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# A natural transfer function space for linear discrete time-invariant and scale-invariant systems

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**Abstract**—In a previous work, we have defined the scale shift for a discrete-time signal and introduced a family of linear scale-invariant systems in connection with character-automorphic Hardy spaces. In this paper, we prove a Beurling-Lax theorem for such Hardy spaces of order 2. We also study an interpolation problem in these spaces, as a first step towards a finite dimensional implementation of a scale invariant system. Our approach uses a characterization of character-automorphic Hardy spaces of order 2 in terms of classical de Branges Rovnyak spaces.

## I. INTRODUCTION

The self-similarity property is widely studied in the literature in the framework of stochastic process theory [1], [2], [3], and in the framework of systems theory [4], [5], [6]. In stochastic process theory the property is seen as a weighted form of stationarity in scale while in the systems theory approach, it is interpreted as a scale invariance. The scale shift, defined for a signal  $x(t)$  by the operator  $\alpha \mapsto x(\alpha t)$  thus plays a central rôle in the definition of self-similarity. Though simple and straightforward in the continuous-time domain, this operator is not well defined for discrete-time signal. In this paper, we use the definition given in [7]. Therein, a family of linear discrete both time- and scale-invariant systems is introduced in connection with character-automorphic Hardy spaces. In this paper, we prove a Beurling-Lax theorem for such Hardy spaces of order 2. We also study an interpolation problem in these spaces, as a first step towards a finite dimensional implementation of a scale invariant system. Our approach uses a characterization of character-automorphic Hardy spaces of order 2 in terms of classical de Branges Rovnyak spaces [8].

### A. Scale shift for discrete-time signals

Let  $f \in \mathbf{L}_1(\mathbb{R}_+)$  (that is, a continuous time signal), with Laplace transform  $F(s)$ ,  $\Re(s) \geq 0$ . As it is well known, for every  $\alpha = 1/\beta > 0$ , the Laplace transform of  $f(\beta t)$  is  $\sqrt{\alpha}F(\alpha s)$ . Therefore, time scaling has the same form both in the time and frequency domains. This remark is the starting point to define the scaling operator for discrete-time signals. Let  $\theta$  be given such that  $|\theta| < \frac{\pi}{2}$ . Consider the Möbius transformation

$$G_\theta(s) = \frac{e^{i\theta} - s}{e^{-i\theta} + s},$$

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which maps conformally the open right half-plane  $\mathbb{C}_+$  onto the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Then, the scale shift

$$S_\alpha : s \mapsto S_\alpha(s) = \alpha s, \quad \alpha > 0$$

translates in the unit disc, into the hyperbolic transformation [7]

$$\gamma_{\{\alpha\}} = G_\theta \circ S_\alpha \circ G_\theta^{-1}. \quad (1)$$

Any transformation of this form maps the unit open disc (resp. the unit circle) into itself.

Conversely, the following lemma is true.

*Lemma 1.1:* For each hyperbolic transformation

$$\gamma(z) = \frac{\gamma_1 z + \gamma_2}{\bar{\gamma}_2 z + \bar{\gamma}_1}, \quad (2)$$

there exist  $\alpha_\gamma > 0$ ,  $\theta_\gamma$  with  $|\theta_\gamma| < \frac{\pi}{2}$  and  $\xi_\gamma$  such that

$$e^{i\xi_\gamma} \gamma(z) = (G_{\theta_\gamma} \circ S_{\alpha_\gamma} \circ G_{\theta_\gamma}^{-1})(e^{i\xi_\gamma} z). \quad (3)$$

In particular,  $\alpha_\gamma$  is given by the multiplier of the transformation

*Proof:* We assume that  $\gamma$  is normalized such that  $|\gamma_1|^2 - |\gamma_2|^2 = 1$ . Since  $\gamma$  is hyperbolic, we have the relation [9]

$$\frac{\gamma(z) - \xi_1}{\gamma(z) - \xi_2} = K \frac{z - \xi_1}{z - \xi_2} \quad (4)$$

where  $\xi_1 = \frac{\sqrt{[\Re(\gamma_1)]^2 - 1 + i\Im(\gamma_1)}}{\Re(\gamma_1)}$  and  $\xi_2 = -\frac{\bar{\lambda}_\gamma}{\gamma_2}$  are the two fixed points of  $\gamma$ . The positive constant  $K$  is called the multiplier of the transformation [9] and is given by  $K + \frac{1}{K} = 4[\Re(\gamma_1)]^2 - 2$ . Noting that  $|\xi_1| = |\xi_2| = 1$ , one may rearrange (4) to obtain

$$\frac{\lambda_\gamma - \bar{\lambda}_\gamma e^{i\xi_\gamma} \gamma(z)}{1 + e^{i\xi_\gamma} \gamma(z)} = K \frac{\lambda_\gamma - \bar{\lambda}_\gamma e^{i\xi_\gamma} z}{1 + e^{i\xi_\gamma} z}, \quad (5)$$

where  $e^{i\xi_\gamma} = \frac{\lambda_\gamma}{\gamma_2}$ . Dividing both sides of this equality by  $|\lambda_\gamma|$ , we get (3) by setting  $e^{i\theta_\gamma} = \frac{\lambda_\gamma}{|\lambda_\gamma|}$  and  $\alpha_\gamma = K$ . ■

The set of all linear transformations as in (2) forms a group that we denote by  $\Gamma$ . The lemma then shows that the action of  $\Gamma$  on  $\mathbb{D}$  is equivalent to the scale operator on  $\mathbb{C}_+$ . Therefore, we define below the discrete-time frequency domain scale shift by the action of the (hyperbolic) group of

automorphisms of  $\mathbb{D}$ . Given a discrete-time signal  $\{x_n\}_{n \geq 0}$ , consider its  $\mathcal{Z}$  transform  $X(z) = \sum_{n=0}^{\infty} x_n z^n$ , which we assume convergent in a neighborhood of the origin. The scale  $\alpha = \alpha_\gamma$  shift of the sequence  $\{x_n\}_{n \geq 0}$  is the sequence  $\{x_n(\gamma)\}_{n \geq 0}$  defined via the equation

$$X_\gamma(z) = \frac{1}{\bar{\gamma}_2 z + \bar{\gamma}_1} X(\gamma(z)) = \sum_{n \geq 0} x_n(\gamma) z^n. \quad (6)$$

It is useful to note that this operator also makes sense for vector-valued functions.

### B. Scale-invariant systems

A wide class of causal discrete time-invariant linear systems can be given in terms of convolution in the form

$$y_n = \sum_{m=0}^n h_{n-m} x_m, \quad n = 0, 1, \dots, \quad (7)$$

where  $\{h_n\}$  is the impulse response and where the input sequence  $\{x_m\}$  and output sequence  $\{y_m\}$  are requested to belong to some pre-assigned sequences spaces. The  $\mathcal{Z}$  transform of the sequence  $\{h_n\}$ , that is  $H(z) = \sum_{n=0}^{\infty} z^n h_n$ , is called the *transfer function* of the system, and there are deep relationships between properties of  $H(z)$  and of the system; see [10] for a survey. In particular, it is well known that if the system is asymptotically stable, then  $H(z)$  belongs to the classical Hardy space of order 2.

In this paper, we are interested in linear discrete-time systems which, in addition to the time invariance, are also invariant under a scale shift. Scale-invariance is defined [7] similarly to time-invariance: a scale shift in the input sequence induces the same scale shift on the corresponding output sequence. In terms of the  $\mathcal{Z}$  transforms, this would directly mean that for all  $\gamma \in \Gamma$ ,

$$Y_\gamma(z) = H(\gamma(z))X_\gamma(z) = H(z)X_\gamma(z). \quad (8)$$

Therefore, scale-invariance implies that the transfer function of the system be  $\Gamma$ -periodic. Now a function  $f$  satisfying  $f \circ \gamma = f$ , for all  $\gamma \in \Gamma$  is said to be *automorphic* with respect to  $\Gamma$ . This makes sense only for discrete groups (see [9]).

In the following, we will be interested in the character-automorphic Hardy spaces of order 2. These are the natural transfer function spaces for the LTI and scale-invariant systems. In a previous study, see [8], we have given a characterization of these spaces in terms of associated classical de Branges Rovnyak spaces. In section II, we use this approach to prove a Beurling-Lax theorem and we study interpolation in these Hardy spaces in section III. Leech's theorem and the characterization of de Branges Rovnyak spaces given in [11, Theorem 3.1.2, p. 85] play an important role in the arguments.

In the remaining of the paper, the complex conjugation of is denoted par  $*$  and no longer by  $\bar{\phantom{x}}$ .

## II. A NATURAL TRANSFER FUNCTION SPACE

### A. Definitions

Now on, we discretise the scale axis and consider that  $\Gamma$  is a Fuchsian group of Widom type (with no elliptic element) [12]. We denote by  $\mathfrak{z}$  its uniformizing map and by  $\widehat{\Gamma}$  its dual group, *i.e.* the group of unimodular characters. Recall that a character  $\alpha$  is a function defined on  $\Gamma$  and satisfying:

$$|\alpha(\gamma)| = 1 \quad \text{and} \quad \alpha(\gamma \circ \varphi) = \alpha(\gamma)\alpha(\varphi), \quad \forall \gamma, \varphi \in \Gamma.$$

A function  $f$  satisfying  $f \circ \gamma = \alpha(\gamma)f$ ,  $\forall \gamma \in \Gamma$  for  $\alpha \in \widehat{\Gamma}$ , is called *character-automorphic* with respect to  $\Gamma$ . Given a character  $\alpha$  of  $\widehat{\Gamma}$ , the character-automorphic Hardy space  $\mathcal{H}_2^\alpha(\mathbb{D})$  of order 2 is the space of character-automorphic functions which belong to the classical Hardy space  $\mathcal{H}_2(\mathbb{D})$ . Its reproducing kernel has been characterized in [13, Lemma 4.4.2 p. 387] as follows:

$$k^\alpha(z, \omega) = c(\alpha) \frac{\frac{k^{\alpha\mu}(z,0)}{b(z)} k^\alpha(\omega, 0)^* - \left(\frac{k^{\alpha\mu}(\omega,0)}{b(\omega)}\right)^* k^\alpha(z, 0)}{\mathfrak{z}(z) - \mathfrak{z}(\omega)^*} \quad (9)$$

where  $c(\alpha) = \frac{\mathfrak{z}(0)b(0)}{k^{\alpha\mu}(0,0)} > 0$ . In (9),  $b$  is the Green's function of  $\Gamma$ , and the character associated to the Green's function is denoted by  $\mu$ . Formula (9) expresses that the kernel is *structured*, and of the form of the kernels studied in the papers [14], [15]. It depends on a  $\mathbb{C}^{1 \times 2}$ -valued function of one variable. Using (9) we proved in [8] that there exists a Schur function  $\mathcal{S}_\alpha$ , associated to the de Branges space  $\mathcal{H}(\mathcal{S}_\alpha)$ , such that

$$\mathcal{H}_2^\alpha(\mathbb{D}) = \left\{ F(z) = \frac{A^\alpha(z)}{1 - i\mathfrak{z}(z)} f(\sigma(z)) ; f \in \mathcal{H}(\mathcal{S}_\alpha) \right\} \quad (10)$$

where  $A^\alpha(z) = \sqrt{c(\alpha)} \left( \frac{k^{\alpha\mu}(z,0)}{b(z)} + ik^\alpha(z, 0) \right)$  and  $\sigma(z) = \frac{1+i\mathfrak{z}(z)}{1-i\mathfrak{z}(z)}$ , and with the norm

$$\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} = \|f\|_{\mathcal{H}(\mathcal{S}_\alpha)}.$$

We close this subsection with the

*Definition 2.1:* A causal linear time-invariant system is called scale-invariant with respect to  $\Gamma$ , if its transfer function is an element of  $\mathcal{H}_2^\alpha(\mathbb{D})$  for some character  $\alpha$ . Note that such a system is not rational, unless  $\Gamma$  is a finite group.

### B. The shift operator in $\mathcal{H}_2^\alpha(\mathbb{D})$

Set  $\mathfrak{p} \in \mathbb{D}$  and given a function  $f$  analytic in  $\mathbb{D}$ , consider the operators

$$(R_{\mathfrak{p}}f)(\lambda) = \frac{f(\lambda) - f(\mathfrak{p})}{\lambda - \mathfrak{p}}.$$

These operators  $R_{\mathfrak{p}}$  satisfy the resolvent equation

$$R_{\mathfrak{p}} - R_{\mathfrak{q}} = (\mathfrak{p} - \mathfrak{q})R_{\mathfrak{p}}R_{\mathfrak{q}}.$$

Let  $m(z) = \frac{A_\alpha(z)}{1-i\mathfrak{z}(z)}$ . The isomorphism

$$F(z) = m(z)f(\sigma(z)) \quad (11)$$

between the de Branges space  $\mathcal{H}(\mathcal{S}_\alpha)$  and the character-automorphic Hardy space allows one to define the following operators:

$$\mathcal{R}_p F = m(z)(R_p f)(\sigma(z)).$$

As we may directly check, these operators  $\mathcal{R}_p$  also satisfy a resolvent equation. Therefore, they can be written in the form

$$\mathcal{R}_p = (\mathbf{T} - p)^{-1}.$$

Here  $\mathbf{T}$  is not an operator in general. It is a linear relation and is given by

$$(\mathbf{T}F)(z) = \sigma(z)F(z) + m(z)c_F,$$

where  $c_F$  is such that

$$\sigma(z)F(z) + m(z)c_F \in \mathcal{H}_2^\alpha(\mathbb{D}).$$

### C. A Beurling theorem in $\mathcal{H}_2^\alpha(\mathbb{D})$

The classical Beurling theorem (see for instance [16, Théorème 17.21 p. 330]) gives a characterization of the closed subspaces  $\mathcal{M}$  of the Hardy space  $\mathcal{H}_2(\mathbb{D})$  of the unit disk  $\mathbb{D}$ : *Any such subspace is of the form  $\mathcal{M} = j\mathcal{H}_2(\mathbb{D})$ , where  $j$  is an inner function* (the case of vector-valued functions was first considered by Lax; see [17] for a discussion and references). The orthogonal complement of  $\mathcal{M}$  is the reproducing kernel Hilbert space with reproducing kernel  $K_j(z, w) = (1 - j(z)j(w)^*)/(1 - zw^*)$ . If one replaces  $j$  by a Schur function  $s$ , that is, by a function analytic and contractive in  $\mathbb{D}$ , the kernel  $k_s(z, w)$  is still positive in  $\mathbb{D}$ . Its associated reproducing kernel Hilbert space was denoted in the preceding subsection by  $\mathcal{H}(\mathcal{S})$ . These spaces are called de Branges Rovnyak spaces, and originate with the work [18]. When allowing  $\mathcal{S}$  to be vector-valued, they have been fully characterized in [11, Theorem 3.1.2]. They are contractively included, but in general not isometrically included, in  $\mathcal{H}_2(\mathbb{D})$ .

*Theorem 2.2:* A Hilbert space  $\mathcal{M}$  is contractively included in  $\mathcal{H}_2^\alpha(\mathbb{D})$ , and invariant under  $\mathcal{R}_p$ , and satisfies the inequality

$$\|\mathcal{R}_0 F\|_{\mathcal{M}}^2 \leq \|F\|_{\mathcal{M}}^2 - |F(0)|^2 \quad (12)$$

if, and only if, its reproducing kernel is of the form

$$\frac{A_\alpha(z)}{\sqrt{2}} \frac{1 - \mathcal{R}(\sigma(z))\mathcal{R}(\sigma(w))^* A_\alpha(w)^*}{-i(\mathfrak{z}(z) - \mathfrak{z}(w)^*)} \frac{A_\alpha(w)^*}{\sqrt{2}},$$

where  $\mathcal{R}$  is a vector-valued Schur function such that

$$\mathcal{S}_\alpha = \mathcal{R}\mathcal{R}_1, \quad (13)$$

with  $\mathcal{R}_1$  also a vector-valued Schur function.

*Remark 1:* The inequality(12) is automatically satisfied if  $\mathcal{M}$  is isometrically included in  $\mathcal{H}_2^\alpha(\mathbb{D})$ .

*Proof:* of 2.2 Associated to the space  $\mathcal{M}$ , there is, by the isomorphism (11), a Hilbert space  $\mathcal{M}_\alpha$  which is  $\mathcal{R}_p$ -invariant and satisfying

$$\|\mathcal{R}_0 f\|_{\mathcal{M}_\alpha}^2 \leq \|f\|_{\mathcal{M}_\alpha}^2 - |f(0)|^2. \quad (14)$$

Using [11, Theorem 3.1.2], we see that the reproducing kernel of  $\mathcal{M}_\alpha$  is of the form

$$\frac{1 - \mathcal{R}(\lambda)\mathcal{R}(p)^*}{1 - \lambda p^*},$$

where  $\mathcal{R}$  is a vector-valued Schur function. Since  $\mathcal{M}_\alpha$  is contractively included in  $\mathcal{H}(\mathcal{S}_\alpha)$ , the kernel

$$\frac{\mathcal{R}(\lambda)\mathcal{R}(p)^* - \mathcal{S}_\alpha(\lambda)\mathcal{S}_\alpha(p)^*}{1 - \lambda p^*},$$

is positive. We therefore get the factorization (13) by using the Leech theorem. See [19, Theorem 2, p. 134] and [20, Example 1, p. 107]. We conclude by using the isomorphism (11) and the fact that (see equation (10)) the reproducing kernel of the space  $\mathcal{H}_2^\alpha(\mathbb{D})$  is

$$\begin{aligned} k^\alpha(z, w) &= \frac{A_\alpha(z)}{1 - i\mathfrak{z}(z)} \frac{1 - \mathcal{S}_\alpha(\sigma(z))\mathcal{S}_\alpha(\sigma(w))^*}{1 - \sigma(z)\sigma(w)^*} \frac{A_\alpha(w)^*}{1 + i\mathfrak{z}(w)^*} \\ &= \frac{A_\alpha(z)}{\sqrt{2}} \frac{1 - \mathcal{S}_\alpha(\sigma(z))\mathcal{S}_\alpha(\sigma(w))^*}{-i(\mathfrak{z}(z) - \mathfrak{z}(w)^*)} \frac{A_\alpha(w)^*}{\sqrt{2}}. \end{aligned}$$

Reversing these different arguments allows one to establish the converse.  $\blacksquare$

## III. INTERPOLATION

As we already mention, the elements of the character-automorphic Hardy space  $\mathcal{H}_2^\alpha(\mathbb{D})$  are not rational functions unless we consider a finite number of possible scale shifts. Since the corresponding systems are of infinite dimension, a finite dimensional approximation step is necessary before their implementation. This is the motivation of the interpolation problem studied in this section.

To proceed, denote by

$$\mathcal{F} = \{z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \in \Gamma, \gamma \neq id\} \quad (15)$$

the *normal fundamental domain* of  $\Gamma$  with respect to 0: there is no transformation in  $\Gamma$ , which sends one point of  $\mathcal{F}$  into another point of  $\mathcal{F}$ .

So, we consider the following interpolation problem: *Given  $N$  complex numbers  $F_i$  and  $N$  points  $z_i \in \mathcal{F} \cap \mathbb{D}$ , describe the set of all functions  $F \in \mathcal{H}_2^\alpha(\mathbb{D})$  with  $\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} \leq 1$  satisfying*

$$F(z_i) = F_i, \quad i = 1, \dots, N, \quad (16)$$

Note that this problem is different from the interpolation problems considered by Abrahamse in [21] and by Kupin and Yuditskii in [13]. These studies were interested in finding multipliers having given values at prescribed points while here we have the constraint that  $F$  must belong to  $\mathcal{H}_2^\alpha(\mathbb{D})$ .

The function

$$\mathcal{P}_\alpha = \frac{(1 - \mathcal{S}_\alpha)}{(1 + \mathcal{S}_\alpha)} \quad (17)$$

is analytic and has positive real part in  $\mathbb{D}$ . By the Herglotz representation theorem, we can write:

$$\mathcal{P}_\alpha = ic_\alpha + \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\sigma_\alpha(t),$$

with  $c_\alpha \in \mathbb{R}$  and  $d\sigma_\alpha$  is a positive measure  $[0, 2\pi)$ . Defining

$$\mathcal{Q}_\alpha(\lambda) \triangleq \frac{\sqrt{2}}{(1 + \mathcal{S}_\alpha(\lambda))} = \frac{1 + \mathcal{P}_\alpha(\lambda)}{\sqrt{2}}$$

where the second equality follows directly from (17), we see that can also write

$$\frac{\mathcal{P}_\alpha(\lambda) + \mathcal{P}_\alpha(\mathbf{p})^*}{1 - \lambda\mathbf{p}^*} = \mathcal{Q}_\alpha(\lambda) \frac{1 - \mathcal{S}_\alpha(\lambda)\mathcal{S}_\alpha(\mathbf{p})^*}{1 - \lambda\mathbf{p}^*} \mathcal{Q}_\alpha(\lambda)^*. \quad (18)$$

Now this equation (18) implies that the operator of multiplication by  $\mathcal{Q}_\alpha(\lambda)$  is a unitary transformation from the reproducing kernel space  $\mathcal{L}(\mathcal{P}_\alpha)$ , with kernel

$$\frac{\mathcal{P}_\alpha(\lambda) + \mathcal{P}_\alpha(\mathbf{p})^*}{1 - \lambda\mathbf{p}^*}, \quad \lambda, \mathbf{p} \in \mathbb{D}$$

to the space  $\mathcal{H}(\mathcal{S}_\alpha)$ . Moreover  $\mathcal{L}(\mathcal{P}_\alpha)$  is the set of functions of the form

$$x(\lambda) = \int_0^{2\pi} \frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it} - \lambda}, \quad (19)$$

where  $h$  belongs to the closure (subsequently denoted  $\mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$ ) of the set of functions  $1/(1 - e^{it}w^*)$  ( $|w| < 1$ ) in  $\mathbf{L}_2(d\sigma_\alpha)$ . We therefore have the following:

**Proposition 3.1:**  $F \in \mathcal{H}_2^\alpha(\mathbb{D})$  if, and only if,

$$F(z) = \frac{A^\alpha(z)}{1 - i\mathfrak{z}(z)} \frac{1 + \mathcal{S}_\alpha(\sigma(z))}{\sqrt{2}} \int_0^{2\pi} \frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it} - \sigma(z)} \quad (20)$$

with  $h \in \mathbf{H}_2(d\sigma_\alpha)$ .

The interpolation problem in the character-automorphic Hardy space then reduces to an interpolation problem in  $\mathcal{L}(\mathcal{P}_\alpha)$ , or more precisely, to an orthogonal projection in  $\mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$ :

**Problem 1:** Let  $\lambda_\ell = \sigma(z_\ell)$  and

$$x_\ell = \frac{1 - \mathfrak{z}(z_\ell)}{A^\alpha(z_\ell)} \frac{\sqrt{2}F_\ell}{1 + \mathcal{S}_\alpha(\lambda_\ell)}, \quad \ell = 1, \dots, N.$$

Find all  $h \in \mathcal{H}_2(\mathbb{D}, d\sigma_\alpha)$  such that

$$\int_0^{2\pi} \frac{e^{it}h(t)d\sigma_\alpha(t)}{e^{it} - \lambda_\ell} = x_\ell, \quad \ell = 1, \dots, N.$$

Now this is a classical Hilbert space problem: it admits a solution with minimum norm, corresponding to a function  $h_{\min}$  of the form

$$h_{\min}(t) = \sum_{\ell=1}^N \frac{c_\ell}{e^{it} - \lambda_\ell},$$

and any other solution has the form

$$h_{\min} + h, \quad h \perp h_{\min}.$$

We thus deduce the description of all  $x$  of the form (19), and hence a description of the functions  $F$  by formula (20).

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