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Vincent Acary

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Investigations toward higher resolution time–stepping schemes for NonSmooth Multibody Systems (NSMBS)

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Journée CSMA, Nantes , April 3, 2008

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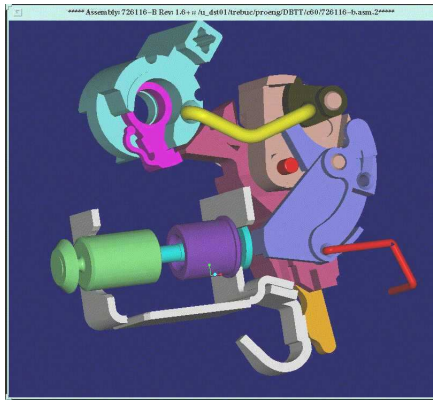
Applications

Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



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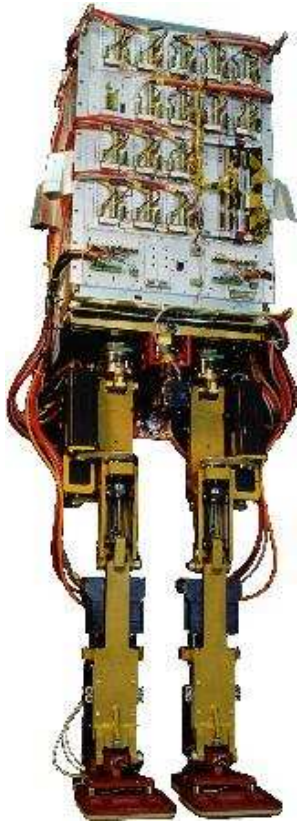
Bipedal Robot INRIA BIPOP



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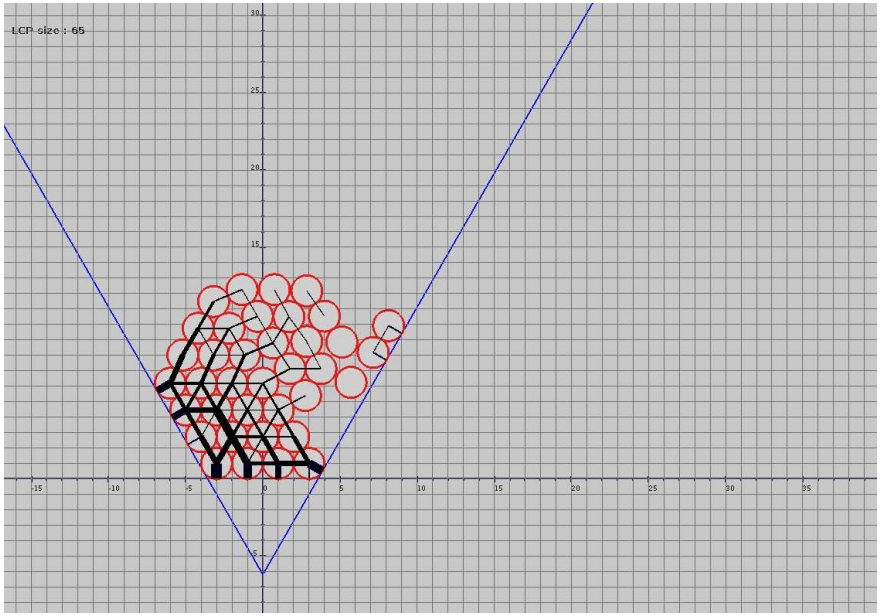
Stack of beads with perturbation



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State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

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Objectives & means

Objectives

Design nonsmooth event capturing methods with

- same properties as standard methods (robustness, accumulation, . . .)
- Higher resolution (ratio error/computational cost)
- Higher order of accuracy

Means

1. Adaptive time–step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

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NonSmooth Multibody Systems (NSMBS)

General definition

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + G(t,q)\lambda & \text{(1a)} \\ \dot{q} = v & \text{(1b)} \\ w = g(t,q,v) & \text{(1c)} \\ 0 \in S(w,\lambda,t) + T(w,\lambda,t) & \text{(1d)} \\ v^+ = \mathcal{F}(v^-,q,t) & \text{(1e)} \end{cases}$$

- $S : \mathbb{R}^{m\times m} \times \mathbb{R} \mapsto \mathbb{R}^{m\times m}$ continuously differentiable mapping
- $T : \mathbb{R}^{m\times m} \times \mathbb{R} \rightsquigarrow \mathbb{R}^{m\times m}$ multivalued mapping with a closed graph.

With scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + G(q)\,\lambda \\ \dot{q} = v \\ 0 \leqslant y = g(q) \perp \lambda \geqslant 0 \\ v^+ = \mathcal{F}(v^-,q,t) \end{cases} \tag{2}$$

where $G(q) = \nabla g(q)$

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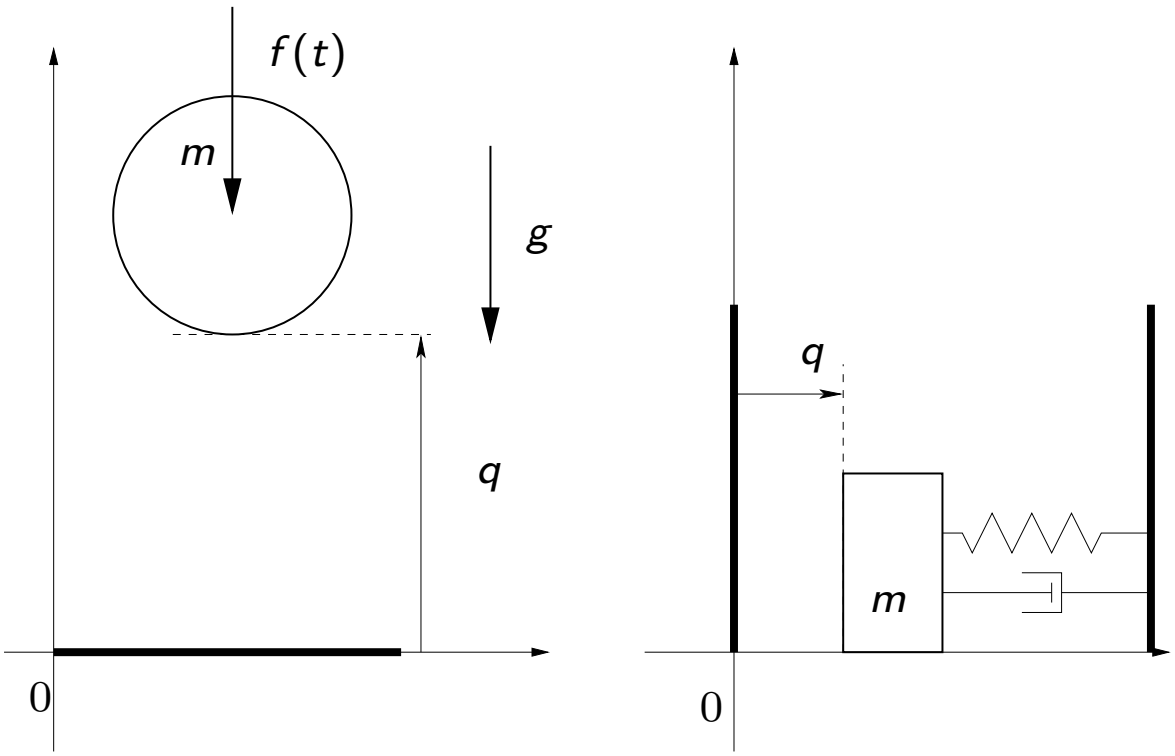
NonSmooth Multibody Systems (NSMBS)

Academic examples I



(a) Bouncing ball example

(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

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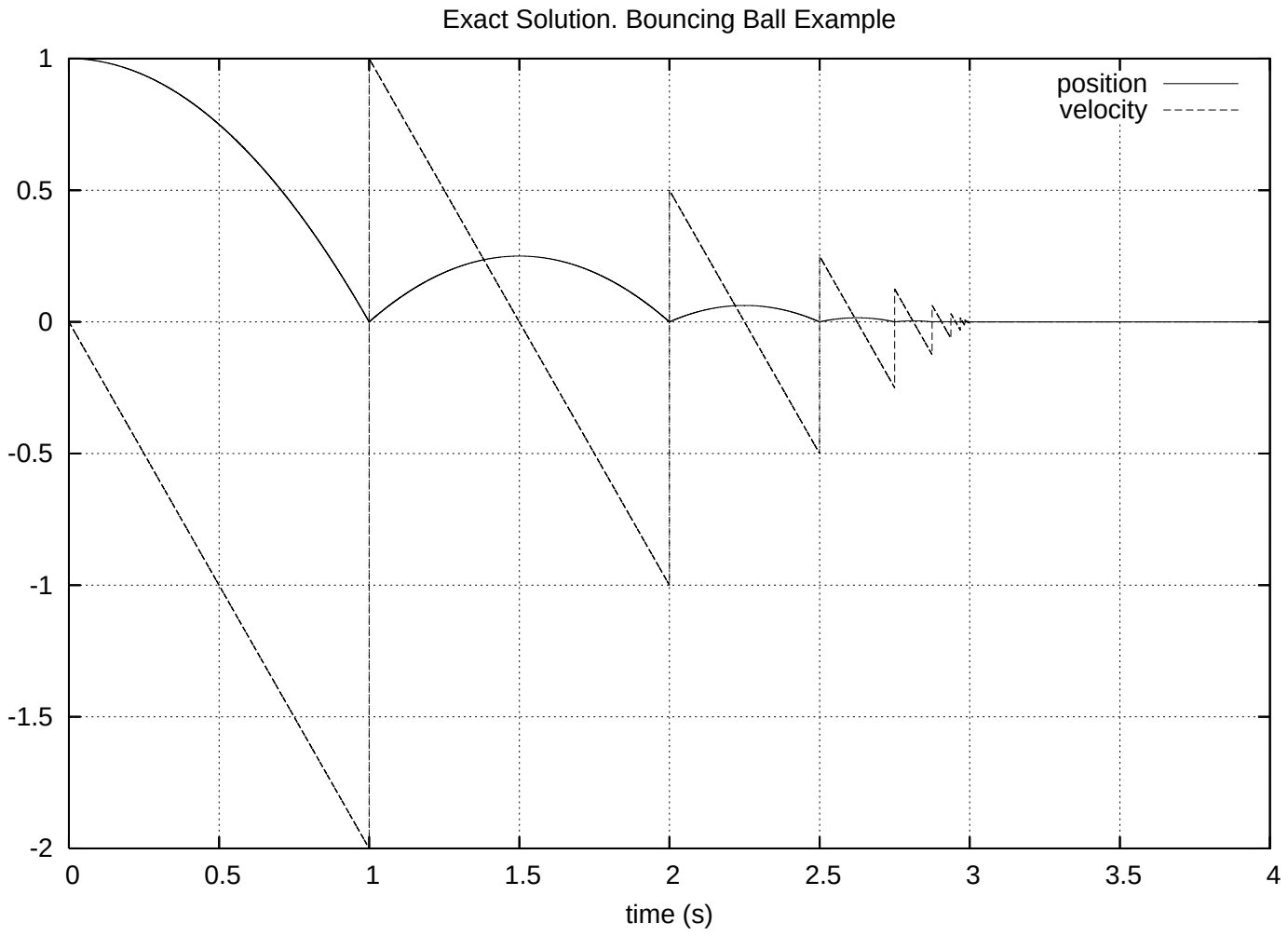

Figure: Analytical solutions. Bouncing ball example]

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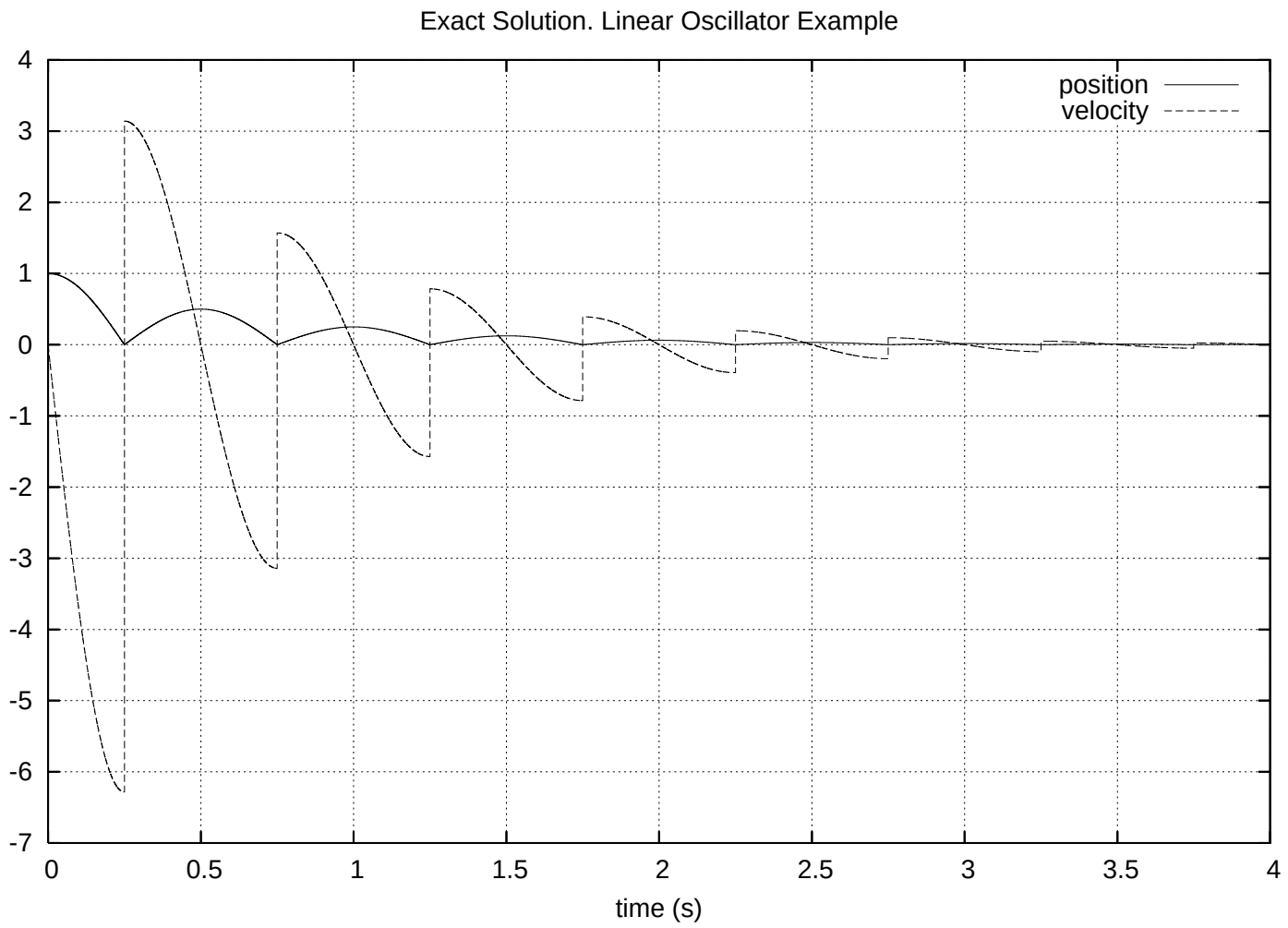

Figure: Analytical solutions. Linear Oscillator

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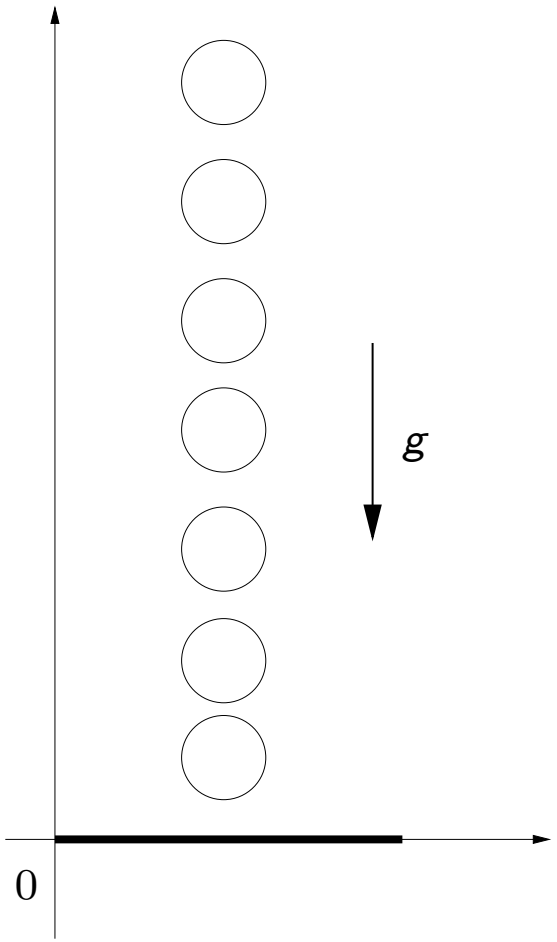
NonSmooth Multibody Systems (NSMBS)

Academic examples II



(a) N Bouncing balls example

Figure: Academic test examples

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Moreau's Time stepping scheme

Principle

$$
\begin{cases}
M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, & \text{(3a)} \\
q_{k+1} = q_k + hv_{k+\theta}, & \text{(3b)} \\
U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & \text{(3c)} \\
-P_{k+1} \in \partial\psi_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), & \text{(3d)} \\
\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & \text{(3e)}
\end{cases}
$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proofs of order

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Schatzman–Paoli's Time stepping scheme

Principle

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1})-h^2F(t_{k+\theta},q_{k+\theta},v_{k+\theta})=p_{k+1}, & (4a) \\ v_{k+1}=\dfrac{q_{k+1}-q_{k-1}}{2h}, & (4b) \\ -p_{k+1}\in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (4c) \end{cases}$$

where N_K defined the normal cone to K.
For $K=\{q\in\mathbb{R}^n, y=g(q)\geqslant 0\}$

$$0\leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right)\perp\nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right)P_{k+1}\geqslant 0 \qquad (5)$$

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

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Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^\star(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)])\}. \quad (6)$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \quad (7)$$

Defining the excess of separation between two graphs by

$$e(gr^\star(f), gr^\star(g)) = \sup_{(t,x)\in gr^\star(f)} \inf_{(s,y)\in gr^\star(g)} d((t,x),(s,y)), \quad (8)$$

the Hausdorff distance between two filled-in graphs $h^\star$ is defined by

$$h^\star(gr^\star(f), gr^\star(g)) = \max\{e(gr^\star(f), gr^\star(g)), e(gr^\star(g), gr^\star(f))\}. \quad (9)$$

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Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^{N} |f_i - f(t_i)| \tag{10}$$

In the same way, the $p - norm$ can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^{N} |f_i - f(t_i)|^p \right)^{1/p} \tag{11}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{12}$$

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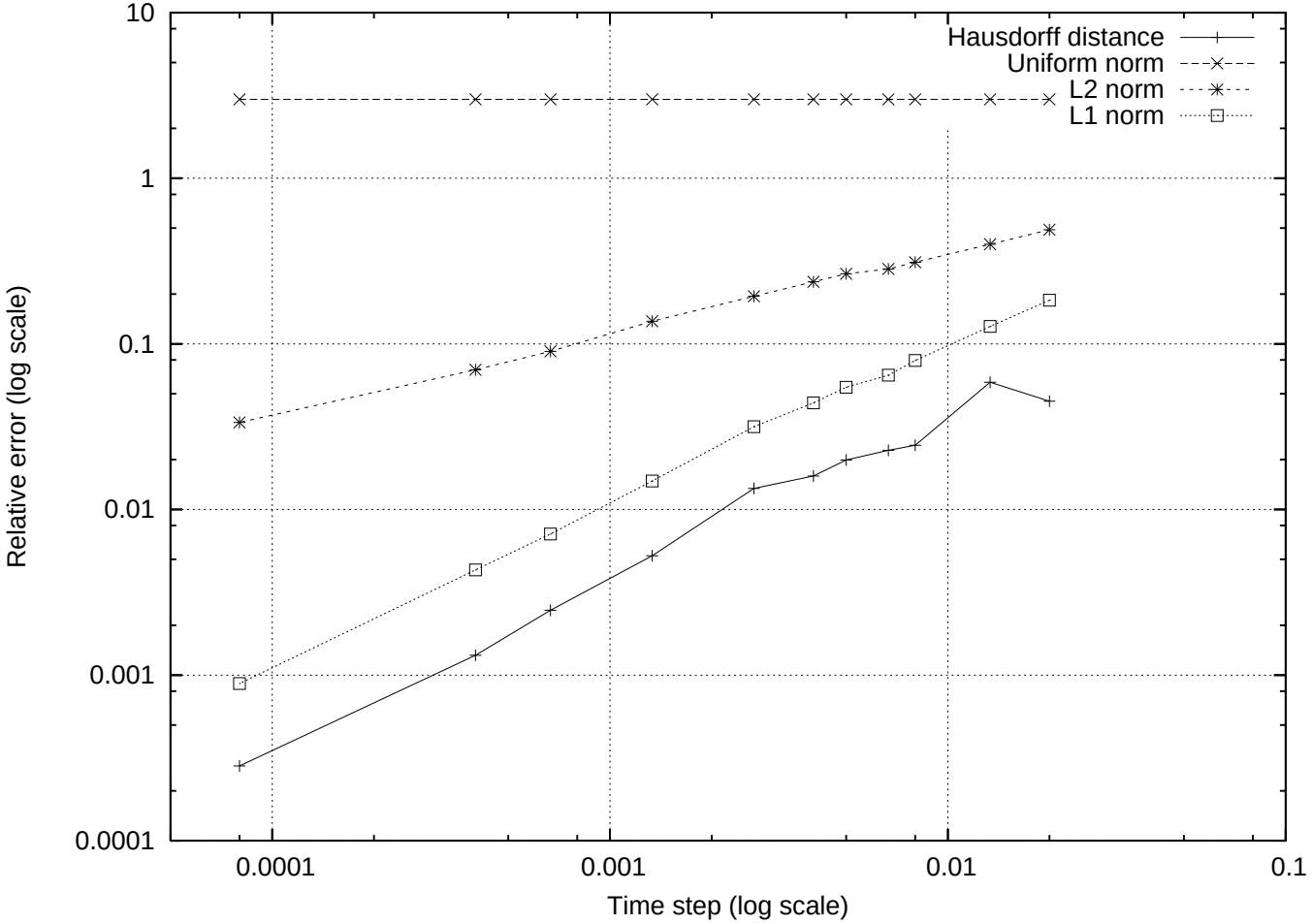
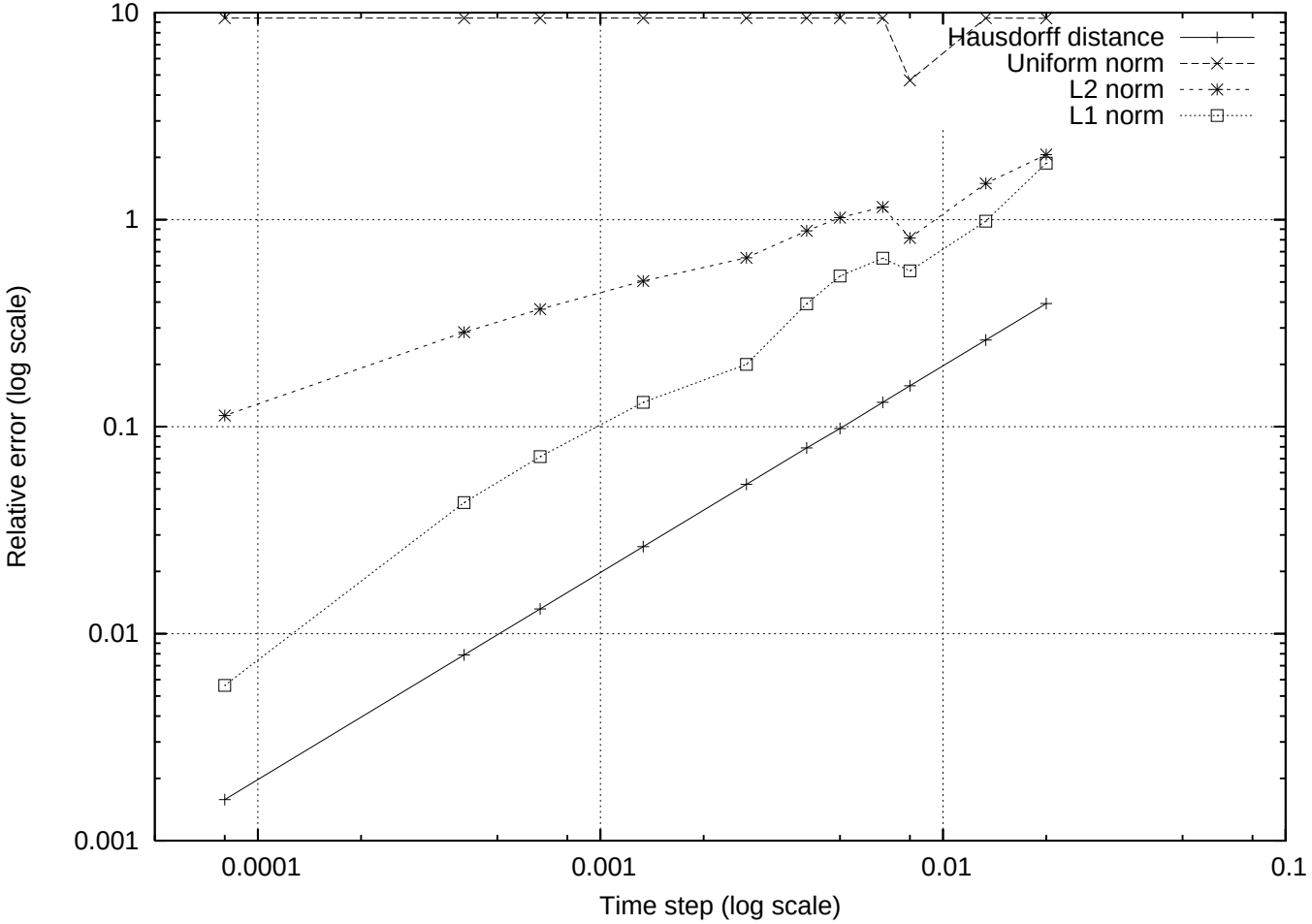
Empirical order of convergence. Moreau's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

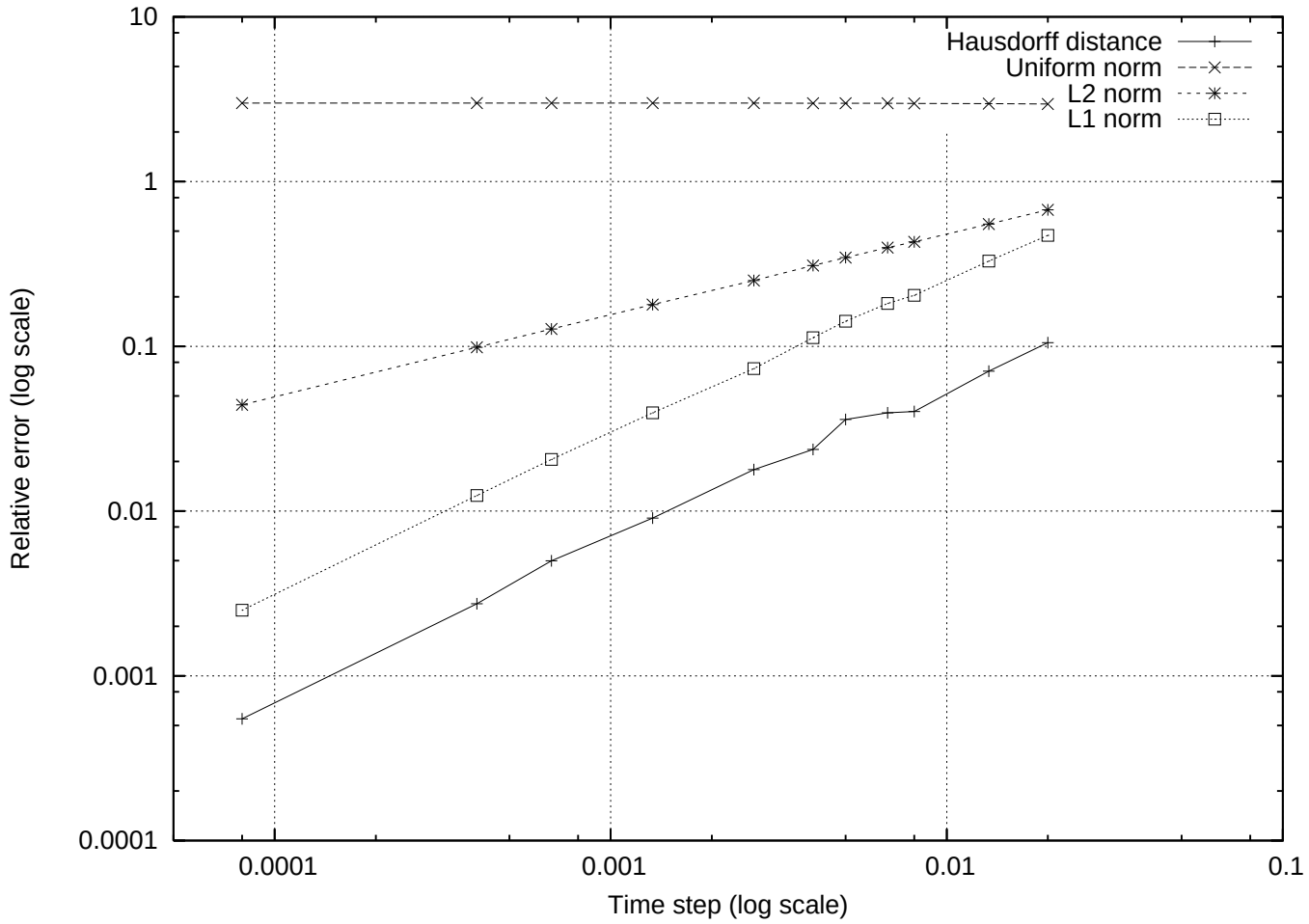
Empirical order of convergence. Moreau's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

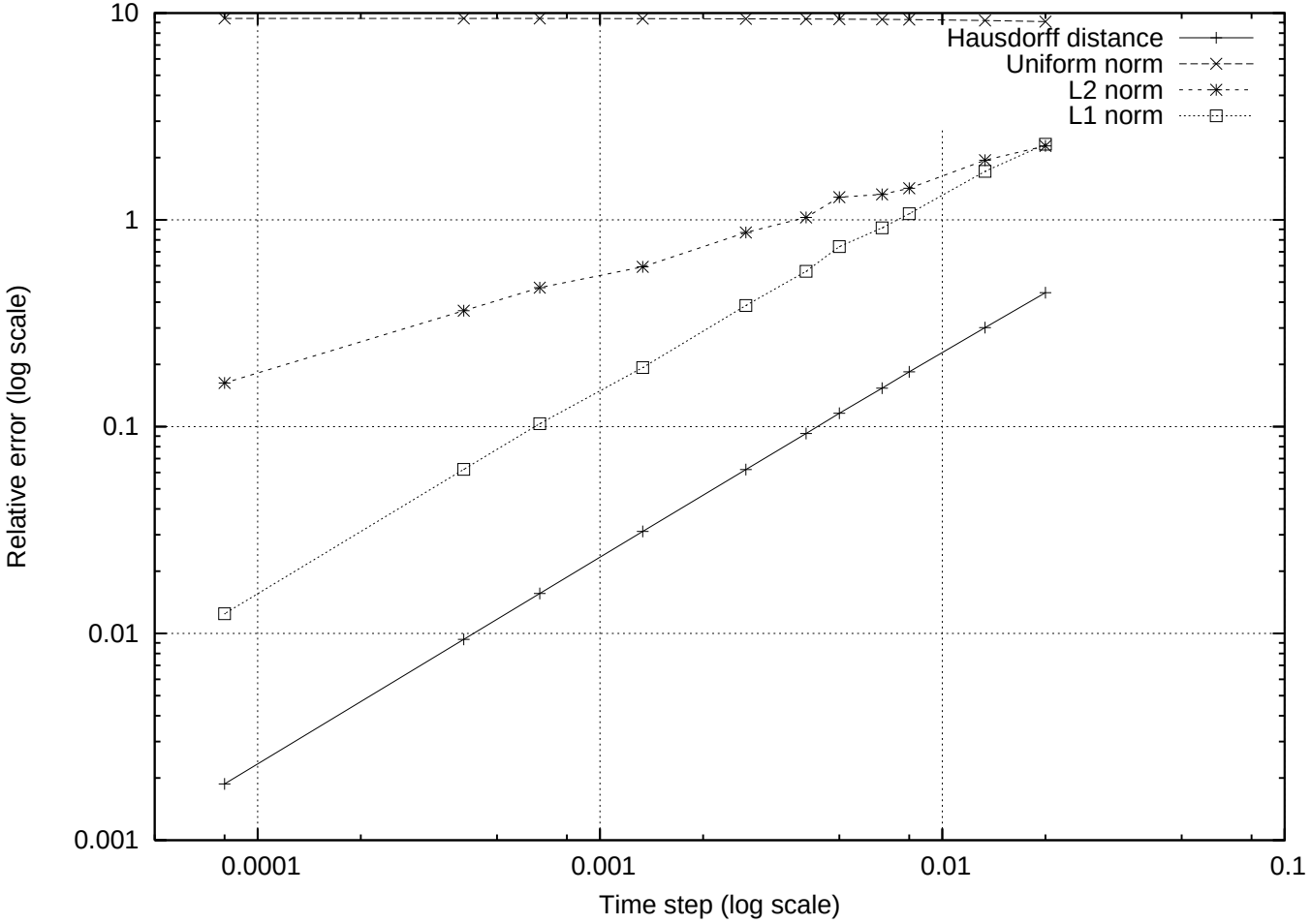
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

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Smooth ODEs

One–step numerical solvers for ODEs

Let us consider a ODE

$$\dot{x} = f(x, t), \tag{13}$$

where f is a mapping with sufficient regularity.
The one–step time–stepping method over the time–step $[t_k, t_{k+1} = t_k + h]$ is generically denoted by

$$x_{k+1} = x_k + h\Phi(t_k, h, x_k). \tag{14}$$

Order of consistency

The one–step time–stepping method is said to be consistent if $\Phi(t, 0, x, x) = f(x, t)$ and has a consistency order p if there exists a constant C such that

$$e_{k+1} = x(t_{k+1}) - x_{k+1} = Ch^{p+1} + \mathcal{O}(h^{p+2}), \tag{15}$$

assuming that $x_k = x(t_k)$.
If the time–stepping method has an order of consistency p and converges, then the global order of convergence is p,

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Smooth ODEs

Basic practical error evaluation

1. Two "small" time steps of size $h/2$ $\Longrightarrow x_{1/2}$.
2. One "big" time-step h $\Longrightarrow x_1$.

$$\begin{aligned} e_1 &= x(t_0+h) - x_1 = C\ h^{p+1} + \mathcal{O}(h^{p+2}), \\ e_{1/2} &= x(t_0+h) - x_{1/2} = 2C\ (h/2)^{p+1} + \mathcal{O}(h^{p+2}). \end{aligned} \tag{16}$$

This procedure permits us to evaluate the constant C and to obtain and a local error estimate such that:

$$e_2 = x(t_0+h) - x_2 = \frac{x_{1/2} - x_1}{2^p - 1} + \mathcal{O}(h^{p+2}). \tag{17}$$

Enhanced practical error evaluation

- Runge–Kutta Embedded pairs (Dormand-Price, Felhberg)
- Milne's device
- Nordsieck's method

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Smooth ODEs

Automatic control of the time–step

$$\|e_k\| \leqslant etol = atol + rtol \circ \max(x_0, x_k) \tag{18}$$

The measure of the error is given by

$$\text{error} = \|e_k \circ invtol\| \tag{19}$$

with $invtol = [1/etol_i, i = 1 \ldots n]$. The optima step size is then obtained by

$$h_{\text{opt}} = h(\frac{1}{\text{error}})^{1/(p+1)} \tag{20}$$

Usually, the step size is not allowed to decrease of to increase too fast, thanks to the following heuristic rule

$$h_{\text{new}} = h \min(\alpha_{max}, \max(\alpha_{min}, \alpha(\frac{1}{\text{error}})^{1/(p+1)})) \tag{21}$$

where α, α_{min} and α_{max} are some user parameters.

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Local error estimates for the Moreau's time–stepping

Notation

$$e = x(t_k + h) - x_{k+1} = \begin{bmatrix} e_v \\ e_q \end{bmatrix} = \begin{bmatrix} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{bmatrix} \tag{22}$$

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One impact at time $t_* \in (t_k, t_{k+1}]$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dI = P\delta_{t_*}, \text{with } P = G(t_*)p. \tag{23}$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \tag{24}$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geqslant 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \tag{25}$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} - U^\alpha_k = -(1+e)U^\alpha_k\} \tag{26}$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1}]$ without persistent contact, i.e. , $di = p\delta_{t_*}$.

1. *If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{array}{rcl} e_v & = & K_v h + \mathcal{O}(h^2) \\ e_q & = & K_q h + \mathcal{O}(h^2) \end{array} \tag{27}$$

2. *If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{array}{rcl} e_v & = & K_v + \mathcal{O}(h) \\ e_q & = & K_q h + \mathcal{O}(h^2) \end{array} \tag{28}$$

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Smooth Lagrange multiplier in persistent contact without impact in $(t_k, t_{k+1}]$

Assumption

$$di = \lambda(t)dt, \tag{29}$$

or equivalently

$$dI = \Lambda(t)dt, \text{with } \Lambda(t) = G(t)\lambda(t). \tag{30}$$

Notation

$$\mathcal{I}_\Lambda(t) = \{\alpha \in \mathcal{I}, \Lambda^\alpha(t) \geqslant 0, U^{\alpha,+}(t) = U^{\alpha,-}(t) = 0\} \tag{31}$$

$$\mathcal{I}_{\Lambda,k+1} = \{\alpha \in \mathcal{I}, \Lambda^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} = U^\alpha_k = 0\} \tag{32}$$

Lemma

Assuming that $\mathcal{I}_\Lambda(t) = \mathcal{I}_{\Lambda,k+1}$ for all $t \in (t_k, t_{k+1}]$. The local order of consistency of the scheme is one that is

$$\begin{array}{rcl} e_v & = & Kh^2 + \mathcal{O}(h^3) \\ e_q & = & K_q h^2 + \mathcal{O}(h^3) \end{array} \tag{33}$$

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Local error estimates for the Moreau's time–stepping

Other cases

- *One impact and smooth Lagrange multiplier* The same result holds ad in first Lemma.
- *losing contact event (take–off) without impact* The order of the time–integration scheme depends on the regularity of the contact forces (at least continuous).
- *Finite accumulation* The order of the time–integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

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Practical error estimates for the Moreau's time–stepping

Order "0" case

Standard error estimates do not apply for Order 0.
We propose to extend it to the order 0 of consistency by assuming that the constant can be evaluated by

$$C = \frac{2(e_1 - e_{1/2})}{h} \tag{34}$$

and the local error estimate by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \tag{35}$$

The adaptive time–step control exposed for smooth ODE is then apply directly.

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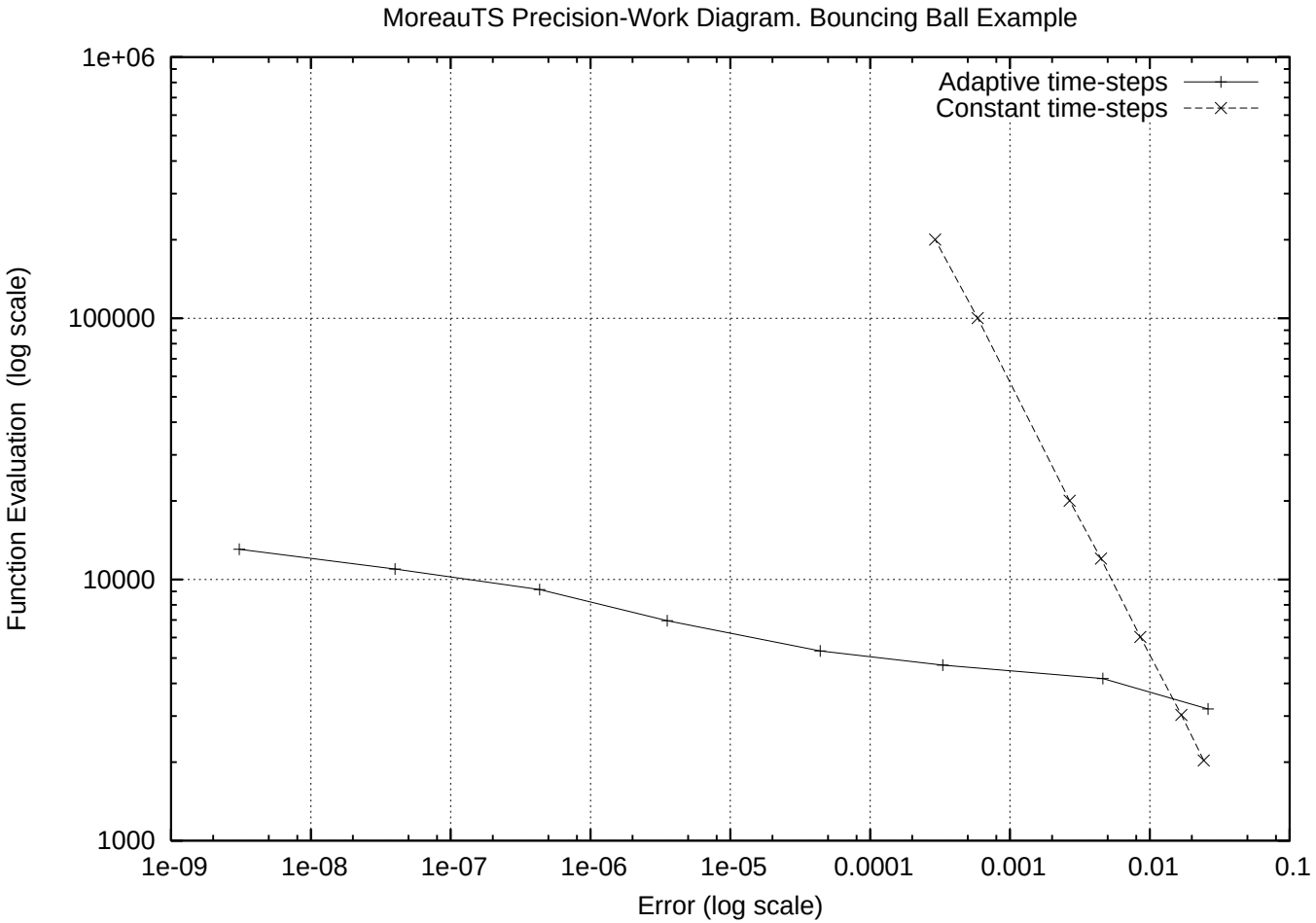
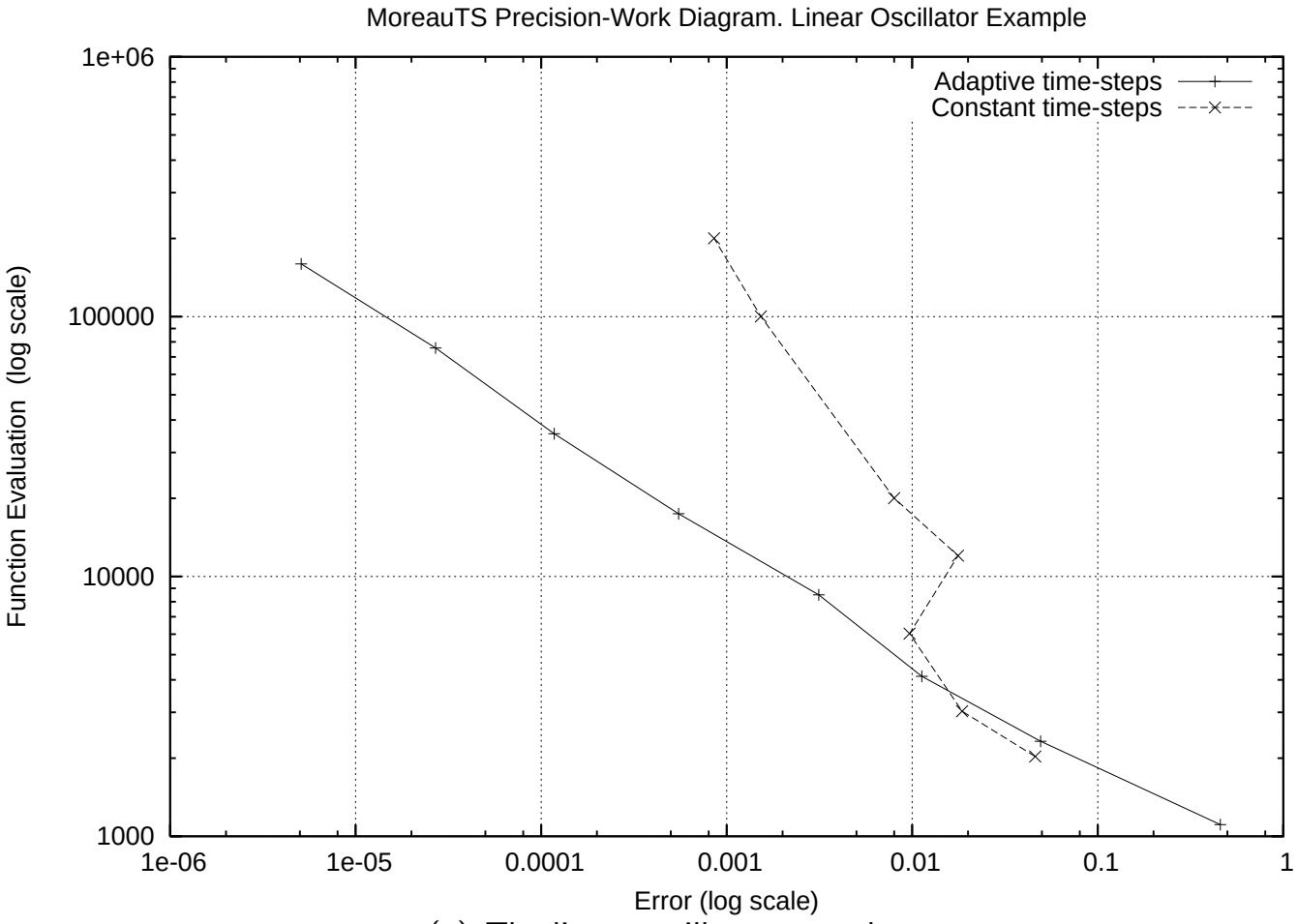
Order "0" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

Order "0" time–step adjustment for the Moreau's time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

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Sizing the error in the violation of constraints

The violation of constraints is sized by the following rule:

$$e_{\text{violation}} = \| \min(0, g(q)) \circ invtol \|_\infty \tag{36}$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by $e_{\text{violation}}$ when an nonsmooth vent occurs, the step size adjustment is implemented by the means of the following error estimation

$$\text{error} = \max(e_{\text{violation}}, \|e_k \circ invtol\|_\infty) \tag{37}$$

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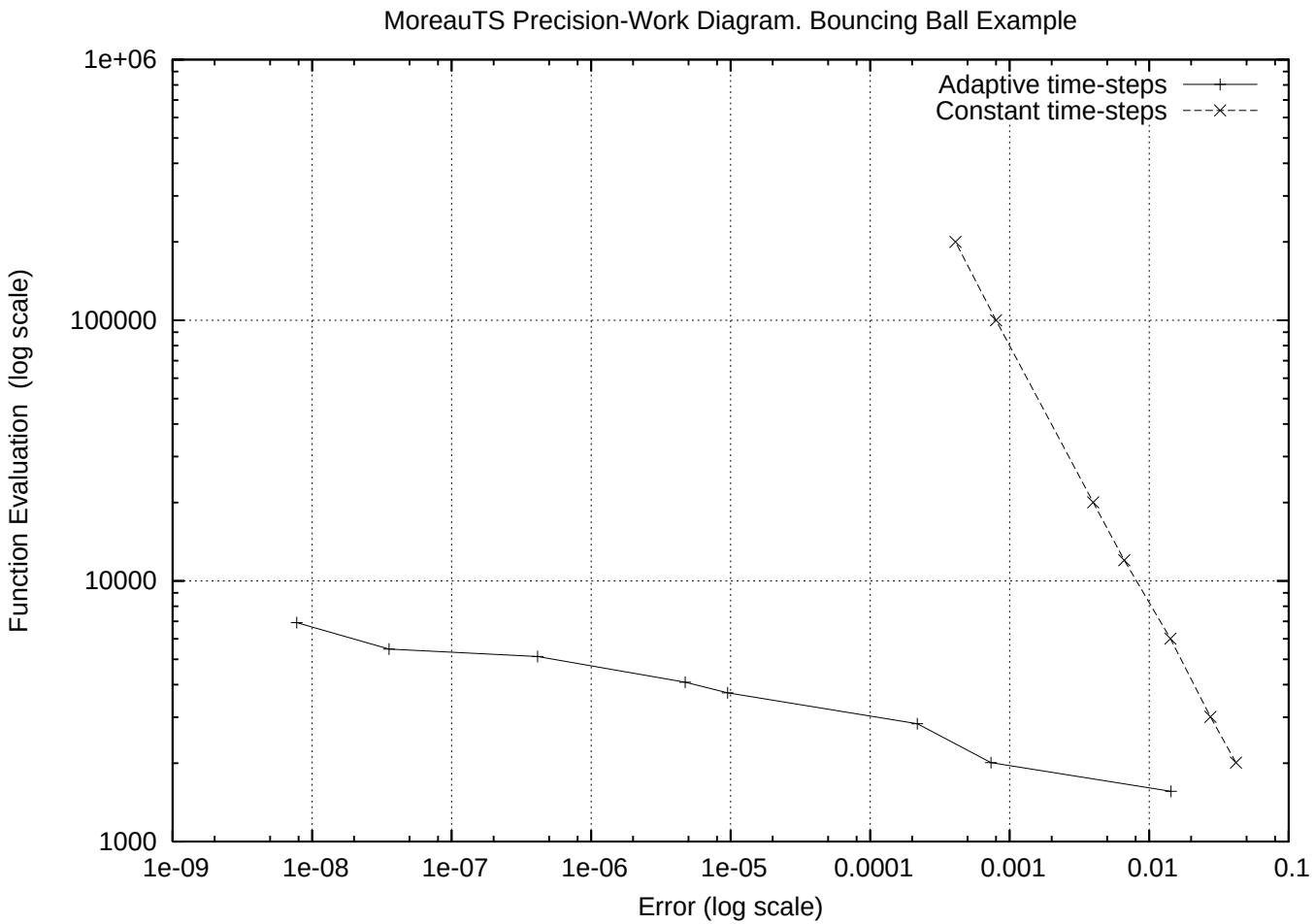

(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

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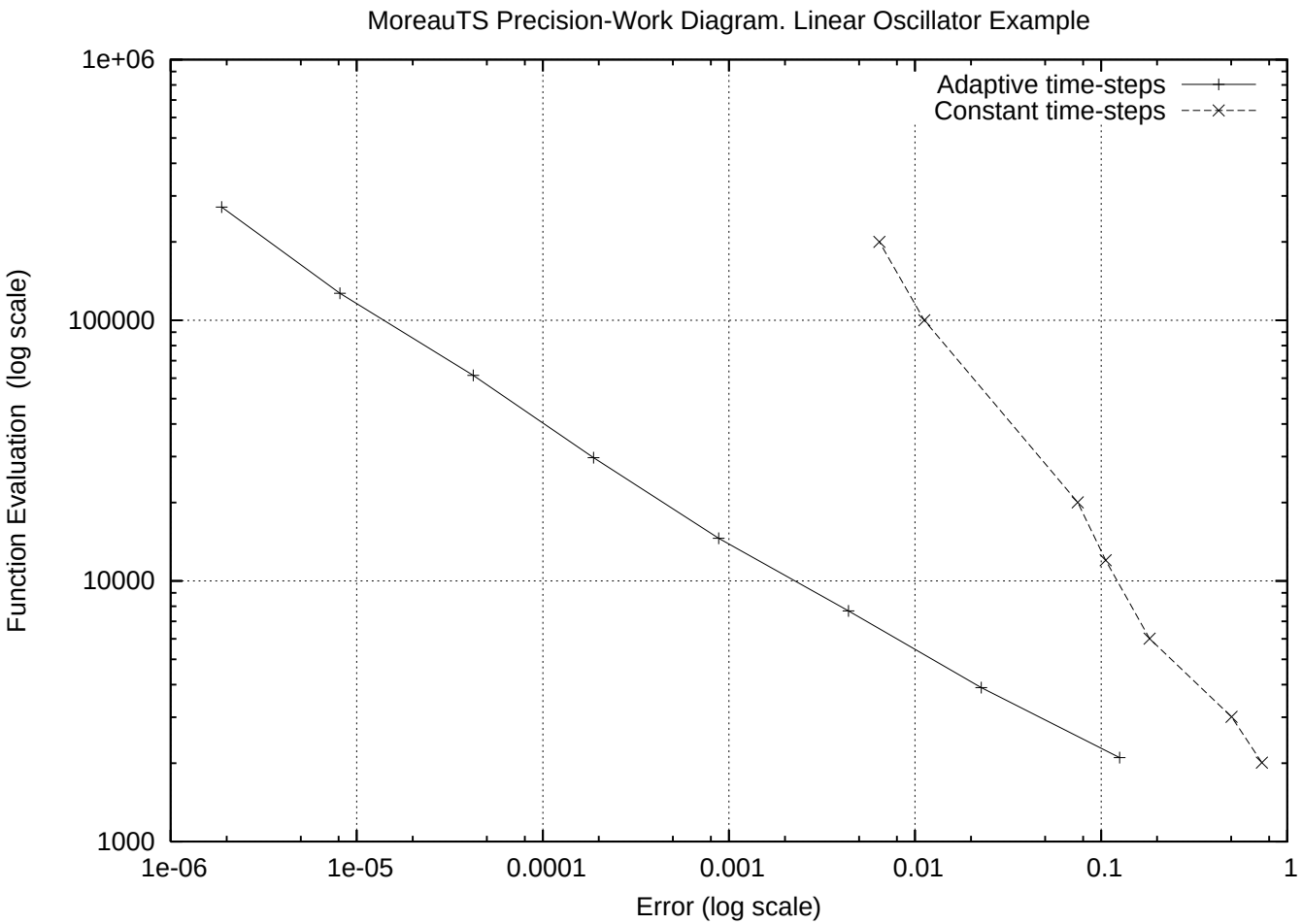

(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

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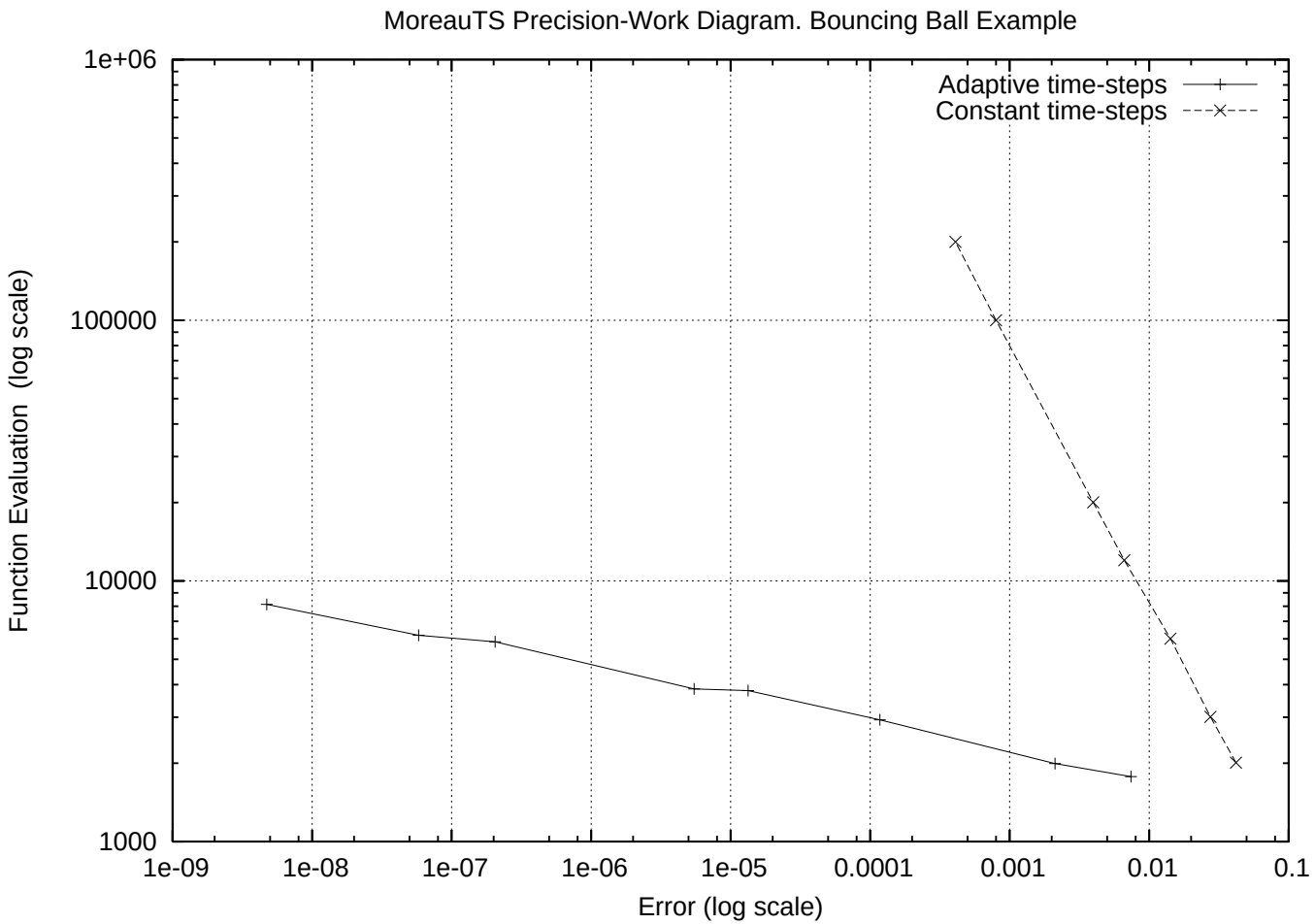
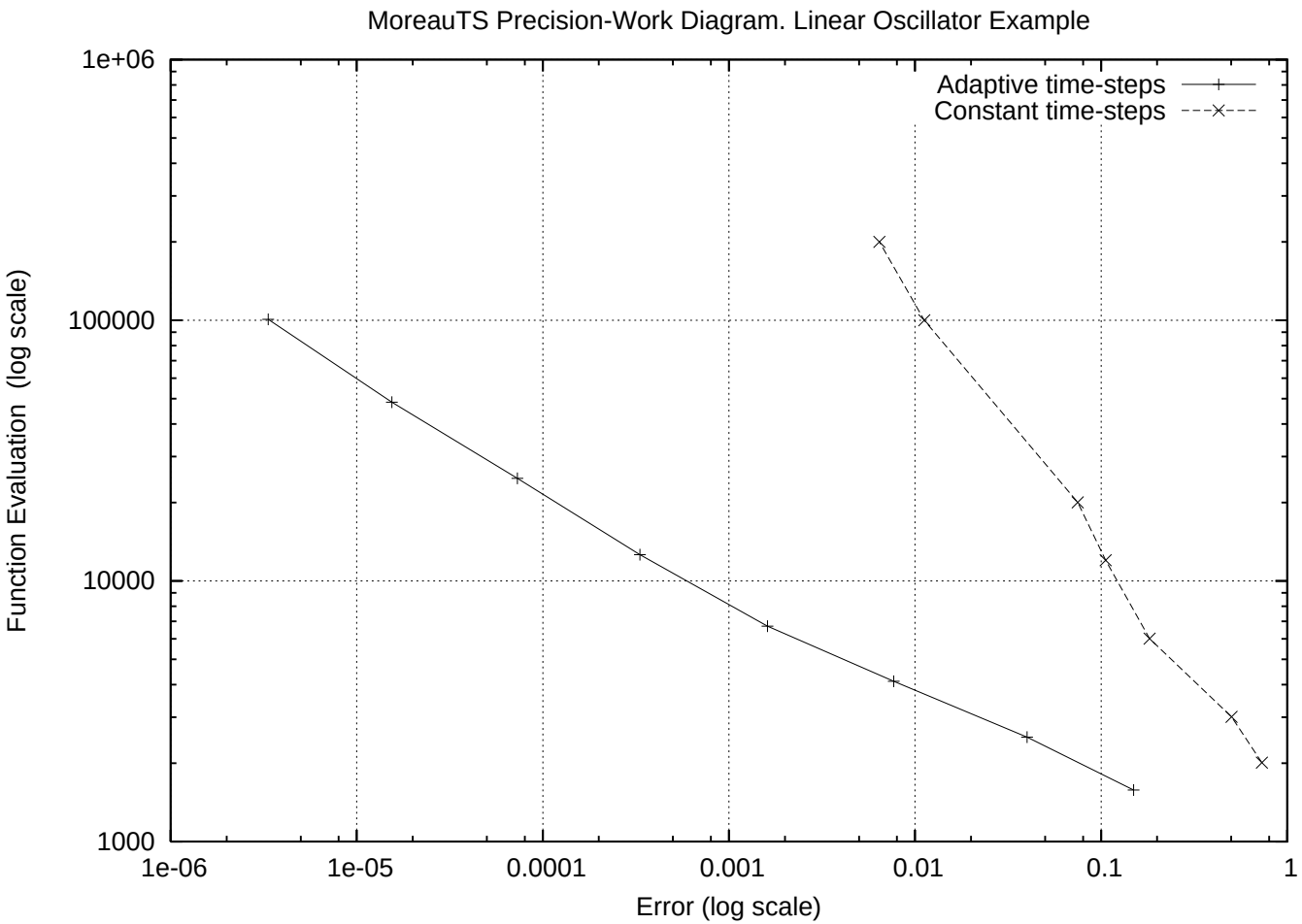


(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Results on two academic test examples



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Variable order approach. Principle

Guess the order of consistency of the integration at each step.
Adapt the practical error estimation

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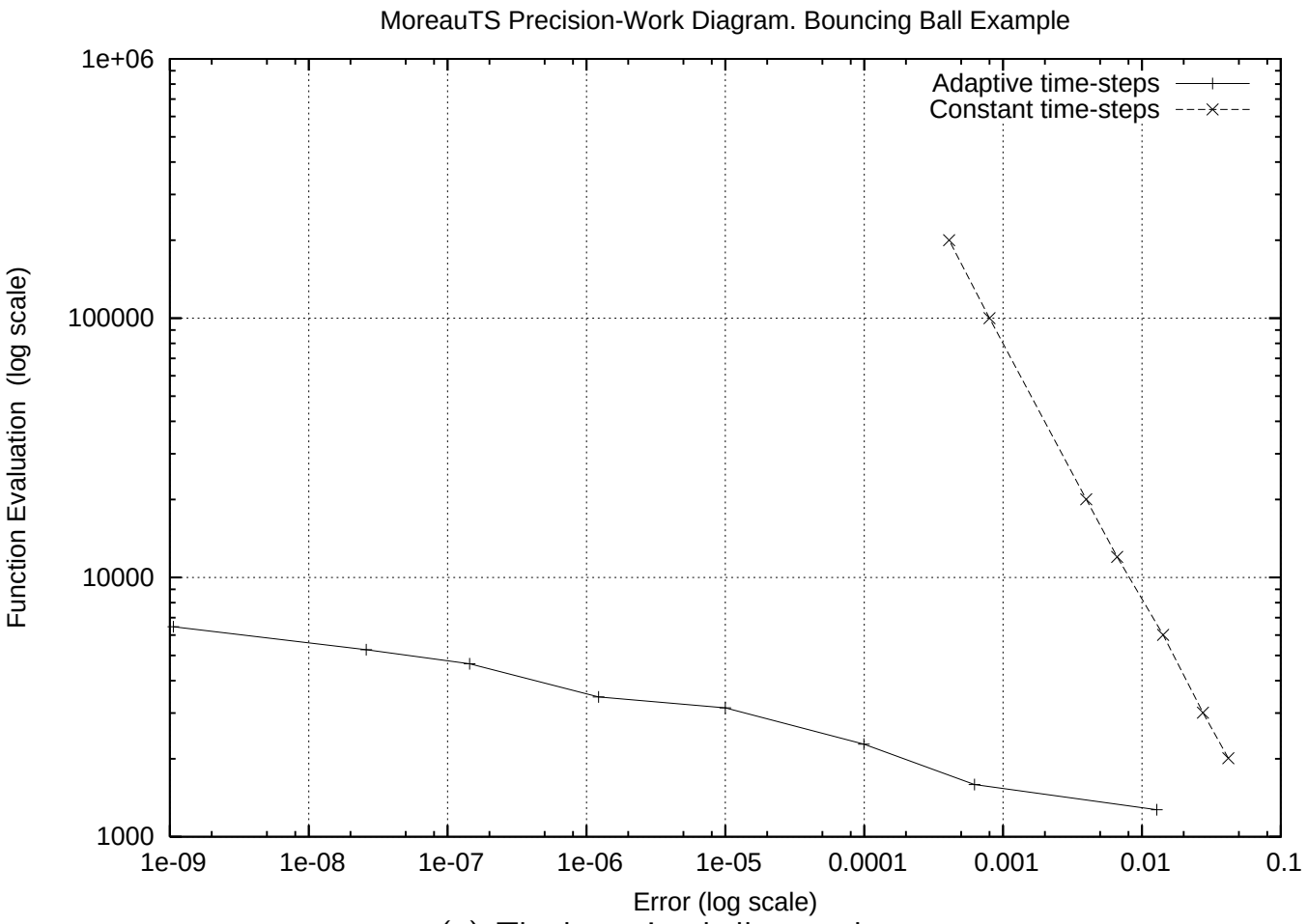

(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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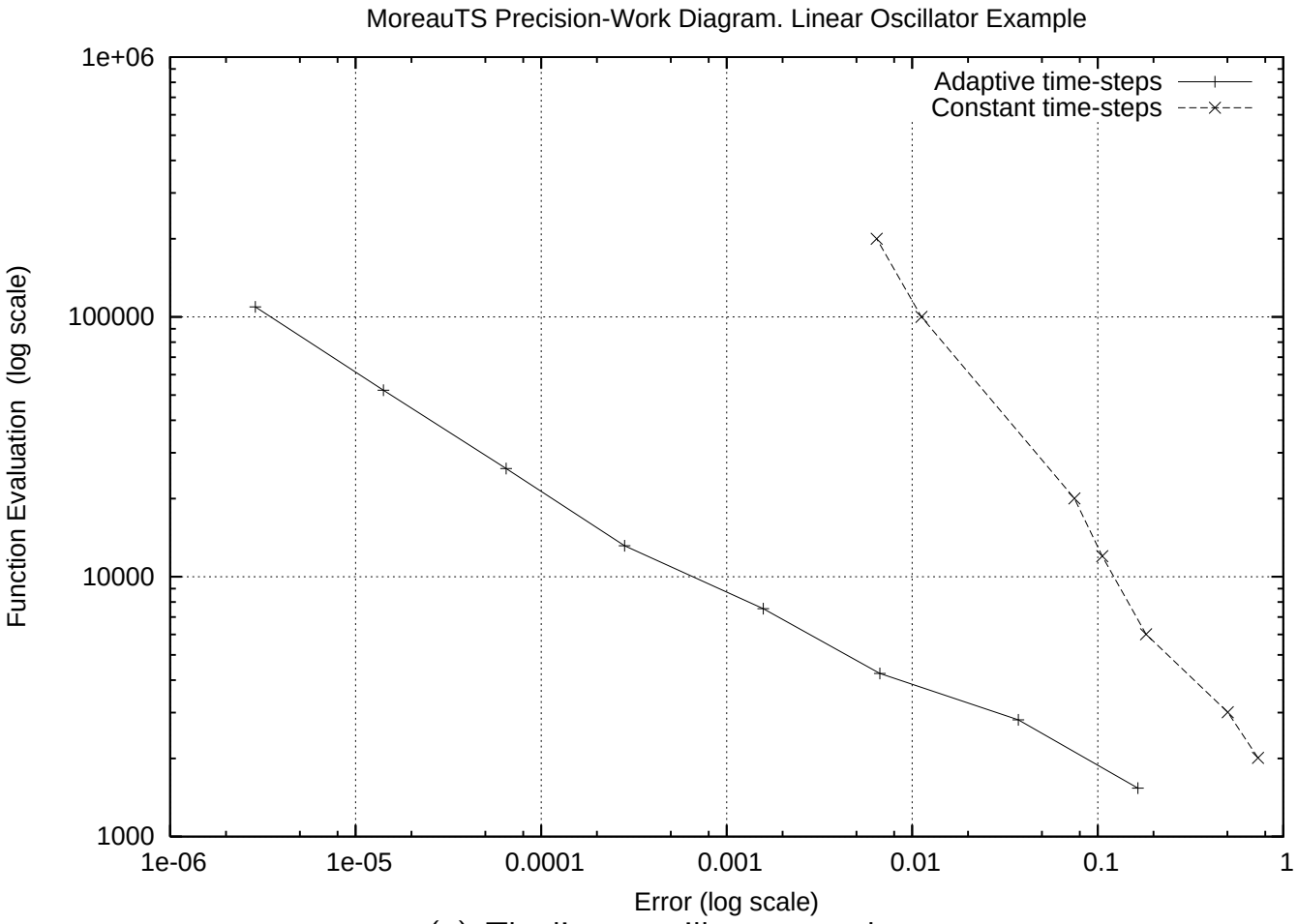

(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Higher Order Time–stepping schemes

Background

Work of Mannshardt (1978) on time–integration schemes of any order for ODEs with discontinuities (with tranversality assumption)

Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE solvers of order p
- Perform a rough location of the event inside the time step of length h Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \tag{38}$$

  Dichotomy, Newton, Local Interpolants, Dense output,. . .
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

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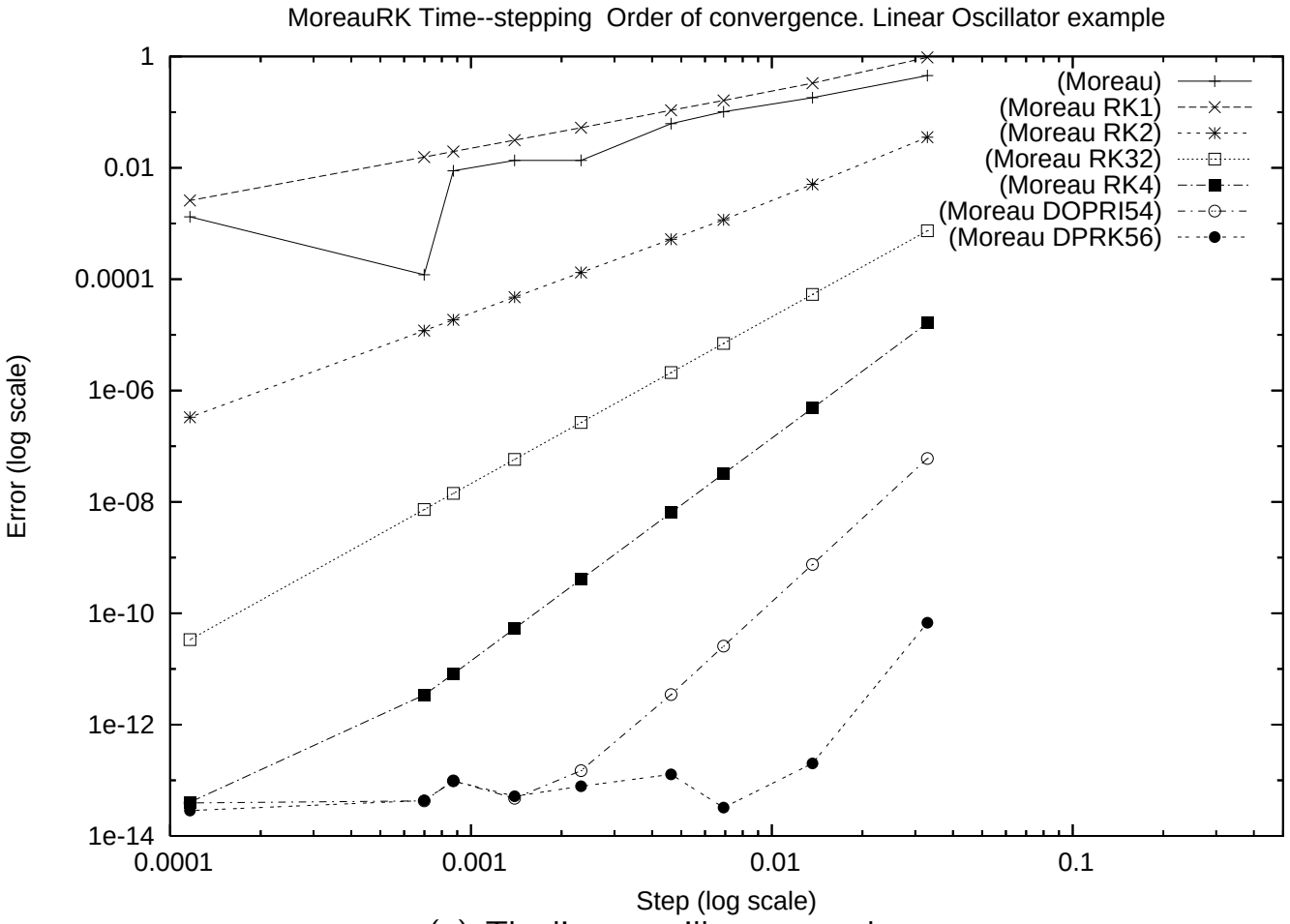

(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Higher Order Time–stepping schemes

Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

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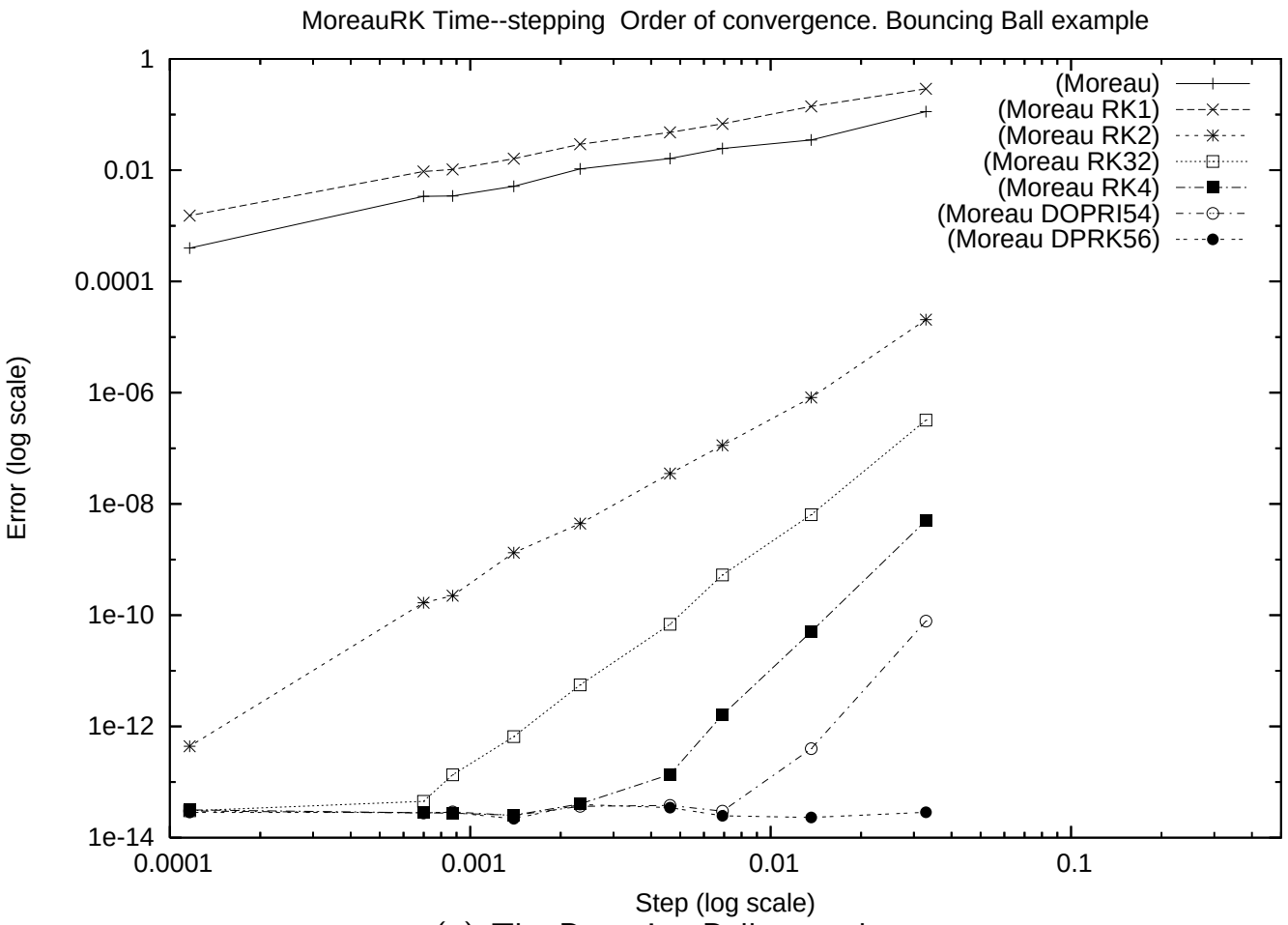

(a) The Bouncing Ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Splitting–based methods.

Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x,t) + g(x,t) \tag{39}$$

A example of splitting–based method is given by the following procedure

1. Perform the integration of f on $[t_k, t_{k+1}]$ to obtain $\tilde{x}(t_{k+1})$ that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x,t)\,dt \tag{40}$$

2. Perform the integration of g on $[t_k, t_{k+1}]$ with initial value $\tilde{x}(t_{k+1})$ to obtain $\hat{x}(t_{k+1})$ that is

$$\hat{x}(t_{k+1}) \quad = \quad \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x,t)\,dt \tag{41}$$

Properties

- $x(t_k+1) \neq \hat{x}(t_{k+1})$ is the general case. (except special linear case, constant dynamics, ...)
- $\hat{x}_(t_{k+1}) \rightarrow x(t_{k+1})$ when $t_{k+1} \rightarrow t_k$

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Splitting–based methods.

Splitting–based for Moreau scheme without continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v), \\ \dot{q} = v, \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{42}$$

yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{43}$$

and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

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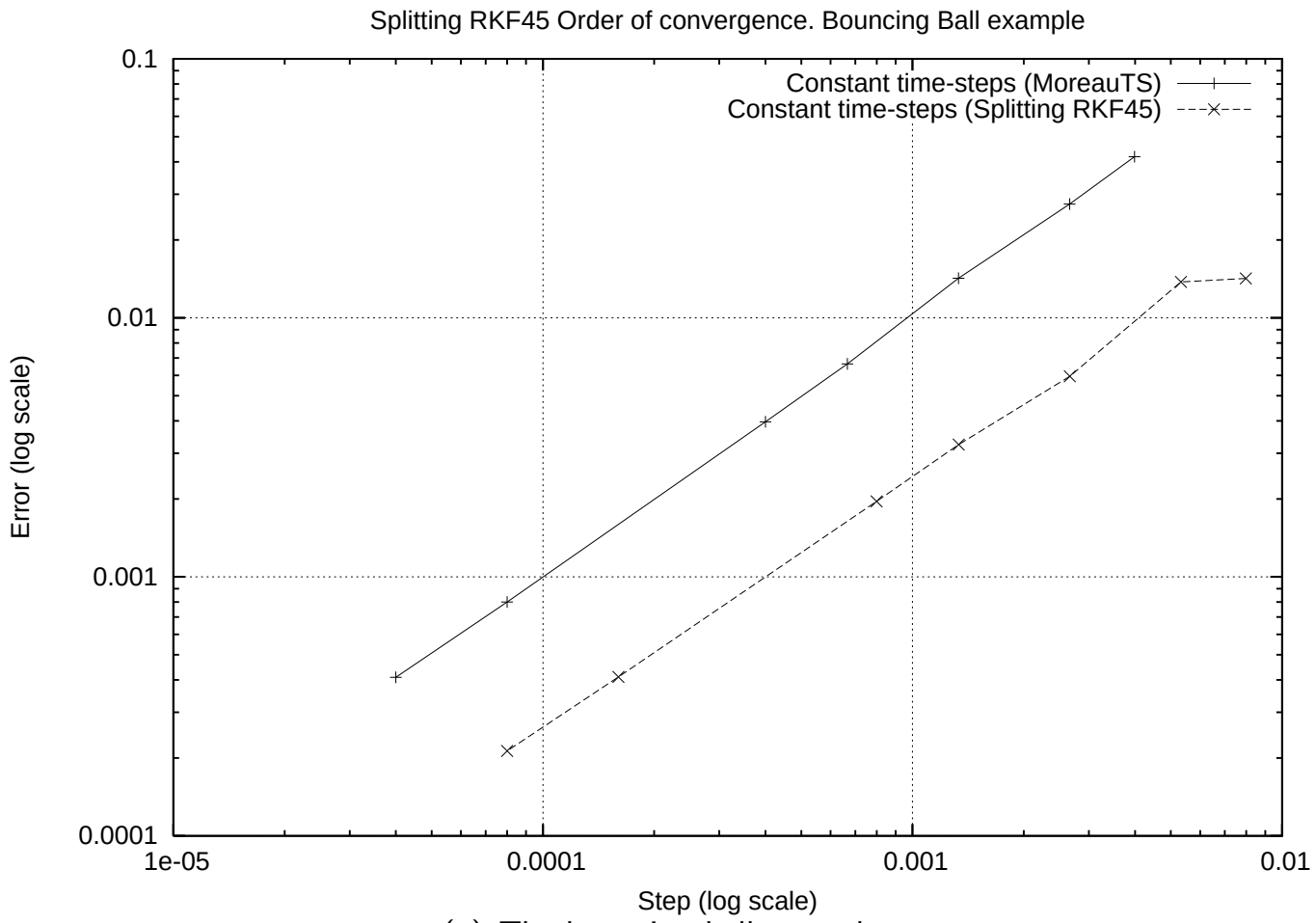
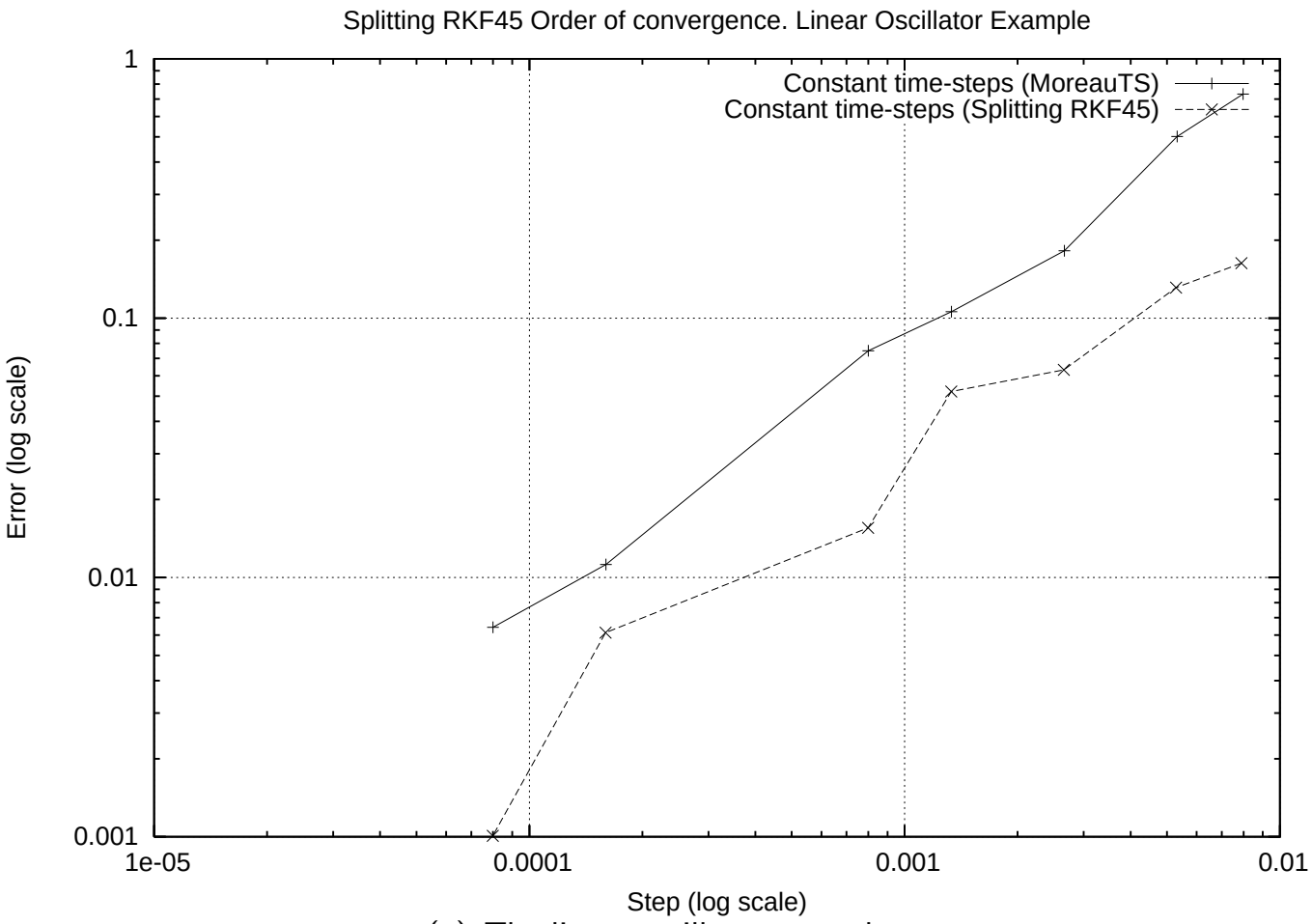
Splitting–based methods with constants time–step.



(a) The bouncing ball example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods with constants time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

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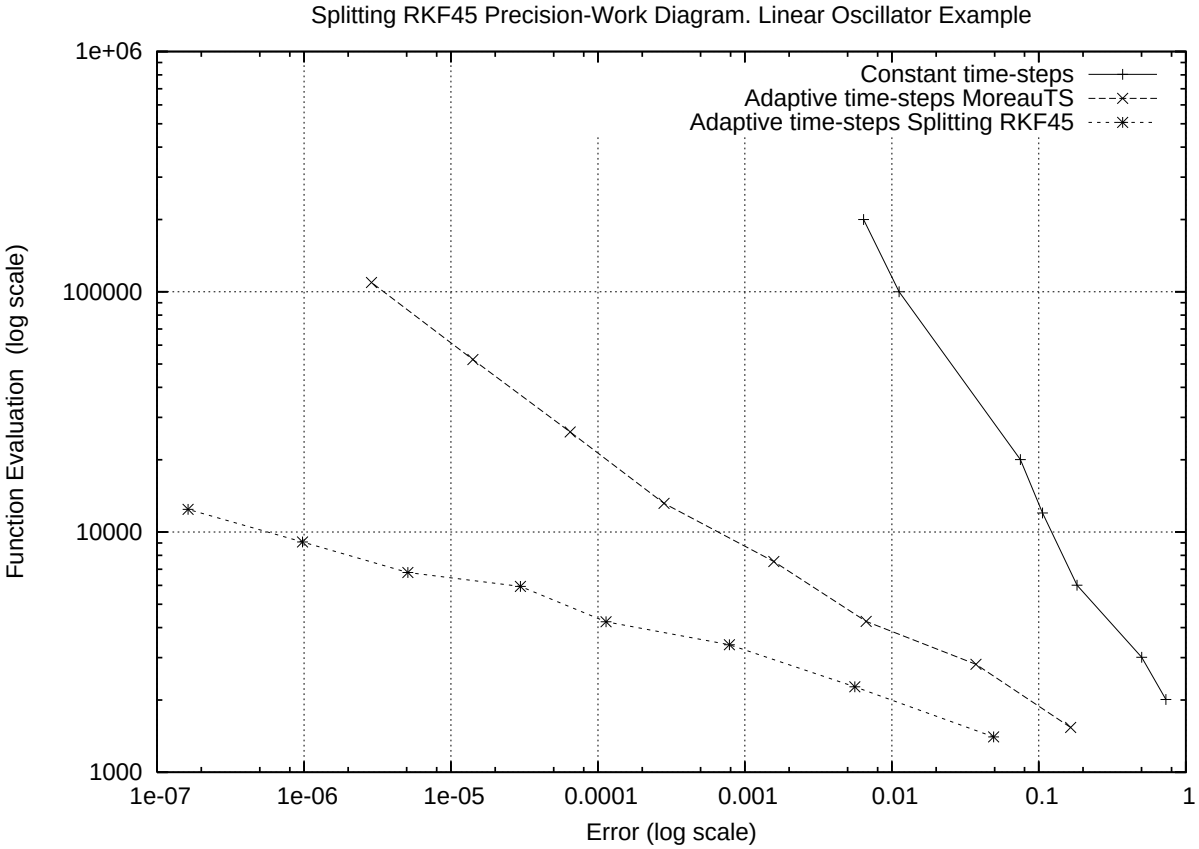
Splitting–based methods with adaptive time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods.

Splitting–based for Moreau scheme with continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t)) \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{44}$$

  yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{45}$$

  and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

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Adaptive time–step strategies

- Higher resolution schemes
- Work with finite accumulation of events

Higher order schemes

- Schemes of any orders
- Work with finite accumulation of events

Splitting based methods

- Higher resolution schemes
- Work with finite accumulation of events

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- Theoretical works on orders and practical error estimations
- Adaptive time–step strategies on the higher order time–stepping schemes.
- Improve the pre–detection process of the event and the order of discontinuity
- Test on nonsmooth and nonlinear mechanical systems.
- Adapt the schemes with a step without external forces when the Moreau's scheme is used
- Other types of time–stepping schemes ...

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Thank you for your attention.

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