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# Investigations toward higher resolution time-stepping schemes for NonSmooth MultiBody Systems (NSMBS)

Vincent Acary

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# Investigations toward higher resolution time-stepping schemes for NonSmooth Multibody Systems (NSMBS)

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Journée CSMA, Nantes , April 3, 2008

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

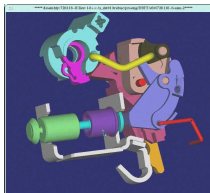
Splitting based Schemes

Conclusions & Perspectives

References

## Mechanical systems with contact, impact and friction

### Simulation of Circuit breakers (INRIA/Schneider Electric)



Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

# Applications

## Mechanical systems with contact, impact and friction

### Bipedal Robot INRIA BIPOP



Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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#### Introduction & Motivations

##### Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

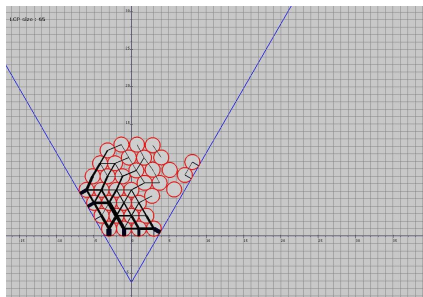
Splitting based Schemes

Conclusions & Perspectives

References

## Mechanical systems with contact, impact and friction

### Stack of beads with perturbation



### Introduction & Motivations

#### Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

Numerical time-integration methods for Nonsmooth Multibody systems (NSMBS):

## Nonsmooth event capturing methods (Time-stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

## Nonsmooth event tracking methods (Event-driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

# Objectives & means

## Objectives

Design nonsmooth event capturing methods with

- ▶ same properties as standard methods (robustness, accumulation, ...)
- ▶ Higher resolution (ratio error/computational cost)
- ▶ Higher order of accuracy

## Means

1. Adaptive time-step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

## General definition

$$\left\{ \begin{array}{l} M(q)\dot{v} = F(t, q, v) + G(t, q)\lambda \quad (1a) \\ \dot{q} = v \quad (1b) \\ w = g(t, q, v) \quad (1c) \\ 0 \in S(w, \lambda, t) + T(w, \lambda, t) \quad (1d) \\ v^+ = \mathcal{F}(v^-, q, t) \quad (1e) \end{array} \right.$$

- ▶  $S : \mathbb{R}^{m \times m} \times \mathbb{R} \mapsto \mathbb{R}^{m \times m}$  continuously differentiable mapping
- ▶  $T : \mathbb{R}^{m \times m} \times \mathbb{R} \rightsquigarrow \mathbb{R}^{m \times m}$  multivalued mapping with a closed graph.

## With scleronomous holonomic perfect unilateral constraints

$$\left\{ \begin{array}{l} M(q)\dot{v} = F(t, q, v) + G(q)\lambda \\ \dot{q} = v \\ 0 \leq y = g(q) \perp \lambda \geq 0 \\ v^+ = \mathcal{F}(v^-, q, t) \end{array} \right. \quad (2)$$

where  $G(q) = \nabla g(q)$



# NonSmooth Multibody Systems (NSMBS)

## Academic examples I

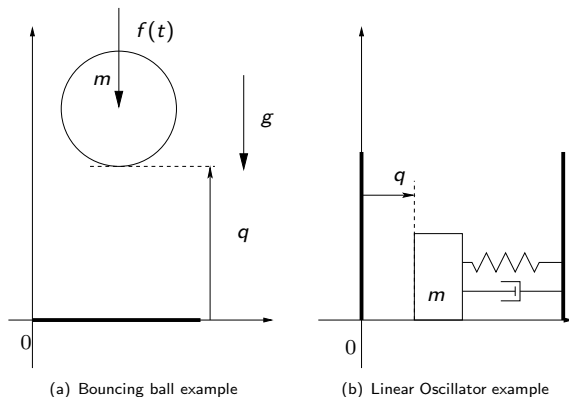


Figure: Academic test examples with analytical solutions

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

# NonSmooth Multibody Systems (NSMBS)

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

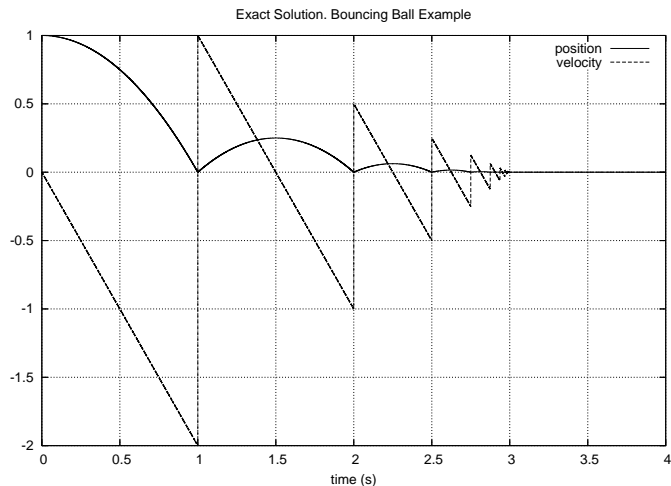


Figure: Analytical solutions. Bouncing ball example]

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Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

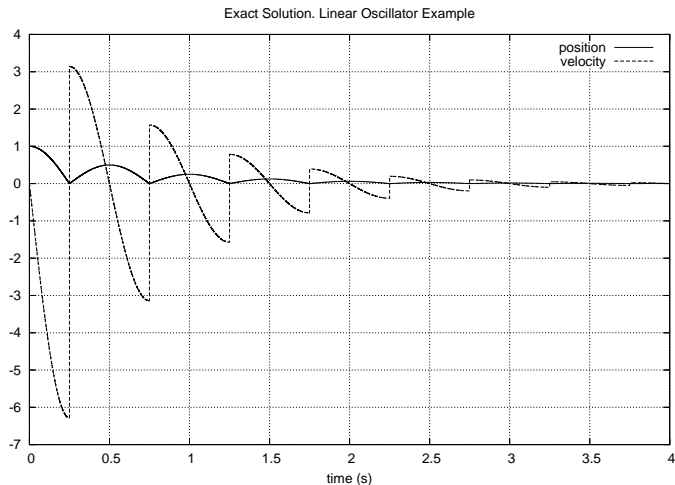
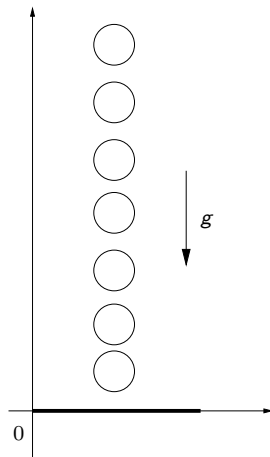


Figure: Analytical solutions. Linear Oscillator

# NonSmooth Multibody Systems (NSMBS)

## Academic examples II



(a) N Bouncing balls example

Figure: Academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

# Moreau's Time stepping scheme

## Principle

$$\left\{ \begin{array}{l} M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, \end{array} \right. \quad (3a)$$

$$q_{k+1} = q_k + hv_{k+\theta}, \quad (3b)$$

$$U_{k+1} = G^T(q_{k+\theta})v_{k+1} \quad (3c)$$

$$-P_{k+1} \in \partial\psi_{T_{\mathbb{R}_+^m}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), \quad (3d)$$

$$\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0, 1]. \quad (3e)$$

with  $\theta \in [0, 1], \gamma \geq 0$  and  $x_{k+\alpha} = (1 - \alpha)x_{k+1} + \alpha x_k$  and  $\tilde{y}_{k+\gamma}$  is a prediction of the constraints.

## Properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proofs of order

## Principle

$$\left\{ \begin{array}{l} M(q_k + 1)(q_{k+1} - 2q_k + q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, \quad (4a) \\ v_{k+1} = \frac{q_{k+1} - q_{k-1}}{2h}, \quad (4b) \\ -p_{k+1} \in N_K \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right), \quad (4c) \end{array} \right.$$

where  $N_K$  defined the normal cone to  $K$ .

For  $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right) \perp \nabla g \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right) P_{k+1} \geq 0 \quad (5)$$

## Properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proof of order

## Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^*(f) = \{(t, x) \in [0, T] \times \mathbb{R}^n, 0 \leq t \leq T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (6)$$

Such graphs are closed bounded subsets of  $[0, T] \times \mathbb{R}^n$ , hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t, x), (s, y)) = \max\{|t - s|, \|x - y\|\}. \quad (7)$$

Defining the excess of separation between two graphs by

$$e(gr^*(f), gr^*(g)) = \sup_{(t,x) \in gr^*(f)} \inf_{(s,y) \in gr^*(g)} d((t, x), (s, y)), \quad (8)$$

the Hausdorff distance between two filled-in graphs  $h^*$  is defined by

$$h^*(gr^*(f), gr^*(g)) = \max\{e(gr^*(f), gr^*(g)), e(gr^*(g), gr^*(f))\}. \quad (9)$$



An equivalent grid-function norm to the function norm in  $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^N |f_i - f(t_i)| \quad (10)$$

In the same way, the  $p$ -norm can be defined by

$$\|e\|_p = \left( h \sum_{i=0}^N |f_i - f(t_i)|^p \right)^{1/p} \quad (11)$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

Global order of convergence.

## Definition

A one-step time-integration scheme is of order  $q$  for a given norm  $\|\cdot\|$  if there exists a constant  $C$  such that

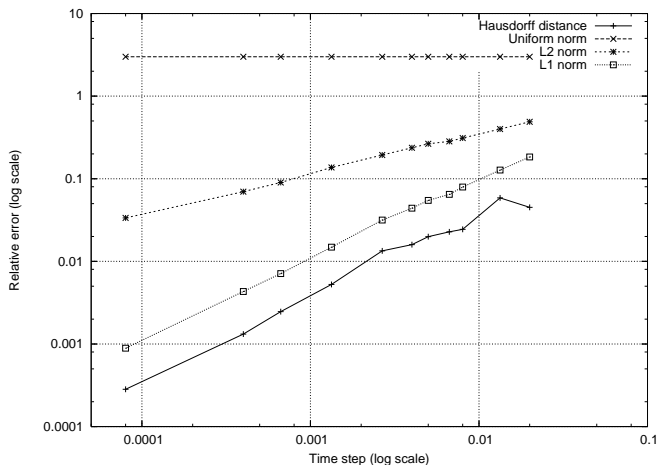
$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \quad (12)$$

# Empirical order of convergence. Moreau's time-stepping scheme

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping schemes](#)

**Background**

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

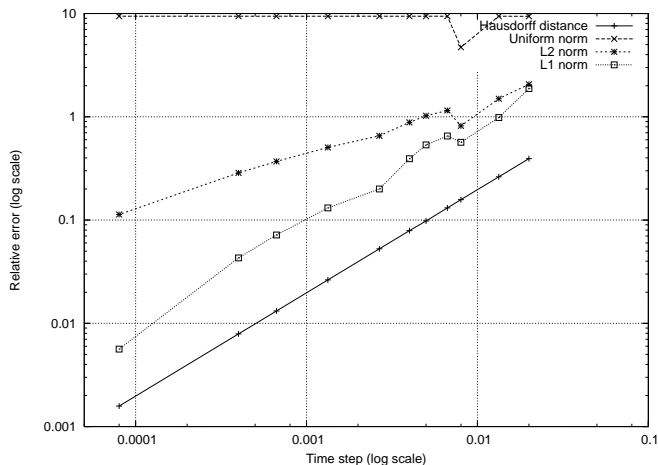
[References](#)

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Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The linear oscillator example

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Introduction & Motivations

Outline

Background time-stepping  
schemes

**Background**

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

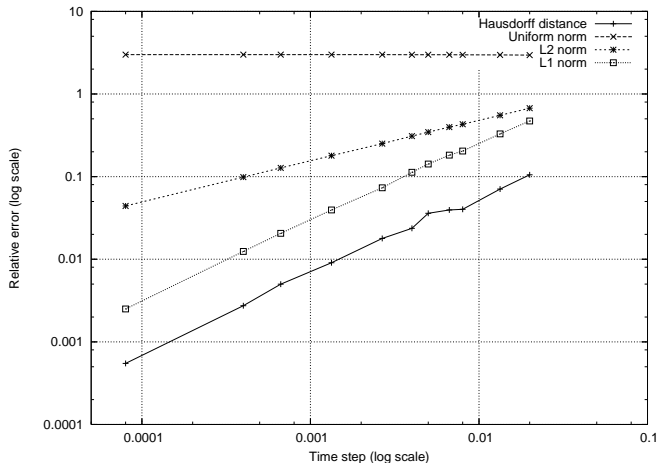
References

# Empirical order of convergence. Schatzman–Paoli's time-stepping scheme

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The bouncing ball example

**Figure:** Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping schemes](#)

**Background**

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

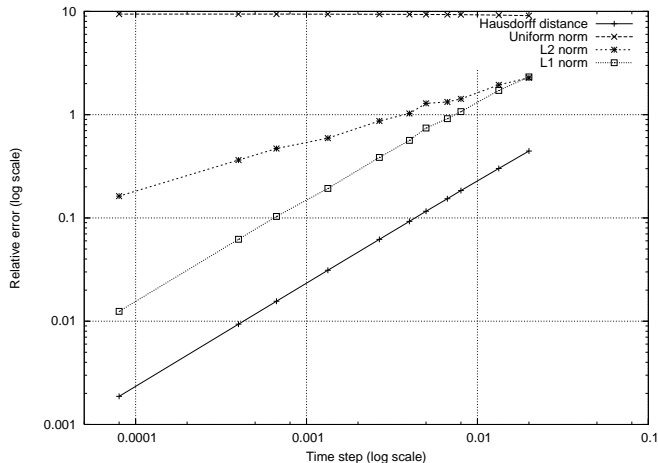
[References](#)

# Empirical order of convergence. Schatzman–Paoli's time-stepping scheme

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The linear oscillator example

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Introduction & Motivations

Outline

Background time-stepping  
schemes

**Background**

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

Introduction & Motivations

Outline

Background time-stepping  
schemes

Background

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

## One-step numerical solvers for ODEs

Let us consider a ODE

$$\dot{x} = f(x, t), \quad (13)$$

where  $f$  is a mapping with sufficient regularity.

The one-step time-stepping method over the time-step  $[t_k, t_{k+1} = t_k + h]$  is generically denoted by

$$x_{k+1} = x_k + h\Phi(t_k, h, x_k). \quad (14)$$

## Order of consistency

The one-step time-stepping method is said to be consistent if  $\Phi(t, 0, x, x) = f(x, t)$  and has a consistency order  $p$  if there exists a constant  $C$  such that

$$e_{k+1} = x(t_{k+1}) - x_{k+1} = Ch^{p+1} + \mathcal{O}(h^{p+2}), \quad (15)$$

assuming that  $x_k = x(t_k)$ .

If the time-stepping method has an order of consistency  $p$  and converges, then the global order of convergence is  $p$ ,

## Basic practical error evaluation

1. Two “small” time steps of size  $h/2 \implies x_{1/2}$ .
2. One “big” time-step  $h \implies x_1$ .

$$\begin{aligned}e_1 &= x(t_0 + h) - x_1 = C h^{p+1} + \mathcal{O}(h^{p+2}), \\e_{1/2} &= x(t_0 + h) - x_{1/2} = 2C (h/2)^{p+1} + \mathcal{O}(h^{p+2}).\end{aligned}\quad (16)$$

This procedure permits us to evaluate the constant  $C$  and to obtain and a local error estimate such that:

$$e_2 = x(t_0 + h) - x_2 = \frac{x_{1/2} - x_1}{2^p - 1} + \mathcal{O}(h^{p+2}).\quad (17)$$

## Enhanced practical error evaluation

- ▶ Runge–Kutta Embedded pairs (Dormand-Price, Fehlberg)
- ▶ Milne’s device
- ▶ Nordsieck’s method



## Automatic control of the time-step

$$\|e_k\| \leq etol = atol + rtol \circ \max(x_0, x_k) \quad (18)$$

The measure of the error is given by

$$\text{error} = \|e_k \circ invtol\| \quad (19)$$

with  $invtol = [1/etol_i, i = 1 \dots n]$ . The optima step size is then obtained by

$$h_{opt} = h \left( \frac{1}{\text{error}} \right)^{1/(p+1)} \quad (20)$$

Usually, the step size is not allowed to decrease or to increase too fast, thanks to the following heuristic rule

$$h_{new} = h \min(\alpha_{max}, \max(\alpha_{min}, \alpha \left( \frac{1}{\text{error}} \right)^{1/(p+1)})) \quad (21)$$

where  $\alpha$ ,  $\alpha_{min}$  and  $\alpha_{max}$  are some user parameters.

# Local error estimates for the Moreau's time-stepping

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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## Notation

$$\mathbf{e} = \mathbf{x}(t_k + h) - \mathbf{x}_{k+1} = \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_q \end{bmatrix} = \begin{bmatrix} \mathbf{v}^+(t_k + h) - \mathbf{v}_{k+1} \\ \mathbf{q}(t_k + h) - \mathbf{q}_{k+1} \end{bmatrix} \quad (22)$$

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Smooth ODE time  
integration

**Local error estimates for the  
Moreau's Time-stepping  
scheme**

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

# One impact at time $t_* \in (t_k, t_{k+1})$

## Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dl = P\delta_{t_*}, \quad \text{with } P = G(t_*)p. \quad (23)$$

## Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \quad (24)$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geq 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \quad (25)$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P_{k+1}^\alpha \geq 0, U_{k+1}^\alpha - U_k^\alpha = -(1+e)U_k^\alpha\} \quad (26)$$

## Lemma

Let us assume that we have only one elastic impact at time  $t_* \in (t_k, t_{k+1})$  without persistent contact, i.e. ,  $di = p\delta_{t_*}$ .

1. If  $\mathcal{I}_* = \mathcal{I}_p$ , then the local order of consistency of the scheme is given by

$$\begin{aligned} e_v &= K_v h + \mathcal{O}(h^2) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \quad (27)$$

2. If  $\mathcal{I}_* \neq \mathcal{I}_p$ , then the local order of consistency of the scheme is given by

$$\begin{aligned} e_v &= K_v + \mathcal{O}(h) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \quad (28)$$

# Smooth Lagrange multiplier in persistent contact without impact in $(t_k, t_{k+1}]$

## Assumption

$$di = \lambda(t)dt, \quad (29)$$

or equivalently

$$dI = \Lambda(t)dt, \text{ with } \Lambda(t) = G(t)\lambda(t). \quad (30)$$

## Notation

$$\mathcal{I}_\Lambda(t) = \{\alpha \in \mathcal{I}, \Lambda^\alpha(t) \geq 0, U^{\alpha,+}(t) = U^{\alpha,-}(t) = 0\} \quad (31)$$

$$\mathcal{I}_{\Lambda,k+1} = \{\alpha \in \mathcal{I}, \Lambda_{k+1}^\alpha \geq 0, U_{k+1}^\alpha = U_k^\alpha = 0\} \quad (32)$$

## Lemma

Assuming that  $\mathcal{I}_\Lambda(t) = \mathcal{I}_{\Lambda,k+1}$  for all  $t \in (t_k, t_{k+1}]$ . The local order of consistency of the scheme is one that is

$$\begin{aligned} e_v &= Kh^2 + \mathcal{O}(h^3) \\ e_q &= K_q h^2 + \mathcal{O}(h^3) \end{aligned} \quad (33)$$

# Local error estimates for the Moreau's time-stepping

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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## Other cases

- ▶ *One impact and smooth Lagrange multiplier* The same result holds ad in first Lemma.
- ▶ *losing contact event (take-off) without impact* The order of the time-integration scheme depends on the regularity of the contact forces (at least continuous).
- ▶ *Finite accumulation* The order of the time-integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Smooth ODE time  
integration

Local error estimates for the  
Moreau's Time-stepping  
scheme

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

# Practical error estimates for the Moreau's time-stepping

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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## Order “0” case

Standard error estimates do not apply for Order 0.

We propose to extend it to the order 0 of consistency by assuming that the constant can be evaluated by

$$C = \frac{2(e_1 - e_{1/2})}{h} \quad (34)$$

and the local error estimate by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \quad (35)$$

The adaptive time-step control exposed for smooth ODE is then apply directly.

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

**An “Order 0” approach**

A control based on violation

Variable order approach

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

# Order “0” time-step adjustment for the Moreau’s time-stepping

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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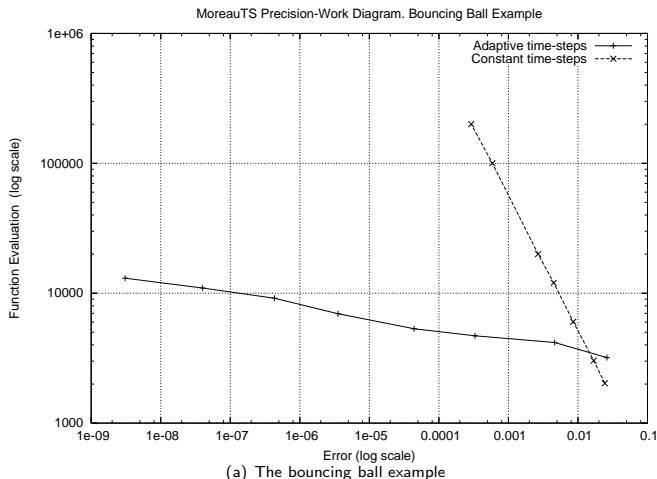


Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

An “Order 0” approach  
A control based on violation  
Variable order approach

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

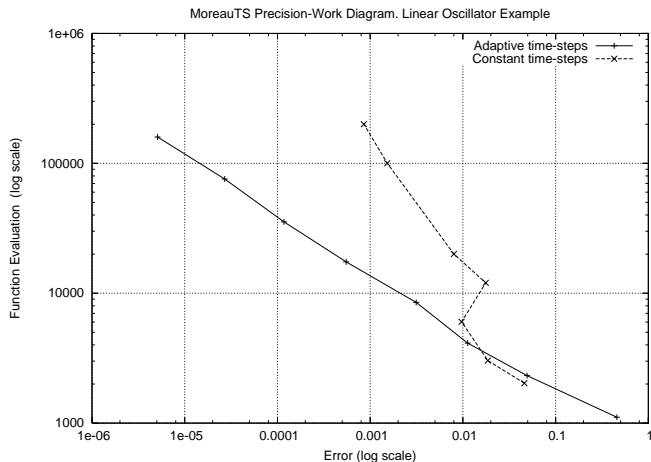
References

# Order “0” time-step adjustment for the Moreau’s time-stepping

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

**An “Order 0” approach**  
A control based on violation  
Variable order approach

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References



# Sizing the error in the violation of constraints

The violation of constraints is sized by the following rule:

$$e_{\text{violation}} = \|\min(0, g(q)) \circ \text{invtol}\|_{\infty} \quad (36)$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by  $e_{\text{violation}}$  when a nonsmooth event occurs, the step size adjustment is implemented by the means of the following error estimation

$$\text{error} = \max(e_{\text{violation}}, \|e_k \circ \text{invtol}\|_{\infty}) \quad (37)$$

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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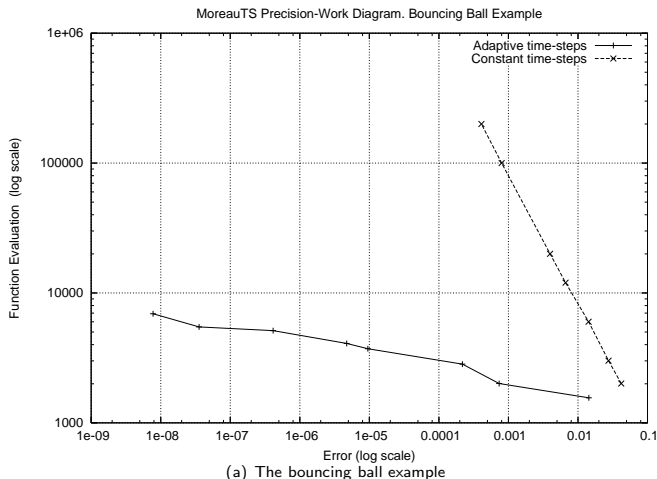


Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[An "Order 0" approach](#)

**[A control based on violation](#)**

[Variable order approach](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

Vincent Acary

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[An "Order 0" approach](#)

**[A control based on violation](#)**

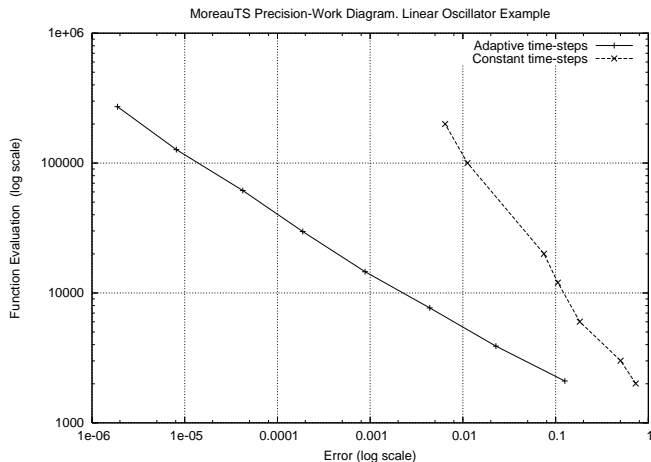
[Variable order approach](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)



(a) The linear oscillator example

**Figure:** Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

Vincent Acary

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

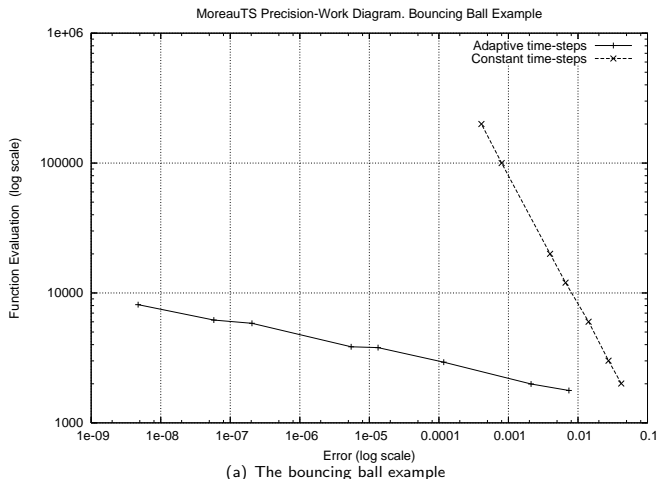
An "Order 0" approach  
**A control based on violation**  
Variable order approach

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)



**Figure:** Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

Vincent Acary

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

An "Order 0" approach

**A control based on violation**

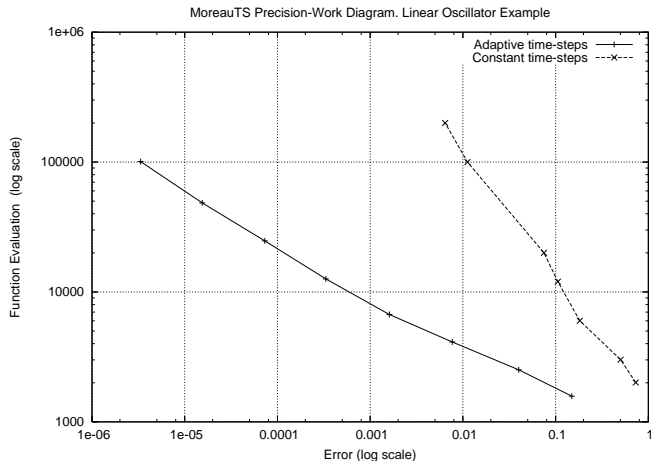
Variable order approach

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)



(a) The linear oscillator example

**Figure:** Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

# Variable order approach. Principle

Guess the order of consistency of the integration at each step.  
Adapt the practical error estimation

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

An "Order 0" approach

A control based on violation

**Variable order approach**

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

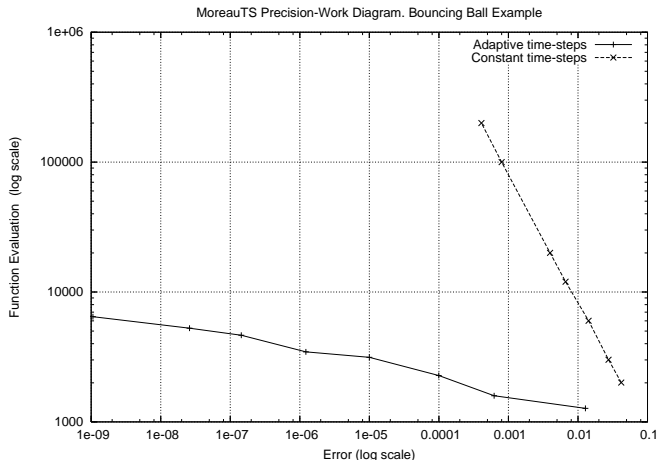
References

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The bouncing ball example

[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[An "Order 0" approach](#)

[A control based on violation](#)

[Variable order approach](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)

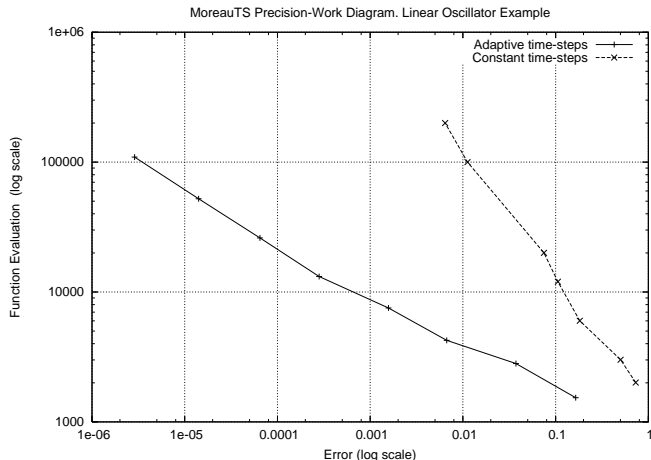
Figure: Precision Work diagram for the Moreau's time-stepping scheme.

# Results on two academic test examples

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The linear oscillator example

[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[An "Order 0" approach](#)  
[A control based on violation](#)

**[Variable order approach](#)**

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)

Figure: Precision Work diagram for the Moreau's time-stepping scheme.



## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

An "Order 0" approach

A control based on violation

**Variable order approach**

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

References

## Background

Work of Mannshardt (1978) on time-integration schemes of any order for ODEs with discontinuities (with transversality assumption)

## Principle

- ▶ Let us assume only one event per time-step at instants  $t_*$ .
- ▶ Choose any ODE solvers of order  $p$
- ▶ Perform a rough location of the event inside the time step of length  $h$   
Find an interval  $[t_a, t_b]$  such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \quad (38)$$

Dichotomy, Newton, Local Interpolants, Dense output,...

- ▶ Perform an integration on  $[t_k, t_a]$  with the ODE solver of order  $p$
- ▶ Perform an integration on  $[t_a, t_b]$  with Moreau's time-stepping scheme
- ▶ Perform an integration on  $[t_b, t_{k+1}]$  with the ODE solver of order  $p$

# Results on the linear oscillator

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

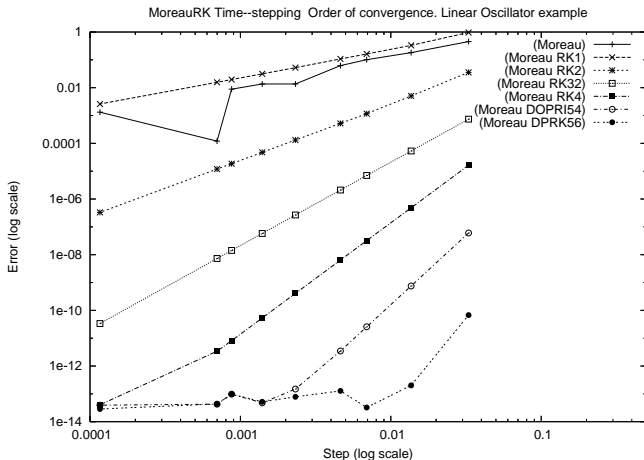
[Higher Order Schemes](#)

**Principle**

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

# Higher Order Time-stepping schemes

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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## Finite accumulation

- ▶ Repeat the whole process on the remaining part of the interval  $[t_b, t_k]$
- ▶ By induction, repeat this process up to the end of the original time step.

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

**Principle**

Splitting based Schemes

Conclusions & Perspectives

References

# Results on the Bouncing Ball

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

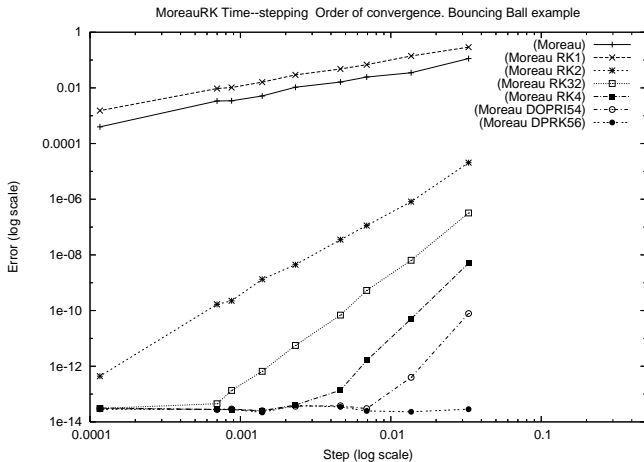
[Higher Order Schemes](#)

**Principle**

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)



(a) The Bouncing Ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

# Splitting-based methods.

## Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x, t) + g(x, t) \quad (39)$$

A example of splitting-based method is given by the following procedure

1. Perform the integration of  $f$  on  $[t_k, t_{k+1}]$  to obtain  $\tilde{x}(t_{k+1})$  that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x, t) dt \quad (40)$$

2. Perform the integration of  $g$  on  $[t_k, t_{k+1}]$  with initial value  $\tilde{x}(t_{k+1})$  to obtain  $\hat{x}(t_{k+1})$  that is

$$\hat{x}(t_{k+1}) = \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x, t) dt \quad (41)$$

## Properties

- ▶  $x(t_k + 1) \neq \hat{x}(t_{k+1})$  is the general case. (except special linear case, constant dynamics, ...)
- ▶  $\hat{x}(t_{k+1}) \rightarrow x(t_{k+1})$  when  $t_{k+1} \rightarrow t_k$

## Splitting-based for Moreau scheme without continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v), \\ \dot{q} = v, \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (42)$$

yielding to the approximations  $q_1 = q(t_{k+1})$  and  $v_1 = v(t_{k+1})$  which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{\mathbb{R}_+}(y)(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (43)$$

and leads to the approximation  $q_{k+1} = q(t_{k+1})$  and  $v_{k+1} = v(t_{k+1})$ .

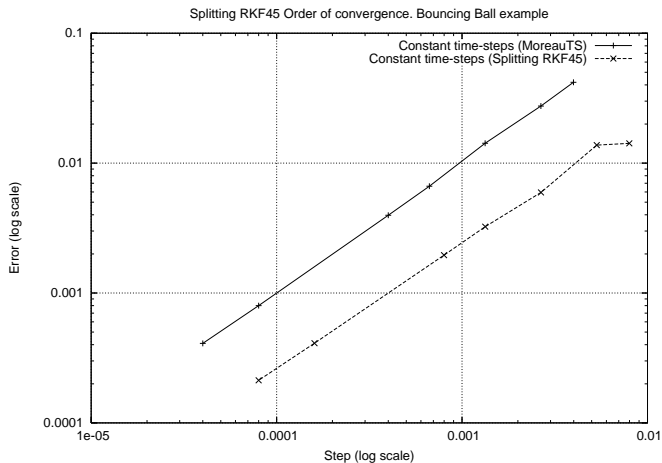


# Splitting-based methods with constants time-step.

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
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(a) The bouncing ball example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

**Principle**

Conclusions & Perspectives

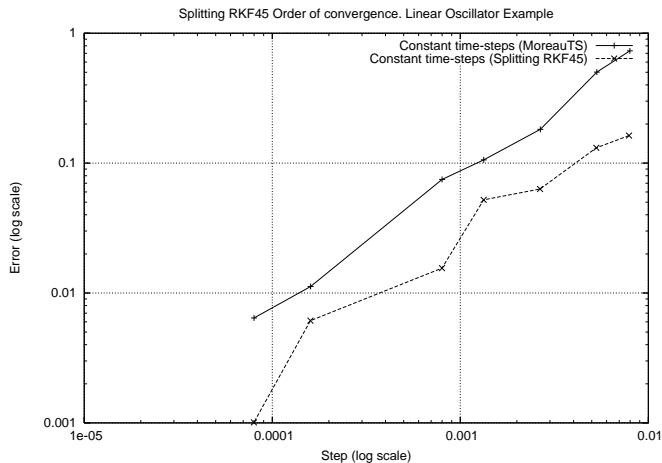
References

# Splitting-based methods with constants time-step.

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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(a) The linear oscillator example

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

**Principle**

Conclusions & Perspectives

References

# Splitting-based methods with adaptive time-step.

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

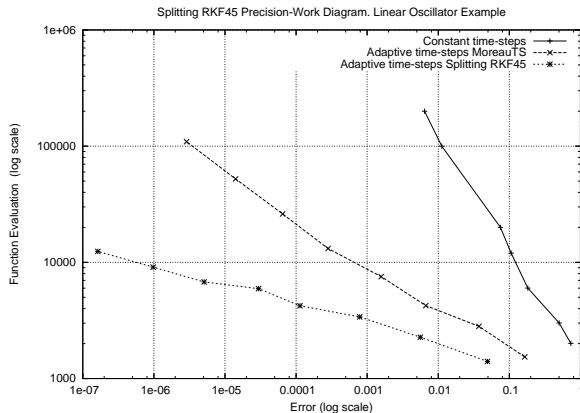
[Higher Order Schemes](#)

[Splitting based Schemes](#)

**Principle**

[Conclusions & Perspectives](#)

[References](#)



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

# Splitting-based methods.

## Splitting-based for Moreau scheme with continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{\mathbb{R}_+}(y)(\dot{y}(t)) \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (44)$$

yielding to the approximations  $q_1 = q(t_{k+1})$  and  $v_1 = v(t_{k+1})$  which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{\mathbb{R}_+}(y)(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (45)$$

and leads to the approximation  $q_{k+1} = q(t_{k+1})$  and  $v_{k+1} = v(t_{k+1})$ .

## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

# Conclusions

## Adaptive time-step strategies

- ▶ Higher resolution schemes
- ▶ Work with finite accumulation of events

## Higher order schemes

- ▶ Schemes of any orders
- ▶ Work with finite accumulation of events

## Splitting based methods

- ▶ Higher resolution schemes
- ▶ Work with finite accumulation of events

Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

**Conclusions**

Perspectives

References

- ▶ Theoretical works on orders and practical error estimations
- ▶ Adaptive time-step strategies on the higher order time-stepping schemes.
- ▶ Improve the pre-detection process of the event and the order of discontinuity
- ▶ Test on nonsmooth and nonlinear mechanical systems.
- ▶ Adapt the schemes with a step without external forces when the Moreau's scheme is used
- ▶ Other types of time-stepping schemes ...

## Introduction & Motivations

### Background time-stepping schemes

Background

### Adaptive time-step size time-stepping scheme

Smooth ODE time integration

Local error estimates for the Moreau's Time-stepping scheme

### Adaptive time-step strategies

An "Order 0" approach

A control based on violation

Variable order approach

### Time-stepping schemes of any order

Principle

### Splitting based Time-stepping schemes

Principle

### Conclusions & Perspectives

Conclusions

Perspectives

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

Conclusions

**Perspectives**

References



Thank you for your attention.

Introduction & Motivations

Outline

Background time-stepping  
schemes

Adaptive schemes

Adaptive time-step strategies

Higher Order Schemes

Splitting based Schemes

Conclusions & Perspectives

Conclusions

**Perspectives**

References

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Investigations toward  
higher resolution  
time-stepping schemes  
for NonSmooth Multibody  
Systems (NSMBS)

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[Introduction & Motivations](#)

[Outline](#)

[Background time-stepping  
schemes](#)

[Adaptive schemes](#)

[Adaptive time-step strategies](#)

[Higher Order Schemes](#)

[Splitting based Schemes](#)

[Conclusions & Perspectives](#)

[References](#)