle covering for all requests satisfying the DRC constraint is the same as finding a covering of the edge set of  $I$ , such that each cycle in the covering satisfies the DRC in  $G$ . Such a covering will be called a DRC-covering of  $I$  relatively to  $G$ .

As an illustration, let  $G$  be  $C_4$  and  $I$  be  $K_4$  (See Figure 1). A first covering is given by the two  $C_4$ 's  $(1, 2, 3, 4, 1)$  and  $(1, 3, 4, 2, 1)$  (See Figure 1.(c)), but there does not exist an edge disjoint routing for the cycle  $(1, 3, 4, 2, 1)$ . In counterpart, the covering given in Figure 1.(d) by the  $C_4$   $(1, 2, 3, 4, 1)$  and the two  $C_3$ 's  $(1, 2, 4, 1)$  and  $(1, 3, 4, 1)$ , satisfy the edge disjoint routing property.

Our aim is to minimize the cost of the network ; the cost function is a complicated one, but for the particular case, considered here, where  $G = C_n$ , it corresponds to minimize the number of cycles in the covering.

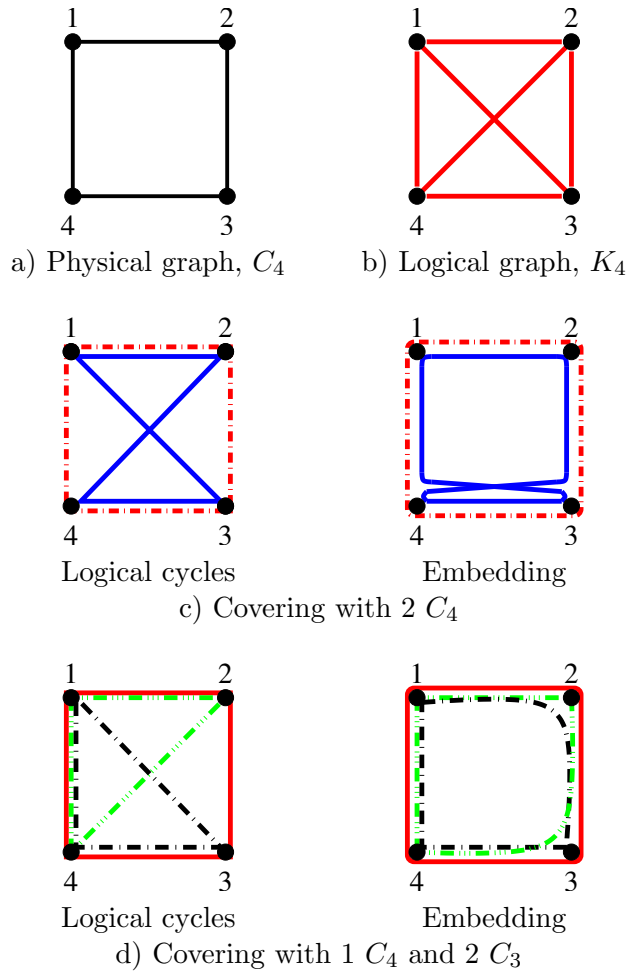


Figure 1: Cycle covering example.

In summary, we want to find the minimum number of cycles in a DRC-covering of  $K_n$  (or  $K_n^*$ ) relatively to the cycle  $C_n$  (or  $C_n^*$ ). As a variation of the problem, we can also add some restriction to the cycles in the covering, for example, we can consider the case when the size of the cycles is uniform or is bounded.

We denote  $\rho(n)$  the minimum number of cycles needed in such a DRC-covering of  $K_n$  and similarly we define  $\rho_k(n)$  for the case when the cycle size is restricted to be  $k$ . We prove the following results :

**Theorem 3.3.** *When  $n = 2p + 1$ ,  $\rho(n) = \frac{n^2-1}{8} = \frac{p(p+1)}{2}$ . Furthermore, the DRC-covering of  $K_{2p+1}$  consists of  $p$   $C_3$  and  $\frac{p(p-1)}{2}$   $C_4$ .*

**Theorem 3.4.** *When  $n = 2p$ ,  $p \geq 2$ ,  $\rho(n) = \left\lceil \frac{n^2+4}{8} \right\rceil = \left\lceil \frac{p^2+1}{2} \right\rceil$ . Furthermore, when  $n = 4q$ ,  $q \geq 2$ , a DRC-covering of  $K_{4q}$  consists of 4  $C_3$  and  $2q^2 - 3$   $C_4$ , and when  $n = 4q + 2$ ,  $q \geq 1$ , a DRC-covering of  $K_{4q+2}$  consists of 2  $C_3$  and  $2q^2 + 2q - 1$   $C_4$ .*

**Theorem 3.8.** *When  $n = 2p + 1$ ,  $p \geq 3$ ,  $\rho_4(2p + 1) = \frac{p(p+1)}{2} + 1$  ; when  $n = 4q$ ,  $q \geq 2$ ,  $\rho_4(4q) = 2q^2 + 1$ , and when  $n = 4q + 2$ ,  $q \geq 1$ ,  $\rho_4(4q + 2) = 2q^2 + 2q + 2$ .*

Let  $\rho^*(n)$  be the minimum number of directed cycles needed in a DRC-covering of  $K_n^*$  and we also define  $\rho_4^*(n)$  similarly. We have the following results.

**Theorem 4.1.** *When  $n = 2p + 1$ ,  $\rho^*(n) = p^2 + p$ . Furthermore, a DRC-covering of  $K_{2p+1}$  consists of  $2p$   $\vec{C}_3$  and  $p^2 - p$   $\vec{C}_4$ .*

**Theorem 4.2.** *When  $n = 2p$ ,  $\rho^*(n) = p^2$ . Furthermore, we have a DRC-covering of  $K_{2p}$  with  $p$   $\vec{C}_2$  and  $p^2 - p$   $\vec{C}_4$ , and another covering with  $2p$   $\vec{C}_3$  and  $p^2 - 2p$   $\vec{C}_4$ .*

**Theorem 4.3.** *When  $n = 2p + 1$ ,  $p \geq 3$ ,  $\rho_4^*(2p + 1) = p^2 + p + 2$  ; when  $n = 2p$ ,  $p \geq 4$ ,  $\rho_4^*(2p) = p^2 + 2$ .*

**Remark** Theorem 3.3 and 3.4 were presented in SPAA conference [3].

## 2 Related results

Our problem has not yet been studied in the literature. However, without the disjoint routing constraint, there are some known results for finding the covering of  $K_n$  or  $K_n^*$  by cycles. The following four results determine the covering numbers when the cycle sizes are three and four.

**Theorem 2.1** ([8, 9]) *The minimum number of 3-cycles required to cover the edges of  $K_n$  is  $\left\lceil \frac{n}{3} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil$ .*

**Theorem 2.2** ([2]) *The minimum number of 4-cycles required to cover the edges of  $K_n$  is*

$$\left\lceil \frac{n}{4} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil + \epsilon(n) \quad \text{where } \epsilon(n) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.3** ([2]) *The minimum number of  $\vec{C}_3$  required to cover the arcs of  $K_n^*$  is  $\frac{n(n-1)}{3}$  if  $n \equiv 0$  or  $1 \pmod{3}$ , except for  $n = 6$  for which it is 12, and  $\frac{n(n-1)+4}{3}$  if  $n \equiv 2 \pmod{3}$ .*

**Theorem 2.4 ([2])** *The minimum number of  $\vec{C}_4$  required to cover the arcs of  $K_n^*$  is  $\left\lceil \frac{n(n-1)}{4} \right\rceil$ ,  $n > 4$ .*

A related problem has been considered in the applications of optical networks where one wants to find the minimum number of colors (wavelengths), denoted  $w(G, I)$ , required to color the edges of the logical graph  $I$  such that the paths associated to the requests with the same color are edge (arc) disjoint in the physical graph  $G$  (see [1]). In the undirected case for  $G = C_n$ , the problem is similar as finding a DRC cycle covering of  $K_n$ ; indeed, we can associate to each cycle of a DRC-covering a color and conversely, we can build a cycle from the paths with the same color, as in  $C_n$  edge disjoint paths are also vertex disjoint. So our results give the values of  $w(C_n, K_n)$ . In the directed case,  $w(C_n^*, K_n^*)$  has been determined in [4, 11]. But our results are stronger; indeed the paths of [4, 11] with the same color are not vertex disjoint and furthermore the subnetworks induced by a color are not small cycles.

### 3 Undirected case

Let the vertices of the cycle  $C_n$  be labeled with integers modulo  $n$ , represented by the set  $\{0, 1, \dots, n-1\}$ . A  $C_k$  satisfies DRC if and only if its vertices can be ordered cyclically modulo  $n$ , that is if the vertices can be written  $(a_1, a_2, \dots, a_k)$  with  $0 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n-1$ . As an example, in Figure 2, the cycle  $(0, 2, 3, 6, 0)$  satisfies DRC, but the cycle  $(0, 4, 3, 6, 0)$  does not satisfy it.

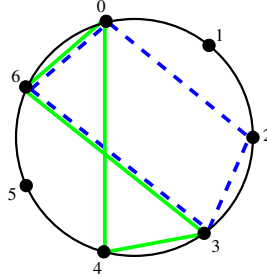


Figure 2: Disjoint Routing Constraint.

We will first give the lower bounds for  $\rho(n)$  and  $\rho_4(n)$  and then construct the coverings which attain the lower bounds. We then extend in the next section the results to the directed cases.

#### 3.1 Lower bounds

**Proposition 3.1** *When  $n = 2p + 1$ ,  $p \geq 1$ ,  $\rho(2p + 1) \geq \frac{p(p+1)}{2}$ , and when  $n = 2p$ ,  $p \geq 2$ ,  $\rho(2p) \geq \frac{p^2+1}{2}$ .*

**Proof:** Let  $C^j$ ,  $1 \leq j \leq \rho(n)$ , be the cycles of a DRC-covering of  $K_n$  (remark that the cycles do not necessarily have the same length). The disjoint routing property implies that the vertices of any  $C^j$  are cyclically ordered modulo  $n$ . Thus  $C^j$  can be written  $(a_1^j, a_2^j, \dots, a_{k_j}^j, a_1^j)$ , with  $0 \leq a_1^j \leq a_2^j \leq \dots \leq a_{k_j}^j \leq n-1$ .

Let  $\delta_i^j = a_{i+1}^j - a_i^j$ ,  $1 \leq i \leq k_j - 1$ , and  $\delta_{k_j}^j = n + a_1^j - a_{k_j}^j$ . The disjoint routing property implies  $\sum_i \delta_i^j = n$ .

For an edge  $\{x, y\}$  of  $K_n$  with  $x < y$ , we call difference of the edge the value  $y - x$  if  $y - x \leq n/2$  or  $x + n - y$  otherwise (it corresponds to the distance between  $x$  and  $y$  on a cycle of length  $n$ ).

In the odd case,  $n = 2p + 1$ , the covering must contain for every  $d$ ,  $1 \leq d \leq p$  the  $n$  edges of difference  $d$ . Each difference corresponds to a  $\delta_i^j$ , with  $\delta_i^j = d$  or  $n - d$ . Thus,  $\sum_{i,j} \delta_i^j \geq \sum_{d=1}^p nd = n \frac{p(p+1)}{2}$ . Remind that  $\sum_i \delta_i^j = n$ . Consequently, if the covering contains  $\rho(n)$  cycles, we have  $n\rho(n) \geq n \frac{p(p+1)}{2}$  and hence,  $\rho(n) \geq \frac{p(p+1)}{2}$ .

In the even case,  $n = 2p$ , the covering must contain  $n$  edges of difference  $d$ , where  $1 \leq d \leq p - 1$ , and  $\frac{n}{2} = p$  edges of difference  $p$ . Furthermore, since the degree of the nodes in  $K_n$  is odd (equal to  $n - 1$ ) and the degree of the nodes of a cycle is even (equal to 2), the covering must contain extra edges (i.e. in each vertex, there is an edge covered at least twice). Thus, there are at least  $p$  extra edges of difference at least 1 in the covering. Consequently,  $\sum_{i,j} \delta_i^j \geq \left(\sum_{d=1}^{p-1} nd\right) + pp + p = p(p^2 + 1)$  and if the covering contains  $\rho(n)$  cycles, we obtain  $n\rho(n) = 2p\rho(n) \geq p(p^2 + 1)$  and therefore,  $\rho(n) \geq \frac{p^2+1}{2}$ .  $\square$

Note that for both odd and even cases, the length of the cycles involved in a DRC-covering of  $K_n$  has no influence on the lower bound of  $\rho(n)$ . We show, in this paper, that the optimal solution can be obtained by using cycles of both length 3 and 4. Also, from Theorem 2.1, we can see that by using only cycles of length 3, it will not give the optimal solution.

**Proposition 3.2** *When  $n = 2p + 1$ ,  $p \geq 2$ ,  $\rho_4(2p + 1) \geq \frac{p(p+1)}{2} + 1$ ; when  $n = 4q$ ,  $q \geq 2$ ,  $\rho_4(4q) \geq 2q^2 + 1$ , and when  $n = 4q + 2$ ,  $q \geq 1$ ,  $\rho_4(4q + 2) \geq 2q^2 + 2q + 2$ .*

**Proof:** In the odd case,  $n = 2p + 1$ , we know by Proposition 3.1 that  $\rho_4(2p + 1) \geq \rho(2p + 1) = \frac{p(p+1)}{2}$ . If there was an equality, as  $4 \frac{p(p+1)}{2} = \frac{2p(2p+1)}{2} + p$ ,  $p$  extra edges are used because the covering is not a decomposition. So  $\sum_{i,j} \delta_i^j \geq (2p + 1) \frac{p(p+1)}{2} + p$ , as the sum of the distances of the  $p$  extra edges is at least  $p$ . Therefore,  $(2p + 1)\rho_4(2p + 1) \geq (2p + 1) \frac{p(p+1)}{2} + p$  which implies  $\rho_4(2p + 1) \geq \frac{p(p+1)}{2} + 1$ .

In the even case, for  $n = 4q$ , the bound is that of Proposition 3.1. For  $n = 4q + 2$ , by Proposition 3.1, we have  $\rho_4(4q + 2) \geq \rho(4q + 2) = 2q^2 + 2q + 1$ . If  $\rho_4(4q + 2) = 2q^2 + 2q + 1$ , then, as  $4(2q^2 + 2q + 1) = \frac{(4q+2)(4q+1)}{2} + 2q + 3$ ,  $2q + 3$  extra edges are used, and so  $\sum_{i,j} \delta_i^j \geq (2q + 1)^3 + 2q + 3$ , and  $(4q + 2)\rho_4(4q + 2) \geq (2q + 1)^3 + 2q + 3$ , which implies  $\rho_4(4q + 2) \geq 2q^2 + 2q + 1 + \frac{2}{4q+2} > 2q^2 + 2q + 1$ .  $\square$

### 3.2 Minimum DRC covering

**Theorem 3.3** *When  $n = 2p + 1$ ,  $\rho(n) = \frac{n^2-1}{8} = \frac{p(p+1)}{2}$ . Furthermore, a DRC-covering of  $K_{2p+1}$  consists of  $p$   $C_3$  and  $\frac{p(p-1)}{2}$   $C_4$ .*

**Proof:** (By induction on  $p$ )  $K_3$  is covered using one  $C_3$ . Thus, the theorem is true when  $p = 1$ .

Suppose now that the theorem is true for  $K_{2p+1}$ . We will show that it is also true for  $n = 2p + 3$ . For that, let us arrange the vertices of  $K_{2p+3}$  in the following order :  $A, 0, 1, \dots, p - 1, B, p, \dots, 2p$ .

We build a DRC-covering of  $K_{2p+3}$  from a DRC-covering of  $K_{2p+1}$  as follows. The cycles of a DRC-covering of  $K_{2p+3}$  will be

- the  $p(p+1)/2$  cycles of a DRC-covering of the  $K_{2p+1}$  on vertices  $0, 1, \dots, p-1, p, \dots, 2p$ ,
- the  $p$   $C_4$ 's of a DRC-decomposition of the  $K_{2p,2}$  constructed between vertices  $0, \dots, p-1, p+1, \dots, 2p$  on one side and vertices  $A$  and  $B$  on the other side, namely  $(A, p-1-i, B, p+1+i, A)$ ,  $0 \leq i \leq p-1$ . Note that these cycles satisfy DRC.
- the  $C_3$   $(A, B, p, A)$ .

One can check that each edge of  $K_{2p+3}$  is covered by exactly one of these cycles and altogether we have  $p(p+1)/2 + p + 1 = (p+1)(p+2)/2$  cycles. Furthermore, there are exactly  $p+1$   $C_3$ 's and  $p(p+1)/2$   $C_4$ 's.  $\square$

In Figure 3, we show a covering of  $K_5$  obtained in that way. Let us call the vertices of  $K_5$   $A, 0, B, 1, 2$  in that order. The DRC-covering of  $K_5$  consists of the unique cycle  $(0, 1, 2, 0)$  of the covering of  $K_3$ , plus the  $C_4$   $(A, 0, B, 2, A)$  of a DRC-decomposition of the  $K_{2,2}$  constructed between vertices  $A$  and  $B$  and vertices  $0$  and  $2$ , plus the  $C_3$   $(A, B, 1, A)$ .

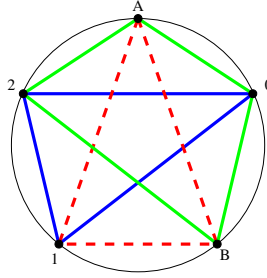


Figure 3: Covering of  $K_5$  obtained from the covering of  $K_3$ .

**Theorem 3.4** When  $n = 2p$ ,  $p \geq 2$ ,  $\rho(n) = \left\lceil \frac{n^2+4}{8} \right\rceil = \left\lceil \frac{p^2+1}{2} \right\rceil$ . Furthermore, when  $n = 4q$ ,  $q \geq 2$ , a DRC-covering of  $K_{4q}$  consists of 4  $C_3$  and  $2q^2 - 3$   $C_4$ , and when  $n = 4q+2$ ,  $q \geq 1$ , a DRC-covering of  $K_{4q+2}$  consists of 2  $C_3$  and  $2q^2 + 2q - 1$   $C_4$ .

A covering of  $K_4$  with one  $C_3$  and 2  $C_4$  is given in Fig 1. So  $\rho(4) = 3$ .

In order to prove this theorem for  $n \geq 6$ , we first need to prove some lemmas.

**Lemma 3.5**  $K_6$  can be covered by 2  $C_3$  and 3  $C_4$ .

**Proof:** The covering is given by the two  $C_3$ :  $(0, 1, 3, 0)$  and  $(0, 1, 4, 0)$ , plus three  $C_4$ :  $(0, 2, 4, 5, 0)$ ,  $(1, 2, 3, 5, 1)$  and  $(2, 3, 4, 5, 2)$ , as shown in Figure 4. Furthermore, there are three edges,  $\{0, 1\}$ ,  $\{2, 3\}$  and  $\{4, 5\}$ , covered exactly twice (they form a perfect matching).  $\square$

**Lemma 3.6** If there exists a DRC-covering of  $K_{4q+2}$  with  $\rho(4q+2) = 2q^2 + 2q + 1$  cycles, then there exists a DRC-covering of  $K_{4q+4}$  with  $\rho(4q+4) = 2q^2 + 4q + 3$  cycles.



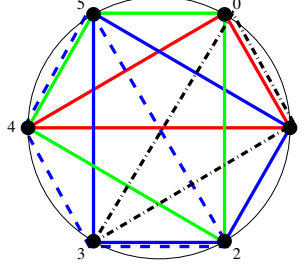


Figure 4:  $K_6$

**Proof:** Let the vertices of  $K_{4q+4}$  be  $A, 0, 1, \dots, 2q, B, 2q+1, \dots, 4q+1$  and arrange them in this order.

We build a DRC-covering of  $K_{4q+4}$  from a DRC-covering of  $K_{4q+2}$  as follows. The cycles of a DRC-covering of  $K_{4q+4}$  will be

- the  $2q^2 + 2q + 1$  cycles of a DRC-covering of the  $K_{4q+2}$  on vertices  $0, 1, \dots, 2q, 2q+1, \dots, 4q+1$ ,
- the  $2q$   $C_4$ 's of a DRC-decomposition of the  $K_{4q,2}$  constructed on vertices  $1, \dots, 2q, 2q+1, \dots, 4q$  on one side, and vertices  $A$  and  $B$  on the other side, namely  $(A, i, B, 2q+i, A)$ ,  $1 \leq i \leq 2q$ ,
- the 2 triangles  $(A, 0, B, A)$  and  $(A, B, 4q+1, A)$ .

One can check that every edge of  $K_{4q+4}$  is covered by one of these cycles and that altogether we have  $2q^2 + 2q + 1 + 2q + 2 = 2q^2 + 4q + 3 = \left\lceil \frac{(2q+2)^2 + 1}{2} \right\rceil$  cycles. Furthermore, there are exactly 4  $C_3$ 's in the covering (2 from a DRC-covering of  $K_{4q+2}$  and the 2 extra  $C_3$ 's).  $\square$

To illustrate this proof, we indicate in Figure 5 the cycles involved in the covering of  $K_8$ .

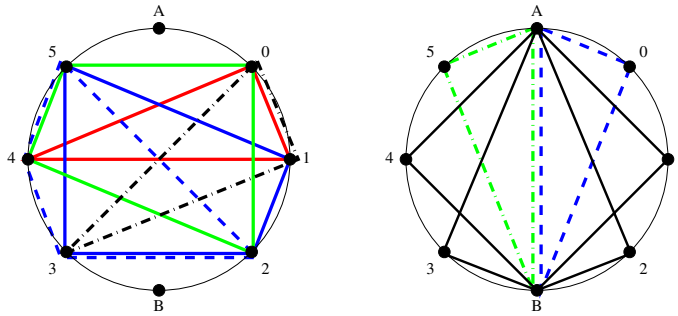


Figure 5: Cycles involved in the covering of  $K_8$ .

**Lemma 3.7** *If there exists a DRC-covering of  $K_{4q+2}$  with  $\rho(4q+2) = 2q^2 + 2q + 1$  cycles, then there exists a DRC-covering of  $K_{4q+6}$  with  $\rho(4q+6) = 2q^2 + 6q + 5$  cycles.*

**Proof:** We will prove a stronger lemma, imposing some extra properties in the decomposition which will be kept in the construction.

Let us suppose that there exists a DRC-covering of  $K_{4q+2}$ , where the nodes  $0, 1, \dots, 4q+1$  are cyclically ordered, with the following properties :

- the edges (of the perfect matching)  $(0, 1), (2, 3), \dots, (4q, 4q+1)$  are covered exactly twice, while other edges are covered exactly once.
- the edge  $\{0, 1\}$  belongs to the  $C_3(0, 1, x, 0)$ , for some  $x$  different from  $0, 1$ .

We will show that there exists a DRC-covering of  $K_{4q+6}$  with the same properties. Note that these properties are satisfied by the covering of  $K_6$  of Lemma 3.5 (with  $x$  being either 3 or 4).

Let the vertices of  $K_{4q+6}$  be  $0, A, B, 1, \dots, 2q+1, C, D, 2q+2, \dots, 4q+1$  and arrange them in this order. The cycles of a DRC-covering of  $K_{4q+6}$  will be

- the  $2q^2 + 2q$  cycles of the covering of  $K_{4q+2}$  except the  $C_3(0, 1, x, 0)$ ,
- the  $2q$   $C_4$ 's  $(A, i, C, f(i), A)$ , with  $2 \leq i \leq 2q+1$  and where  $f$  is a bijection from  $\{2, 3, \dots, 2q+1\}$  to  $\{2q+2, \dots, 4q+1\}$ ,
- the  $2q+1$   $C_4$ 's  $(B, j, D, g(j), B)$ ,  $1 \leq j \leq 2q+1$ , and where  $g$  is a bijection from  $\{1, 2, \dots, 2q+1\}$  to  $\{2q+2, \dots, 4q+1, 0\}$ .
- the 3  $C_4$ 's  $(A, B, C, D, A)$ ,  $(0, A, 1, x, 0)$ ,  $(B, 1, C, D, B)$  and the  $C_3(0, A, C, 0)$ ,

One can check that each edge of  $K_{4q+6}$  is covered by one of these cycles and that altogether, we have  $2q^2 + 2q + 2q + 2q + 1 + 3 + 1 = 2q^2 + 6q + 5 = \left\lceil \frac{(2q+2)^2 + 1}{2} \right\rceil$  cycles. Furthermore, there are still exactly 2  $C_3$ 's in the covering. Also, the edges  $\{0, A\}, \{B, 1\}, \{2, 3\}, \dots, \{2q, 2q+1\}, \{C, D\}, \{2q+2, 2q+3\}, \dots, \{4q, 4q+1\}$  (corresponding to a perfect matching) are covered twice, while other edges are covered only once. Moreover, the edge  $\{0, A\}$ , which is covered twice, appears in the  $C_3(0, A, C, 0)$ . Let us relabel the vertices of  $K_{4q+6}$  as follows :  $A$  (resp.  $B$ ) becomes 1 (resp. 2),  $i$ , for  $1 \leq i \leq 2q+1$ , becomes  $i+2$ ,  $C$  (resp.  $D$ ) becomes  $2q+4$  (resp.  $2q+5$ ), and  $j$ ,  $2q+2 \leq j \leq 4q+1$  becomes  $j+4$ . Therefore,  $K_{4q+6}$  satisfies the induction properties.  $\square$

**Remark** For the proof of Theorem 3.8, we note that the two  $C_3$ 's of the decomposition of  $K_6$  are  $(0, 1, 3, 0)$  and  $(0, 1, 4, 0)$ . So assuming that  $x = 3$  and after  $q-1$  steps of induction, the two  $C_3$ 's of  $K_{4q+2}$ , if  $q \geq 2$ , become  $(0, 1, 2q+2, 0)$  and  $(0, 2q-1, 4q, 0)$ . Furthermore, we have a cycle  $(1, i_0, 2q+2, 4q, 1)$  (corresponding to the  $C_4(A, i, C, f(i), A)$  with  $f(i_0) = 4q$  in above construction).

Now, we are able to prove Theorem 3.4.

**Proof of Theorem 3.4 :** (By induction)

The theorem is true for  $n = 6$  as shown by Lemma 3.5. Note that the covering of  $K_6$  satisfies the two extra properties needed in the proof of Lemma 3.7. So using Lemma 3.7, one can build by induction a DRC-covering of  $K_{4q+2}$ ,  $q \geq 1$ , by  $\rho(4q+2) = 2q^2 + 2q + 1$  cycles. Then, using Lemma 3.6, one can build a DRC-covering of  $K_{4q+4}$ ,  $q \geq 1$ , by  $\rho(4q+4) = 2q^2 + 4q + 3$  cycles.

So Theorem 3.4 is proved.  $\square$

**Theorem 3.8** When  $n = 2p + 1$ ,  $p \geq 3$ ,  $\rho_4(2p + 1) = \frac{p(p+1)}{2} + 1$  ; when  $n = 4q$ ,  $q \geq 2$ ,  $\rho_4(4q) = 2q^2 + 1$ , and when  $n = 4q + 2$ ,  $q \geq 1$ ,  $\rho_4(4q + 2) = 2q^2 + 2q + 2$ .

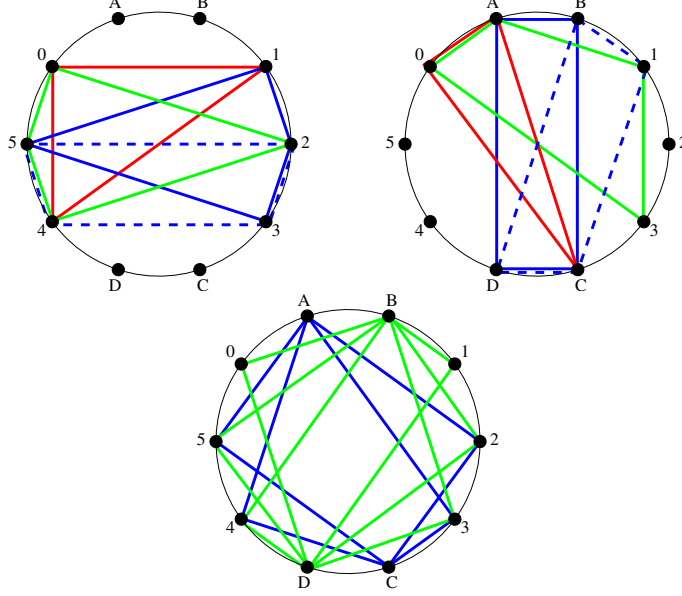


Figure 6: Cycles involved in the covering of  $K_{10}$ .

**Proof:**

**Case 1 :**  $n = 2p + 1$ . The proof is similar to that of Theorem 3.3, and by induction, we will prove that there exists a  $C_4$ -covering of  $K_{2p+1}$  with vertices labeled  $0, 1, \dots, 2p$ , containing a  $C_4$  of the form  $(0, p - 1, p, p + 1, 0)$  and the edge  $\{p - 1, p\}$  covered twice.

Note that it is true for  $p = 3$  as we have a covering of  $K_7$  with the 7  $C_4$ 's  $(i, i + 1, i + 2, i + 5, i)$ ,  $0 \leq i \leq 6$ , containing in particular the  $C_4$   $(2, 3, 4, 0, 2)$  and with the edge  $\{2, 3\}$  covered twice. Suppose that the induction hypothesis is true for  $K_{2p+1}$ . We will show that it is true for  $K_{2p+3}$ . Let the vertices of  $K_{2p+3}$  be  $A, 0, 1, \dots, p - 1, B, p, \dots, 2p$  and arrange them in this order. The  $C_4$ 's of a DRC-covering of  $K_{2p+3}$  will be :

- the  $\frac{p(p+1)}{2}$   $C_4$ 's of the covering of  $K_{2p+1}$ , except the  $C_4$   $(0, p - 1, p, p + 1, 0)$ ,
- the  $p - 1$   $C_4$ 's of a DRC decomposition of  $K_{2p-2,2}$ , constructed between vertices  $0, 1, \dots, p - 2, p + 2, \dots, 2p$  on one side, and the vertices  $A$  and  $B$  on the other side, namely the  $C_4$ 's  $(A, p - 1 - i, B, p + 1 + i, A)$ ,  $1 \leq i \leq p - 1$ ,
- the 3  $C_4$ 's  $(0, p - 1, B, p + 1, 0)$ ,  $(A, p - 1, B, p, A)$  and  $(A, B, p, p + 1, A)$ .

One can check that all the edges are covered and there are altogether  $\frac{p(p+1)}{2} + p - 1 + 3 = \frac{(p+1)(p+2)}{2} + 1$   $C_4$ 's. The covering contains the cycle  $(A, p - 1, B, p, A)$  which becomes, if we label the vertices of  $K_{2p+3}$  from 0 to  $2p + 2$ , the cycle  $(0, p, p + 1, p + 2, 0)$  ; furthermore the edge  $\{p - 1, B\}$  which becomes  $\{p, p + 1\}$  after relabeling, is covered twice.

**Case 2 :**  $n = 4q + 2$ ,  $q \geq 1$ . Note that in the proof of Lemma 3.7, we obtain a covering of  $K_{4q+2}$  with  $2q^2 + 2q - 1$   $C_4$ 's and two  $C_3$ 's which are (see remark)  $(0, 1, 2q + 2, 0)$  and  $(0, 2q - 1, 4q, 0)$ . Delete these  $C_3$ 's and consider the 3  $C_4$ 's  $(0, 1, 2q + 2, 4q, 0)$ ,  $(0, 2q - 1, 4q, 4q + 1, 0)$  and  $(0, \alpha, \beta, 2q + 2, 0)$ , with  $0 < \alpha < \beta < 2q + 2$ . They contain all the edges of the deleted  $C_3$ 's. Thus, we have a covering of  $K_{4q+2}$  with  $2q^2 + 2q - 1 + 3 = 2q^2 + 2q + 2$   $C_4$ 's.

**Case 3 :**  $n = 4q + 4$ . We assume that  $q > 1$ , and let us prove that  $\rho_4(4q + 4) = 2q^2 + 4q + 3$ . Like in the proof of Lemma 3.6, let the vertices be  $A, 0, 1, \dots, 2q, B, 2q + 1, \dots, 4q + 1$ . The proof of Lemma 3.6 also gives a decomposition of  $K_{4q+4}$  into the  $2q^2 + 2q - 1$   $C_4$ 's plus the 2  $C_3$ 's of the covering of  $K_{4q+2}$ , plus  $2q + 1$   $C_4$ 's of the decomposition of the  $K_{4q+2,2}$  (with vertices  $0, 1, \dots, 4q + 1$  on one side and  $A, B$  on the other side), and plus the edge  $\{A, B\}$ . Using the remark of the proof of Lemma 3.7, we have a decomposition with  $2q^2 + 4q$   $C_4$ 's plus the edge  $\{A, B\}$ , plus the  $C_3$ 's  $(0, 1, 2q + 2, 0)$  and  $(0, 2q - 1, 4q, 0)$ , and we know that one  $C_4$  is of the form  $(1, i_0, 2q + 2, 4q, 1)$ . Let us replace the edge  $\{A, B\}$ , the 2  $C_3$ 's and this  $C_4$  by the following 4  $C_4$ 's which cover all their edges :  $(A, 0, 2q - 1, B, A)$ ,  $(0, 1, 2q + 2, 4q, 0)$ ,  $(0, 1, i_0, 2q + 2, 0)$  and  $(1, \alpha, 2q - 1, 4q, 1)$ , for some  $1 < \alpha < 2q - 1$ , for example  $\alpha = 2$  (which supposes  $q > 1$ ). Hence, if  $q \geq 2$ , we have a covering with  $2q^2 + 4q + 3$  cycles.

Finally, if  $q = 1$  a covering of  $K_8$  is given by the following 9  $C_4$ 's :  $(i, i + 1, i + 4, i + 5, i)$ ,  $0 \leq i \leq 3$ ,  $(0, 1, 2, 6, 0)$ ,  $(1, 2, 4, 5, 1)$ ,  $(0, 2, 3, 4, 0)$ ,  $(3, 4, 6, 7, 3)$  and  $(1, 3, 5, 7, 1)$ . Note that the edges  $\{0, 1\}$  and  $\{4, 5\}$  are covered twice.  $\square$

**Remark** For  $n = 5$ ,  $\rho_4(5) = 5$ . Indeed in any  $C_4$ , there is exactly one edge  $\{i, j\}$  such that  $\delta_i^j = 2$  since the only way to have  $\sum_i \delta_i^j = 5$  is to have 3 edges such that  $\delta_i^j = 1$  and one with  $\delta_i^j = 2$  ; therefore,  $\rho_4(5) \geq 5$ . Finally a covering of  $K_5$  is given by the 5  $C_4$ 's :  $(i, i + 1, i + 2, i + 4, i)$ ,  $0 \leq i \leq 4$ .

## 4 Directed Case

We will now denote by  $\rho^*(n)$  the minimum number of directed cycles needed in a DRC-covering of  $K_n^*$ .

**Theorem 4.1** *When  $n = 2p + 1$ ,  $\rho^*(n) = p^2 + p$ . Furthermore, a DRC-covering of  $K_{2p+1}^*$  consists of  $2p$   $\vec{C}_3$  and  $p^2 - p$   $\vec{C}_4$ .*

**Proof:** A proof similar to that of Theorem 3.3 gives  $\rho^*(n) \geq p^2 + p$ . Given a DRC-covering of  $K_{2p+1}$ , obtained in Theorem 3.3 using  $p$   $C_3$ 's and  $\frac{p^2-p}{2}$   $C_4$ 's, we replace each  $C_4$  by two opposite  $\vec{C}_4$  and each  $C_3$  by two opposite  $\vec{C}_3$ . Thus, we obtain a DRC-covering using exactly  $p^2 + p$  directed cycles, consisting of  $2p$   $\vec{C}_3$  and  $p^2 - p$   $\vec{C}_4$ .  $\square$

**Theorem 4.2** *When  $n = 2p$ ,  $\rho^*(n) = p^2$ . Furthermore, we have a DRC-covering of  $K_{2p}^*$  with  $p$   $\vec{C}_2$  and  $p^2 - p$   $\vec{C}_4$ , and another covering with  $2p$   $\vec{C}_3$  and  $p^2 - 2p$   $\vec{C}_4$ .*

**Proof:** Using a similar proof to that of Proposition 3.1, one can check that the lower bound is  $\rho^*(2p) \geq p^2$  (we have no more the degree condition).

We will prove Theorem 4.2 by induction on  $p$ . First, a DRC-covering of  $K_4^*$  is given by the 4  $\vec{C}_3$   $(0, 1, 2, 0)$ ,  $(0, 2, 3, 0)$ ,  $(0, 3, 1, 0)$  and  $(1, 3, 2, 1)$ , and another is given by the 2  $\vec{C}_4$   $(0, 1, 2, 3, 0)$  and  $(0, 3, 2, 1, 0)$  plus the 2  $\vec{C}_2$   $(0, 2, 0)$  and  $(1, 3, 1)$ .

Now, suppose that  $\rho^*(2p) = p^2$  and let us prove that  $\rho^*(2p + 2) = (p + 1)^2$ . Let the vertices of  $K_{2p+2}^*$  be  $A, 0, 1, \dots, p - 1, B, p, \dots, 2p - 1$  and arrange them in this order. A directed DRC-covering is given by the  $p^2$  cycles of the covering of  $K_{2p}^*$ , plus the  $2p$   $\vec{C}_4$   $(A, i, B, p + i, A)$  and  $(A, p + i, B, i, A)$ ,  $0 \leq i \leq p - 1$ , and plus the  $\vec{C}_2$   $(A, B, A)$ . Thus, we have a covering of  $K_{2p+2}^*$  with

$p^2 + 2p + 1 = (p + 1)^2$  cycles. If the  $p^2$  cycles of the covering of  $K_{2p}^*$  are  $p \vec{C}_2$  and  $p^2 - p \vec{C}_4$ , we obtain a covering of  $K_{2p+2}^*$  with  $p + 1 \vec{C}_2$  and  $p^2 + p \vec{C}_4$ .

Furthermore, one can replace the  $\vec{C}_4 (A, 0, B, p, A)$  and the  $\vec{C}_2 (A, B, A)$  by the  $2 \vec{C}_3 (A, 0, B, A)$  and  $(B, p, A, B)$  to obtain a covering without  $\vec{C}_2$ , if we start from a covering of  $K_{2p}^*$  without  $\vec{C}_2$ .  $\square$

**Theorem 4.3** *When  $n = 2p + 1$ ,  $p \geq 3$ ,  $\rho_4^*(2p + 1) = p^2 + p + 2$ ; when  $n = 2p$ ,  $p \geq 4$ ,  $\rho_4^*(2p) = p^2 + 2$ .*

**Proof: Case 1 :**  $n = 2p + 1$ . Suppose that  $\rho_4^*(n) = p^2 + p + 1$ . Then, as  $4(p^2 + p + 1) = 2p(2p + 1) + 2p + 4$ ,  $2p + 4$  extra edges are used and so  $\sum_{i,j} \delta_i^j \geq (2p + 1)p(p + 1) + 2p + 4$ , which implies  $\rho_4^*(n) \geq p(p + 1) + \frac{2p+4}{2p+1} > p^2 + p + 1$  (recall that the sum of the distances over each cycle is  $2p + 1$ ).

A covering with  $p^2 + p + 2 \vec{C}_4$ 's can be deduced from that of Theorem 3.8 by replacing each  $C_4$  of this covering by two opposite  $\vec{C}_4$ .

**Case 2 :**  $n = 2p$ . Suppose that  $\rho_4^*(n) = p^2 + 1$ . Then, as  $4(p^2 + 1) = 2p(2p - 1) + 2p + 4$ ,  $2p + 4$ extra edges are used and so  $\sum_{i,j} \delta_i^j \geq 2p(p - 1)p + 2p^2 + 2p + 4$ , which implies  $\rho_4^*(n) \geq p^2 + \frac{2p+4}{2p} > p^2 + 1$ .

A covering of  $K_8^*$  by 18  $\vec{C}_4$ 's is deduced from that of  $K_8$  by 9  $C_4$ 's, given in the proof of Theorem 3.8, by replacing each  $C_4$  of this covering by two opposite  $\vec{C}_4$ . Note that this covering contains the  $\vec{C}_4 (0, 1, 4, 5, 0)$ , which is of the form  $(0, 1, p, p + 1, 0)$ , and that the arcs  $(0, 1)$  and  $(4, 5)$  are covered twice.

We will prove by induction that there exists a covering of  $K_{2p}^*$  with  $p^2 + 2$  directed cycles containing the  $\vec{C}_4 (0, 1, p, p + 1, 0)$  and with the arcs  $(0, 1)$  and  $(p, p + 1)$  covered twice. That is true for  $2p = 8$  as we have seen before. Suppose that it is true for  $K_{2p}^*$  and let the vertices of  $K_{2p+2}^*$  be  $0, A, 1, \dots, p, B, p + 1, \dots, 2p - 1$  and arrange them in this order. A  $\vec{C}_4$ 's DRC-covering of  $K_{2p+2}^*$  is given by the  $p^2 + 1$  cycles of the covering of  $K_{2p}^*$  excluding the  $\vec{C}_4 (0, 1, p, p + 1, 0)$ , plus the  $2p \vec{C}_4 (A, i, B, p + i, A)$  and  $(A, p + i, B, i, A)$ ,  $1 \leq i \leq p$ , and plus the  $2 \vec{C}_4 (A, 1, p, B, A)$  and  $(B, p + 1, 0, A, B)$ . Thus, we have a covering of  $K_{2p+2}^*$  with  $p^2 + 1 + 2p + 2 = p^2 + 2p + 3 = (p + 1)^2 + 2$  cycles. Furthermore, this covering contains the cycle  $(0, A, B, p + 1, 0)$  which becomes after relabeling  $(0, 1, p + 1, p + 2, 0)$ , and the arcs  $(0, A)$  and  $(B, p + 1)$  are covered twice.  $\square$

Note that the above covering of  $K_n^*$  also gives an optimal covering of  $2K_n$  as the lower bound was derived using  $2K_n$ .

## 5 Conclusion

The problem of the design of a survivable WDM network was considered as an extension of the classical edge covering problem by adding of the disjoint routing constraint. In particular, we have studied the case of a physical ring network with the all-to-all ( $K_n$ ) communication instance. For this design problem, we give a solution with the optimal number of cycles as sub-networks.

Recently, in [5], we have extended the results to  $G$  being a torus (instead of  $C_n$ ). It will be interesting to look further into other WDM networks like tree of rings and grids. One can also consider other communications instances like  $\lambda K_n$ ,  $\lambda K_{m,n}$  or general logical graphs.

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