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Abstract: We propose a new analysis for the PSPG method applied to the transient Stokes' problem. Stability and convergence are obtained under different conditions on the discretization parameters depending on the approximation used in space. For the pressure we prove optimal stability and convergence only in the case of piecewise affine approximation under the standard condition on the time-step.

Key-words: Transient Stokes' equations, finite element methods, PSPG stabilization, time discretization, Ritz-projection.

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Analyse de la méthode PSPG pour l'équation de Stokes transitoire

Résumé : Nous proposons une nouvelle analyse pour la méthode PSPG appliquée à l'équation de Stokes transitoire. On montre la stabilité et la convergence de la méthode sous différentes conditions et selon le type d'approximation en space. Pour la pression, stabilité et convergence optimale sont établies dans le cas d'approximations affines par morceaux, sous une condition standard de type CFL parabolique inverse.

Mots-clés : Équation de Stokes transitoire, méthode d'éléments finis, stabilisation PSPG, discrétisation en temps, projection de Ritz.

1 Introduction

The pressure stabilized Petrov-Galerkin (PSPG) method, as introduced by Hughes *et al.* in [11], is a popular tool for the approximation of the Stokes' problem using equal order interpolation for velocities and pressures. In spite of its extensive use, so far it has been analyzed for the transient case only for the velocities, using the backward Euler scheme and under an inverse CFL-condition. If τ and h denote the discretization parameters, in time and space respectively, and ν denotes the dynamic viscosity, the condition writes $h^2 \leq \nu\tau$. To our best knowledge, the first reference where this condition appeared was in the work by Picasso and Rappaz [12] (see also [1]). The small time step instability was thoroughly investigated by Bochev *et al.* in [5, 4], where they examined the algebraic properties of the system matrices. Their conclusion was that regularity of the system imposed an inverse parabolic CFL condition. They also observed numerically, in [3], that for higher polynomial order the instability polluted the velocity approximation as well.

Our aim in this paper is to consider the transient Stokes' equations discretized using Petrov-Galerkin pressure stabilization and derive global in time stability, using the variational framework. From this different viewpoint we arrive at similar conclusions as [4], but with global stability bounds that we then use to derive optimal order error estimates. The estimates for the velocities also give insight in a possible mechanism for the observed loss of accuracy in the velocity approximation for higher order polynomials reported in [3].

The case of symmetric stabilization methods for the transient Stokes' problem was treated in [7]. There we proved that, for symmetric stabilizations, the small time-step instability can be circumvented by using a particular discrete initial data given by the Ritz-projection associated to the discrete Stokes' operator. Our analysis herein shows that also for the PSPG method the small time-step pressure instability stems from the initial data, however it can not be cured using the Ritz-projection. The reason for this is a coupling between the time-derivative and the pressure gradient, appearing in the estimate for the acceleration, resulting in the factor $h^2/(\nu\tau)$ in the stability estimate.

For the global in time stability estimate for the velocities, we observe that the properties of the PSPG method change drastically depending on what approximation spaces are used. Indeed, depending on the regularity of the discretization space, different conditions must be imposed in order for stability to hold. For high order polynomial spaces, it appears that energy contributions from gradient jumps over element faces may interact with the time-derivative of the velocity and destabilize the solution for small time-step size. Since the gradient jumps are small for smooth solutions this instability may difficult to observe numerically. We give one example of a computation where the velocity approximation diverges as the time-step is reduced.

The main theoretical results for the velocities are as follows:

- The backward Euler method (BDF1): stability and optimal convergence hold unconditionally for piecewise affine approximation and when $\tau \geq h^2/\nu$ for higher polynomial order.

- Crank-Nicolson and the second order backward differentiation method (BDF2) are unconditionally stable and have optimally convergent velocity approximation for piecewise affine approximation. For higher polynomial order stability and convergence holds under the condition $\tau \geq h/\nu^{\frac{1}{2}}$.

- All $C^1(\Omega)$ approximation spaces, such as the one obtained using NURBS, result in unconditionally stable PSPG methods for all the time-discretization schemes proposed above.

In particular this means that for piecewise affine approximation or C^1 approximation, the system matrix that must be inverted for every timestep is regular independent of the discretization parameters. We only give the analysis for stability and convergence of the velocities in the case of the backward differentiation formulas. The extension to the Crank-Nicolson scheme is straightforward.

For the pressure, stability and optimal convergence (in the natural norm) are proven, under the standard condition $\tau \geq h^2/\nu$, for piecewise affine approximation spaces and the BDF1 scheme. The extension to the BDF2 method is straightforward using the same techniques. Nevertheless, the case of high order polynomials or the Crank-Nicolson method remains open.

Although incomplete and possibly not sharp for higher polynomial order, we hope that these result will bring some new insights in the dynamics of the PSPG-method applied to the transient Stokes' problem. In particular it is interesting to notice that the choice of space discretization seems important for the stability of the discretization of the transient problem.

The remainder of this paper is organized as follows. In the next section we introduce the continuous problem and fixe some notation. The fully discrete schemes are presented in section 3. Sections 4 and 5 are respectively devoted to the stability and the convergence analysis. Finally, section 6 presents a numerical example illustrating the velocity instability, in the small time-step limit, predicted by the theory for higher order polynomials.

2 Problem setting

Let Ω be a convex domain in $\mathbb{R}^d$ ($d = 2$ or 3) with a polyhedral boundary $\partial\Omega$. For $T > 0$ we consider the problem of solving, for $\boldsymbol{u} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and $p : \Omega \times (0,T) \longrightarrow \mathbb{R}$, the following time-dependent Stokes problem:

$$\left\{\begin{aligned} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}, &\quad \text{in} \quad \Omega \times (0,T),\\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, &\quad \text{in} \quad \Omega \times (0,T),\\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on} \quad \partial\Omega \times (0,T),\\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, &\quad \text{in} \quad \Omega. \end{aligned}\right. \tag{1}$$

Here, $\boldsymbol{f} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ stands for the source term, $\boldsymbol{u}_0 : \Omega \longrightarrow \mathbb{R}^d$ for the initial velocity and $\nu > 0$ for a given constant viscosity. In order to introduce a variational setting for (1) we consider the following standard velocity and pressure spaces

$$V \stackrel{\text{def}}{=} [H_0^1(\Omega)]^d, \quad H \stackrel{\text{def}}{=} [L^2(\Omega)]^d, \quad Q \stackrel{\text{def}}{=} L_0^2(\Omega),$$

normed with

$$\|\boldsymbol{v}\|_H \stackrel{\text{def}}{=} (\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}}, \quad \|\boldsymbol{v}\|_V \stackrel{\text{def}}{=} \|\nu^{\frac{1}{2}}\boldsymbol{\nabla}\boldsymbol{v}\|_H, \quad \|q\|_Q \stackrel{\text{def}}{=} \|\nu^{-\frac{1}{2}} q\|_H,$$

where $(\cdot,\cdot)$ denotes the standard L^2-inner product in Ω. The standard seminorm of $H^k(\Omega)$ will be denoted by $|\cdot|_k$.

The transient Stokes' problem may be formulated in weak form as follows: For all $t>0$, find $\boldsymbol{u}(t)\in V$ and $p(t)\in Q$ such that

$$\begin{cases} (\partial_t \boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) + b(p,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), & \text{a.e. in} \quad (0,T), \\ b(q,\boldsymbol{u}) = 0, & \text{a.e. in} \quad (0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, & \text{a.e. in} \quad \Omega, \end{cases} \tag{2}$$

for all $\boldsymbol{v}\in V$, $q\in Q$ and with $a(\boldsymbol{u},\boldsymbol{v}) \stackrel{\text{def}}{=} (\nu\boldsymbol{\nabla}\boldsymbol{u},\boldsymbol{\nabla}\boldsymbol{v})$, $\quad b(p,\boldsymbol{v}) \stackrel{\text{def}}{=} -(p,\boldsymbol{\nabla}\cdot\boldsymbol{v})$.

From these definitions, the following classical coercivity and continuity estimates hold:

$$a(\boldsymbol{v},\boldsymbol{v}) \geq \|\boldsymbol{v}\|_V^2, \quad a(\boldsymbol{u},\boldsymbol{v}) \leq \|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V, \quad b(\boldsymbol{v},q) \leq \|\boldsymbol{v}\|_V\|q\|_Q, \tag{3}$$

for all $\boldsymbol{u},\boldsymbol{v}\in V$ and $q\in Q$. It is known that if $\boldsymbol{f}\in C^0([0,T];H)$ and that $\boldsymbol{u}_0\in V\cap H_0(\mathrm{div};\Omega)$ problem (2) admits a unique solution $(\boldsymbol{u},p)$ in $L^2(0,T;V)\times L^2(0,T;Q)$ with $\partial_t\boldsymbol{u}\in L^2(0,T;V')$ (see, *e.g.*, [8]). Throughout this paper, C stands for a generic positive constant independent of the discretization parameters. We also use the notation $a\lesssim b$ meaning $a\leq Cb$.

3 Space and time discretization

In this section we discretize problem (2) in space and in time, using the pressure stabilized Petrov-Galerkin method and a (first or second order) backward difference formula.

We introduce the approximation space W_h with optimal approximation properties. The approximation space could either consist of finite element functions, with $W_h\subset C^0(\bar\Omega)$, or approximation spaces with higher regularity such as NURBS, with $W_h\subset C^1(\bar\Omega)$. In the finite element case, let $\mathcal{T}_h$ denote a conforming, shape regular triangulation of Ω consisting of the triangles K and

$$W_h \stackrel{\text{def}}{=} \{v_h\in H^1(\Omega) : v_h|_K\in P_k(K),\, \forall K\in\mathcal{T}_h\}. \tag{4}$$

For NURBS with continuous derivatives, we refer the reader to [2] for a precise definition of the space and its properties.

The discrete spaces for velocities and pressure respectively are given by $V_h \stackrel{\text{def}}{=} [W_h]^d\cap V$ and $Q_h \stackrel{\text{def}}{=} W_h\cap Q$. The discrete time-derivative $\partial_{\tau,\kappa}\boldsymbol{u}_h^n$ is chosen either as the first,

$$\kappa=1,\ \partial_{\tau,1}\boldsymbol{u}_h^n \stackrel{\text{def}}{=} (\boldsymbol{u}^n-\boldsymbol{u}^{n-1})/\tau,$$

or second

$$\kappa=2,\ \partial_{\tau,2}\boldsymbol{u}_h^n \stackrel{\text{def}}{=} (3\boldsymbol{u}^n-4\boldsymbol{u}^{n-1}+\boldsymbol{u}^{n-2})/(2\tau)$$

order backward difference formula. The formulation then reads: Find $(\boldsymbol{u}_h^n,p_h^n)\in V_h\times Q_h$ such that

$$(\partial_{\tau,\kappa}\boldsymbol{u}_h^n+\boldsymbol{\nabla}p_h^n,\boldsymbol{w}_h) - (\nu\Delta\boldsymbol{u}_h^n,\delta\boldsymbol{\nabla}q_h)_h + a(\boldsymbol{u}_h^n,\boldsymbol{v}_h) + b(q_h,\boldsymbol{u}_h^n) = (\boldsymbol{f}^n,\boldsymbol{w}_h) \tag{5}$$

for all $(\boldsymbol{v}_h,q_h)\in V_h\times Q_h$ and where $\boldsymbol{w}_h=\boldsymbol{v}_h+\delta\boldsymbol{\nabla}q_h$, $(\cdot,\cdot)_h$ denotes the element-wise L^2-scalar product and $\delta=\gamma\frac{h^2}{\nu}$, with $\gamma>0$ a free dimensionless parameter.

For the convergence analysis below we introduce the following Ritz-projection: Find $\big(\boldsymbol{R}_h(\boldsymbol{u},p), P_h(\boldsymbol{u},p)\big) \in V_h \times Q_h$,

$$\begin{aligned} a(\boldsymbol{R}_h(\boldsymbol{u},p),\boldsymbol{v}_h) + b(\boldsymbol{R}_h(\boldsymbol{u},p),\boldsymbol{v}_h) - b(q_h,\boldsymbol{R}_h(\boldsymbol{u},p)) \\ - \big(-\Delta \boldsymbol{R}_h(\boldsymbol{u},p) + \boldsymbol{\nabla} P_h(\boldsymbol{u},p), \delta \boldsymbol{\nabla} q_h \big)_h = a(\boldsymbol{u},\boldsymbol{v}_h) + b(p,\boldsymbol{v}_h) - b(q_h,\boldsymbol{u}) \\ - (-\Delta \boldsymbol{u} + \boldsymbol{\nabla} p, \delta \boldsymbol{\nabla} q_h) \end{aligned} \tag{6}$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$. It is known (see [13] in the finite element case and [2] for the NURBS case) that the Ritz projection satisfies the following a priori error estimate, for $j = 0, 1$:

$$\begin{aligned} \|\partial_t^j(\boldsymbol{R}_h(\boldsymbol{u},p) - \boldsymbol{u})\|_H + h(\|\partial_t^j(\boldsymbol{R}_h(\boldsymbol{u},p) - \boldsymbol{u})\|_V \\ + \delta^{\frac{1}{2}}\|\partial_t^j \boldsymbol{\nabla}(P_h(\boldsymbol{u},p) - p)\|_H + \nu^{-\frac{1}{2}}\|\partial_t^j(P_h(\boldsymbol{u},p) - p)\|_H) \\ \lesssim h^{k+1}(|\partial_t^j \boldsymbol{u}|_{k+1} + |\partial_t^j p|_k). \end{aligned}$$

For the stability analysis of affine pressure approximations, we shall also consider a reduced Ritz-projection $\big(\widetilde{\boldsymbol{R}}_h\boldsymbol{u}, \widetilde{P}_h\boldsymbol{u}\big) \in V_h \times Q_h$, obtained from (6) by omitting the term $(\Delta \boldsymbol{u}, \delta \boldsymbol{\nabla} q_h)$ and taking $p = 0$ in the right hand side, that is,

$$\begin{aligned} a(\widetilde{\boldsymbol{R}}_h\boldsymbol{u},\boldsymbol{v}_h) + b(\widetilde{\boldsymbol{R}}_h\boldsymbol{u},\boldsymbol{v}_h) - b(q_h,\widetilde{\boldsymbol{R}}_h\boldsymbol{u}) - (\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}, \delta\boldsymbol{\nabla} q_h)_h \\ = a(\boldsymbol{u},\boldsymbol{v}_h) - b(q_h,\boldsymbol{u}) \end{aligned} \tag{7}$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$. In this case the following stability estimate holds:

$$\|\widetilde{\boldsymbol{R}}_h\boldsymbol{u}\|_V^2 + \|\delta^{\frac{1}{2}}\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}\|_H^2 \lesssim \|\boldsymbol{u}\|_V^2. \tag{8}$$

4 Stability analysis

In this section we state the main stability results of this paper. We first prove an *a priori* stability bound for the velocity approximations obtained from (5), under the conditions given in the introduction. Then, in the case of affine approximations in space ($k = 1$) and the BDF1 scheme in time ($\kappa = 1$), we derive a stability estimate for the pressure under the standard inverse parabolic CFL condition.

4.1 Velocity

The velocity stability analysis for (5) draws from earlier ideas applied to the SUPG method for the transient transport problem, proposed in [6]. Since the mass-matrix is non-symmetric global stability is obtained not by the standard choice of test functions, $\boldsymbol{v}_h = \boldsymbol{u}_h^n$, $q_h = p_h^n$, but with a perturbation added to the test function for the velocities. Indeed, as we will see below stability is obtained by taking $\boldsymbol{v}_h = \boldsymbol{u}_h^n + \delta\partial_{\tau,\kappa}\boldsymbol{u}_h^n$

Lemma 4.1 *Let $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=\kappa}^N$ denote the solution of* (5) *and assume that, depending on W_h, the conditions on h and τ given in the introduction are satisfied, then there holds:*

$$\|\boldsymbol{u}_h^N\|_H^2 + \tau\sum_{n=\kappa}^{N}(\delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2 + \|\boldsymbol{u}_h^n\|_V^2) \lesssim \tau\sum_{n=\kappa}^{N}\|\boldsymbol{f}^n\|_Q^2 + \sum_{n=0}^{\kappa-1}\|\boldsymbol{u}_h^n\|_H^2.$$

Proof. First take $\boldsymbol{v}_h = \boldsymbol{u}_h^n$ and $q_h = p_h^n$, so that $\boldsymbol{w}_h = \boldsymbol{u}_h^n + \delta \boldsymbol{\nabla} p_h^n$ and

$$(\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n, \boldsymbol{u}_h^n + \delta\boldsymbol{\nabla} p_h^n) - \delta(\nu\Delta\boldsymbol{u}_h^n, \boldsymbol{\nabla} p_h^n)_h + \|\boldsymbol{u}_h^n\|_V^2 = (\boldsymbol{f}^n, \boldsymbol{u}_h^n).$$

Now test with $\boldsymbol{v}_h = \delta\partial_{\tau,\kappa}\boldsymbol{u}_h^n$ and $q_h = 0$,

$$(\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n, \delta\partial_{\tau,\kappa}\boldsymbol{u}_h^n) + \delta(\nu\boldsymbol{\nabla}\boldsymbol{u}_h^n, \boldsymbol{\nabla}\partial_{\tau,\kappa}\boldsymbol{u}_h^n) = (\boldsymbol{f}^n, \delta\partial_{\tau,\kappa}\boldsymbol{u}_h^n).$$

Summing the two equations yields for $n \geq \kappa$

$$\begin{aligned}(2\kappa)^{-1}\partial_{\tau,1}\|\boldsymbol{u}_h^n\|_H^2 + \frac{\kappa-1}{4}(\partial_{\tau,1}\|\tilde{\boldsymbol{u}}_h^n\|_H^2 + \|\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^{n-1}\|_H^2) + \underbrace{\frac{2-\kappa}{2}\tau\|\partial_{\tau,1}\boldsymbol{u}^n\|_H^2}_{I_0} \\ + \|\boldsymbol{u}_h^n\|_V^2 + \delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2 - \underbrace{\delta(\nu\Delta\boldsymbol{u}_h^n, \boldsymbol{\nabla} p_h^n)_h}_{I_1} \\ + \delta(\nu\boldsymbol{\nabla}\boldsymbol{u}_h^n, \boldsymbol{\nabla}\partial_{\tau,\kappa}\boldsymbol{u}_h^n) = (\boldsymbol{f}^n, \boldsymbol{u}_h^n + \delta(\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n)). \quad (9)\end{aligned}$$

Where the notation $\tilde{\boldsymbol{u}}_h^n = 2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}$ is used in the contribution from the BDF2-scheme and

$$\partial_{\tau,1}\|\boldsymbol{u}_h^n\|_H^2 \stackrel{\text{def}}{=} \tau^{-1}(\|\boldsymbol{u}_h^n\|_H^2 - \|\boldsymbol{u}_h^{n-1}\|_H^2).$$

When $\Delta\boldsymbol{u}_h^n|_K \neq 0$, the term I_1 in the left hand side needs to be controlled. Note that the following holds

$$(\nu\Delta\boldsymbol{u}_h^n, \boldsymbol{\nabla} p_h^n)_h = \delta(\nu\Delta\boldsymbol{u}_h^n, \partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n)_h - \delta(\nu\Delta\boldsymbol{u}_h^n, \partial_{\tau,\kappa}\boldsymbol{u}_h^n)_h$$

and after an integration by parts in the second term in the right hand side

$$\begin{aligned}(\nu\Delta\boldsymbol{u}_h^n, \boldsymbol{\nabla} p_h^n)_h = \delta(\nu\Delta\boldsymbol{u}_h^n, \partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n)_h \\ + \delta(\nu\boldsymbol{\nabla}\boldsymbol{u}_h^n, \boldsymbol{\nabla}\partial_{\tau,\kappa}\boldsymbol{u}_h^n) + \underbrace{\sum_F \delta\nu\int_{F\in\mathcal{F}_i} [\![\boldsymbol{\nabla}\boldsymbol{u}_h^n \cdot \boldsymbol{n}]\!]\partial_{\tau,\kappa}\boldsymbol{u}_h^n \mathrm{d}s}_{I_2}, \quad (10)\end{aligned}$$

where $\mathcal{F}_i$ stands for the set of interior faces F of $\mathcal{T}_h$. For the first term in the right hand side of (10) we may use, Cauchy-Schwarz inequality, an inverse inequality and the arithmetic geometric inequality to write

$$\delta(\nu\Delta\boldsymbol{u}_h^n, \partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n)_h \geq -\frac{C_i^2\delta^{\frac{1}{2}}\nu^{\frac{1}{2}}}{4\epsilon h}\|\boldsymbol{u}_h^n\|_V^2 - \epsilon\delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2.$$

Choosing ϵ and γ small enough we see that the term can be absorbed in the left hand side of (9). The term that remains to control is I_2.

For $C^1(\bar{\Omega})$ approximation spaces the term I_2 is zero. Hence for piecewise linear approximation ($I_1 = 0$) and C^1 NURBS we conclude by multiplying (9) with τ, summing over the time levels and applying the Cauchy-Schwarz inequality, the arithmetic-geometric inequality and the Poincaré inequality in the right hand side:

$$|(\boldsymbol{f}^n, \boldsymbol{u}_h^n + \delta(\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n))| \leq \left(\delta + \frac{C_{\mathrm{P}}^2}{\nu}\right)\|\boldsymbol{f}^n\|_H^2 + \frac{1}{4}\|\boldsymbol{u}_h^n\|_V^2 + \frac{1}{4}\|\delta^{\frac{1}{2}}(\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n)\|_H^2.$$

The resulting contributions $\delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2$and $\|\boldsymbol{u}\|_V^2$ may be absorbed in the left hand side of (9).

In the case of high order finite element approximation, i.e. $k \geq 2$, different approaches must be used for $\kappa = 1$ and $\kappa = 2$. For BDF1, we apply the Cauchy-Schwarz inequality in I_2 followed by a trace and inverse inequality to deduce the upper bound

$$\begin{aligned}\delta(\nu\boldsymbol{\nabla}\boldsymbol{u}_h^n, \boldsymbol{\nabla}\partial_{\tau,\kappa}\boldsymbol{u}_h^n) - (\nu\Delta\boldsymbol{u}_h^n, \boldsymbol{\nabla} p_h^n)_h \geq -\frac{C\delta\nu}{h^2}\|\boldsymbol{u}_h\|_V^2 \\ -\frac{\delta}{2}\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2 - \delta\|\partial_{\tau,1}\boldsymbol{u}_h^n\|_H^2. \quad (11)\end{aligned}$$

Choosing the coefficient γ, in δ, sufficiently small and under the inverse CFL-condition $\delta \leq \tau$, for BDF1, we have the estimate

$$\partial_{\tau,1}\|\boldsymbol{u}^n\|_H^2 + \delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2 + \|\boldsymbol{u}_h\|_V^2 \lesssim \frac{C_p^2}{\nu}\|\boldsymbol{f}^n\|_H^2. \quad (12)$$

For BDF2 the term $\frac{1}{2}\tau\|\partial_{\tau,1}\boldsymbol{u}^n\|_H^2$ is not present in the left hand side and the dissipation resulting from the BDF2 scheme, $\frac{1}{4}\|\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^{n-1}\|_H^2$, may not be used to obtain stability. Assume then the stronger inverse CFL-condition $\delta^{\frac{1}{2}} \leq \tau$ to obtain

$$\sum_{F\in\mathcal{F}_i}\delta\nu\int_F [\![\boldsymbol{\nabla}\boldsymbol{u}_h^n\cdot\boldsymbol{n}]\!]\partial_{\tau,2}\boldsymbol{u}_h^n \mathrm{d}s \leq \frac{C\delta\nu}{h^2}\|\boldsymbol{u}_h^n\|_V^2 + \delta\|\partial_{\tau,2}\boldsymbol{u}_h^n\|_H^2 \leq C\gamma\|\boldsymbol{u}_h^n\|_V^2 + \frac{\delta}{2\epsilon}\|\partial_{\tau,2}\boldsymbol{u}_h^n\|_H^2.$$

Since $\delta^{\frac{1}{2}} \leq \tau$, it follows that

$$\delta\|\partial_{\tau,2}\boldsymbol{u}_h^n\|_H^2 \leq \frac{\gamma}{\mu}\sum_{j=0}^{2}\|\boldsymbol{u}_h^{n-j}\|_H^2,$$

and for γ small enough we obtain the stability estimate

$$\begin{aligned}\partial_{\tau,1}\|\boldsymbol{u}^n\|_H^2 + \partial_{\tau,1}\|\tilde{\boldsymbol{u}}_h^n\|_H^2 + \|\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^n\|_H^2 + \delta\|\partial_{\tau,\kappa}\boldsymbol{u}_h^n + \boldsymbol{\nabla} p_h^n\|_H^2 + \|\boldsymbol{u}_h\|_V^2 \\ \lesssim \frac{C_p^2}{\nu}\|\boldsymbol{f}^n\|_H^2 + \frac{\gamma}{\mu}\sum_{j=0}^{1}\|\boldsymbol{u}^{n-j}\|_H^2. \quad (13)\end{aligned}$$

We multiply the equations (12) and (13) with τ and sum over $n = \kappa, \ldots, N$ to conclude. For BDF2 we apply Gronwall's lemma.

4.2 Pressure

For the pressure stability analysis below, we shall make use of the following modified inf-sup condition.

Lemma 4.2 *For all $q_h \in Q_h$ and $\boldsymbol{\xi}_h \in V_h$ we have*

$$\|q_h\|_Q \lesssim \sup_{\boldsymbol{v}_h\in V_h}\frac{|(q_h, \boldsymbol{\nabla}\cdot\boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_V} + \gamma^{-\frac{1}{2}}\|\delta^{\frac{1}{2}}(\boldsymbol{\nabla} q_h + \boldsymbol{\xi}_h)\|_H. \quad (14)$$

Proof. Let $q_h \in Q_h$, it is known (see, *e.g.*, [9, Corollary 2.4]) that there exists $\boldsymbol{v}_q \in H^1_0(\Omega)$ such that $\boldsymbol{\nabla} \cdot \boldsymbol{v}_q = \nu^{-1} q_h$ and $\|\boldsymbol{v}_q\|_V \lesssim \|q_h\|_Q$. Hence, from the orthogonality and approximation properties of the L^2-projection π_h, onto V_h, we have

$$\begin{aligned}\|q_h\|_Q^2 =& -(q_h, \boldsymbol{\nabla} \cdot (\boldsymbol{v}_q - \pi_h \boldsymbol{v}_q)) - (q_h, \boldsymbol{\nabla} \cdot \pi_h \boldsymbol{v}_q) = (\boldsymbol{\nabla} q_h + \boldsymbol{\xi}_h, \boldsymbol{v}_q - \pi_h \boldsymbol{v}_q) \\ & - (q_h, \boldsymbol{\nabla} \cdot \pi_h \boldsymbol{v}_q) \\ \leq & \frac{C}{\gamma^{\frac{1}{2}}} \|\delta^{\frac{1}{2}} (\boldsymbol{\nabla} q_h + \boldsymbol{\xi}_h)\|_H \|q_h\|_Q - (q_h, \boldsymbol{\nabla} \cdot \pi_h \boldsymbol{v}_q),\end{aligned}$$

which completes the proof by noting that $\|\pi_h \boldsymbol{v}_q\|_V \lesssim \|\boldsymbol{v}_q\|_V$.

The next lemma states the stability of affine PSPG pressure approximations with the BDF1 scheme, in the natural norm.

Lemma 4.3 *Assume that* $\boldsymbol{f} \in C^1(0,T;H)$ *and let* $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=1}^N$ *be the solution of* (5) *with* $\kappa = 1$ *and* $k = 1$ *in* (4)*.*

1. *Assume that* $\boldsymbol{u}_0 \in [H^2(\Omega)]^d$ *and let* $\boldsymbol{u}_h^0 = \mathcal{I}_h \boldsymbol{u}_0$*, where* $\mathcal{I}_h$ *is the Lagrange interpolation operator onto* V_h*. Then we have*

$$\begin{aligned}&\tau \left(1 - C_{\mathcal{I}}^2 \frac{h^2}{\nu\tau}\right) \|p_h^1\|_Q^2 + \tau \sum_{n=2}^N \|p_h^n\|_Q^2 \\ &\qquad \lesssim \left(1 + \frac{\delta}{\tau}\right) \|\boldsymbol{\nabla} \boldsymbol{u}_h^0\|_H^2 + \|\boldsymbol{f}^1\|_Q^2 + \frac{1}{\nu} |\boldsymbol{u}_0|_{2,\Omega}^2 \\ &+ \tau \sum_{n=1}^N \left(\|\boldsymbol{u}_h^n\|_V^2 + \|\delta^{\frac{1}{2}} (\boldsymbol{\nabla} p_h^n + \partial_{\tau,1} \boldsymbol{u}_h^n)\|_H^2 + \frac{\delta}{\nu^2} \|\partial_{\tau,1} \boldsymbol{f}^n\|_H^2 + \|\boldsymbol{f}^n\|_Q^2 \right).\end{aligned} \tag{15}$$

2. *Assume that* $\boldsymbol{u}_0 \in [H^1(\Omega)]^d$ *and let* $\boldsymbol{u}_h^0 = \widetilde{\boldsymbol{R}}_h \boldsymbol{u}_0$*. Then we have*

$$\begin{aligned}&\tau \sum_{n=1}^N \|p_h^n\|_Q^2 \lesssim \left(1 + \frac{\delta}{\tau}\right) \|\boldsymbol{\nabla} \boldsymbol{u}_h^0\|_H^2 + \|\boldsymbol{f}^1\|_Q^2 \\ &+ \tau \sum_{n=1}^N \left(\|\boldsymbol{u}_h^n\|_V^2 + \|\delta^{\frac{1}{2}} (\boldsymbol{\nabla} p_h^n + \partial_{\tau,1} \boldsymbol{u}_h^n)\|_H^2 + \frac{\delta}{\nu^2} \|\partial_{\tau,1} \boldsymbol{f}^n\|_H^2 + \|\boldsymbol{f}^n\|_Q^2 \right).\end{aligned} \tag{16}$$

As a result, owing to Lemma 4.1, the pressure is stable in the natural discrete $L^2(0,T;L^2(\Omega))$*-norm under the inverse parabolic CFL-condition* $h^2 \lesssim \nu\tau$*.*

Proof. From (14) and (5), we have

$$\tau \sum_{n=1}^N \|p_h^n\|_Q^2 \lesssim \tau \sum_{n=1}^N \left(\|\delta^{\frac{1}{2}} (\boldsymbol{\nabla} p_h^n + \partial_{\tau,1} \boldsymbol{u}_h^n)\|_H^2 + \|\partial_{\tau,1} \boldsymbol{u}_h^n\|_Q^2 + \|\boldsymbol{u}_h^n\|_V^2 + \|\boldsymbol{f}^n\|_Q^2 \right). \tag{17}$$

As a result, we only need to estimate the acceleration

$$\tau \sum_{n=1}^N \|\partial_{\tau,1} \boldsymbol{u}_h^n\|_Q^2.$$

To this aim, we test (5) with $\boldsymbol{v}_h = \nu^{-1}\tau\partial_{\tau,1}\boldsymbol{u}_h^n$ and $q_h = 0$, multiply by 2τ and sum over $n = 1,\ldots,N$. This yields

$$\tau\sum_{n=1}^{N}\|\partial_{\tau,1}\boldsymbol{u}_h^n\|_Q^2 + \|\boldsymbol{\nabla}\boldsymbol{u}_h^N\|_H^2 - 2\frac{\tau}{\nu}\sum_{n=1}^{N}(p_h^n,\nabla\cdot\partial_{\tau,1}\boldsymbol{u}_h^n) \leq \|\boldsymbol{\nabla}\boldsymbol{u}_h^0\|_H^2 + \tau\sum_{n=1}^{N}\|\boldsymbol{f}^n\|_Q^2. \tag{18}$$

We must now show that the term

$$-\frac{\tau}{\nu}\sum_{n=1}^{N}(p_h^n,\nabla\cdot\partial_{\tau,1}\boldsymbol{u}_h^n) = \underbrace{-\frac{\tau}{\nu}\sum_{n=2}^{N}(p_h^n,\nabla\cdot\partial_{\tau,1}\boldsymbol{u}_h^n)}_{I_1}\underbrace{-\frac{\tau}{\nu}(p_h^1,\nabla\cdot\partial_{\tau,1}\boldsymbol{u}_h^n)}_{I_2}, \tag{19}$$

can be appropriately bounded. From (5) with $\boldsymbol{v}_h = 0$, for $n \geq 2$ we have

$$-(q_h,\nabla\cdot\partial_{\tau,1}\boldsymbol{u}_h^n) = \big(\partial_{\tau,1}(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n - \boldsymbol{f}^n),\delta\nabla q_h\big).$$

Hence, taking $q_h = \nu^{-1}\tau p_h^n$ and after summation over $n = 2,\ldots,N$, we get

$$\begin{aligned} I_1 = \underbrace{\frac{\tau}{\nu}\sum_{n=2}^{N}\big(\partial_{\tau,1}(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n - \boldsymbol{f}^n),\delta(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n)\big)}_{I_3} \\ \underbrace{-\frac{\tau}{\nu}\sum_{n=2}^{N}\big(\partial_{\tau,1}(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n - \boldsymbol{f}^n),\delta\partial_{\tau,1}\boldsymbol{u}_h^n\big)}_{I_4}. \end{aligned} \tag{20}$$

Term I_3 can be lower bounded as follows:

$$\begin{aligned} I_3 &= \frac{\tau}{\nu}\sum_{n=2}^{N}(\partial_{\tau,1}(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n),\delta(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n)) - \frac{\tau}{\nu}\sum_{n=2}^{N}(\partial_{\tau,1}\boldsymbol{f}^n,\delta(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n)) \\ &\geq \frac{\tau}{2\nu}\sum_{n=2}^{N}\partial_{\tau,1}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n)\|_H^2 - \frac{\tau}{\nu}\sum_{n=2}^{N}(\partial_{\tau,1}\boldsymbol{f}^n,\delta(\partial_{\tau,1}\boldsymbol{u}_h^n + \nabla p_h^n)) \\ &\geq -\frac{1}{2\nu}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}^1 + \nabla p_h^1)\|_H^2 - \frac{\delta}{2\nu^2}\tau\sum_{n=2}^{N}\|\partial_{\tau,1}\boldsymbol{f}^n\|_H^2 - \frac{\tau}{2}\sum_{n=2}^{N}\|\delta^{\frac{1}{2}}(\nabla p_h^n + \partial_{\tau,1}\boldsymbol{u}_h^n)\|_H^2. \end{aligned} \tag{21}$$

For I_4 we note that, from (5) with $q_h = 0$, we have $(n \geq 2)$

$$-\big(\partial_{\tau,1}(\partial_{\tau,1}\boldsymbol{u}^n + \nabla p_h^n - \boldsymbol{f}^n),\boldsymbol{v}_h\big) = (\nu\nabla\partial_{\tau,1}\boldsymbol{u}_h^n,\nabla\boldsymbol{v}_h).$$

Hence, testing this expression with $\boldsymbol{v}_h = \nu^{-1}\tau\partial_{\tau,1}\boldsymbol{u}^n$ and summing over $n = 2,\ldots,N$, yields

$$I_4 = \frac{\delta}{\nu}\tau\sum_{n=2}^{N}\|\partial_{\tau,1}\boldsymbol{u}_h^n\|_V^2 \geq 0. \tag{22}$$

Therefore, inserting the estimations of (21) and (22) into (20), gives

$$I_1 \geq -\frac{1}{2\nu}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}^1 + \nabla p_h^1)\|_H^2 - \frac{\delta}{2\nu^2}\tau\sum_{n=2}^{N}\|\partial_{\tau,1}\boldsymbol{f}^n\|_H^2 - \frac{\tau}{2}\sum_{n=2}^{N}\|\delta^{\frac{1}{2}}(\nabla p_h^n + \partial_{\tau,1}\boldsymbol{u}_h^n)\|_H^2. \tag{23}$$

It now remains to derive a bound for I_2, which corresponds to the first time-step. We have

$$I_2 = \underbrace{-\frac{1}{\nu}(p_h^1, \nabla \cdot \boldsymbol{u}_h^1)}_{I_5} + \underbrace{\frac{1}{\nu}(p_h^1, \nabla \cdot \boldsymbol{u}_h^0)}_{I_6}. \tag{24}$$

Term I_5 can be controlled using an argument similar to the one used to estimate I_1. We first note that, from (5) with $\boldsymbol{v}_h = 0$, for $n = 1$ we have

$$-(q_h, \nabla \cdot \boldsymbol{u}_h^1) = \big(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1 - \boldsymbol{f}^1, \delta \nabla q_h\big).$$

Hence, taking $q_h = \nu^{-1} p_h^1$, we get

$$I_5 = \underbrace{\frac{1}{\nu}\left(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1 - \boldsymbol{f}^1, \delta(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1)\right)}_{I_7} \underbrace{-\frac{1}{\nu}\left(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1 - \boldsymbol{f}^1, \delta\partial_{\tau,1}\boldsymbol{u}_h^1\right)}_{I_8}. \tag{25}$$

Term I_7 is estimated as follows

$$\begin{aligned} I_7 =& \frac{1}{\nu}\left(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1, \delta(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1)\right) - \frac{1}{\nu}\left(\boldsymbol{f}^1, \delta(\partial_{\tau,1}\boldsymbol{u}_h^1 + \nabla p_h^1)\right) \\ \geq& \frac{3}{4\nu}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}^1 + \nabla p_h^1)\|_H^2 - \|\boldsymbol{f}^1\|_Q^2. \end{aligned} \tag{26}$$

For I_8, we take $n = 1$, $q_h = 0$ and $\boldsymbol{v}_h = \delta\nu^{-1}\partial_{\tau,1}\boldsymbol{u}_1^1$ in (5), which yields

$$I_8 = \frac{\delta}{\nu}(\nu \boldsymbol{\nabla}\boldsymbol{u}_h^1, \boldsymbol{\nabla}\partial_{\tau,1}\boldsymbol{u}_1^1) = \frac{\delta}{2\nu}\partial_{\tau,1}\|\boldsymbol{u}_h^1\|_V^2 + \frac{\delta\tau}{2\nu}\|\partial_{\tau,1}\boldsymbol{u}_h^1\|_V^2 \geq -\frac{\delta}{2\tau\nu}\|\boldsymbol{u}_h^0\|_V^2. \tag{27}$$

The estimation of I_6 depends on the choice of the discrete initial velocity $\boldsymbol{u}_h^0$. Let us first consider the case $\boldsymbol{u}_h^0 = \mathcal{I}_h^0\boldsymbol{u}_0$. Since $\boldsymbol{\nabla} \cdot \boldsymbol{u}_0 = 0$, we have

$$I_6 = \frac{1}{\nu}\left(p_h^1, \boldsymbol{\nabla} \cdot (\mathcal{I}_h^0\boldsymbol{u}_0 - \boldsymbol{u}_0)\right) \leq \frac{C_{\mathcal{I}}^2}{2\nu}h^2\|p_h^1\|_Q^2 + \frac{1}{2}|\boldsymbol{u}_0|_{2,\Omega}^2. \tag{28}$$

Therefore, inserting the estimations of (25), (30), (25) and (27) into (24), gives

$$I_2 \geq \frac{3}{4\nu}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}^1 + \nabla p_h^1)\|_H^2 - \|\boldsymbol{f}^1\|_Q^2 - \frac{\delta}{2\tau\nu}\|\boldsymbol{u}_h^0\|_V^2 - \frac{C_{\mathcal{I}}^2}{2\nu}h^2\ \|p_h^1\|_Q^2 - \frac{1}{2}|\boldsymbol{u}_0|_{2,\Omega}^2. \tag{29}$$

The stability estimate (15) then follows by applying (18) to (17) and inserting (23) and (29) into (19).

We will now choose the initial data as the reduced Ritz-projection (7) and show that this choice allows for less regular initial data. If $\boldsymbol{u}_h^0 = \widetilde{\boldsymbol{R}}_h\boldsymbol{u}_0$, we use (7) with $\boldsymbol{v}_h = \boldsymbol{0}$, and estimate term I_6 as follows

$$\begin{aligned} I_6 =& \frac{1}{\nu}(p_h^1, \boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{R}}_h\boldsymbol{u}_0) = \frac{\delta}{\nu}(\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}_0, \boldsymbol{\nabla}p_h^1) \\ =& \frac{\delta}{\nu}(\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}_0, \boldsymbol{\nabla}p_h^1 + \partial_{\tau,1}\boldsymbol{u}^1) - \frac{\delta}{\nu}(\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}_0, \partial_{\tau,1}\boldsymbol{u}^1) \\ \geq& -\frac{1}{\nu}\left(1 + \frac{\delta}{2\tau}\right)\|\delta^{\frac{1}{2}}\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}_0\|_H^2 - \frac{1}{2\nu}\|\delta^{\frac{1}{2}}(\boldsymbol{\nabla}p_h^1 + \partial_{\tau,1}\boldsymbol{u}^1)\|_H^2 - \frac{\tau}{2}\|\partial_{\tau,1}\boldsymbol{u}^1\|_Q^2, \end{aligned} \tag{30}$$

and hence, (29) becomes

$$I_2 \geq \frac{3}{4\nu}\|\delta^{\frac{1}{2}}(\partial_{\tau,1}\boldsymbol{u}^1 + \nabla p_h^1)\|_H^2 - \|\boldsymbol{f}^1\|_Q^2 \\ - \frac{\delta}{2\tau\nu}\|\boldsymbol{u}_h^0\|_V^2 - \frac{1}{\nu}\left(1+\frac{\delta}{2\tau}\right)\|\delta^{\frac{1}{2}}\boldsymbol{\nabla}\widetilde{P}_h\boldsymbol{u}_0\|_H^2 - \frac{\tau}{2}\|\partial_{\tau,1}\boldsymbol{u}^1\|_Q^2 \quad (31)$$

the stability estimate (16) then follows by applying (18) to (17) and inserting (23) and (31) into (19), after having applied the stability estimate for the Ritz projection (8).

We conclude this subsection with a series of remarks.

- Lemma 4.3 shows that the PSPG small time-step pressure instability has its origins in the initial velocity approximation, as for symmetric stabilization methods [7].
- However, in opposition to the case of symmetric stabilization methods, the specific Ritz-projection (7) does not remove the pressure instability. The reason for this is the coupling between the time-derivative and the pressure gradient, appearing in (30), which leads to the factor δ/τ in the initial data contribution of (16).
- Lemma 4.3 also shows that the initialization with the Ritz-projection allows pressure stability with less regular initial data.

5 Convergence analysis

The next theorem provides an optimal *a priori* error estimate for the velocities.

Theorem 5.1 *Let* $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=\kappa}^N$ *denote the solution of* (5)*. Then, under the stability conditions given in the introduction, we have*

$$\|\boldsymbol{u}_h^N - \boldsymbol{u}(t^N)\|_H^2 + \tau\sum_{n=\kappa}^N \|\boldsymbol{u}_h^n - \boldsymbol{u}(t^n)\|_V^2 \\ + \tau\sum_{n=\kappa}^N\left(\delta\big\|\partial_{\tau,\kappa}(\boldsymbol{u}_h^n - \boldsymbol{u}(t^n)) + \boldsymbol{\nabla}(p_h^n - p(t^n))\big\|_H^2 + \|\boldsymbol{u}_h^n\|_V^2\right) \\ \leq \widehat{C}\left(h^{2k}+\tau^{2\kappa}\right) + \sum_{j=0}^{\kappa-1}\|\boldsymbol{u}_h^j - \boldsymbol{u}(t^j)\|_H^2,$$

where $\widehat{C}$ *depends on Sobolev norms of* $\boldsymbol{u}$ *and* p *in a fashion detailed below.*

Proof. Decompose the error as $\boldsymbol{u}-\boldsymbol{u}_h = \boldsymbol{u}-\boldsymbol{R}_h(\boldsymbol{u},p)+\boldsymbol{R}_h(\boldsymbol{u},p)-\boldsymbol{u}_h = \boldsymbol{\eta}+\boldsymbol{\theta}$ and $p-p_h = p-P_h(\boldsymbol{u},p)+P_h(\boldsymbol{u},p)-p_h = \zeta-\xi_h$. Consider the discrete error injected in the formulation (5), using Galerkin orthogonality in the first equality and the properties of the Ritz projection in the second we have

$$
\begin{aligned}
&(\partial_{\tau,\kappa}\boldsymbol{\theta}_h^n + \boldsymbol{\nabla}\xi_h, \boldsymbol{w}_h) - (\nu\Delta\boldsymbol{\theta}^n, \delta\boldsymbol{\nabla} q_h)_h + a(\boldsymbol{\theta}^n, \boldsymbol{v}_h) + b(q_h, \boldsymbol{\theta}) \\
&\quad = \big(\partial_{\tau,\kappa}\boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \partial_t \boldsymbol{u}(t^n) + \boldsymbol{\nabla}\big(P_h(\boldsymbol{u}(t^n), p(t^n)) - p(t^n)\big), \boldsymbol{w}_h\big) \\
&\qquad\qquad - \big(\nu\Delta(\boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \boldsymbol{u}(t^n)), \delta\boldsymbol{\nabla} q_h\big)_h \\
&\quad + a\big(\boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \boldsymbol{u}(t^n), \boldsymbol{v}_h\big) + b\big(q_h, \boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \boldsymbol{u}(t^n)\big) \\
&\qquad = \big(\partial_{\tau,\kappa}\boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \partial_{\tau,\kappa}\boldsymbol{u}(t^n) + \partial_{\tau,\kappa}\boldsymbol{u}(t^n) - \partial_t\boldsymbol{u}(t^n), \boldsymbol{w}_h\big). \qquad (32)
\end{aligned}
$$

It follows that the functions $\{\boldsymbol{\theta}^n, \xi^n\}_{n=1}^N$ satisfies the formulation (5) with $\boldsymbol{\theta}^0 = \boldsymbol{R}_h\big(\boldsymbol{u}_0, p(0)\big) - \boldsymbol{u}_h^0$ and source term

$$
\boldsymbol{f}^n = \partial_{\tau,\kappa}\boldsymbol{R}_h(\boldsymbol{u}(t^n), p(t^n)) - \partial_{\tau,\kappa}\boldsymbol{u}(t^n) + \partial_{\tau,\kappa}\boldsymbol{u}(t^n) - \partial_t\boldsymbol{u}(t^n).
$$

Applying now the stability estimate of Lemma 4.1 to the perturbation equation (32) we may write

$$
\|\boldsymbol{\theta}^N\|_H^2 + \tau\sum_{n=\kappa}^N (\delta\|\partial_{\tau,\kappa}\boldsymbol{\theta}^n + \boldsymbol{\nabla}\xi_h\|_H^2 + \|\boldsymbol{\theta}^n\|_V^2) \lesssim \tau\sum_{n=\kappa}^N \|\partial_{\tau,\kappa}\boldsymbol{\eta}^n + \partial_{\tau,\kappa}\boldsymbol{u}(t^n) - \partial_t\boldsymbol{u}(t^n)\|_Q^2.
$$

We conclude by applying the estimates for the Ritz projection and standard truncation error estimates yielding

$$
\begin{aligned}
&\tau\sum_{n=\kappa}^N \|\partial_{\tau,\kappa}\boldsymbol{\eta}^n + \partial_{\tau,\kappa}\boldsymbol{u}(t^n) - \partial_t\boldsymbol{u}(t^n)\|_Q^2 \\
&\qquad\qquad \lesssim h^{2(k+1)}\nu^{-1}\int_0^T |\partial_t\boldsymbol{u}(t)|_{k+1}^2 \mathrm{d}t + \tau^\kappa\nu^{-1}\int_0^T \|\partial_t^{\kappa+1}\boldsymbol{u}(t)\|_H^2 \mathrm{d}t.
\end{aligned}
$$

An optimal *a priori* error estimate for the pressure follows in a similar fashion, by combining the above result with the pressure stability estimate provided by Theorem 4.3 applied to the perturbation equation (32). The result is stated in the next corollary.

Corollary 5.2 *Let $\{(\boldsymbol{u}_h^n, p_h^n)\}_{n=1}^N$ denote the solution of* (5) *with $\boldsymbol{u}_h^0 = \widetilde{\boldsymbol{R}}_h\boldsymbol{u}_0$, $\kappa = 1$ and $k = 1$ in* (4). *Assume that $h^2 \lesssim \nu\tau$. Then, there holds*

$$
\tau\sum_{n=1}^N \|p_h^n - p(t^n)\|_Q^2 \leq \widetilde{C}\left(h^2 + \tau^2\right).
$$

where $\widetilde{C}$ depends on Sobolev norms of $\boldsymbol{u}$ and p.

6 Numerical example

The pressure instability has been thoroughly investigated in [4]. The velocity instability indicated by the theory above however has not to our best knowledge been reported in the literature. This is probably because since it is triggered by a coupling between the jump of the gradient and the time-derivative, in many cases the method remains stable since the gradient jumps are so small for high order elements that the term remains bounded by the viscous dissipation and the dissipation of the time-discretization scheme.

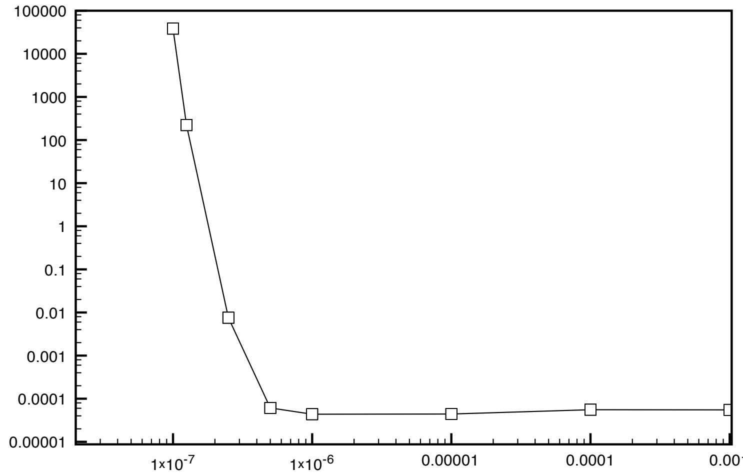

Figure 1: $l^\infty((0,200\tau);L^2(\Omega))$-norm error of the velocities versus time step-size τ

Here we give an example of the instability when using the Crank-Nicolson scheme in time and third order polynomial approximation for both velocities and pressure. The package FreeFem++ was used [10]. The computational domain was the unit square, with ten elements on each side and a Delaunay triangulation. The source term and boundary data were chosen so that

$$\boldsymbol{u}_0 = \left[\begin{array}{c} \cos(t)\sin(\pi x-0.7)\sin(\pi y+0.2) \\ \cos(t)\cos(\pi x-0.7)\sin(\pi y+0.2) \end{array}\right],$$
$$p = \cos(t)(\sin(x)\cos(y)+(\cos(1)-1)\sin(1)).$$

Starting from the timestep $\tau = 10^{-3}$ we ran a sequence of simulations with decreasing time step. For each timestep-size we took 200 steps, so that the final time actually decreased for smaller steps. The error in the $l^\infty((0,200\tau);L^2(\Omega))$ norm against timestep-size is reported in Figure 1. Note that for step-sizes down to $\tau = 10^{-6}$ the method remains stable on this (shrinking) time interval. For $\tau < 10^{-6}$ however a brutal loss of stability is observed. In the shown example $\gamma = 0.075$, however we tried values of γ down to $\gamma = 5.0\times 10^{-5}$ and the only effect was to postpone the onset of instability. Needless to say for such a small γ the pressure is unstable.

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