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An Algorithm for Constructing the Convex Hull of a Set of Spheres in Dimension d^*

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Abstract

We present an algorithm which computes the convex hull of a set of n spheres in dimension d in time $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. It is worst-case optimal in three dimensions and in even dimensions. The same method can also be used to compute the convex hull of a set of n homothetic convex objects of \mathbb{E}^d . If the complexity of each object is constant, the time needed in the worst case is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

Keywords: Computational Geometry, Convex Hull, Spheres

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1 Introduction

We present an algorithm which computes the convex hull of a set of n spheres in dimension d in time $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. It is worst-case optimal in three dimensions and in even dimensions. It can also be used to compute the convex hull of a set of n homothetic convex objects of \mathbb{E}^d .

Though the complexity and the computation of the convex hull of a set of points in any dimensions is a problem which has been studied extensively, only a few results about the convex hull of a set of spheres are known. The previous results, which are given below, are only for the case $d = 2$ and 3 , and, as far as we know, there were no results about the computation of the convex hull of a set of homothetic objects.

The convex hull of a set of spheres is the smallest convex body that contains the spheres. In two dimensions, the boundary of such a convex hull consists of line segments and arcs of circles. In three dimensions, the convex hull boundary is composed of three different kinds of facets (see Figure 1).

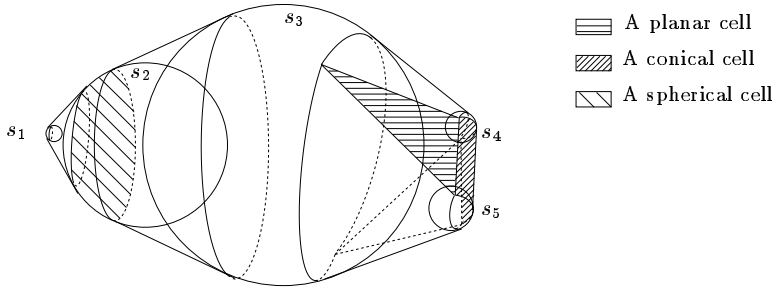


Figure 1: The convex hull of a set of spheres in 3 dimensions

- Planar facets, which are triangles included in planes tangent to three spheres.
- Conical facets, which are parts of cones tangent to two spheres.
- Spherical facets, which are parts of the spheres.

In the plane, the convex hull of a set of disks can be computed in $O(n \log n)$ time (see [6]) which is optimal. In three-dimensional space, the complexity of the convex hull of a set of n spheres is $\Theta(n^2)$ in the worst case, even for collections of pairwise disjoint spheres [7] (see Section 2 below). The convex hull of a set of n spheres in \mathbb{E}^3 can be computed in $O(nh)$ time [1], where h is the size of the output (i.e. of the convex hull). In the case where all spheres have the same radius, the convex hull of a set of spheres in \mathbb{E}^d can be easily deduced from the convex hull of the centers of the spheres (see Section 3.1), which can be computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

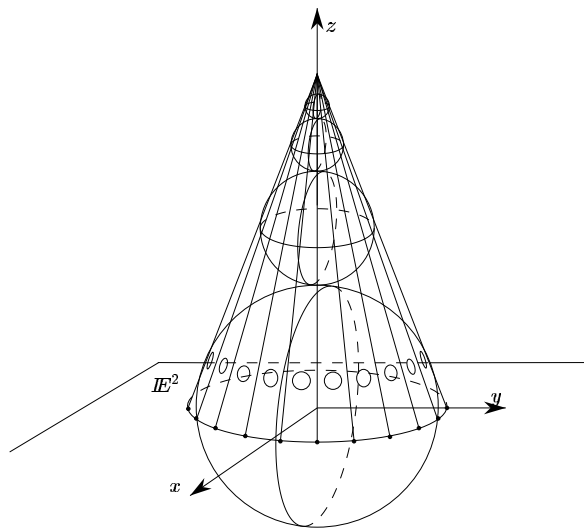
In dimension d , the boundary of the convex hull is composed of d different kinds of facets. Let a *supporting hyperplane of a set* be a hyperplane H which has a non empty

intersection with the set and such that the whole set is included in one of the closed halfspaces limited by H . Let a *supporting halfspace of a set* be a halfspace containing the set and limited by a supporting hyperplane of the set. The convex hull of a set of spheres in \mathbb{E}^d is the intersection of the supporting halfspaces of the set of spheres. A *facet of circularity i* ($0 \leq i \leq d - 1$) is a maximal connected portion of the boundary of the convex hull consisting of points where the supporting hyperplanes are tangent to a given set of $(d - i)$ spheres. For example, in dimension 3, the planar facets have circularity 0, the conical facets have circularity 1, the spherical facets have circularity 2.

The boundary of the convex hull of a set of spheres is the union of the closure of facets of circularity $0, 1, 2, \dots, d - 1$. The boundary of the convex hull is represented by the adjacency graph of these facets.

The paper is organized as follows: In the next Section we give a lower bound on the complexity of the convex hull of a set of n spheres. In Section 3 we present the algorithm to compute this convex hull and we show in Section 4 that it is optimal in three dimensions and in even dimensions. In Section 5 we extend our results to homothetic convex objects.

2 Lower Bounds



• n points on a circle

Figure 2: A set of spheres whose convex hull has size $\Theta(n^2)$

In dimension 3, let us take n points, considered as spheres of radius 0, on a circle in the (x, y) -plane and take a point above this plane, on the z -axis. The convex hull of these $n+1$ points is a pyramid. Now add n spheres having non-zero radii and centered on the z -axis, such that each sphere intersects each facet of this pyramid but none of its edges (see Figure 2). The complexity of the convex hull of this set of $2n+1$ spheres is $\Omega(n^2)$.

By the upper bound theorem [5], the complexity of the convex hull of a set of n points in dimension d is $O(n^{\lfloor \frac{d}{2} \rfloor})$ in the worst case. This bound is tight for cyclic polytopes. A point can be considered as a sphere of radius 0. Therefore, the complexity of the convex hull of a set of n spheres is at least equal to the complexity of the convex hull of a set of points, thus is $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$.

We conjecture that the complexity of the convex hull of a set of n spheres is $\Omega(n^{\lceil \frac{d}{2} \rceil})$.

3 The Algorithm

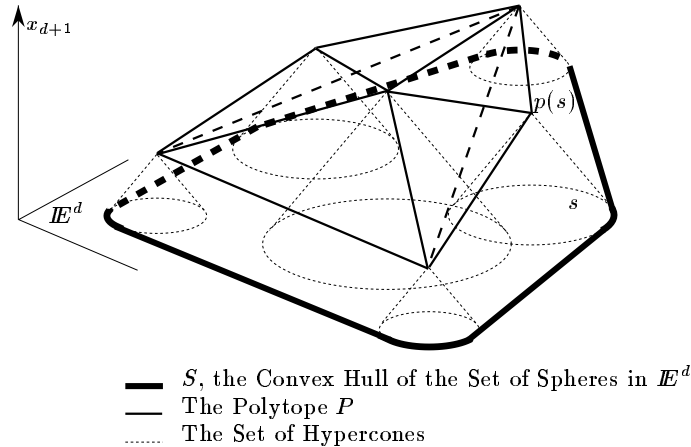


Figure 3: Embedding the spheres in \mathbb{E}^{d+1}

We first introduce some notations, then recall some of the properties of duality, and finally give the algorithm that computes the convex hull of a set of spheres.

3.1 Notations and Preliminaries

Let S be the *convex hull of a set of n spheres* $\{s_1, \dots, s_n\}$ in \mathbb{E}^d . We embed \mathbb{E}^d in \mathbb{E}^{d+1} so that the hyperplane $\{x_{d+1} = 0\}$ of \mathbb{E}^{d+1} contains all the spheres. The $(d+1)$ -th axis will be called the vertical axis, and in the sequel, the expression *above* will refer to the $(d+1)$ -th coordinate. Let s be a sphere in \mathbb{E}^d with center (x_1, \dots, x_d) and radius r . Let p be the mapping that associates to s the *point* $p(s)$ in \mathbb{E}^{d+1} such that:

$$p : s \rightarrow p(s) = (x_1, \dots, x_d, r)$$

Let P be the *convex hull of the set of points* $\{p(s_1), \dots, p(s_n)\}$ of \mathbb{E}^{d+1} . Let λ_0 be a half lower hypercone with arbitrary apex, vertical axis and angle at the apex $\pi/4$.

For a sphere s in \mathbb{E}^d , let $\lambda(s)$ be the translated copy of λ_0 , with apex $p(s)$. Notice that the intersection between the hypercone $\lambda(s)$ and the hyperplane $\{x_{d+1} = 0\}$ is identical to the sphere s . Let Λ be the *convex hull of the set* $\{\lambda(s_1), \dots, \lambda(s_n)\}$ of the n half lower hypercones of \mathbb{E}^{d+1} associated to the n spheres s_1, \dots, s_n (see Figure 3). The intersection between Λ and the hyperplane $\{x_{d+1} = 0\}$ is equal to S .

Let O' be a point inside P .

Theorem 1 *Any hyperplane of \mathbb{E}^d supporting S is the intersection with $\{x_{d+1} = 0\}$ of a unique hyperplane H of \mathbb{E}^{d+1} satisfying the three properties:*

1. H supports P ,
2. H is the translated copy of a hyperplane tangent to λ_0 along one of its generatrices,
3. H is above O' .

Conversely, let H be a hyperplane of \mathbb{E}^{d+1} satisfying the above three properties. Its intersection with the hyperplane $\{x_{d+1} = 0\}$ is a hyperplane of \mathbb{E}^d supporting S .

Proof: Through any point of S of circularity i passes a hyperplane H which supports Λ along a generatrix of at least $d - i$ of the hypercones $\lambda(s_1), \dots, \lambda(s_n)$.

This means that H supports P and is the translated copy of a hyperplane tangent to λ_0 . As H supports Λ , it is above O' .

Conversely, if an hyperplane H supports P and is above O' , it is above P . As H is also the translated copy of a hyperplane tangent to λ_0 , it supports Λ , along a generatrix of at least one of the hypercones $\lambda(s_1), \dots, \lambda(s_n)$. Its intersection with $\{x_{d+1} = 0\}$ is a hyperplane of \mathbb{E}^d supporting S . \square

In the case where all spheres have the same radius, it is easy to see that the convex hull of a set of spheres in \mathbb{E}^d can be obtained by *growing the faces* of the convex hull of the centers of the spheres, i.e. the convex hull of the spheres is exactly the Minkowski Sum of the convex hull of the centers and of a sphere of radius r . Notice that in this case, all the apices of the cones lie on the same horizontal plane $\{x_{d+1} = r\}$, and the growing mechanism can be interpreted as sweeping a plane $\{x_{d+1} = t\}$, t varying from r to 0. Therefore, the complexity drops to $O(n^{\lfloor \frac{d}{2} \rfloor})$ and the running time to $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$.

3.2 Duality

We use *duality* to convert properties 1, 2 and 3 of the above theorem into simpler ones. Let us recall that O' is a point inside P . Let O' be the origin of a new coordinate system, whose axis are parallel to the axis of the previous coordinate system. New coordinates are denoted with a prime: $X' = (x'_1, \dots, x'_{d+1})$. Polarity with respect to O' is a one-to-one transformation which maps points of \mathbb{E}^{d+1} distinct from O' to hyperplanes of

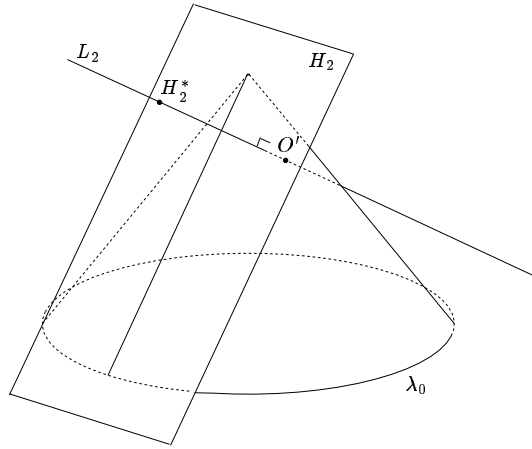


Figure 4: Pole of a hyperplane

\mathbb{E}^{d+1} which do not contain O' . Let M be a point of \mathbb{E}^{d+1} distinct from O' . M^* , the polar hyperplane of M , is defined by the following relation:

$$M^* = \{X' \in \mathbb{E}^{d+1} \mid M.X' = 1\}$$

H^* , the pole of a hyperplane H not containing O' , is defined by

$$H^*.X' = 1, \quad \forall X' \in H$$

We have $(M^*)^* = M$ and $(H^*)^* = H$. Let H^{*-} be the halfspace bounded by H and containing O' .

Let the *polar set* of a set of hyperplanes be the set of poles of these hyperplanes.

Proposition

1. The polytope $P^* = p(s_1)^{* -} \cap \dots \cap p(s_n)^{* -}$ of \mathbb{E}^{d+1} is dual to the polytope P , i.e. there is a bijection between the l -faces of P and the $(d-l)$ -faces of P^* which reverses the relation of inclusion. Each hyperplane supporting P along a l -face F has its polar point on the corresponding $(d-l)$ -face of F^* of P^* .
2. The polar set of the hyperplanes which are translated copies of the hyperplanes tangent to λ_0 is a hypercone K with apex O' , a vertical axis, and an angle at the apex equal to $\pi/4$.
3. The polar set of the hyperplanes above O' is the half space $x'_{d+1} > 0$.

Proof:

The first assertion is well known.

Second assertion: Let H_2 be a hyperplane tangent to λ_0 (see Figure 4). H_2^* , the pole of H_2 , belongs to the line L_2 issued from O' and normal to H_2 . The polar set of the hyperplanes parallel to H_2 is L_2 . The angle of H_2 with the vertical axis is $\pi/4$. Therefore, the angle between L_2 and the vertical axis is $\pi/4$. In order for H_2^* to be well defined, O' may be anywhere inside or outside the hypercone, but not on the hypercone.

As H_2 moves around the hypercone λ_0 , staying tangent to it, L_2 moves on a hypercone K with apex O' , axis $O'x'_{d+1}$, angle at the apex $\pi/4$. K is a hypersurface of \mathbb{E}^{d+1} . The polar set of all the hyperplanes tangent to λ_0 and of the translated copies of these hyperplanes is the hypercone K .

Third assertion: Let H_3 be a hyperplane which lies above O' . Its equation is $\sum_{i=1}^{d+1} h'_i x'_i = 1$. Its intersection with the vertical axis, x'_{d+1} , is such that $h'_{d+1} x'_{d+1} = 1$. As this intersection is above O' , we have $x'_{d+1} > 0$ and thus $h'_{d+1} > 0$. As the coordinates of H_3^* are (h'_1, \dots, h'_{d+1}) , H_3^* is in the halfspace $\{x'_{d+1} > 0\}$. Hence, the polar set of the hyperplanes lying above O' is the half space $\{x'_{d+1} > 0\}$. \square

P^*	S
\mathbb{E}^{d+1}	\mathbb{E}^d
i -face ($1 \leq i \leq d$)	facet of circularity $(d-i)$ ($0 \leq d-i \leq d-1$)
1-face	facet of circularity $d-1$ =part of a sphere
d -face	facet of circularity 0=planar facet

Table 1: correspondence between faces of P and facets of S

The polar set of the hyperplanes supporting the convex hull of the set of points P , tangent to at least one hypercone of Λ along a generatrix and above O' is

$$P^* \cap K \cap \{x'_{d+1} > 0\}$$

Let us consider the intersection of a i -face of P^* with K . The polar hyperplane of each point of this intersection supports P along a $(d - i)$ -face and Λ along $(d - i)$ generatrices of $(d - i)$ hypercones. The polar set of this intersection is a family of hyperplanes whose intersection with $\{x_{d+1} = 0\}$ is a facet of S of circularity $(d - i)$ (see Table 1).

3.3 The Algorithm

1. Compute the convex hull P and choose a point O' inside P
2. Compute the polytope P^* dual to P with respect to O' .
3. Compute the intersection between P^* , the hypercone K and the half space $\{x'_{d+1} > 0\}$.
4. Compute the incidence graph of the facets of S from the incidence graph of the faces of P^* intersecting K and the halfspace $\{x_{d+1} > 0\}$.

4 Complexity Analysis

Chazelle has shown that the convex hull of a set of n points in dimension $d + 1$ can be computed in optimal time $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n)$ (see [3]). Simpler randomized algorithms to compute the convex hull of a set of points can be found in [2, 4, 8]. The complexity of computing the polytope P^* dual to P is a linear function of the complexity of P . The complexity of computing the intersection of P^* with K and with $\{x'_{d+1} > 0\}$ is a linear function of the complexity of P^* since K and $\{x'_{d+1} > 0\}$ have constant complexity. The complexity of computing the incidence graph of the facets of S from the incidence graph of the faces of P^* intersecting K and the half space $\{x_{d+1} > 0\}$ is a linear function of the complexity of P^* . Hence, the total amount of time needed to compute S is a linear function of the amount of time needed to compute P . Therefore, the time needed in the worst case to compute S , the convex hull of the set of spheres, is

$$O(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n) = O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$$

This is optimal in three and in even dimensions.

The given reduction from the problem of computing S , the convex hull of a set of spheres in \mathbb{E}^d , to that of computing P , the contour of the convex hull of a set of points in \mathbb{E}^{d+1} , does not preserve the output size. The complexity of P can exceed that of S . Therefore, using output-sensitive algorithms does not help here.

5 Extension to Homothetic Convex Objects

The algorithm for spheres generalizes to a set of homothetic convex objects having the same orientation. The case of non-convex homothetic objects can be reduced to the case of homothetic convex objects by taking the convex hull of each object. More precisely, let us take a convex object c of \mathbb{E}^d and let c_i ($1 \leq i \leq n$) be a convex object obtained from c by some homothety and some translation. We compute *the convex hull* C of the set of convex objects $\{c_1, \dots, c_n\}$. The main point is that the hypercone K with angle at the apex $\pi/4$ is now replaced by a more *general hypercone* G , which is no longer circular.

Let us associate a half lower hypercone $\lambda(c)$ of \mathbb{E}^{d+1} to c by taking an apex $p(c)$ above the object such that the vertical projection of the apex on \mathbb{E}^d is inside the convex object. $\lambda(c)$ is the half hypercone consisting of the half lines joining $p(c)$ and a point of c . Now we may associate to any object homothetic to c a hypercone $\lambda(c_i)$ which is a translated copy of $\lambda(c)$, such that $\{\lambda(c_i) \cap (x_{d+1} = 0)\} = c_i$,

As before P is the convex hull of $\{p(c_1), \dots, p(c_n)\}$ and Λ the convex hull of $\{\lambda(c_1), \dots, \lambda(c_n)\}$.

Arguments similar to those of Section 3 can be used. If we replace the half lower hypercone λ_0 by $\lambda(c)$, we have the following theorem:

Theorem 2 *Any hyperplane of \mathbb{E}^d supporting C is the intersection with $\{x_{d+1} = 0\}$ of a unique hyperplane H of \mathbb{E}^{d+1} satisfying the three properties:*

1. H supports P
2. H is the translated copy of a hyperplane tangent to $\lambda(c)$ along one of its generatrices.
3. H is above O' .

Conversely, let H be a hyperplane of \mathbb{E}^{d+1} satisfying the above three properties. Its intersection with the hyperplane $\{x_{d+1} = 0\}$ is a hyperplane of \mathbb{E}^d supporting C .

The dual of the set of hyperplanes H satisfying Condition 2 is now a general hypercone G with apex O' , which is no longer circular.

The algorithm of Section 3 can be used if we replace the hypercone K by G .

We assume that the convex objects c have constant complexity. Hence the complexity of the hypercone G is constant. The complexity analysis remains the same as for spheres. Replacing K by G does not change the complexity of the algorithm since G has constant complexity. Therefore, the convex hull of n homothetic convex objects of constant complexity in dimension d can be computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ time.

For example, the time needed to compute the convex hull of n homothetic ellipsoids in dimension d is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

Let us notice that, similarly to the case of spheres having the same radii, the convex hull of a set of translates of a given convex object can be easily computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

6 Conclusion

In this paper, we have reduced by a suitable geometric transform the problem of constructing the convex hull of n spheres in d -space to the problem of computing the intersection of a $(d + 1)$ -polytope with n facets with a hypercone. We have shown that the convex hull of n spheres in dimension d can be computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ time in the worst case, which is optimal in dimension 3 and in even dimensions. We conjecture that the algorithm is optimal in all dimensions.

We have extended these results to homothetic convex objects: If each object has constant complexity, the time needed in the worst case to compute their convex hull is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. Computing the convex hull of general ellipsoids or convex objects in dimension $d \geq 3$ remains an open problem.

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