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# Tensor approach to mixed high-order moments of absorbing Markov chains

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Tensor approach to mixed high-order moments of  
absorbing Markov chains*

Danil Nemirowsky

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Thème COM

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*apport  
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## **Tensor approach to mixed high-order moments of absorbing Markov chains**

Danil Nemirovsky

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**Abstract:** Moments of absorbing Markov chain are considered. First moments and non-mixed second moments are determined in classical textbooks such as the book of J. Kemeny and J. Snell “Finite Markov Chains”. The reason is that the first moments and the non-mixed second moments can be easily expressed in a matrix form. Since the representation of mixed moments of higher orders in matrix form is not straightforward, if ever possible, they were not calculated. The gap is filled by this paper. Tensor approach to the mixed high-order moments is proposed and closed-form expressions for the moments are discovered.

**Key-words:** Absorbing Markov Chain, high-order moment, tensor

## **Une approche tensorielle pour le calcul des moments mixtes d'ordre supérieur des chaînes de Markov avec absorption**

**Résumé :** Cet article s'intéresse au calcul des moments d'une chaîne de Markov absorbante. Les premiers et les seconds moments non-mixtes sont déterminés dans les livres classiques, tel que le livre de J. Kemeny et J. Snell "Finite Markov Chains". La raison en est que les premiers et les seconds moments non-mixtes s'expriment facilement sous forme matricielle, ce qui n'est pas le cas des moments mixtes d'ordre arbitraire. Le fossé est comblé dans cet article où grâce à une approche tensorielle des formules explicites pour les moments supérieurs mixtes d'ordre arbitraire sont obtenues.

**Mots-clés :** Chaînes de Markov absorbantes, moments d'ordre supérieur, tenseur.

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## 1 Introduction

Let us consider an absorbing Markov chain and let matrix  $P$  be its transition matrix. By renumbering the states we can decompose matrix  $P$  in the following way

$$P = \begin{pmatrix} I & 0 \\ S & Q \end{pmatrix},$$

where submatrix  $Q$  is a substochastic matrix corresponding to transient states. Let  $T$  be the set of transient states and  $\bar{T}$  be the set of absorbing states. We can define a fundamental matrix  $Z$  of the absorbing Markov chain

$$Z = (I - Q)^{-1} = I + Q + Q^2 + \dots$$

Fundamental matrix  $Z = \{z_{ij}\}_{i,j \in T}$  has the following probabilistic interpretation.

**Definition 1.1** Define  $N_j$  to be a function giving the total number of times before absorption that the absorbing Markov chain visits a transient state  $j$ .

Let us denote by  $E_i [N_j]$  the first moment of function  $N_j$  assuming that the Markov chain starts at state  $i$ , where  $i, j \in T$ . Then

$$Z = \{E_i [N_j]\}_{i,j \in T}$$

as it is noted in [1]. Non-mixed second moments  $E_i [N_j^2]$  can also be found [1] with the help of matrix  $Z$  as

$$\{E_i [N_j^2]\}_{i,j \in T} = Z(2Z_{dg} - I),$$

where  $Z_{dg}$  is the same matrix as  $Z$ , but all off-diagonal elements are set to zero.

However, the mixed second moments and the mixed higher-order moments are not so easy to calculate, but we manage it in the further discussion.

## 2 Mixed second moments in matrix form

First we consider mixed second moments  $E_i[N_j N_k]$  to show that calculation of them is not straightforward in matrix form.

Let us denote by  $u_j^k$  the indicator function  $1_{\{X_k=j\}}$ . We note that  $N_j = \sum_{\varphi=0}^{\infty} u_j^{\varphi}$ .

**Theorem 2.1**  $E_i[N_j N_k]$  is finite.

**Proof.** When we have proved the statement, it justifies our algebra with series below.

$$E_i[N_j N_k] = E_i \left[ \left( \sum_{\varphi=0}^{\infty} u_j^{\varphi} \right) \left( \sum_{\psi=0}^{\infty} u_k^{\psi} \right) \right] = E_i \left[ \sum_{\varphi=0}^{\infty} \sum_{\psi=0}^{\infty} u_j^{\varphi} u_k^{\psi} \right] = \sum_{\varphi=0}^{\infty} \sum_{\psi=0}^{\infty} E_i \left[ u_j^{\varphi} u_k^{\psi} \right].$$

$E_i \left[ u_j^{\varphi} u_k^{\psi} \right]$  is the probability that the process is in state  $j$  at step  $\varphi$  and in state  $k$  at step  $\psi$ , starting in state  $i$ .

We need to consider three cases

- Let  $\varphi = \psi$ . If states  $j$  and  $k$  are equal then we have that  $E_i \left[ u_j^{\varphi} u_k^{\varphi} \right] = p_{ij}^{(\varphi)}$ ; if states  $j$  and  $k$  are not equal then we have that  $E_i \left[ u_j^{\varphi} u_k^{\varphi} \right] = 0$  since the process cannot be in two different states at the same step  $\varphi = \psi$ . Hence, we write  $E_i \left[ u_j^{\varphi} u_k^{\varphi} \right] = p_{ij}^{(\varphi)} \delta_{jk}$ , where  $\delta_{jk}$  here and further is the Kronecker symbol.
- Let  $\varphi < \psi$ , and let  $d_1 = \psi - \varphi$ . Then,  $E_i \left[ u_j^{\varphi} u_k^{\psi} \right]$  is the probability that the process is in state  $j$  at step  $\varphi$ , and in state  $k$  at step  $\varphi + d_1$ . Hence,  $E_i \left[ u_j^{\varphi} u_k^{\psi} \right] = p_{ij}^{(\varphi)} p_{jk}^{(d_1)}$ .
- Let  $\varphi > \psi$ , and let  $d_2 = \varphi - \psi$ . Then,  $E_i \left[ u_j^{\varphi} u_k^{\psi} \right]$  is the probability that the process is in state  $k$  at step  $\psi$ , and in state  $j$  at step  $\psi + d_2$ . Hence,  $E_i \left[ u_j^{\varphi} u_k^{\psi} \right] = p_{ik}^{(\psi)} p_{kj}^{(d_2)}$ .

We proceed as following.

$$\begin{aligned} E_i[N_j N_k] &= \sum_{\varphi=0}^{\infty} \sum_{\psi=0}^{\infty} E_i \left[ u_j^{\varphi} u_k^{\psi} \right] = \\ &= \sum_{\varphi=0}^{\infty} \left( \sum_{\psi=0}^{\varphi-1} E_i \left[ u_j^{\varphi} u_k^{\psi} \right] + E_i \left[ u_j^{\varphi} u_k^{\varphi} \right] + \sum_{\psi=\varphi+1}^{\infty} E_i \left[ u_j^{\varphi} u_k^{\psi} \right] \right) = \\ &= \sum_{\varphi=0}^{\infty} \left( \sum_{\psi=0}^{\varphi-1} p_{ik}^{(\psi)} p_{kj}^{(\varphi-\psi)} + p_{ij}^{(\varphi)} \delta_{jk} + \sum_{\psi=\varphi+1}^{\infty} p_{ij}^{(\varphi)} p_{jk}^{(\psi-\varphi)} \right). \end{aligned}$$

According to [1, Corollary 3.1.2], there are numbers  $b > 0$ ,  $0 < d < 1$  such that  $p_{ij}^\varphi \leq bd^\varphi$ , and we can give the following estimate.

$$\begin{aligned}
E_i [N_j N_k] &\leq \sum_{\varphi=0}^{\infty} \left( \sum_{\psi=0}^{\varphi-1} (bd^\psi) (bd^{\varphi-\psi}) + bd^\varphi \delta_{jk} + \sum_{\psi=\varphi+1}^{\infty} (bd^\varphi) (bd^{\psi-\varphi}) \right) = \\
&= \sum_{\varphi=0}^{\infty} \left( b^2 \sum_{\psi=0}^{\varphi-1} d^\varphi + bd^\varphi \delta_{jk} + b^2 \sum_{\psi=\varphi+1}^{\infty} d^\psi \right) = \\
&= \sum_{\varphi=0}^{\infty} \left( b^2 \varphi d^\varphi + bd^\varphi \delta_{jk} + b^2 d^{\varphi+1} \sum_{\psi=0}^{\infty} d^\psi \right) = \sum_{\varphi=0}^{\infty} \left( b^2 \varphi d^\varphi + bd^\varphi \delta_{jk} + b^2 \frac{d^{\varphi+1}}{1-d} \right) = \\
&= \sum_{\varphi=0}^{\infty} b^2 \varphi d^\varphi + \sum_{\varphi=0}^{\infty} bd^\varphi \delta_{jk} + b^2 \frac{1}{1-d} \sum_{\varphi=0}^{\infty} d^{\varphi+1} = b^2 \sum_{\varphi=0}^{\infty} \varphi d^\varphi + b \delta_{jk} \frac{1}{1-d} + b^2 \frac{d}{(1-d)^2}.
\end{aligned}$$

Since  $\frac{(\varphi+1)d^{\varphi+1}}{\varphi d^\varphi} = \frac{\varphi+1}{\varphi} d \rightarrow d < 1$ , when  $\varphi \rightarrow \infty$ , the series  $\sum_{\varphi=0}^{\infty} \varphi d^\varphi$  converges. This completes the proof. ■

Now that we have proved that  $E_i[N_j N_k]$  is finite, let us calculate its value. We define matrix  $\Lambda(i)$  as

$$\Lambda_{jk}(i) = E_i[N_j N_k].$$

**Theorem 2.2** *The matrix of the mixed second order moment is given by*

$$\Lambda(i) = \sum_{\nu \in T} z_{i\nu} (D(\nu)Z + ZD(\nu) - D(\nu)),$$

where matrix  $D(\nu)$  is defined by

$$D_{jk}(\nu) = \begin{cases} 1, & \text{if } \nu = j = k, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** Let us calculate  $E_\nu [N_j N_k]$ . Following the principal idea of [1, Theorem 3.3.3] we ask where the process can go in one step, from its starting position  $\nu$ . It can go to state  $\varphi$  with probability  $p_{\nu\varphi}$ . If the new state is absorbing, then we can never reach states  $j$  or  $k$  again, and the only possible contribution is from the initial state, which is  $\delta_{\nu j} \delta_{\nu k}$ . If the new state is transient then we will be in state  $j$   $\delta_{\nu j}$  times from the initial state, and  $N_j$  times from the later steps, and we will be in state  $k$   $\delta_{\nu k}$  times from the initial state, and  $N_k$  times from the later steps. Let us denote by  $T$  the set of



transient states, and by  $\bar{T}$  the set of absorbing states. We have

$$\begin{aligned}
E_\nu [N_j N_k] &= \sum_{\varphi \in \bar{T}} p_{\nu\varphi} \delta_{\nu j} \delta_{\nu k} + \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi [(N_j + \delta_{\nu j})(N_k + \delta_{\nu k})] = \\
&= \sum_{\varphi \in \bar{T}} p_{\nu\varphi} \delta_{\nu j} \delta_{\nu k} + \sum_{\varphi \in T} p_{\nu\varphi} (E_\varphi [N_j N_k] + \delta_{\nu j} E_\varphi [N_k] + E_\varphi [N_j] \delta_{\nu k} + \delta_{\nu j} \delta_{\nu k}) = \\
&= \sum_{\varphi \in T} p_{\nu\varphi} (E_\varphi [N_j N_k] + \delta_{\nu j} E_\varphi [N_k] + E_\varphi [N_j] \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k} = \\
&= \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi [N_j N_k] + \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} E_\varphi [N_k] + E_\varphi [N_j] \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}. \tag{1}
\end{aligned}$$

We recall that  $z_{\varphi j} = E_\varphi [N_j]$ . Let us denote  $\varepsilon(\varphi, j, k) = E_\varphi [N_j N_k]$ . Let us continue as follows

$$\begin{aligned}
E_\nu [N_j N_k] &= \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi [N_j N_k] + \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} E_\varphi [N_k] + E_\varphi [N_j] \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}. \\
\varepsilon(\nu, j, k) - \sum_{\varphi \in T} p_{\nu\varphi} \varepsilon(\varphi, j, k) &= \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} z_{\varphi k} + z_{\varphi j} \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}. \\
\sum_{\varphi \in T} (\delta_{\nu\varphi} - p_{\nu\varphi}) \varepsilon(\varphi, j, k) &= \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} z_{\varphi k} + z_{\varphi j} \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}.
\end{aligned}$$

Let us multiply the last expression by  $z_{i\nu}$  and sum over  $\nu$ .

$$\sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T} (\delta_{\nu\varphi} - p_{\nu\varphi}) \varepsilon(\varphi, j, k) = \sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} z_{\varphi k} + z_{\varphi j} \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}.$$

Next let us consider the lefthand side of the expression. Let us fix  $j, k$  for the moment. We can reformulate the lefthand side in matrix terms. We can consider  $\varepsilon(\varphi, j, k)$  as a vector indexed by  $\varphi$ , let say  $\varepsilon(\varphi, j, k) = \lambda_\varphi(j, k)$ . Let us form matrices  $Q = \{p_{\nu\varphi}\}_{\nu, \varphi \in T}$ ,  $Z = \{z_{\varphi j}\}_{\varphi, j \in T}$ , and  $I = \{\delta_{\nu\varphi}\}_{\nu, \varphi \in T}$  is the identity matrix. One can see that the lefthand side can be formulated as

$$Z(I - Q)\lambda(j, k),$$

and, since  $Z = (I - Q)^{-1}$ , we have

$$Z(I - Q)\lambda(j, k) = \lambda(j, k),$$

or, written in a component form,

$$\sum_{\nu \in T} z_{i\nu} \sum_{\varphi \in T} (\delta_{\nu\varphi} - p_{\nu\varphi}) \varepsilon(\varphi, j, k) = \varepsilon(i, j, k).$$

Now we consider the righthand side

$$\begin{aligned}
&\sum_{\varphi \in T} p_{\nu\varphi} (\delta_{i j} z_{\varphi k} + z_{\varphi j} \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k} = \\
&\delta_{\nu j} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi k} + \delta_{\nu k} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi j} + \delta_{\nu j} \delta_{\nu k}. \tag{2}
\end{aligned}$$

One can see that  $\delta_{\nu j}\delta_{\nu k} = D_{jk}(\nu)$  and  $\delta_{\nu j} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi k} = \{D(\nu)QZ\}_{jk}$ , and  $\delta_{ik} \sum_{\varphi \in T} p_{\nu\varphi} z_{\varphi j} = \{QZD(\nu)\}_{jk}$ . Hence, we can write (2) in a matrix form

$$D(\nu)QZ + QZD(\nu) + D(\nu).$$

Let us analyse the last expression

$$\begin{aligned} D(\nu)QZ + QZD(\nu) + D(\nu) &= D(\nu)(Z - I) + (Z - I)D(\nu) + D(\nu) = \\ &= D(\nu)Z - D(\nu) + ZD(\nu) - D(\nu) + D(\nu) = D(\nu)Z + ZD(\nu) - D(\nu). \end{aligned}$$

Thus, we can complete the proof by concluding that

$$\Lambda(i) = \sum_{\nu \in T} z_{i\nu} (D(\nu)Z + ZD(\nu) - D(\nu)).$$

■

One can see that we have to consider the mixed second moments either as a vector  $\lambda(j, k)$  depending on two indices or as a matrix  $\Lambda(i)$  depending on one index in the above proof. We need this trick because of poverty of matrix operations. In contrast to matrix approach, calculation of the mixed second moments and the mixed high-order moments is natural in tensor form, as we shall show below.

### 3 Introduction to tensors

We give a brief introduction to basic facts from tensor theory which we shall use in further sections. We do not present tensor theory in its completeness, we just define what we need for our application to the mixed high-order moments. Interested reader is referred to [2, 3] for more details.

Tensors are a generalization of such notions as vector and linear operator. Firstly, let us remind the notion of vector.

We consider a vector as an objective quantity having a magnitude and a direction. The vector does not depend on the way we describe the world. We denote the vector under consideration by  $\mathbf{a}$ . If we fix a coordinate system with its basis,  $(e_1, e_2, \dots, e_n)$ , we can represent the vector as an array of real numbers, coordinates of the vector,  $(a^1, a^2, \dots, a^n)$ ,

$$\mathbf{a} = \sum_{i=1}^n a^i e_i.$$

When we change the basis or the coordinate system, we recalculate the coordinates by certain rules, but the vector itself does not change, see Figure 1. Let us assume now that we know only the vector and we do not know the coordinates of the vector. How can we determine them? It turns out that we can find a vector for each coordinate multiplying which by vector  $\mathbf{a}$  by inner product we determine certain coordinate. Let  $e^i$  is such a vector for coordinate  $a^i$ .

$$a^i = \mathbf{a} \cdot e^i, \quad i = 1, \dots, n.$$

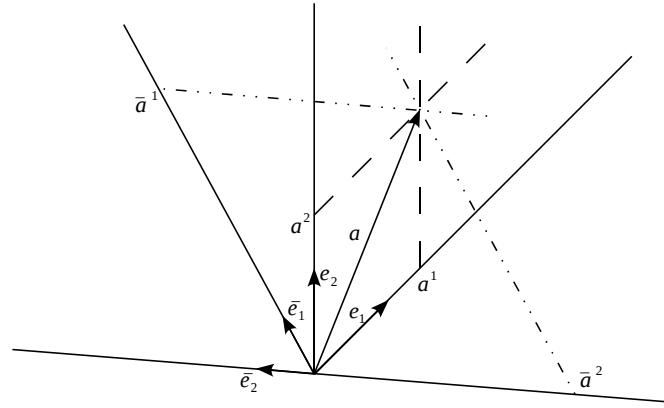


Figure 1: Change of a basis of a coordinate system. At the figure we have vector  $\mathbf{a}$ , basis  $(e_1, e_2)$ , coordinates of vector  $\mathbf{a}$  in the basis  $(a^1, a^2)$ , new basis  $(\bar{e}_1, \bar{e}_2)$ , coordinates of vector  $\mathbf{a}$  in the new basis  $(\bar{a}^1, \bar{a}^2)$ .

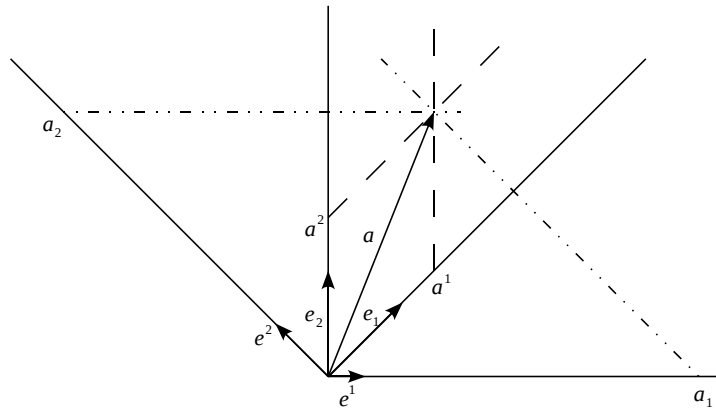


Figure 2: Dual basis. At the figure we have vector  $\mathbf{a}$ , basis  $(e_1, e_2)$ , coordinates of vector  $\mathbf{a}$  in the basis  $(a^1, a^2)$ , dual basis  $(e^1, e^2)$ , coordinates of vector  $\mathbf{a}$  in the dual basis  $(a_1, a_2)$ .

Vectors  $e^i$  are linear independent and, hence, form other basis which is called dual basis. Dual basis  $(e^1, e^2, \dots, e^n)$  relates to basis  $(e_1, e_2, \dots, e_n)$  as

$$e^i \cdot e_j = \delta_j^i,$$

where  $\delta_j^i$  is the Kronecker symbol. As in any other basis, we can find coordinates of vector  $\mathbf{a}$  in the dual basis,  $(a_1, a_2, \dots, a_n)$ , see Figure 2.

$$\mathbf{a} = \sum_{i=1}^n a_i e^i.$$

The coordinates of the vector in the main basis are called contravariant components of the vector. The coordinates of the vector in the dual basis are called covariant components of the vector. The notions “contravariant” and “covariant” are justified by the fact that when we change basis we use different rules to recalculate covariant and contravariant coordinates of the vector. Further, we shall always write covariant components with subscripts and contravariant components with superscripts. One can see that one-dimensional array of real numbers is enough to determine a vector.

But there are other entities for which one-dimensional array of real numbers is not enough. They are linear operators, linear mappings of a vector space to another vector space. If we fix coordinate systems in the vector spaces we can express a linear operator  $\mathcal{A}$  by a matrix, say matrix  $A$ . If we change the basis, we recalculate entries of matrix  $A$  obtaining another matrix  $\bar{A}$ , but the both matrices correspond to same linear operator  $\mathcal{A}$ .

$$A = \{a_j^i\} \quad \bar{A} = \{\bar{a}_j^i\}.$$

Matrices corresponding to linear operator  $\mathcal{A}$  can be written in the main basis and/or the dual basis. Let matrices  $A, B, C$  correspond to linear operator  $\mathcal{A}$ , but they are expressed in different bases.

$$A = \{a_j^i\} \quad B = \{b_{ij}\} \quad C = \{c^{ij}\}.$$

Components of matrix  $A$  are one-time contravariant and one-time covariant, components of matrix  $B$  are twice contravariant, components of matrix  $C$  are twice covariant, but all the matrices corresponds to same linear operator  $\mathcal{A}$ . One can see that a linear operator can be expressed by a two-dimensional array of real numbers.

But there are entities which cannot be represented by a two-dimensional array of real numbers. They are multilinear operators which are also called tensors. Let us express a tensor  $\mathcal{A}$  by components which are  $n$ -times contravariant and  $m$ -times covariant. The order of tensor is  $n + m$  and its component form is given by

$$a_{h_1 h_2 \dots h_m}^{i_1 i_2 \dots i_n}.$$

Let us introduce tensor operators which we need for further development. Tensor product  $\otimes$  of a tensor  $\mathcal{A}$  which is  $n$ -times contravariant and  $m$ -times covariant and a tensor  $\mathcal{B}$  which is  $s$ -times

contravariant and  $t$ -times covariant is a tensor  $\mathcal{C}$  which is  $n + s$ -times contravariant and  $m + t$ -times covariant (3).

$$\mathcal{A} \otimes \mathcal{B} = \mathcal{C}, \quad (3)$$

where components of tensor  $\mathcal{C}$  in some basis can be found by formula

$$a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} b_{h_1 h_2 \dots h_t}^{p_1 p_2 \dots p_s} = c_{k_1 k_2 \dots k_m h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n p_1 p_2 \dots p_s},$$

where indices  $i_1, i_2, \dots, i_n, k_1, k_2, \dots, k_m, p_1, p_2, \dots, p_s$  and  $h_1, h_2, \dots, h_t$  take all possible values. Further we shall write tensor product  $\otimes$  as

$$a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} \otimes b_{h_1 h_2 \dots h_t}^{p_1 p_2 \dots p_s} = c_{k_1 k_2 \dots k_m h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n p_1 p_2 \dots p_s},$$

assuming that indices  $i_1, i_2, \dots, i_n, k_1, k_2, \dots, k_m, p_1, p_2, \dots, p_s$  and  $h_1, h_2, \dots, h_t$  take all possible values.

In some cases we need to consider only components of tensors having same indices in tensor product (4).

$$a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} \otimes b_{h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n} = a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} b_{h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n} = c_{k_1 k_2 \dots k_m h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n}. \quad (4)$$

Also let us define tensor contraction  $\odot$  by formula in (5).

$$a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} \odot b_{h_1 h_2 \dots h_t}^{k_1 k_2 \dots k_m} = \sum_{k_1} \sum_{k_2} \dots \sum_{k_m} a_{k_1 k_2 \dots k_m}^{i_1 i_2 \dots i_n} b_{h_1 h_2 \dots h_t}^{k_1 k_2 \dots k_m} = c_{h_1 h_2 \dots h_t}^{i_1 i_2 \dots i_n}. \quad (5)$$

We note that tensor contraction is equivalent to matrix product if the matrices are written in one-time contravariant and one-time covariant components.

We shall use tensors and the tensor operations in application to the mixed second moments and the mixed high-order moments.

## 4 Mixed second moments in tensor form

Having introduced tensor operation in the previous section, we shall show that the mixed second moments can be calculated from the tensor point of view without tricks which we used in the matrix form.

We denote  $\varepsilon_j^i = E_i [N_j]$  and  $\varepsilon_{jk}^i = E_i [N_j N_k]$ , where  $\varepsilon_j^i$  and  $\varepsilon_{jk}^i$  we consider as tensors.

**Theorem 4.1** *The mixed second moments are given by*

$$\varepsilon_{jk}^i = \varepsilon_\nu^i \odot (\varepsilon_j^\nu \otimes \delta_k^\nu + \varepsilon_k^\nu \otimes \delta_j^\nu - \delta_k^\nu \otimes \delta_j^\nu).$$

**Proof.** We begin the proof as in Theorem 2.2 arriving to expression (1).

$$E_\nu [N_j N_k] = \sum_{\varphi \in T} p_{\nu\varphi} E_\varphi [N_j N_k] + \sum_{\varphi \in T} p_{\nu\varphi} (\delta_{\nu j} E_\varphi [N_k] + E_\varphi [N_j] \delta_{\nu k}) + \delta_{\nu j} \delta_{\nu k}.$$

Now we rewrite the above expression in tensor form. Hence,  $E_\varphi [N_j] = \varepsilon_j^\varphi$ ,  $E_i [N_j N_k] = \varepsilon_{jk}^i$ . We consider matrix  $Q = \{p_{\nu\varphi}\}_{\nu,\varphi \in T}$  as tensor  $q_\varphi^\nu$ . Kroneker symbol  $\delta_{\nu k}$  we treat as  $\delta_k^\nu$  tensor.

$$\begin{aligned}\varepsilon_{jk}^\nu &= q_\varphi^\nu \odot \varepsilon_{jk}^\varphi + q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu + \varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu, \\ \varepsilon_{jk}^\nu - q_\varphi^\nu \odot \varepsilon_{jk}^\varphi &= q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu + \varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu.\end{aligned}$$

Since  $\varepsilon_{jk}^\nu = \delta_\varphi^\nu \odot \varepsilon_{jk}^\varphi$ , we write

$$(\delta_\varphi^\nu - q_\varphi^\nu) \odot \varepsilon_{jk}^\varphi = q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu + \varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu. \quad (6)$$

Let us multiply the above expression from left by  $\varepsilon_\nu^i$  by tensor product with contraction.

$$\varepsilon_\nu^i \odot (\delta_\varphi^\nu - q_\varphi^\nu) \odot \varepsilon_{jk}^\varphi = \varepsilon_\nu^i \odot (q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu + \varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu).$$

Since tensor  $\varepsilon_\nu^i$  corresponds to matrix  $Z$ , and tensor  $(\delta_\varphi^\nu - q_\varphi^\nu)$  corresponds to matrix  $I - Q$ , and  $Z = (I - Q)^{(-1)}$ , we obtain that

$$\varepsilon_\nu^i \odot (\delta_\varphi^\nu - q_\varphi^\nu) \odot \varepsilon_{jk}^\varphi = \delta_\varphi^i \odot \varepsilon_{jk}^\varphi = \varepsilon_{jk}^i.$$

Now we can continue using the following observation. Since tensor  $q_\varphi^\nu$  corresponds to matrix  $Q$ , and tensor  $\varepsilon_j^\varphi$  corresponds to matrix  $Z$ , and  $QZ = Z - I$ , then  $q_\varphi^\nu \odot \varepsilon_j^\varphi = \varepsilon_j^\nu - \delta_j^\nu$ .

$$\begin{aligned}\varepsilon_{jk}^i &= \varepsilon_\nu^i \odot (q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu + \varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu) = \\ &= \varepsilon_\nu^i \odot (q_\varphi^\nu \odot (\varepsilon_j^\varphi \otimes \delta_k^\nu) + q_\varphi^\nu \odot (\varepsilon_k^\varphi \otimes \delta_j^\nu) + \delta_j^\nu \otimes \delta_k^\nu) = \\ &= \varepsilon_\nu^i \odot ((q_\varphi^\nu \odot \varepsilon_j^\varphi) \otimes \delta_k^\nu + (q_\varphi^\nu \odot \varepsilon_k^\varphi) \otimes \delta_j^\nu + \delta_j^\nu \otimes \delta_k^\nu) = \\ &= \varepsilon_\nu^i \odot ((\varepsilon_j^\nu - \delta_j^\nu) \otimes \delta_k^\nu + (\varepsilon_k^\nu - \delta_k^\nu) \otimes \delta_j^\nu + \delta_j^\nu \otimes \delta_k^\nu) = \\ &= \varepsilon_\nu^i \odot (\varepsilon_j^\nu \otimes \delta_k^\nu - \delta_j^\nu \otimes \delta_k^\nu + \varepsilon_k^\nu \otimes \delta_j^\nu - \delta_k^\nu \otimes \delta_j^\nu + \delta_j^\nu \otimes \delta_k^\nu) = \\ &= \varepsilon_\nu^i \odot (\varepsilon_j^\nu \otimes \delta_k^\nu + \varepsilon_k^\nu \otimes \delta_j^\nu - \delta_k^\nu \otimes \delta_j^\nu).\end{aligned}$$

Concluding that

$$\varepsilon_{jk}^i = \varepsilon_\nu^i \odot (\varepsilon_j^\nu \otimes \delta_k^\nu + \varepsilon_k^\nu \otimes \delta_j^\nu - \delta_k^\nu \otimes \delta_j^\nu),$$

we complete the proof. ■

One can see that we used the natural tensor operations to calculate the mixed second moments in the above proof and we need not the trick with representation of the moments as in matrix form.

## 5 Auxiliary combinatorial result

Before we deal with the mixed high-order moments, we need an auxiliary combinatorial result.

Let  $M$  be a finite set of elements of any nature with cardinality  $m$ . Let  $M = \{k_0, k_1, \dots, k_{m-1}\}$ .

Let us enumerate all the combinations of the elements of set  $M$  having length  $j$  and let us index them by  $\psi$ , where  $j = 0, \dots, m$  and  $\psi = 0, \dots, \binom{m}{j} - 1$ . Let us define a function  $f(M, j, \psi)$ . Value  $f(M, j, \psi)$  is the combination of the elements of set  $M$  having length  $j$  and index  $\psi$ . Let us denote  $\bar{f}(M, j, \psi) = M \setminus f(M, j, \psi)$ .

Let us consider  $f(M, j, \psi)$ , where  $\psi = 0, \dots, \binom{m}{j} - 1$ . Since the order of elements in combination  $f(M, j, \psi)$  does not matter, we can assume any order. Let  $f(M, j, \psi) = \{k_{\omega_0}, k_{\omega_1}, \dots, k_{\omega_{j-1}}\}$ , where  $\omega_x = 0, \dots, m-1$ ,  $x = 0, \dots, j-1$ . We shall assume that  $\omega_0 \leq \omega_1 \leq \dots \leq \omega_{j-1}$ . According to [4] we can calculate  $\psi$  as

$$\psi = \sum_{x=0}^{j-1} \binom{\omega_x}{x+1},$$

where  $\binom{a}{b} = 0$ , if  $a < b$ . Such indexing provide lexico-graphic ordering to combinations  $f(M, j, \psi)$ . It means that, for example, when  $m = 3$  and  $j = 2$ , combinations will be ordered like this:  $k_0k_1, k_0k_2, k_1k_3$ . We need the lexico-graphic ordering only to prove Proposition 5.1 below, although the proposition holds for any ordering. In any other discussion we assume any, but fixed, ordering.

Let us denote by  $A$  the set of all combinations of elements of set  $M$  with length  $\varkappa$ .

$$A = \left\{ f(M, \varkappa, \rho) \mid \rho = 0, \dots, \binom{m}{\varkappa} - 1 \right\}.$$

Let us denote by  $B$  the following multiset.

$$B = \left\{ f(f(M, j, \psi), \varkappa, \chi) \mid \psi = 0, \dots, \binom{m}{j} - 1, \chi = 0, \dots, \binom{j}{\varkappa} - 1 \right\}.$$

One can see that multiset  $B$  consists of the same elements as set  $A$ . Let us establish precise relation between set  $A$  and multiset  $B$ .

**Proposition 5.1**  *$B$  is a multiset of the elements of set  $A$  and each element of set  $A$  is taken  $\binom{m-\varkappa}{m-j}$  times.*

**Proof.** See Appendix. ■

The auxiliary combinatorial result plays an important role in our treatment of the mixed high-order moment which we consider in the next section.

## 6 Mixed high-order moments

Let us now consider the mixed moments of higher order. Before we formulate the mixed high-order moments in tensor formalism, let us prove that the moments are finite.

Let us consider the conditional moment generating function of the absorbing Markov chain,  $M_i(y) = E_i [e^{\sum y_j N_j}]$ , where summation is performed over all states of the Markov chain and

the process starts at a transient state  $i$ . We need to prove that the moment generating function is analytical in the origin.

Let us define vector  $\zeta$  and matrix  $\Theta = \{\vartheta_{ik}\}_{i,k \in T}$

$$\begin{aligned}\zeta_i &= e^{y_i} \left( 1 - \sum_{k \in T} p_{ik} \right), \\ \vartheta_{ik} &= \delta_{ik} - e^{y_i} p_{ik}.\end{aligned}$$

**Proposition 6.1** *If all  $y_i$  are small enough, the moment generating function  $M(y)$  is give by*

$$M(y) = \Theta^{-1} \zeta.$$

**Proof.** We ask where the process can go in one step, from its starting position  $i$ .

$$\begin{aligned}E_i \left[ e^{\sum y_j N_j} \right] &= \sum_{k \in \bar{T}} p_{ik} e^{y_i} + \sum_{k \in T} p_{ik} E_k \left[ e^{\sum_{j \neq i} y_j N_j + y_i (N_i + 1)} \right] = \\ &= \sum_{k \in \bar{T}} p_{ik} e^{y_i} + \sum_{k \in T} p_{ik} E_k \left[ e^{\sum y_j N_j + y_i} \right] = \\ &= \sum_{k \in \bar{T}} p_{ik} e^{y_i} + \sum_{k \in T} p_{ik} E_k \left[ e^{\sum y_j N_j} e^{y_i} \right] = \\ &= e^{y_i} \left( 1 - \sum_{k \in T} p_{ik} \right) + e^{y_i} \sum_{k \in T} p_{ik} E_k \left[ e^{\sum y_j N_j} \right].\end{aligned}$$

And we solve the above equation in the following way.

$$\begin{aligned}E_i \left[ e^{\sum y_j N_j} \right] &= e^{y_i} \left( 1 - \sum_{k \in T} p_{ik} \right) + e^{y_i} \sum_{k \in T} p_{ik} E_k \left[ e^{\sum y_j N_j} \right], \\ E_i \left[ e^{\sum y_j N_j} \right] - e^{y_i} \sum_{k \in T} p_{ik} E_k \left[ e^{\sum y_j N_j} \right] &= e^{y_i} \left( 1 - \sum_{k \in T} p_{ik} \right), \\ \sum_{k \in T} (\delta_{ik} - e^{y_i} p_{ik}) E_k \left[ e^{\sum y_j N_j} \right] &= e^{y_i} \left( 1 - \sum_{k \in T} p_{ik} \right).\end{aligned}\tag{7}$$

Then, we can rewrite (7) in matrix form

$$\Theta M(y) = \zeta.$$

Let us show that matrix  $\Theta$  is invertible. Let us denote by  $t = |T|$  and by  $\Xi$  the matrix

$$\Xi = \begin{pmatrix} e^{y_1} & 0 & \dots & 0 \\ 0 & e^{y_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{y_t} \end{pmatrix}.$$



We express matrix  $\Theta$  as

$$\Theta = I - \Xi Q,$$

where  $Q$  corresponds to transient states of the absorbing Markov chain. Since matrix  $\Xi$  is diagonal, then matrix  $\Xi Q$  is matrix  $Q$  whose rows are multiplied by diagonal elements of matrix  $\Xi$ . Matrix  $Q$  is substochastic, hence,

$$Q\mathbf{1} = q,$$

where  $q = (q_1, q_2, \dots, q_t)^T$ , and  $0 \leq q_i \leq 1, \forall i = 1, \dots, t$ , and  $\exists i : q_i < 1$ . Then,

$$\Xi Q\mathbf{1} = q = (e^{y_1} q_1, e^{y_2} q_2, \dots, e^{y_t} q_t)^T.$$

Matrix  $\Xi Q$  is substochastic, if  $0 \leq e^{y_i} q_i \leq 1, \forall i = 1, \dots, t$ , and  $\exists i : e^{y_i} q_i < 1$ . Therefore, if  $y_i \leq -\ln q_i, \forall i = 1, \dots, t$ , and  $\exists i : y_i < -\ln q_i$ , matrix  $\Theta$  is invertible.

And we can determine the conditional moment generating function by

$$M(y) = \Theta^{-1} \zeta.$$

■

One can see that the conditional moment generating function  $M(y)$  is analytical at the origin, and, hence, there exist all the mixed high-order moments and they are finite.

We denote  $\varepsilon_j^i = E_i [N_j]$ ,  $\varepsilon_{jk}^i = E_i [N_j N_k]$ ,  $\dots$ ,  $\varepsilon_{k_0 k_1 \dots k_{m-1}}^i = E_i \left[ \prod_{j=0}^{m-1} N_{k_j} \right]$ , where  $m$  is a natural number. Let us denote  $M = \{k_0, k_1, \dots, k_{m-1}\}$ . The cardinality of set  $M$  is  $m$ . We call set  $M$  the basis set.

We have got the mixed moments of higher order in tensor representation. Since the product is a commutative operation, the order of indeces  $k_0 k_1 \dots k_{m-1}$  in  $\varepsilon_{k_0 k_1 \dots k_{m-1}}^i$  does not matter, and we can write  $\varepsilon_{k_0 k_1 \dots k_{m-1}}^i = \varepsilon_M^i$ .

Let us denote  ${}_0 a_{k_0 k_1 \dots k_{m-1}}^\nu = \bigotimes_{l=0}^{m-1} \delta_{k_l}^\nu$ . Let us define tensor  ${}_j a_{k_0 k_1 \dots k_{m-1}}^\nu$  as follows:

$${}_j a_{k_0 k_1 \dots k_{m-1}}^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu. \quad (8)$$

Since index  $\psi$  passes all possible values, the order of indeces  $k_0 k_1 \dots k_{m-1}$  in  ${}_j a_{k_0 k_1 \dots k_{m-1}}^\nu$  does not matter, and we can write  ${}_j a_{k_0 k_1 \dots k_{m-1}}^\nu = {}_j a_M^\nu$ . We note that  ${}_m a_M^\nu = \varepsilon_M^\nu$ .

Let us define tensor  ${}_j b_{k_0 k_1 \dots k_{m-1}}^{\varphi\nu}$  as follows:

$${}_j b_{k_0 k_1 \dots k_{m-1}}^{\varphi\nu} = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^\varphi \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu. \quad (9)$$

Since index  $\psi$  passes all possible values, the order of indeces  $k_0 k_1 \dots k_{m-1}$  in  ${}_j b_{k_0 k_1 \dots k_{m-1}}^{\varphi\nu}$  does not matter, and we can write  ${}_j b_{k_0 k_1 \dots k_{m-1}}^{\varphi\nu} = {}_j b_M^{\varphi\nu}$ . We note that  ${}_m b_M^{\varphi\nu} = \varepsilon_M^\varphi$ ,  ${}_0 b_M^{\varphi\nu} = {}_0 a_M^\nu$ , and, in general,  ${}_j b_M^{\varphi\nu} = {}_j a_M^\nu$ .

Let us define tensor  ${}_{\varkappa j}c_{k_0 k_1 \dots k_{m-1}}^\nu$ , where  $\varkappa \leq j$ , as follows:

$${}_{\varkappa j}c_{k_0 k_1 \dots k_{m-1}}^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} {}_{\varkappa}a_{f(M,j,\psi)}^\nu \otimes {}_0a_{\bar{f}(M,j,\psi)}^\nu. \quad (10)$$

Since index  $\psi$  passes all possible values, the order of indices  $k_0 k_1 \dots k_{m-1}$  in  ${}_{\varkappa j}c_{k_0 k_1 \dots k_{m-1}}^\nu$  does not matter, and we can write  ${}_{\varkappa j}c_{k_0 k_1 \dots k_{m-1}}^\nu = {}_{\varkappa j}c_M^\nu$ .

**Proposition 6.2** *The following formula takes place:*

$${}_0 j c_M^\nu = \binom{m}{j} {}_0 a_M^\nu.$$

**Proof.**

$${}_0 j c_M^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} {}_0 a_{f(M,j,\psi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} {}_0 a_M^\nu = \binom{m}{j} {}_0 a_M^\nu. \quad \blacksquare$$

**Proposition 6.3** *The following formula takes place:*

$${}_{j j} c_M^\nu = {}_j a_M^\nu.$$

**Proof.**

$${}_{j j} c_M^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} {}_j a_{f(M,j,\psi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu = \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu = {}_j a_M^\nu. \quad \blacksquare$$

Propositions 6.2 and 6.3 are the particular cases of the following proposition.

**Proposition 6.4** *The following formula takes place:*

$${}_{\varkappa j} c_M^\nu = \binom{m-\varkappa}{m-j} {}_{\varkappa} a_M^\nu.$$

**Proof.** We can write  ${}_{\varkappa j}c_M^\nu$  as follows

$$\begin{aligned} {}_{\varkappa j} c_M^\nu &= \sum_{\psi=0}^{\binom{m}{j}-1} {}_{\varkappa} a_{f(M,j,\psi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu = \\ &= \sum_{\psi=0}^{\binom{m}{j}-1} \sum_{\chi=0}^{\binom{j}{\varkappa}-1} \varepsilon_{f(M,j,\psi),\varkappa,\chi}^\nu \otimes {}_0 a_{\bar{f}(f(M,j,\psi),\varkappa,\chi)}^\nu \otimes {}_0 a_{\bar{f}(M,j,\psi)}^\nu. \end{aligned} \quad (11)$$

We can write  ${}_{\varkappa}a_M^\nu$  as follows:

$${}_{\varkappa}a_M^\nu = \sum_{\psi=0}^{\binom{m}{\varkappa}-1} \varepsilon_{f(M,\varkappa,\psi)}^\nu \otimes_0 a_{\bar{f}(M,\varkappa,\psi)}^\nu. \quad (12)$$

Since operations “+” and “ $\otimes$ ” are commutative, to prove the statement of the proposition it is enough to show that every term of summation (12) can be found in summation (11)  $\binom{m-\varkappa}{m-j}$  times.

Indices  $f(f(M, j, \psi), \varkappa, \chi)$  in (11) defined in multiset  $B$  from Section 5 and indices  $f(M, \varkappa, \psi)$  in (12) defined in set  $A$  from Section 5. Therefore, we apply Proposition 5.1 and complete the proof. ■

**Theorem 6.1** *The mixed high-order moments of the absorbing Markov chain is given by*

$$\varepsilon_M^i = \varepsilon_\nu^i \odot \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} {}_{\varkappa}a_M^\nu. \quad (13)$$

**Proof.** Let us assume that the theorem is proven for smaller values of  $m$ , particularly for  $m = 2$  Theorem 4.1.

We start with non-tensor representation of the high-order mixed moments,  $E_i \left[ \prod_{j=0}^{m-1} N_{k_j} \right]$ .

Let us calculate  $E_i \left[ \prod_{j=0}^{m-1} N_{k_j} \right]$ . Following the approach of [1, Theorem 3.3.3] we ask where the process can go in one step, from its starting position  $i$ . It can go to state  $\varphi$  with probability  $p_{i\varphi}$ . If the new state is absorbing, then we can never reach states  $k_j$ , where  $j = 0, \dots, m-1$ , again, and the only possible contribution is from the initial state, which is  $\prod_{j=0}^{m-1} \delta_{ik_j}$ . If the new state is transient then we will be in state  $k_j$   $\delta_{ik_j}$  times from the initial state, and  $N_{k_j}$ , where  $j = 0, \dots, m-1$ , times from the later steps. We have

$$\begin{aligned} E_i \left[ \prod_{j=0}^{m-1} N_{k_j} \right] &= \sum_{\varphi \in \bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \sum_{\varphi \in T} p_{i\varphi} E_\varphi \left[ \prod_{j=0}^{m-1} (N_{k_j} + \delta_{ik_j}) \right] = \\ &= \sum_{\varphi \in \bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \\ &+ \sum_{\varphi \in T} p_{i\varphi} \left( E_\varphi \left[ \prod_{\kappa \in M} N_\kappa \right] + \sum_{\psi=0}^{\binom{m-1}{m-1}-1} E_\varphi \left[ \prod_{\kappa \in f(M, m-1, \psi)} N_\kappa \right] \prod_{\kappa \in \bar{f}(M, m-1, \psi)} \delta_{i\kappa} + \right. \\ &+ \dots + \sum_{\psi=0}^{\binom{m}{j}-1} E_\varphi \left[ \prod_{\kappa \in f(M, j, \psi)} N_\kappa \right] \prod_{\kappa \in \bar{f}(M, j, \psi)} \delta_{i\kappa} + \dots + \\ &\left. + \sum_{\psi=0}^{\binom{m}{1}-1} E_\varphi \left[ \prod_{\kappa \in f(M, 1, \psi)} N_\kappa \right] \prod_{\kappa \in \bar{f}(M, 1, \psi)} \delta_{i\kappa} + \prod_{\kappa \in M} \delta_{i\kappa} \right). \end{aligned}$$

We note that

$$\sum_{\varphi \in \bar{T}} p_{i\varphi} \prod_{j=0}^{m-1} \delta_{ik_j} + \sum_{\varphi \in T} p_{i\varphi} \prod_{\kappa \in M} \delta_{i\kappa} = \prod_{\kappa \in M} \delta_{i\kappa} \left( \sum_{\varphi \in \bar{T}} p_{i\varphi} + \sum_{\varphi \in T} p_{i\varphi} \right) = \prod_{\kappa \in M} \delta_{i\kappa}.$$

And we continue

$$E_i \left[ \prod_{j=0}^{m-1} N_{k_j} \right] = \prod_{\kappa \in M} \delta_{i\kappa} + \sum_{\varphi \in T} p_{i\varphi} \left( E_{\varphi} \left[ \prod_{\kappa \in M} N_{\kappa} \right] + \sum_{j=1}^{m-1} \sum_{\psi=0}^{\binom{m}{j}-1} E_{\varphi} \left[ \prod_{\kappa \in f(M,j,\psi)} N_{\kappa} \right] \prod_{\kappa \in \bar{f}(M,j,\psi)} \delta_{i\kappa} \right).$$

Let us rewrite the last expression in tensor form. We will use index  $\nu$  in place of  $i$  for further development. We note that  $\prod_{\kappa \in \bar{f}(M,j,\psi)} \delta_{\nu\kappa}$  is represented in tensor form as  $\otimes_{\kappa \in \bar{f}(M,j,\psi)} \delta_{\nu\kappa}^{\nu} = {}_0a_{\bar{f}(M,j,\psi)}^{\nu}$ .

$$\begin{aligned} \varepsilon_M^{\nu} &= {}_0a_M^{\nu} + q_{\varphi}^{\nu} \odot \left( \varepsilon_M^{\varphi} + \sum_{j=1}^{m-1} \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^{\varphi} \otimes {}_0a_{\bar{f}(M,j,\psi)}^{\nu} \right) = \\ &= q_{\varphi}^{\nu} \odot \varepsilon_M^{\varphi} + q_{\varphi}^{\nu} \odot \sum_{j=1}^{m-1} j b_M^{\varphi\nu} + {}_0a_M^{\nu}. \end{aligned}$$

Let us consider  $q_{\varphi}^{\nu} \odot \sum_{j=1}^{m-1} j b_M^{\varphi\nu} + {}_0a_M^{\nu}$  and, in particular,  $q_{\varphi}^{\nu} \odot j b_M^{\varphi\nu}$  as one term of the summation.

$$q_{\varphi}^{\nu} \odot j b_M^{\varphi\nu} = q_{\varphi}^{\nu} \odot \sum_{\psi=0}^{\binom{m}{j}-1} \varepsilon_{f(M,j,\psi)}^{\varphi} \otimes {}_0a_{\bar{f}(M,j,\psi)}^{\nu}.$$

Next we consider  $q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi}$  and proceed further by induction.

$$q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi} = q_{\varphi}^{\nu} \odot \varepsilon_{\mu}^{\varphi} \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa}a_{f(M,j,\psi)}^{\mu}.$$

Since  $q_{\varphi}^{\nu} \odot \varepsilon_{\mu}^{\varphi} = \varepsilon_{\mu}^{\nu} - \delta_{\mu}^{\nu}$ .

$$\begin{aligned} q_{\varphi}^{\nu} \odot \varepsilon_{f(M,j,\psi)}^{\varphi} &= (\varepsilon_{\mu}^{\nu} - \delta_{\mu}^{\nu}) \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa}a_{f(M,j,\psi)}^{\mu} = \\ &= \varepsilon_{f(M,j,\psi)}^{\nu} - \delta_{\mu}^{\nu} \odot \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa+1} {}_{\varkappa}a_{f(M,j,\psi)}^{\mu} = \\ &= \varepsilon_{f(M,j,\psi)}^{\nu} + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} {}_{\varkappa}a_{f(M,j,\psi)}^{\nu}. \end{aligned}$$

Now we come back to  $q_\varphi^\nu \odot_j b_M^{\varphi\nu}$ .

$$\begin{aligned}
q_\varphi^\nu \odot_j b_M^{\varphi\nu} &= \sum_{\psi=0}^{\binom{m}{j}-1} \left( \varepsilon_{f(M,j,\psi)}^\nu + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} \varkappa a_{f(M,j,\psi)}^\nu \right) \otimes_0 a_{\bar{f}(M,j,\psi)}^\nu = \\
&= {}_j a_M^\nu + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} \sum_{\psi=0}^{\binom{m}{j}-1} \varkappa a_{f(M,j,\psi)}^\nu \otimes_0 a_{\bar{f}(M,j,\psi)}^\nu = \\
&= {}_j a_M^\nu + \sum_{\varkappa=0}^{j-1} (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu = \\
&= \sum_{\varkappa=0}^j (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu.
\end{aligned}$$

Now we come back to  $q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu$ .

$$\begin{aligned}
q_\varphi^\nu \odot \sum_{j=1}^{m-1} {}_j b_M^{\varphi\nu} + {}_0 a_M^\nu &= \sum_{j=1}^{m-1} \sum_{\varkappa=0}^j (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu + {}_0 a_M^\nu = \\
&= \sum_{j=1}^{m-1} \sum_{\varkappa=1}^j (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu + \sum_{j=1}^{m-1} (-1)^j {}_j c_M^\nu + {}_0 a_M^\nu = \\
&= \sum_{\varkappa=1}^{m-1} \sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu + \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} {}_0 a_M^\nu.
\end{aligned}$$

We note that  $\sum_{j=0}^m (-1)^j \binom{m}{j} = 0$ , and, therefore,

$$\sum_{j=0}^{m-1} (-1)^{j-\varkappa} \binom{m}{j} = (-1)^{m+1}. \quad (14)$$

Let us consider  $\sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \varkappa {}_j c_M^\nu$ . We recall that  $\varkappa {}_j c_M^\nu = \binom{m-\varkappa}{m-j} \varkappa a_M^\nu$  according to Proposition 6.4, and we consider  $\sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \binom{m-\varkappa}{m-j}$ .

$$\begin{aligned}
\sum_{j=\varkappa}^{m-1} (-1)^{j-\varkappa} \binom{m-\varkappa}{m-j} &= \sum_{j=0}^{m-\varkappa-1} (-1)^j \binom{m-\varkappa}{m-j-\varkappa} = \sum_{j=0}^{m-\varkappa-1} (-1)^j \binom{m-\varkappa}{j} = \\
&= \sum_{j=0}^{m-\varkappa} (-1)^j \binom{m-\varkappa}{j} - (-1)^{m-\varkappa} \binom{m-\varkappa}{m-\varkappa} = (-1)^{m-\varkappa+1}.
\end{aligned}$$

Hence, for  $q_\varphi^\nu \odot \sum_{j=1}^{m-1} j b_M^{\varphi\nu} + {}_0 a_M^\nu$  we have

$$q_\varphi^\nu \odot \sum_{j=1}^{m-1} j b_M^{\varphi\nu} + {}_0 a_M^\nu = \sum_{\varkappa=1}^{m-1} (-1)^{m-\varkappa+1} \varkappa a_M^\nu + (-1)^{m+1} {}_0 a_M^\nu = \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} \varkappa a_M^\nu.$$

And, finally, we obtain

$$\begin{aligned} \varepsilon_M^\nu &= q_\varphi^\nu \odot \varepsilon_M^\varphi + q_\varphi^\nu \odot \sum_{j=1}^{m-1} j b_M^{\varphi\nu} + {}_0 a_M^\nu = \\ &= q_\varphi^\nu \odot \varepsilon_M^\varphi + \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} \varkappa a_M^\nu. \end{aligned}$$

Next we consider  $\varepsilon_M^\nu - q_\varphi^\nu \odot \varepsilon_M^\varphi$ .

$$\varepsilon_M^\nu - q_\varphi^\nu \odot \varepsilon_M^\varphi = (\delta_\varphi^\nu - q_\varphi^\nu) \odot \varepsilon_M^\varphi,$$

and multiplying by  $\varepsilon_\nu^i$  from left, and recalling that  $\varepsilon_\nu^i \odot (\delta_\varphi^\nu - q_\varphi^\nu) = \delta_\varphi^i$  we have

$$\varepsilon_\nu^i \odot (\delta_\varphi^\nu - q_\varphi^\nu) \odot \varepsilon_M^\varphi = \delta_\varphi^i \odot \varepsilon_M^\varphi = \varepsilon_M^i.$$

We complete the proof with

$$\varepsilon_M^i = \varepsilon_\nu^i \odot \sum_{\varkappa=0}^{m-1} (-1)^{m-\varkappa+1} \varkappa a_M^\nu.$$

■

One can see that tensor formalism allows us to calculate the mixed high-order moments by compact formula. The mixed high-order moments are determined by the moments of lower orders.

## 7 Conclusion

We considered the mixed high-order moments of an absorbing Markov chain. While the first moments and the non-mixed second moment can be expressed in matrix form, it can hardly be done for the mixed high-order moments. Using tensor formalism, we developed compact expression for the mixed high-order moments.

## 8 Acknowledgements

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### Appendix: Proof of Proposition 5.1

Let us consider the following set

$$D(M, j, \varkappa) = \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m}{j} - 1, \chi = 0, \dots, \binom{j}{\varkappa} - 1 \right\}.$$

It is the set of elements of multiset  $B$  equipped by its indeces, therefore, we can distinguish equal elements of multiset  $B$  and compose the set. We shall write  $D(M, j, \varkappa) = D$  if it does not produce any ambiguity.

Let us consider the following set

$$G = \left\{ (f(M, \varkappa, \rho), (\rho, \iota)) \mid \rho = 0, \dots, \binom{m}{\varkappa} - 1, \iota = 0, \dots, \binom{m - \varkappa}{m - j} - 1 \right\}.$$

It is the set of the elements of set  $A$  equipped with its index  $\rho$  and auxiliary index  $\iota$ . Due to index  $\iota$ , each element of set  $A$  is repeated  $\binom{m - \varkappa}{m - j}$  times in set  $G$ .

To prove the proposition we need to show that there is one-to-one mapping from  $(\rho, \iota)$  to  $(\psi, \chi)$ , which we denote by  $(\psi, \chi)(\rho, \iota)$ , such that

$$D = F,$$

where

$$F(M, j, \varkappa) = \left\{ (f(M, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m}{\varkappa} - 1, \iota = 0, \dots, \binom{m - \varkappa}{m - j} - 1 \right\}.$$

We shall write  $F(M, j, \varkappa) = F$  if it does not produce any ambiguity.

First of all, we prove that the cardinalities of sets  $D$  and  $F$  are equal. The cardinality of set  $D$  is equal to

$$|D| = \binom{m}{j} \binom{j}{\varkappa} = \frac{m!}{j!(m-j)!} \frac{j!}{\varkappa!(j-\varkappa)!} = \frac{m!}{(m-j)!\varkappa!(j-\varkappa)!}.$$

And the cardinality of set  $F$  is equal to

$$|F| = \binom{m}{\varkappa} \binom{m - \varkappa}{m - j} = \frac{m!}{\varkappa!(m-\varkappa)!} \frac{(m - \varkappa)!}{(m - j)!(m - j - m + \varkappa)!} = \frac{m!}{\varkappa!(m - j)!(\varkappa - j)!}.$$

Hence, one can see that  $|D| = |F|$  and the one-to-one mapping can be potentially established. Therefore, the definition of set  $F$  is valid. We should prove that  $D = F$ .

We shall assume lexicographic ordering of combinations discussed above in the further development of the proof.

We shall continue the proof using the mathematical induction. We lead the induction by the cardinality of set  $M$ . Hence, let us prove the base of induction.

- Let  $m = 1$  and  $M = \{k_0\}$ . We have following options for  $(j, \varkappa)$ :  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ . Let us consider each option.

$$- (j, \varkappa) = (0, 0)$$

$$\begin{aligned} D(M, 0, 0) &= \\ &= \left\{ (f(f(M, 0, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{1}{0} - 1, \chi = 0, \dots, \binom{0}{0} - 1 \right\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$\begin{aligned} F(M, 0, 0) &= \\ &= \left\{ (f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{1}{0} - 1, \iota = 0, \dots, \binom{1-0}{1-0} - 1 \right\} = \\ &= \{(\emptyset, (\psi, \chi)(0, 0))\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$- (j, \varkappa) = (1, 0)$$

$$\begin{aligned} D(M, 1, 0) &= \\ &= \left\{ (f(f(M, 1, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{1}{1} - 1, \chi = 0, \dots, \binom{1}{0} - 1 \right\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$\begin{aligned} F(M, 1, 0) &= \\ &= \left\{ (f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{1}{0} - 1, \iota = 0, \dots, \binom{1-0}{1-1} - 1 \right\} = \\ &= \{(\emptyset, (\psi, \chi)(0, 0))\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$- (j, \varkappa) = (1, 1)$$

$$\begin{aligned} D(M, 1, 1) &= \\ &= \left\{ (f(f(M, 1, \psi), 1, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{1}{1} - 1, \chi = 0, \dots, \binom{1}{1} - 1 \right\} = \\ &= \{(\{k_0\}, (0, 0))\}. \end{aligned}$$



$$\begin{aligned}
F(M, 1, 1) &= \\
&= \left\{ (f(M, 1, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{1}{1} - 1, \iota = 0, \dots, \binom{1-1}{1-1} - 1 \right\} = \\
&= \{(\{k_0\}, (\psi, \chi)(0, 0))\} = \\
&= \{(\{k_0\}, (0, 0))\}.
\end{aligned}$$

- Let  $m = 2$  and  $M = \{k_0, k_1\}$ . We have following options for  $(j, \varkappa)$ :  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(2, 0)$ ,  $(2, 1)$ ,  $(2, 2)$ . Let us consider each option.

–  $(j, \varkappa) = (0, 0)$

$$\begin{aligned}
D(M, 0, 0) &= \\
&= \left\{ (f(f(M, 0, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{0} - 1, \chi = 0, \dots, \binom{0}{0} - 1 \right\} = \\
&= \{(\emptyset, (0, 0))\}.
\end{aligned}$$

$$\begin{aligned}
F(M, 0, 0) &= \\
&= \left\{ (f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{0} - 1, \iota = 0, \dots, \binom{2-0}{2-0} - 1 \right\} = \\
&= \{(\emptyset, (\psi, \chi)(0, 0))\} = \\
&= \{(\emptyset, (0, 0))\}.
\end{aligned}$$

–  $(j, \varkappa) = (1, 0)$

$$\begin{aligned}
D(M, 1, 0) &= \\
&= \left\{ (f(f(M, 1, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{1} - 1, \chi = 0, \dots, \binom{1}{0} - 1 \right\} = \\
&= \{(\emptyset, (\psi, \chi)) \mid \psi = 0, \dots, 1, \chi = 0\} = \\
&= \{(\emptyset, (0, 0)), (\emptyset, (1, 0))\}.
\end{aligned}$$

$$\begin{aligned}
F(M, 1, 0) &= \\
&= \left\{ (f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{0} - 1, \iota = 0, \dots, \binom{2-0}{2-1} - 1 \right\} = \\
&= \{(\emptyset, (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \iota = 0, \dots, 1\} = \\
&= \{(\emptyset, (\psi, \chi)(0, 0)), (\emptyset, (\psi, \chi)(0, 1))\} = \\
&= \{(\emptyset, (0, 0)), (\emptyset, (1, 0))\}.
\end{aligned}$$

$$- (j, \varkappa) = (1, 1)$$

$$\begin{aligned} D(M, 1, 1) &= \\ &= \left\{ (f(f(M, 1, \psi), 1, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{1} - 1, \chi = 0, \dots, \binom{1}{1} - 1 \right\} = \\ &= \{(f(f(M, 1, \psi), 1, \chi), (\psi, \chi)) \mid \psi = 0, \dots, 1, \chi = 0\} = \\ &= \{(\{k_0\}, (0, 0)), (\{k_1\}, (1, 0))\}. \end{aligned}$$

$$\begin{aligned} F(M, 1, 1) &= \\ &= \left\{ (f(M, 1, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{1} - 1, \iota = 0, \dots, \binom{2-1}{2-1} - 1 \right\} = \\ &= \{(f(M, 1, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, 1, \iota = 0\} = \\ &= \{(\{k_0\}, (\psi, \chi)(0, 0)), (\{k_1\}, (\psi, \chi)(1, 0))\} = \\ &= \{(\{k_0\}, (0, 0)), (\{k_1\}, (1, 0))\}. \end{aligned}$$

$$- (j, \varkappa) = (2, 0)$$

$$\begin{aligned} D(M, 2, 0) &= \\ &= \left\{ (f(f(M, 2, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{2} - 1, \chi = 0, \dots, \binom{2}{0} - 1 \right\} = \\ &= \{(f(f(M, 2, \psi), 0, \chi), (\psi, \chi)) \mid \psi = 0, \chi = 0\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$\begin{aligned} F(M, 2, 0) &= \\ &= \left\{ (f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{0} - 1, \iota = 0, \dots, \binom{2-0}{2-2} - 1 \right\} = \\ &= \{(f(M, 0, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \iota = 0\} = \\ &= \{(\emptyset, (\psi, \chi)(0, 0))\} = \\ &= \{(\emptyset, (0, 0))\}. \end{aligned}$$

$$- (j, \varkappa) = (2, 1)$$

$$\begin{aligned} D(M, 2, 1) &= \\ &= \left\{ (f(f(M, 2, \psi), 1, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{2} - 1, \chi = 0, \dots, \binom{2}{1} - 1 \right\} = \\ &= \{(f(f(M, 2, \psi), 1, \chi), (\psi, \chi)) \mid \psi = 0, \chi = 0, \dots, 1\} = \\ &= \{(\{k_0\}, (0, 0)), (\{k_1\}, (0, 1))\}. \end{aligned}$$

$$\begin{aligned}
& F(M, 2, 1) = \\
& = \left\{ (f(M, 1, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{1} - 1, \iota = 0, \dots, \binom{2-1}{2-2} - 1 \right\} = \\
& = \{(f(M, 1, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, 1, \iota = 0\} = \\
& = \{(\{k_0\}, (\psi, \chi)(0, 0)), (\{k_1\}, (\psi, \chi)(1, 0))\} = \\
& = \{(\{k_0\}, (0, 0)), (\{k_1\}, (0, 1))\}. \\
- (j, \varkappa) &= (2, 2)
\end{aligned}$$

$$\begin{aligned}
& D(M, 2, 2) = \\
& = \left\{ (f(f(M, 2, \psi), 2, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{2}{2} - 1, \chi = 0, \dots, \binom{2}{2} - 1 \right\} = \\
& = \{(f(f(M, 2, \psi), 2, \chi), (\psi, \chi)) \mid \psi = 0, \chi = 0\} = \\
& = \{(\{k_0, k_1\}, (0, 0))\}.
\end{aligned}$$

$$\begin{aligned}
& F(M, 2, 2) = \\
& = \left\{ (f(M, 2, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{2}{2} - 1, \iota = 0, \dots, \binom{2-2}{2-2} - 1 \right\} = \\
& = \{(f(M, 2, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \iota = 0\} = \\
& = \{(\{k_0, k_1\}, (\psi, \chi)(0, 0))\} = \\
& = \{(\{k_0, k_1\}, (0, 0))\}.
\end{aligned}$$

Having proved the induction base, we continue with the induction step.

Let us consider set  $F$ . Since combinations  $f(M, \varkappa, \rho)$  are ordered in lexico-graphic order, we know that combinations  $f(M, \varkappa, \rho)$  containing element  $k_0$  have indices  $\rho = 0, \dots, \binom{m-1}{\varkappa-1} - 1$ , and we can write

$$\begin{aligned}
F &= \left\{ (f(M, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m-1}{\varkappa-1} - 1, \iota = 0, \dots, \binom{m-\varkappa}{m-j} - 1 \right\} \cup \\
&\cup \left\{ (f(M, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = \binom{m-1}{\varkappa-1}, \dots, \binom{m}{\varkappa} - 1, \iota = 0, \binom{m-\varkappa}{m-j} - 1, \dots \right\} = \\
&= F_1 \cup F_2.
\end{aligned}$$

Combinations  $f(M, \varkappa, \rho)$  as elements of set  $F_1$  contain element  $k_0$  and the combinations of set  $F_2$  do not contain element  $k_0$ .

Let us consider set  $D$ . We again exploit that the combinations are ordered in lexico-graphic order, then combinations  $f(M, j, \psi)$  containing element  $k_0$  have indices  $\psi = 0, \dots, \binom{m-1}{j-1} - 1$ .

Thus, we can write

$$\begin{aligned} D &= \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j}{\varkappa} - 1 \right\} \cup \\ &\cup \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = \binom{m-1}{j-1}, \dots, \binom{m}{j} - 1, \chi = 0, \dots, \binom{j}{\varkappa} - 1 \right\} = \\ &= D_1 \cup D_2. \end{aligned}$$

We do the same with set  $D_1$ . Combinations  $f(f(M, j, \psi), \varkappa, \chi)$  containing element  $k_0$  have indices  $\chi = 0, \dots, \binom{j-1}{\varkappa-1} - 1$ , and we express  $D_1$  as follows

$$\begin{aligned} D_1 &= \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa-1} - 1 \right\} \cup \\ &\cup \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = \binom{j-1}{\varkappa-1}, \dots, \binom{j}{\varkappa} - 1 \right\} = \\ &= D_a \cup D_b. \end{aligned}$$

Hence, we partition set  $D$  as  $D = D_a \cup D_b \cup D_2$ .

Let us prove that  $D_a = F_1$ . We can rewrite  $F_1$  as follows

$$\begin{aligned} F_1 &= \left\{ \{k_0, (f(M \setminus \{k_0\}, \varkappa - 1, \rho)), (\psi, \chi) (\rho, \iota)\} \mid \right. \\ &\quad \left. \rho = 0, \dots, \binom{m-1}{\varkappa-1} - 1, \iota = 0, \dots, \binom{m-\varkappa}{m-j} - 1 \right\}. \end{aligned}$$

Considering set  $D_a$  one can see that each element of the set contains  $k_0$  and all the combinations of the elements of set  $M$  containing element  $k_0$  are counted.

$$\begin{aligned} D_a &= \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa-1} - 1 \right\} = \\ &= \left\{ (\{k_0, f(f(M \setminus \{k_0\}, j-1, \psi), \varkappa-1, \chi)\}, (\psi, \chi)) \mid \right. \\ &\quad \left. \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa-1} - 1 \right\}. \end{aligned}$$

One can see that  $D_a = D(M \setminus \{k_0\}, j-1, \varkappa-1)$ , therefore, by induction, we can conclude that

$$D_a = F_1.$$

Next we shall prove that  $F_2 = D_b \cup D_2$ . One can easily see that

$$F_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi) (\rho, \iota)) \mid \rho = \binom{m-1}{\varkappa-1}, \dots, \binom{m}{\varkappa} - 1, \iota = 0, \dots, \binom{m-\varkappa}{m-j} - 1 \right\},$$

and, renumbering elements,

$$F_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m-1}{\varkappa} - 1, \iota = 0, \dots, \binom{m-\varkappa}{m-j} - 1 \right\}.$$

Let us consider  $D_2$ . Since combinations  $f(M, j, \psi)$  of set  $D_2$  do not contain  $k_0$ , we write, renumbering elements,

$$D_2 = \left\{ (f(f(M \setminus \{k_0\}, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j} - 1, \chi = 0, \dots, \binom{j}{\varkappa} - 1 \right\}.$$

One can see that  $D_2 = D(M \setminus \{k_0\}, j, \varkappa)$ , therefore, we conclude by induction that

$$D_2 = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m-1}{\varkappa} - 1, \iota = 0, \dots, \binom{m-1-\varkappa}{m-1-j} - 1 \right\}.$$

Let us consider  $D_b$ . Renumbering elements of set  $D_b$  we have

$$D_b = \left\{ (f(f(M, j, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa} - 1 \right\}.$$

Since combinations  $f(f(M, j, \psi), \varkappa, \chi)$  do not contain element  $k_0$ , we do not need it in combinations  $f(M, j, \psi)$ , hence, we write

$$D_b = \left\{ (f(f(M, j, \psi) \setminus \{k_0\}, \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa} - 1 \right\},$$

or it is the same as

$$D_b = \left\{ (f(f(M \setminus \{k_0\}, j-1, \psi), \varkappa, \chi), (\psi, \chi)) \mid \psi = 0, \dots, \binom{m-1}{j-1} - 1, \chi = 0, \dots, \binom{j-1}{\varkappa} - 1 \right\}.$$

Now one can see that  $D_2 = D(M \setminus \{k_0\}, j-1, \varkappa)$ , therefore, we conclude by induction that

$$D_b = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m-1}{\varkappa} - 1, \iota = 0, \dots, \binom{m-1-\varkappa}{m-j} - 1 \right\}.$$

We renumber elements of  $D_b$  as follows

$$D_b = \left\{ (f(M \setminus \{k_0\}, \varkappa, \rho), (\psi, \chi)(\rho, \iota)) \mid \rho = 0, \dots, \binom{m-1}{\varkappa} - 1, \iota = \binom{m-1-\varkappa}{m-1-j}, \dots, \binom{m-\varkappa}{m-j} - 1 \right\},$$

and we obtain that

$$D_b \cup D_2 = F_2.$$

Thus, we have

$$D_a \cup D_b \cup D_2 = F_1 \cup F_2,$$

and, consequently,

$$D = F,$$

which completes the proof.

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