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Abstract We address the problem of traffic grooming in WDM rings with all-to-all uniform unitary traffic. We want to minimize the total number of SONET add-drop multiplexers (ADMs) required. This problem corresponds to a partition of the edges of the complete graph into subgraphs, where each subgraph has at most C edges (where C is the grooming ratio) and where the total number of vertices has to be minimized. Using tools of graph and design theory, we optimally solve the problem for practical values and infinite congruence classes of values for a given C. Among others, we give optimal constructions when $C \geq N(N-1)/6$ and results when $C = 12$. We also show how to improve lower bounds by using refined counting techniques, and how to use efficiently an ILP program by restricting the search space.

Keywords: Traffic grooming, graph, design theory, WDM rings.

1. Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys Dutta and Rouskas, 2002b; Modiano and Lin, 2001; Somani, 2001). By using traffic grooming, one can bypass the electronics in the nodes for which there is no traffic sourced or destinated to it and therefore reduce the cost of the network. Typically, in a WDM (Wavelength Division Multiplexing) network, instead of having one SONET Add Drop Multiplexer (ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelength used at that node (the other wavelengths being optically routed without electronic switching).

This problem is different from that of minimizing the transmission cost and in particular the number of wavelengths to be used considered by many authors (see the surveys Beauquier et al., 1997; Dutta and Rouskas, 2000). Indeed, it is known that even for the simpler network which is the unidirectional ring, the number of wavelengths and the number of ADMs cannot be simultaneously minimized (see Gerstel et al., 1998, or Chiu and Modiano, 2000 for uniform traffic).

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Here, we consider the particular case of unidirectional rings (the routing is unique) with static uniform symmetric all-to-all traffic (there is exactly one request of a given size from i to j for each couple (i, j)) and with no possible wavelength conversion.

In that case, for each pair $\{i, j\}$, we associate a circle (or circuit) which contains both the request from i to j and from j to i. If each circle requires only $\frac{1}{C}$ of the bandwidth of a wavelength, we can "groom" C circles on the same wavelength. C is called the *grooming ratio* (or grooming factor). For example, if the request from i to j (and from j to i) is one OC-12 and a wavelength can carry an OC-48, the grooming factor is 4. Given the grooming ratio C and the size N of the ring, the objective is to minimize the total number of (SONET) ADMs used, denoted $A(C, N)$, and so reducing the network cost by eliminating as many ADMs as possible from the "no grooming case".

For example, let $N = 4$; we have 6 circles corresponding to the 6 pairs $\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$. If we don't use grooming, that is if we assign one wavelength per circle, we need 2 ADMs per circle, and thus a total of 12. Suppose now that $C = 4$, that is we can groom 4 circles on one wavelength. One can groom on wavelength 1 the circles associated with $\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 0\}$ requiring 4 ADMs and on wavelength 2 the circles associated with $\{0, 2\}$ and $\{1, 3\}$ requiring 4 ADMs and so a total of 8. A better way is to groom the circles associated with $\{0, 1\}, \{0, 2\}, \{0, 3\}$ using 4 ADMs and those associated with $\{1, 2\}, \{1, 3\}, \{2, 3\}$ using 3 ADMs for a total of 7 ADMs.

Another interesting example is with $N = 9$. We have $R = 36$ circles. Without grooming, we need $A(1, 9) = 72$ ADM's and for grooming factors $C = 3, 12, 36$ we need respectively, $A(3, 9) = 36$, $A(12, 9) = 18$, and $A(36, 9) = 9$ ADM's. For $C = 36$, we groom all the circles on one wavelength. For $C = 12$, let the vertex set be $A_1 \cup A_2 \cup A_3$ with $|A_i| = 3$. $A_i = \{a_i^j,\ j = 1, 2, 3\}$. We can groom on wavelength i, $i = 1, 2, 3$, the 3 circles $\{a_i^j, a_i^{j+1}\}$ and the 6 circles $\{a_i^j, a_{i+1}^k\}$ where all the indices are taken modulo 3. So wavelength i use only 6 ADMs. For $C = 3$, we groom the circles in 12 wavelengths each containing 3 circles of type $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$. Thus, by increasing the grooming factor, we significantly reduce the total amount of ADM's in the network.

The case we consider has been considered by many authors (Chiu and Modiano, 2000; Dutta and Rouskas, 2002a; Gerstel et al., 1998; Gerstel et al., 2000; Hu, 2002; Wan et al., 2000; Wang et al., 2001; Yuan and Fulay, 2002; Zhang and Qiao, 1996; Zhang and Qiao, 2000) and numerical results, heuristics and tables have been given (see for example those in Wang et al., 2001). It presents the advantage of concentrating on the grooming phase (excluding the routing). It can also be applied to groom components of more general connections than two opposite pairs into wavelengths or more general classes. These components are called circles (Chiu and Modiano, 2000; Zhang and Qiao, 2000) or circuits (Wang et al., 2001) or primitive rings (Colbourn and Ling, ; Colbourn and Wan, 2001).

In Bermond and Coudert, we have shown that the problem of minimizing the number of ADMs for the unidirectional ring C_N with a grooming factor C can be expressed as follows: partition the edges of the complete graph on N vertices (K_N) into W subgraphs B_λ, $\lambda = 1, 2, \ldots, W$, having $|E(B_\lambda)|$ edges and $|V(B_\lambda)|$ vertices with $|E(B_\lambda)| \leq C$ and where $\sum_{\lambda=1}^{W} |V(B_\lambda)|$ has to be minimized (the edges of K_N correspond to the circles, the subgraphs B_λ correspond to the wavelengths and a vertex of B_λ corresponds to an ADM).

In Bermond and Coudert, we have also shown the importance of choosing graphs B_λ in the partition with the best ratio $\frac{|E(B_\lambda)|}{|V(B_\lambda)|}$ (see section 1.3). Indeed, if we denote by $\rho_{\max}(C)$ the maximum ratio among all graphs with at most C edges, we have the following lower bound on the minimum number $A(C, N)$ of ADMs: $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

We have also shown using tools of design theory that this lower bound is attained for a given C when N is large enough. That enables to show that the minimum number of ADMs, $A(C, N)$, for unidirectional rings with uniform unitary traffic is not necessarily obtained using the minimum number of wavelengths, disproving conjectures of Chiu and Modiano, 2000 for many values of C (the first one being $C = 7$) and of Hu, 2002 for $C = 16$. For the sake of completeness, these results are recalled in section 1.3.

Here we concentrate our efforts on small values of N giving the exact values of $A(C,N)$ when $C \geq \frac{N(N-1)}{6}$. We also show how to improve lower bounds by using refined counting techniques. For upper bounds we show how to use efficiently design tools to determine $A(4,N)$ ($= \frac{N(N-1)}{2}$ for $N \geq 5$), a result also obtained in Hu, 2002 but our proof is much shorter. We also give results for $C = 12$. Table 1 gives the values of $A(C,N)$ for $N \leq 16$ and some values of C as the table in Wang et al., 2001.

$C \setminus N$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	3	7	12	17	21	31	36	48	57	69	78	95	105	124
4	3	7	10	15	21	28	36	45	55	66	78	91	105	120
12	3	4	5	9	12	16	18	24	30	35	39	47	55-56	60
16	3	4	5	6	11	14	18	20	26	32	36	41	45	53-54
48	3	4	5	6	7	8	9	10	16	19	22	24	30	32
64	3	4	5	6	7	8	9	10	11	15	19	22	25	28

Table 1. $A(C,N)$ for $N \leq 16$ and $C = 3, 4, 12, 16, 48, 64$

2. Notation and reformulation of the problem

We precise here our notation and show how the problem can be formulated in terms of graph partitioning. Although we restrict ourselves to the case of unidirectional rings with uniform static unit traffic, the ideas can be applied to other situations.

- N will denote the number of node of the unidirectional ring $\overrightarrow{C}_N$
- For the unidirectional ring with symmetric traffic, $C_{\{i,j\}}$ will denote a *circle* associated to the pair $\{i,j\}$, that is containing both an unitary request from i to j and from j to i. So $C_{\{i,j\}}$ uses all the arcs of $\overrightarrow{C}_N$.
- R the total number of circles. In the case of unidirectional rings, with uniform unitary traffic, each pair $\{i,j\}$ is associated to a unique circle $C_{\{i,j\}}$ and thus $R = \frac{N(N-1)}{2}$.
- C the grooming ratio (or grooming factor). In Chiu and Modiano, 2000, C indicates the number of circles a wavelength can contain. Similarly, $\frac{1}{C}$ indicates the part of the bandwidth of a wavelength that can be used by a circle. For example, if a wavelength is running at the line rate of OC-N, it can carry $C = \frac{N}{M}$ low speed OC-M. Typical values of C are $C = 3, 4, 8, 12, 16, 48, 64$.
- Let K_N be the complete graph on N vertices where there is an edge $\{i,j\}$ for each pair of vertices $\{i,j\}$; let C_N be the undirected cycle with N nodes.
- B_λ will denote a subgraph of K_N. $V(B_\lambda)$ (resp. $E(B_\lambda)$) denote its vertex (resp. edge) set. In the example of the introduction, B_λ corresponds to a wavelength ; an edge $\{i,j\}$ of B_λ corresponds to a circle $C_{\{i,j\}}$. So a subgraph can be viewed as the set of circles packed in the wavelength. The grooming factor implies that $|E(B_\lambda)| \leq C$. $V(B_\lambda)$ corresponds to the number of (SONET) ADMs used in the wavelength λ ; indeed we have to use an ADM in all the vertices appearing in a circle $C_{\{i,j\}}$ packed in the wavelength λ.

So, the original problem of minimizing the total number $A(C,N)$ of ADMs in a grooming with grooming ratio C, in the unidirectional ring $\overrightarrow{C}_N$ with unitary static uniform traffic, can be stated as follows.

PROBLEM 1 (ADM)

Table 2. Values of $\rho_{\max}(C)$ for small C

C	1	2	3	4	5	6	7	8	9	10
$\rho_{\max}(C)$	$\frac{1}{2}$	$\frac{2}{3}$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	2
C	11	12	13	14	15	16	24	32	48	64
$\rho_{\max}(C)$	2	2	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{5}{2}$	$\frac{5}{2}$	3	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{64}{11}$

Inputs : *a number of nodes N and a grooming ratio C*
Output : *a partition of the edges of K_N into subgraphs B_λ, $\lambda = 1, \dots, W$, such that $|E_\lambda| \leq C$*
Objective : *minimize $\sum_{1 \leq \lambda \leq W} |V_\lambda|$*

Remark: As we said in the introduction, most interest has focused on a different objective function which was to minimize the number W of subgraphs (wavelengths) of the partition. This is an easy problem in this context since $W_{\min} = \lceil \frac{R}{C} \rceil = \left\lceil \frac{N(N-1)}{2C} \right\rceil$.

3. General bounds

3.1 Maximum ratio $\rho_{\max}(C)$

Let $\rho(B_\lambda)$ denote the ratio of a subgraph B_λ, $\rho(B_\lambda) = \frac{|E(B_\lambda)|}{|V(B_\lambda)|}$, and $\rho(m)$ the maximum ratio of a subgraph with m edges. Let $\rho_{\max}(C)$ denote the maximum ratio of subgraphs with $m \leq C$ edges. We have $\rho_{\max}(C) = \max\{\rho(B_\lambda) \mid |E(B_\lambda)| \leq C\} = \max_{m \leq C} \rho(m)$.

$\rho_{\max}(C)$ is given by the following proposition (see Bermond and Coudert, for a proof).

PROPOSITION 2 (BERMOND AND COUDERT,) *If $\frac{k(k-1)}{2} \leq C \leq \frac{(k+1)(k-1)}{2}$, then $\rho_{\max}(C) = \frac{k-1}{2}$ and the value is attained for K_k.*

If $\frac{(k+1)(k-1)}{2} \leq C \leq \frac{(k+1)k}{2}$, then $\rho_{\max}(C) = \frac{C}{k+1}$ and the value is attained for any graph with C edges and $k+1$ vertices.

For the sake of illustration, Table 2 gives the values of $\rho_{\max}(C)$ for small values of C.

3.2 Lower bound

THEOREM 3 *Any grooming of R circles with a grooming factor C needs at least $\frac{R}{\rho_{\max}(C)}$ ADMs.*

Proof: We have $R = \sum_{\lambda=1}^{W} |E(B_\lambda)| \leq \rho_{\max}(C) \sum_{\lambda=1}^{W} |V(B_\lambda)|$. □

In particular, we get the following lower bound

THEOREM 4 (LOWER BOUND) $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

Because of Theorem 4, subgraphs with a ratio equal to $\rho_{\max}(C)$ should be chosen when possible. Note that according to Proposition 2, these subgraphs do not have necessarily exactly C edges and so the minimum is not necessarily attained for $W = W_{\min}$.

For example, let $C = 7$. If a subgraph has 7 edges, its ratio is at most $\frac{7}{5} = 1.4$. But a subgraph with 6 edges can have a ratio $\frac{6}{4} = 1.5$ (and this is attained for K_4). Any other subgraph has a ratio at most $\frac{5}{4}$. So, in an optimal solution for the number of ADMs, we have to use K_4's as subgraphs of the partition and not subgraphs with 7 edges and 5 vertices. But in a solution minimizing the number of wavelengths, we have in contrary to use these last ones. Using that, we were able in Bermond and Coudert, to give counterexamples to a conjecture of Chiu and Modiano, 2000.

PROPOSITION 5 (BERMOND AND COUDERT,) *The conjecture of Chiu and Modiano, 2000 that the minimum number of ADMs, $A(C, N)$, for unidirectional rings $\overrightarrow{C}_N$ with uniform unitary traffic is obtained for $W = W_{\min} = \left\lceil \frac{N(N-1)}{2C} \right\rceil$, is false.*

3.3 Upper bound and optimal results

Our problem looks similar to design theory. Indeed an $(N, k, 1)$-design is nothing else than a partition of the edges of K_N into subgraphs isomorphic to K_k called blocks in this theory. That corresponds to impose in our partitioning problem that all the subgraphs B_λ are isomorphic to K_k. Note that the classical equivalent definition is : given a set of N elements, find a set of blocks such that each block contains k elements and each pair of elements appears in exactly one block (see the handbook Colbourn and Dinitz, 1996).

More generally, a G-design of order N (see Colbourn and Dinitz, 1996 chap. 22 or Bermond et al., 1980 or Bermond and Sotteau, 1975) consists on a partition of the edges of K_N into subgraphs isomorphic to a given graph G. The interest of the existence of a G-design is shown by the following immediate proposition.

PROPOSITION 6 *If there exists a G-design of order N, where G is a graph with at most C edges and ratio $\rho_{\max}(C)$, then $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.*

NECESSARY CONDITIONS 7 (EXISTENCE OF A G-DESIGN) *If there exists a G-design, then*

- *(i)* $\frac{N(N-1)}{2}$ *should be a multiple of $E(G)$*
- *(ii)* $N - 1$ *should be a multiple of the greatest common divisor of the degrees of the vertices of G.*

Wilson, 1976 has shown that these necessary conditions are also sufficient for large N. From that, we obtain

THEOREM 8 *Given C, for an infinite number of values of N, $A(C, N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.*

Unfortunately, the values of N for which Wilson's Theorem applies are very large. However, for small values of C, we can use exact results of design theory. For example, from the existence of a G-design for $G = K_3, K_3 + e, K_4 - e, K_4, K_5 - 3e, K_5 - 2e, K_5 - e, K_5$ and K_6, where $K_p - \alpha e$ (resp. $K_p + \alpha e$) denotes the graph obtained from K_p by deleting (resp. adding) α edges, we obtain

THEOREM 9

- $A(3, N) = \frac{N(N-1)}{2}$ *when* $N \equiv 1$ *or* 3 (*mod* 6)
- $A(4, N) = \frac{N(N-1)}{2}$ *when* $N \equiv 0$ *or* 1 (*mod* 8)
- $A(5, N) = \frac{2N(N-1)}{5}$ *when* $N \equiv 0$ *or* 1 (*mod* 10)
- $A(6, N) = A(7, N) = \frac{N(N-1)}{3}$ *when* $N \equiv 1$ *or* 4 (*mod* 12)
- $A(8, N) = \frac{5N(N-1)}{16}$ *when* $N \equiv 0$ *or* 1 (*mod* 16)
- $A(9, N) = \frac{5N(N-1)}{18}$ *when* $N \equiv 0$ *or* 1 (*mod* 18)
- $A(10, N) = \frac{N(N-1)}{4}$ *when* $N \equiv 1$ *or* 5 (*mod* 20)
- $A(16, N) = \frac{N(N-1)}{5}$ *when* $N \equiv 1$ (*mod* 30)

4. Determination of $A(C,N)$ for $R/3 \le C$

LEMMA 10 *For all* $N \ge 2$*, we have* $A(C,N) \ge A(C+1,N)$*. Furthermore* $A(1,N) = N(N-1)$ *and* $A\left(\frac{N(N-1)}{2}, N\right) = N$.

Proof: When $C = 1$, each subgraph contains 1 circle and 2 ADMs, and thus, $A(1,N) = N(N-1)$. On the other hand, when $C = \frac{N(N-1)}{2}$ all circles fit in the same subgraph and $A\left(\frac{N(N-1)}{2}, N\right) = N$. Finally, it is clear that $A(C,N)$ is an upper bound for $A(C+1,N)$. □

We will now show that except two particular cases $A(C,N) \le 2N$ when $C \ge R/3$. To prove that, we first need to treat in Lemmas 11 and 12 the particular case of $N = 7$, before proving with Theorem 13 the general result.

LEMMA 11 $A(7,7) = 15$.

Proof: By Theorem 4 $A(7,7) \ge \frac{42}{3} = 14$ and the equality could be attained only if there exists a decomposition of K_7 into subgraphs with ratio $3/2$ (that is K_4). Such decomposition does not exist. So $A(7,7) > 14$. The following assignment of circles into three subgraphs show that $A(7,7) = 15$.

Here, we denote by $\{u_1, u_2, \ldots, u_p\}$ the set of edges of the complete graph K_p form on these vertices, and by $\{u_1, u_2, \ldots, u_p | v_1, v_2, \ldots, v_q\}$ the set of edges of a complete bipartite graph $K_{p,q}$ between the nodes $u_1, u_2, \ldots, u_p$ on one side and the nodes $v_1, v_2, \ldots, v_q$ on the other side.

B_i	V_i	$\lvert V_i \rvert$	E_i	$\lvert E_i \rvert$
B_0	$\{0,1,2,3,4\}$	5	$\{0,1,4\} + \{0,1 \mid 2,3\}$	7
B_1	$\{0,1,4,5,6\}$	5	$\{4,5,6\} + \{0,1 \mid 5,6\}$	7
B_2	$\{2,3,4,5,6\}$	5	$\{2,3\} + \{2,3 \mid 4,5,6\}$	7

□

LEMMA 12 $A(8,7) = 14$.

Proof: By Theorem 4 $A(8,7) \ge \left\lceil \frac{5 \times 21}{8} \right\rceil > 13$, and the following assignment of circles into three subgraphs show that $A(8,7) = 14$.

B_i	V_i	$\lvert V_i \rvert$	E_i	$\lvert E_i \rvert$
B_0	$\{0,1,2,3,4\}$	5	$K_5 - \{1 \mid 2,3\}$	8
B_1	$\{0,4,5,6\}$	4	$K_4 - \{0,4\}$	5
B_2	$\{1,2,3,5,6\}$	5	$\{1 \mid 2,3\} + \{1,2,3 \mid 5,6\}$	8

□

THEOREM 13 *When* $C \ge R/3$*,* $A(C,N) \le 2N$*, except when* $N = 4$ *and* $C = 2$*, and when* $N = 7$ *and* $C = 7$.

Proof: 1

Let $N = 3t + h$, where $h = 0, 1$ or 2 ; partition the vertex set into 3 sets V_1, V_2, V_3 such that $|V_1| = t$, $|V_2| = t + \left\lfloor \frac{h}{2} \right\rfloor$, and $|V_3| = t + \left\lceil \frac{h}{2} \right\rceil$.

Let the covering be done with 3 subgraphs B_i, $i = 1, 2, 3$, such that $V(B_i) = V_i \cup V_{i+1}$ (indices modulo 3). So, the total number of vertices is $2N$.

Each subgraph B_i will contain all the edges between V_i and V_{i+1} plus extra edges as follows.

Case 1 : $N = 3t$. In that case, $C \ge \left\lceil \frac{N(N-1)}{6} \right\rceil = \frac{t(3t-1)}{2} = t^2 + \frac{t(t-1)}{2}$. The subgraph B_i contains also all the edges between the vertices of V_i and so, altogether $t^2 + \frac{t(t-1)}{2} \le C$ edges.

Case 2 : $N = 3t+1$. In that case, $|V_1| = |V_2| = t$, $|V_3| = t+1$; $C \geq \frac{t(3t+1)}{2} = t(t+1) + \frac{t(t-1)}{2}$. The subgraph B_2 (resp. B_3) contains $t(t+1) + \frac{t(t-1)}{2} \leq C$ edges, namely the $t(t+1)$ edges between V_2 and V_3 (resp. V_1 and V_3) plus $\frac{t(t-1)}{2}$ extra edges chosen as follows. The extra $\frac{t(t-1)}{2}$ edges of B_2 are chosen among the edges between vertices of V_3. The $\frac{t(t-1)}{2}$ extra edges of B_3 are the remaining t edges between the vertices of V_3 plus $\frac{t(t-3)}{2}$ edges between the vertices of V_1. That is possible only if $t \geq 3$.

B_1 contains the remaining edges between the vertices of V_1 and all the edges between the vertices of V_2, that is $\frac{t(t-1)}{2} - \frac{t(t-3)}{2} + \frac{t(t-1)}{2} = \frac{t(t+1)}{2}$ edges, so altogether $t^2 + \frac{t(t+1)}{2} \leq C$ edges.

When $t = 1$, $A(2,4) = 9 > 8$ and $A(C,4) \leq 8$ for $C \geq 3$; when $t = 2$, $A(7,7) = 15 > 14$ and $A(C,7) = 14$ for $C \geq 8$ (Lemmas 11, 12 and 10).

Case 3 : $N = 3t+2$. In that case, $|V_1| = t$, $|V_2| = |V_3| = t+1$; $C \geq \frac{3t^2+3t}{2} + 1 = t(t+1) + \frac{t(t+1)}{2} + 1$. The subgraphs B_1 (resp. B_3) contains the $t(t+1)$ edges between V_1 and V_2 (resp. V_1 and V_3) plus $\frac{t(t+1)}{2}$ extra edges chosen as follows. For B_3 we chose $\frac{t(t-1)}{2}$ edges between vertices of V_1 plus t edges between vertices of V_3. For B_1 we chose the $\frac{t(t+1)}{2}$ edges between vertices of V_2. B_2 contains the $(t+1)^2$ edges between V_2 and V_3 plus the remaining edges between the vertices of V_3, that is $(t+1)^2 + \frac{t(t-1)}{2} \leq C$ edges. □

Let $\varphi(m) = \min\left\{k \mid \frac{k(k-1)}{2} \geq m\right\}$, that is $\varphi(m) = \left\lceil \frac{1+\sqrt{1+8m}}{2} \right\rceil$ and note that any subgraph with m edges has at least $\varphi(m)$ vertices.

THEOREM 14 *Let $R = \frac{N(N-1)}{2}$. When $C \geq R/3$, we have*

- *When $C \geq R$, $A(C,N) = N$.*
- *When $R/2 \leq C < R$, $A(C,N) = N + \varphi(R-C)$.*
- *When $R/3 \leq C < R/2$, except when $N = 4$ and $C = 2$, and when $N = 7$ and $C = 7$,*

$$A(C,N) = \min \begin{cases} 2N, \\ N + \varphi(C) + \varphi(R-2C), \\ N + \varphi(C) - 1 + \varphi\left(R - C - \frac{(\varphi(C)-1)(\varphi(C)-2)}{2}\right). \end{cases}$$

Proof:

Case 1: $C \geq R$. See Lemma 10.

Case 2: $R/2 \leq C < R$.

Recall that $\varphi(m)$ is the smallest integer k such that $\frac{k(k-1)}{2} \geq m$, and let $\alpha = \varphi(R-C)$. If each vertex belongs to at least 2 subgraphs then $A(C,N) \geq 2N \geq N + \alpha$. So one vertex belongs to exactly one subgraph which should contain the $N-1$ other vertices and at most C edges. To cover the $R-C$ remaining edges, we need a subgraph with at least α vertices. Therefore, $A(C,N) \geq N + \alpha$.

A solution with $N + \alpha$ ADMs is obtained by taking two subgraphs. The first one has α vertices and covers $\frac{\alpha(\alpha-1)}{2}$ edges, where $\frac{\alpha(\alpha-1)}{2} \geq R - C$ by definition of α. The second subgraph contains all the vertices and covers the remaining edges in number less than or equal to C.

Case 3: $R/3 \leq C < R/2$.

a) If each vertex belongs to at least 2 subgraphs then $A(C,N) \geq 2N$.

b) Otherwise one vertex belongs to an unique subgraph B_0 which contains at most C edges. To cover the remaining edges in number at least $R-C$, we need the following lemma.

LEMMA 15 *Let* $k_0 = \varphi(C)$. *When* $C \le m \le 2C$, *we need at least* $\min\{k_0 + \varphi(m-C), k_0 - 1 + \varphi\left(m - \frac{(k_0-1)(k_0-2)}{2}\right)\}$ *vertices to cover the* m *edges.*

Proof: Let $B_1, B_2, \ldots, B_k$ be the subgraphs needed to cover m edges and let B_1 be the subgraph having the maximum number of edges. We consider 3 different cases (the third one using an induction on m).

1) $|V(B_1)| = k_0$. We have $|E(B_1)| \le C$. To cover the remaining edges, in number $\ge m - C$, we need a subgraph with at least $\varphi(m-C)$ vertices. Thus, altogether we need at least $k_0+\varphi(m-C)$ vertices.
2) $|V(B_1)| = k_0 - 1$, then $|E(B_1)| \le \frac{(k_0-1)(k_0-2)}{2}$, it remains to cover at least $m - \frac{(k_0-1)(k_0-2)}{2}$ edges. If $m-\frac{(k_0-1)(k_0-2)}{2} \le C$, the remaining edges are covered using at least $\varphi\left(m - \frac{(k_0-1)(k_0-2)}{2}\right)$ vertices ; otherwise, at least $\varphi(C) + 2$ vertices are required, but $k_0 - 1 + k_0 + 2 = 2k_0 + 1 \ge 2k_0 \ge k_0 + \varphi(m - C)$.
3) $|V(B_1)| = k_1 \le k_0 - 2$. It remains to cover the $m - \frac{k_1(k_1-1)}{2}$ remaining edges
   (a) If $m - \frac{k_1(k_1-1)}{2} \le C$, we need $k_1 + \varphi\left(m - \frac{k_1(k_1-1)}{2}\right) \le k_0 - 1 + \varphi\left(m - \frac{k_0(k_0-1)}{2}\right)$ vertices by convexity of φ;
   (b) Otherwise, if $m - \frac{k_1(k-1)}{2} > C$, by induction the best covering use a subgraph with at least $\varphi(C) - 1$ vertices and so B_1 is not of maximum size ($\Rightarrow$ contradiction).

□

Now we apply the lemma to cover the m edges not in B_0. Recall that $m \ge R - C$ and $R \ge 2C$, so $m \ge C$. If $m \le 2C$, the lower bound follows from the lemma with $m = R - C$. If $m \ge 2C$, we need at least $2\varphi(C) \ge \varphi(C) + \varphi(R - C)$ vertices.

There exists a solution attaining the minimum. Indeed either the minimum is $2N$ and we have seen such solution for $C = \lceil \frac{R}{3} \rceil$. Either, the minimum is attained for $N + \varphi(C) + \varphi(R - 2C) < 2N$ and so $\varphi(C) + \varphi(R - 2C) < N$. In that case we take two subgraphs on disjoint set of vertices, one with $\varphi(C)$ vertices covering C edges and one with $\varphi(R - 2C)$ vertices covering $R - 2C$ edges. The last C edges are covered by a subgraph containing all the N vertices. Finally, if the minimum is attained for $N + \varphi(C) - 1 + \varphi\left(R - C - \frac{(\varphi(C)-1)(\varphi(C)-2)}{2}\right) < 2N$, we can take two subgraphs on disjoint sets of vertices, one with $\varphi(C) - 1$ vertices covering $\frac{(\varphi(C)-1)(\varphi(C)-2)}{2}$ edges and one with $\varphi\left(R - C - \frac{(\varphi(C)-1)(\varphi(C)-2)}{2}\right)$ vertices covering $R - C - \frac{(\varphi(C)-1)(\varphi(C)-2)}{2}$ edges. The C remaining edges are covered by a subgraph containing the N vertices.
□

Applying Theorem 14, we obtained the results of Table 1 for $C = 48$ or 64 and $N \le 16$, and for $C = 12$ and $N \le 9$. More precisely, when $C = 64$ we have for $N \le 11$, $R \le C$ and thus $A(64, N) = N$ and for $12 \le N \le 16$, $R \le 2C$ and so $A(64, N) = N + \varphi(R - 64)$. For example, for $N = 16$ we have $N = 120$, $R - C = 56$, $\varphi(R - C) = 12$ and so $A(64, 16) = 16 + 12 = 28$.

When $C = 48$, we have for $N \le 10$, $R \le C$ and so $A(48, N) = N$, and for $11 \le N \le 14$, $R \le 2C$ and $A(48, N) = N + \varphi(R - 48)$. For $14 \le N \le 16$, we have $A(48, N) \le 2N$ and the minimum is attained for this value. For example for $N = 16$ we have $R = 120$, $\varphi(48) = 11$, $\varphi(120 - 2 \times 48) = \varphi(24) = 8$ and $N+\varphi(C)+\varphi(R-2C) = 16+11+8 = 35 > 32$; furthermore $\varphi(C)-1 = 10$, $\varphi(R-C-(10\times 9)/2) = \varphi(27) = 8$ and the value is $34 > 32$.

In the preceding cases, the minimum for $R/3 \le C \le R/2$ was $2N$. But the other values of Theorem 14 can be attained. For example for $N = 14$, $R = 91$, $C = 45$, $\varphi(45) = 10$, $\varphi(R - 2C) = \varphi(1) = 2$ and so $N + \varphi(C) + \varphi(R - 2C) = 26 < 28 = 2N$.

Another interesting example is the computation of $A(93, 20)$. We have $N = 20$, $R = 190$, $\varphi(93) = 15$, $\varphi(190 - 2 \times 93) = \varphi(4) = 4$ and thus $20 + 15 + 4 = 39 < 40$, but we also have $\varphi(93) - 1 = 14$, $\varphi(190 - 93 - (14 \times 13)/2) = \varphi(6) = 4$ and so $20 + 14 + 4 = 38 < 39$ (the minimum is attained for the third case).

For lower bounds, if N is small, we have also to take into account the fact that subgraphs should have large intersections and so edges are covered many times.

5. Lower bounds

For other values of C and N, we have to use more sophisticated arguments.

PROPOSITION 16 *Let a_i denotes the number of subgraphs of K_n containing i nodes. In any covering of K_N by subgraphs B_j, $|E(B_j)| \leq C$, the following equations are satisfied :*

$$R = \frac{N(N-1)}{2} \leq \sum_{i \geq 2} a_i . \min\left\{C, \frac{i(i-1)}{2}\right\} \quad (1)$$

$$A(C,N) = \sum_{i \geq 2} i.a_i \quad (2)$$

$$\rho_{\max}(C).A(C,N) - R \geq \sum_{i \geq 2} a_i . \left(i.\rho_{\max}(C) - \min\left\{C, \frac{i(i-1)}{2}\right\}\right) \quad (3)$$

Proof: Equation 1 means that all edges are covered at least once and Equation 2 that the total number of nodes is equal to the sum of the number of nodes of the subgraphs. Equation 3 follows straightforward from equations 1 and 2. □

This proposition help us to prove lower bounds. We will see an example in Proposition 17 to prove that $A(12, 10) > 23$.

PROPOSITION 17 $A(12, 10) = 24$.

Proof: We have $R = \frac{N(N-1)}{2} = 45$, $\rho_{\max}(12) = 2$ and thus $A(12, 10) \geq \lceil 45/\rho_{\max}(12) \rceil = 23$. From Proposition 16, we have :

$$R = \frac{N(N-1)}{2} \leq \sum_{i \geq 7} 12a_i + 12a_6 + 10a_5 + 6a_4 + 3a_3 + a_2 \quad (4)$$

$$A(C,N) = \sum_{i \geq 7} i.a_i + 6a_6 + 5a_5 + 4a_4 + 3a_3 + 2a_2 \quad (5)$$

$$2.A(C,N) - R \geq \sum_{i \geq 7} 2(i-6)a_i + 2a_4 + 3a_3 + 3a_2 \quad (6)$$

Note that a_6 and a_5 are not concerned by Equation 6 as both K_5 and $K_6 - 3e$ satisfy $\rho = 2$. Let us first prove that the value 23 cannot be attained. If $A(12, 10) = 23$, then from Equation 6, we have $46 - 45 = 1 \geq \sum_{i \geq 7} 2(i-6)a_i + 2a_4 + 3a_3 + 3a_2$. Therefore $a_i = 0$ for $i \neq 5, 6$, and a solution consists only of K_5's and K_6's. Since $23 = 6a_6 + 5a_5$, we have necessarily $a_6 = 3$ and $a_5 = 1$.

Note that at least one node (in fact 6) belongs to only 2 subgraphs, otherwise $|A(C,N)| \geq 3 \times 10 = 30$. Let node 0 belong to 2 subgraphs. We have to investigate the two following cases :

- If node 0 belongs to subgraphs B_0 and B_1, one with 6 vertices and one with 5 vertices, then w.l.o.g. $V(B_0) = \{0, 1, 2, 3, 4, 5\}$ and $V(B_1) = \{0, 6, 7, 8, 9\}$. Then the two remaining subgraphs B_i, $(i = 2, 3)$ satisfy $|V(B_i) \cap (V(B_0) \cup V(B_1))| = 6$ and $|E(B_i) \cap (E(B_0) \cup E(B_1))| \geq 6$.

Since $a_6 = 3$ and $a_5 = 1$, we have a total of $3 \times 15 + 10 = 55$ edges in all the subgraphs, but for the subgraphs B_2 and B_3 at least $2 \times 6 = 12$ edges are already covered. Thus, the number of edges covered is at most $55 - 12 = 43 < 45$ a contradiction.

- If node 0 belongs to the subgraphs B_0 and B_1, each with 6 vertices, then w.l.o.g. $V(B_0) = \{0, 1, 2, 3, 4, 5\}$, $V(B_1) = \{0, 1, 6, 7, 8, 9\}$ and $|E(B_0) \cap E(B_1)| = 1$. Then the two remaining subgraphs B_2 and B_3 are such that $|V(B_2) \cap (V(B_0) \cup V(B_1))| = 6$, $|V(B_3) \cap (V(B_0) \cup V(B_1))| = 5$, $|E(B_2) \cap (E(B_0) \cup E(B_1))| \geq 6$, and $|E(B_3) \cap (E(B_0) \cup E(B_1))| \geq 4$.

  Since $a_6 = 3$ and $a_5 = 1$, we have a total of $3 \times 15 + 10 = 55$ edges in all the subgraphs, but for the subgraphs B_1, B_2 and B_3 at least $6 + 4 + 1 = 11$ edges are already covered. Thus, we have at most $55 - 11 = 44 < 45$ edges covered, a contradiction.

Thus, $A(12, 10) \geq 24$. The following covering into 4 subgraphs gives that $A(12, 10) = 24$.

B_i	V_i	$\|V_i\|$	E_i	$\|E_i\|$
B_0	$\{0, 1, 2, 3, 4, 5\}$	6	$\{0, 1\} + \{0, 1\|2, 3, 4, 5\} + \{2, 3, 4\}$	12
B_1	$\{0, 1, 6, 7, 8, 9\}$	6	$\{0, 1\|6, 7, 8, 9\} + \{6, 7\|8, 9\}$	12
B_2	$\{2, 3, 4, 5, 6, 7\}$	6	$\{2, 3, 4\|5, 6, 7\} + \{5, 6, 7\}$	12
B_3	$\{2, 3, 4, 5, 8, 9\}$	6	$\{2, 3, 4, 5\|8, 9\} + \{8, 9\}$	9

□

The other lower bounds for $C = 12$ and $N \leq 16$ are obtained in the same way. For constructions, we need to use designs tools as we will see in the next section.

6. Constructions

For small values of C it is possible to give the exact values of $A(C, N)$ for all N. When $C = 3$ it has been done in Bermond and Ceroi, .

THEOREM 18 (BERMOND AND CEROI,)

(i) When N is odd, $A(3, N) = \frac{N(N-1)}{2} + \epsilon$, where $\epsilon = 0$ if $N \equiv 1$ or $3 \pmod 6$, and $\epsilon = 2$ if $N \equiv 5 \pmod 6$;

(ii) When N is even, $A(3, N) = \frac{N(N-1)}{2} + \lceil \frac{N}{4} \rceil + \epsilon$, where $\epsilon = 1$ if $N \equiv 8 \pmod{12}$, and $\epsilon = 0$ otherwise.

The proof uses techniques inspired of design theory. In the even case, the optimal solutions use a lot of K_3's and some $K_{1,3}$ or P_4. Indeed, the degree of K_N being odd, one has to use subgraphs with odd degree. For example, if $n \equiv 0$ or $4 \pmod{12}$, the optimal solution consists of $\frac{N(N-1)}{6} - \frac{N}{4}$ K_3's and $\frac{N}{4}$ $K_{1,3}$. Note that there always exist solutions minimizing both the number of ADMs and the number of subgraphs (wavelengths) so conjecture of Chiu and Modiano, 2000 is true for $C = 3$.

For $C = 4$, the following theorem was given in Hu, 2002. We give here a shorter proof to show how simple partitions can be used.

THEOREM 19 (HU, 2002) *$A(4, 2) = 2$, $A(4, 4) = 7$ and otherwise, $A(4, N) = \frac{N(N-1)}{2}$. Furthermore, the number of subgraphs is the minimum $\left\lceil \frac{N(N-1)}{8} \right\rceil$.*

We first need the following lemma (a particular case of Sotteau, 1981) for which we recall the proof.

LEMMA 20 *When p and q are even, $K_{p,q}$ can be decomposed into $\frac{pq}{4}$ C_4's.*

Proof: Let $p = 2r$ and $q = 2s$ and let the vertices of $K_{p,q}$ be on one side $a_1, a'_1, a_2, a'_2, \ldots, a_r, a'_r$ and on the other side $b_1, b'_1, b_2, b'_2, \ldots, b_s, b'_s$. Then, the $rs = \frac{pq}{4}$ C_4's of the decomposition of $K_{2r,2s}$ are (a_i, b_j, a'_i, b'_j), for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. □

We can now give a short proof to Theorem 19.

Proof: [Theorem 19]

For $N \leq 5$, the results follows form Theorem 14. $A(4,2) = 2$, $A(4,3) = 3$, $A(4,4) = 7$, and $A(4,5) = 10$. For $N \geq 6$, the lower bound follows from Theorem 4 as $\rho_{\max}(4) = 1$.

Now, we can prove the Theorem by induction. More precisely, we can prove that K_N can be decomposed into $\left\lceil \frac{N(N-1)}{8} \right\rceil - \alpha$ C_4's and $K_3 + e$ (the graph obtained by adding an edge and a node to K_3) plus α K_3s, where $\alpha = 0$ if $N \equiv 0$ or $1 \pmod 8$, $\alpha = 1$ if $N \equiv 3$ or $6 \pmod 8$, $\alpha = 2$ if $N \equiv 4$ or $5 \pmod 8$, and $\alpha = 3$ if $N \equiv 2$ or $7 \pmod 8$. So the total number of subgraphs is $W_{\min} = \left\lceil \frac{N(N-1)}{8} \right\rceil$.

The construction can be easily done for $6 \leq N \leq 12$.

Now suppose that the Theorem is true for N, then it is true for $N + 8$. Indeed if N is even, K_{N+8} can be partitioned into a K_N, a K_8 and a $K_{N,8}$. By induction hypothesis, K_N can be decomposed into C_4's, $K_3 + e$ and α K_3's ; K_8 can be decomposed into C_4's and $K_3 + e$; $K_{N,8}$ into C_4's by Lemma 20. So K_{N+8} can be decomposed into C_4's, $K_3 + e$ and α K_3's.

If N is odd, we partition the edge set of K_{N+8} into a K_N and a K_9 having one vertex in common and a $K_{N-1,8}$. By induction hypothesis, K_N can be decomposed into C_4's, $K_3 + e$ and α K_3's ; K_9 and $K_{N-1,8}$ into C_4's. So K_{N+8} can be decomposed into C_4's, $K_3 + e$ and α K_3's. □

For other values of C, more sophisticated tools of design theory have to be used. We give an example for $C = 12$ where we can solve completely the case $N \equiv 1 \pmod 4$.

PROPOSITION 21 *When* $N = 4h + 1$, $A(12, 4h + 1) = (4h + 1)h$.

Proof: As $\rho_{\max}(12) = 2$, $A(C, N) \geq N(N - 1)$, that is $(4h + 1)h$ for $N = 4h + 1$.

Let $v_1, v_2, \ldots, v_l$ be some nonnegative integers; the *complete multipartite graph with class sizes* $v_1, v_2, \ldots, v_l$, denoted $K_{v_1,v_2,\ldots,v_l}$ is defined to be the graph with vertex set $V_1 \cup V_2 \cup \ldots \cup V_l$ where $|V_i| = v_i$, and two vertices $x \in V_i$ and $y \in V_j$ are adjacent if and only if $i \neq j$. For $t > 0$, we denote $K_{g\times t}$ (resp. $K_{g\times t,u}$) $K_{g,g,\ldots,g}$ (resp. $K_{g,g,\ldots,g,u}$) where g occurs t times.

Note that $K_{2,2,2}$ is a graph with 6 vertices and 12 edges (so with $\rho(K_{2,2,2}) = 2$).

By Theorem 1.2.4 pages 189-190 of Colbourn and Dinitz, 1996, we know that when $t \equiv 0$ or $1 \pmod 3$, $K_{t\times 2}$ can be decomposed into $\frac{2t(t-1)}{3} K_3$, and that when $t \equiv 0 \pmod 3$, $K_{t\times 2,4}$ can be decomposed into $\frac{2t(t-1)+8t}{3}$ K_3. It follows that when $t \equiv 0$ or $1 \pmod 3$, $K_{t\times 4}$ can be decomposed into $\frac{2t(t-1)}{3}$ $K_{2,2,2}$, and that when $t \equiv 0 \pmod 3$, $K_{t\times 4,8}$ can be decomposed into $\frac{2t(t-1)+8t}{3} = \frac{2t(t+5)}{3}$ $K_{2,2,2}$.

We are now able to prove the proposition.

- For $h \equiv 0$ or $1 \pmod 3$, let $V = \sum_{i=1}^{h} V_i \cup \{0\}$ with $|V_i| = 4$. Thus, K_N can be partitioned into h K_5 corresponding to the subgraphs B_i constructed on $V_i \cup \{0\}$ and the $K_{h\times 4}$ with classes V_i. Furthermore, $K_{h\times 4}$ can be partitioned into $\frac{2h(h-1)}{3}$ $K_{2,2,2}$'s. So altogether $A(12, N) = 5h + 4h(h-1) = 4h^2 + h$.

- For $h \equiv 2 \pmod 3$, let $V = \sum_{i=1}^{h-2} V_i \cup V_{h-1} \cup \{0\}$ with $|V_i| = 4$ for $i = 1, 2, \ldots, h-2$ and $|V_{h-1}| = 8$. So, K_N can be decomposed into $(h - 2)K_5$ (constructed on $V_i \cup \{0\}$ for $i = 1, 2, \ldots, h - 2$), a K_9 on $V_{h-1} \cup \{0\}$ and a $K_{(h-2)\times 4,8}$ which can be decomposed into $\frac{2(h-2)(h+1)}{3}$ $K_{2,2,2}$, and thus $A(12, N) = 5(h - 2) + 18 + 4(h - 2)(h + 1) = 4h^2 + h$ (using the fact that $A(12, 9) = 18$).

□

7. ILP formulation

We can easily formulate our problem in terms of integer linear programming (ILP) which may be solved using CPLEX.

Let $e^l_{i,j} = 1$ if subgraph B_l contains edge $\{i,j\}$, and 0 otherwise, and let $n^l_i = 1$ if $i \in V(B_l)$. We have

$$\begin{array}{llcl} \forall \{i,j\} \in V, & \sum_l e^l_{i,j} & \geq & 1 \\ & \forall l, \quad e^l_{i,j} & \leq & n^l_i \\ & e^l_{i,j} & \leq & n^l_j \\ \forall l, & \sum_{\{i,j\}\in V} e^l_{i,j} & \leq & \min\left\{C, \frac{|V_l|(|V_l|-1)}{2}\right\} \\ & & & \\ Minimize & \sum_i \sum_l n^l_i & & \end{array}$$

We may add some other constraints to reduce the research space. Let $d = \sum_l \min\left\{C, \frac{|V_l|(|V_l|-1)}{2}\right\} - R$; it corresponds to the number of edges which may appear in more than one subgraph. Let also $x^l_{i,j} = 1$ if $\sum_{k\leq l} e^k_{i,j} \geq 1$ and 0 otherwise, meaning that edge $\{i,j\}$ is contained by at least one of the subgraphs $B_1, B_2, \ldots, B_l$. We have

$$\begin{array}{llcl} \forall l & \forall \{i,j\} \in V, \quad e^l_{i,j} & \leq & x^l_{i,j} \\ & x^{l-1}_{i,j} & \leq & x^l_{i,j} \\ & \sum_{\{i,j\}\in V} \left(\sum_{k\leq l} e^k_{i,j} - x^l_{i,j}\right) & \leq & d \end{array}$$

With these general conditions, we can find solution only for $N \leq 8$. However, we can again limit the research space. For example we can use Proposition 16 to know for a given possible value of $A(C,N)$ what are the sizes of the subgraphs, fix already some subgraphs, etc ... Doing so, we can quickly eliminate some values of $A(C,N)$. We can also know if a given partition is valid or not.

8. Conclusion

In this article, we have solved the problem of traffic grooming in unidirectional WDM rings with uniform unitary traffic for various values of N and C. We have shown how to use graph theory and design tools to either solve the problem or help an ILP program ; that has enabled us to solve optimally the problem for practical values and infinite congruence classes of values for a given C. The tools can be easily extended to uniform but non unitary traffic. Indeed, if we have a request of size r from i to j, it suffices to consider decomposition of the edges of the complete multipartite graph rK_N. We can also extend the ideas to the case of arbitrary traffic, but it requires to partition general graphs and this is known to be a difficult problem in graph theory. However, our tools can be used in an ILP formulation. We can also consider networks different from the unidirectional ring, if we are first able to group the requests into circles (that is the way used in Colbourn and Ling, ; Colbourn and Wan, 2001 for bidirectional rings). Finally, the tools can also be used to groom traffic in a slightly different context, for example , in the RNRT project PORTO our team developed with France Telecom and Alcatel, the traffic was expressed in terms of STM-1 (each one needed one wavelength) and we grouped them into bands or fibers, typically a fiber containing 8 bands of 4 wavelengths (see Huiban et al., 2002).

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