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An Algorithm for Constructing
the Convex Hull of a Set of Spheres
in Dimension \( d \)*

Jean-Daniel Boissonnat†  André Céręzo†  Olivier Devillers†
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Abstract
We present an algorithm which computes the convex hull of a set of \( n \) spheres in dimension \( d \) in time \( O(n^{\frac{d}{2}} + n \log n) \). It is worst-case optimal in three dimensions and in even dimensions. The same method can also be used to compute the convex hull of a set of \( n \) homothetic convex objects of \( E^d \). If the complexity of each object is constant, the time needed in the worst case is \( O(n^{\frac{d}{2}} + n \log n) \).

Keywords: Computational Geometry, Convex Hull, Spheres

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1 Introduction

We present an algorithm which computes the convex hull of a set of $n$ spheres in dimension $d$ in time $O(n^{\lceil d/2 \rceil} + n \log n)$. It is worst-case optimal in three dimensions and in even dimensions. It can also be used to compute the convex hull of a set of $n$ homothetic convex objects of $\mathbb{E}^d$.

Though the complexity and the computation of the convex hull of a set of points in any dimensions is a problem which has been studied extensively, only a few results about the convex hull of a set of spheres are known. The previous results, which are given below, are only for the case $d = 2$ and 3, and, as far as we know, there were no results about the computation of the convex hull of a set of homothetic objects.

The convex hull of a set of spheres is the smallest convex body that contains the spheres. In two dimensions, the boundary of such a convex hull consists of line segments and arcs of circles. In three dimensions, the convex hull boundary is composed of three different kinds of facets (see Figure 1).

![Convex Hull of Spheres](image)

**Figure 1:** The convex hull of a set of spheres in 3 dimensions

- Planar facets, which are triangles included in planes tangent to three spheres.
- Conical facets, which are parts of cones tangent to two spheres.
- Spherical facets, which are parts of the spheres.

In the plane, the convex hull of a set of disks can be computed in $O(n \log n)$ time (see [6]) which is optimal. In three-dimensional space, the complexity of the convex hull of a set of $n$ spheres is $\Theta(n^2)$ in the worst case, even for collections of pairwise disjoint spheres [7] (see Section 2 below). The convex hull of a set of $n$ spheres in $\mathbb{E}^3$ can be computed in $O(nh)$ time [1], where $h$ is the size of the output (i.e. of the convex hull). In the case where all spheres have the same radius, the convex hull of a set of spheres in $\mathbb{E}^d$ can be easily deduced from the convex hull of the centers of the spheres (see Section 3.1), which can be computed in $O(n^{\lceil d/2 \rceil} + n \log n)$.

In dimension $d$, the boundary of the convex hull is composed of $d$ different kinds of facets. Let a *supporting hyperplane of a set* be a hyperplane $H$ which has a non empty
intersection with the set and such that the whole set is included in one of the closed halfspaces limited by $H$. Let a *supporting halfspace of a set* be a halfspace containing the set and limited by a supporting hyperplane of the set. The convex hull of a set of spheres in $E^d$ is the intersection of the supporting halfspaces of the set of spheres. A *facet of circularity* $i$ ($0 \leq i \leq d - 1$) is a maximal connected portion of the boundary of the convex hull consisting of points where the supporting hyperplanes are tangent to a given set of $(d - i)$ spheres. For example, in dimension 3, the planar facets have circularity 0, the conical facets have circularity 1, the spherical facets have circularity 2.

The boundary of the convex hull of a set of spheres is the union of the closure of facets of circularity $0, 1, 2, ..., d - 1$. The boundary of the convex hull is represented by the adjacency graph of these facets.

The paper is organized as follows: In the next Section we give a lower bound on the complexity of the convex hull of a set of $n$ spheres. In Section 3 we present the algorithm to compute this convex hull and we show in Section 4 that it is optimal in three dimensions and in even dimensions. In Section 5 we extend our results to homothetic convex objects.

2 Lower Bounds

![Diagram of a set of spheres in $E^3$ with an arrow indicating a circle](image)

* $n$ points on a circle

**Figure 2:** A set of spheres whose convex hull has size $\Theta(n^2)$
In dimension 3, let us take \( n \) points, considered as spheres of radius 0, on a circle in the \((x,y)\)-plane and take a point above this plane, on the \(z\)-axis. The convex hull of these \( n + 1 \) points is a pyramid. Now add \( n \) spheres having non-zero radii and centered on the \(z\)-axis, such that each sphere intersects each facet of this pyramid but none of its edges (see Figure 2). The complexity of the convex hull of this set of \( 2n+1 \) spheres is \( \Omega(n^2) \).

By the upper bound theorem [5], the complexity of the convex hull of a set of \( n \) points in dimension \( d \) is \( O(n^{\lfloor d/2 \rfloor}) \) in the worst case. This bound is tight for cyclic polytopes. A point can be considered as a sphere of radius 0. Therefore, the complexity of the convex hull of a set of \( n \) spheres is at least equal to the complexity of the convex hull of a set of points, thus is \( \Omega(n^{\lfloor d/2 \rfloor}) \).

We conjecture that the complexity of the convex hull of a set of \( n \) spheres is \( \Omega(n^{\lfloor d/2 \rfloor}) \).

3 The Algorithm

![Diagram of convex hull](image)

**Figure 3**: Embeding the spheres in \( HE^{d+1} \)

We first introduce some notations, then recall some of the properties of duality, and finally give the algorithm that computes the convex hull of a set of spheres.

3.1 Notations and Preliminaries

Let \( S \) be the convex hull of a set of \( n \) spheres \( \{s_1, \ldots, s_n\} \) in \( HE^d \). We embed \( HE^d \) in \( HE^{d+1} \) so that the hyperplane \( \{x_{d+1} = 0\} \) of \( HE^{d+1} \) contains all the spheres. The \((d+1)\)-th axis will be called the vertical axis, and in the sequel, the expression above will refer to the \((d+1)\)-th coordinate. Let \( s \) be a sphere in \( HE^d \) with center \((x_1, \ldots, x_d)\) and radius \( r \). Let \( p \) be the mapping that associates to \( s \) the point \( p(s) \) in \( HE^{d+1} \) such that:

\[
p : s \rightarrow p(s) = (x_1, \ldots, x_d, r)
\]
Let $P$ be the convex hull of the set of points $\{p(s_1), ..., p(s_n)\}$ of $E^{d+1}$. Let $\lambda_0$ be a half lower hypercone with arbitrary apex, vertical axis and angle at the apex $\pi/4$.

For a sphere $s$ in $E^d$, let $\lambda(s)$ be the translated copy of $\lambda_0$, with apex $p(s)$. Notice that the intersection between the hypercone $\lambda(s)$ and the hyperplane $\{x_{d+1} = 0\}$ is identical to the sphere $s$. Let $\Lambda$ be the convex hull of the set $\{\lambda(s_1), ..., \lambda(s_n)\}$ of the $n$ half lower hypercones of $E^{d+1}$ associated to the $n$ spheres $s_1, ..., s_n$ (see Figure 3). The intersection between $\Lambda$ and the hyperplane $\{x_{d+1} = 0\}$ is equal to $S$.

Let $O'$ be a point inside $P$.

**Theorem 1** Any hyperplane of $E^d$ supporting $S$ is the intersection with $\{x_{d+1} = 0\}$ of a unique hyperplane $H$ of $E^{d+1}$ satisfying the three properties:

1. $H$ supports $P$,
2. $H$ is the translated copy of a hyperplane tangent to $\lambda_0$ along one of its generatrices,
3. $H$ is above $O'$.

Conversely, let $H$ be a hyperplane of $E^{d+1}$ satisfying the above three properties. Its intersection with the hyperplane $\{x_{d+1} = 0\}$ is a hyperplane of $E^d$ supporting $S$.

**Proof:** Through any point of $S$ of circularity $i$ passes a hyperplane $H$ which supports $\Lambda$ along a generatrix of at least $d - i$ of the hypercones $\lambda(s_1), ..., \lambda(s_n)$. This means that $H$ supports $P$ and is the translated copy of a hyperplane tangent to $\lambda_0$. As $H$ supports $\Lambda$, it is above $O'$.

Conversely, if an hyperplane $H$ supports $P$ and is above $O'$, it is above $P$. As $H$ is also the translated copy of a hyperplane tangent to $\lambda_0$, it supports $\Lambda$, along a generatrix of at least one of the hypercones $\lambda(s_1), ..., \lambda(s_n)$. Its intersection with $\{x_{d+1} = 0\}$ is a hyperplane of $E^d$ supporting $S$. □

In the case where all spheres have the same radius, it is easy to see that the convex hull of a set of spheres in $E^d$ can be obtained by *growing the faces* of the convex hull of the centers of the spheres, i.e. the convex hull of the spheres is exactly the Minkowski Sum of the convex hull of the centers and of a sphere of radius $r$. Notice that in this case, all the apices of the cones lie on the same horizontal plane $\{x_{d+1} = r\}$, and the growing mechanism can be interpreted as sweeping a plane $\{x_{d+1} = t\}$, $t$ varying from $r$ to 0. Therefore, the complexity drops to $O(n|\mathcal{F}|)$ and the running time to $O(n|\mathcal{F}| + n\log n)$.

### 3.2 Duality

We use duality to convert properties 1, 2 and 3 of the above theorem into simpler ones. Let us recall that $O'$ is a point inside $P$. Let $O'$ be the origin of a new coordinate system, whose axis are parallel to the axis of the previous coordinate system. New coordinates are denoted with a prime: $X' = (x'_1, ..., x'_{d+1})$. Polarity with respect to $O'$ is a one-to-one transformation which maps points of $E^{d+1}$ distinct from $O'$ to hyperplanes of
$\mathbb{E}^{d+1}$ which do not contain $O'$. Let $M$ be a point of $\mathbb{E}^{d+1}$ distinct from $O'$. $M^*$, the polar hyperplane of $M$, is defined by the following relation:

$$M^* = \{ X' \in \mathbb{E}^{d+1} \mid M.X' = 1 \}$$

$H^*$, the pole of a hyperplane $H$ not containing $O'$, is defined by

$$H^*.X' = 1, \ \forall X' \in H$$

We have $(M^*)^* = M$ and $(H^*)^* = H$. Let $H^-$ be the halfspace bounded by $H$ and containing $O'$.

Let the polar set of a set of hyperplanes be the set of poles of these hyperplanes.
Proposition

1. The polytope \( P^* = p(s_1)^* \cap \ldots \cap p(s_n)^* \) of \( \mathbb{E}^{d+1} \) is dual to the polytope \( P \), i.e. there is a bijection between the \( l \)-faces of \( P \) and the \((d-l)\)-faces of \( P^* \) which reverses the relation of inclusion. Each hyperplane supporting \( P \) along a \( l \)-face \( F \) has its polar point on the corresponding \((d-l)\)-face of \( P^* \).

2. The polar set of the hyperplanes which are translated copies of the hyperplanes tangent to \( \lambda_0 \) is a hypercone \( K \) with apex \( O' \), a vertical axis, and an angle at the apex equal to \( \pi/4 \).

3. The polar set of the hyperplanes above \( O' \) is the half space \( x'_{d+1} > 0 \).

Proof:

The first assertion is well known.

Second assertion: Let \( H_2 \) be a hyperplane tangent to \( \lambda_0 \) (see Figure 4). \( H_2^* \), the pole of \( H_2 \), belongs to the line \( L_2 \) issued from \( O' \) and normal to \( H_2 \). The polar set of the hyperplanes parallel to \( H_2 \) is \( L_2 \). The angle of \( H_2 \) with the vertical axis is \( \pi/4 \). Therefore, the angle between \( L_2 \) and the vertical axis is \( \pi/4 \). In order for \( H_2^* \) to be well defined, \( O' \) may be anywhere inside or outside the hypercone, but not on the hypercone.

As \( H_2 \) moves around the hypercone \( \lambda_0 \), staying tangent to it, \( L_2 \) moves on a hypercone \( K \) with apex \( O' \), axis \( O'x'_{d+1} \), angle at the apex \( \pi/4 \). \( K \) is a hypersurface of \( \mathbb{E}^{d+1} \). The polar set of all the hyperplanes tangent to \( \lambda_0 \) and of the translated copies of these hyperplanes is the hypercone \( K \).

Third assertion: Let \( H_3 \) be a hyperplane which lies above \( O' \). Its equation is \( \sum_{i=1}^{d+1} h'_i x'_i = 1 \). Its intersection with the vertical axis, \( x'_{d+1} \), is such that \( h'_{d+1} x'_{d+1} = 1 \). As this intersection is above \( O' \), we have \( x'_{d+1} > 0 \) and thus \( h'_{d+1} > 0 \). As the coordinates of \( H_3^* \) are \( (h'_1, \ldots, h'_{d+1}) \), \( H_3^* \) is in the halfspace \( \{ x'_{d+1} > 0 \} \). Hence, the polar set of the hyperplanes lying above \( O' \) is the half space \( \{ x'_{d+1} > 0 \} \). \( \square \)

<table>
<thead>
<tr>
<th>( P^* )</th>
<th>( \mathbb{E}^{d+1} )</th>
<th>( S )</th>
<th>( \mathbb{E}^d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l )-face ((1 \leq i \leq d))</td>
<td>facet of circularity ((d-i))</td>
<td>((0 \leq d - i \leq d - 1))</td>
<td></td>
</tr>
<tr>
<td>( 1 )-face</td>
<td>facet of circularity ( d - 1 )=part of a sphere</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d )-face</td>
<td>facet of circularity ( 0 )=planar facet</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: correspondence between faces of \( P \) and facets of \( S \)

The polar set of the hyperplanes supporting the convex hull of the set of points \( P \), tangent to at least one hypercone of \( \Lambda \) along a generatrix and above \( O' \) is

\[
P^* \cap K \cap \{ x'_{d+1} > 0 \}
\]
Let us consider the intersection of a $i$-face of $P^*$ with $K$. The polar hyperplane of each point of this intersection supports $P$ along a $(d - i)$-face and $\Lambda$ along $(d - i)$ generatrices of $(d - i)$ hypercones. The polar set of this intersection is a family of hyperplanes whose intersection with $\{x_{d+1} = 0\}$ is a facet of $S$ of circularity $(d - i)$ (see Table 1).

3.3 The Algorithm

1. Compute the convex hull $P$ and choose a point $O'$ inside $P$.

2. Compute the polytope $P^*$ dual to $P$ with respect to $O'$.

3. Compute the intersection between $P^*$, the hypercone $K$ and the half space $\{x'_{d+1} > 0\}$.

4. Compute the incidence graph of the facets of $S$ from the incidence graph of the faces of $P^*$ intersecting $K$ and the halfspace $\{x_{d+1} > 0\}$.

4 Complexity Analysis

Chazelle has shown that the convex hull of a set of $n$ points in dimension $d + 1$ can be computed in optimal time $\Theta(n^{\frac{d+1}{2}} + n \log n)$ (see [3]). Simpler randomized algorithms to compute the convex hull of a set of points can be found in [2, 4, 8]. The complexity of computing the polytope $P^*$ dual to $P$ is a linear function of the complexity of $P$. The complexity of computing the intersection of $P^*$ with $K$ and with $\{x'_{d+1} > 0\}$ is a linear function of the complexity of $P^*$ since $K$ and $\{x'_{d+1} > 0\}$ have constant complexity. The complexity of computing the incidence graph of the facets of $S$ from the incidence graph of the faces of $P^*$ intersecting $K$ and the half space $\{x_{d+1} > 0\}$ is a linear function of the complexity of $P^*$. Hence, the total amount of time needed to compute $S$ is a linear function of the amount of time needed to compute $P$. Therefore, the time needed in the worst case to compute $S$, the convex hull of the set of spheres, is

$$O(n^{\frac{d+1}{2}} + n \log n) = O(n^{\frac{d}{2}} + n \log n)$$

This is optimal in three and in even dimensions.

The given reduction from the problem of computing $S$, the convex hull of a set of spheres in $\mathbb{E}^d$, to that of computing $P$, the contour of the convex hull of a set of points in $\mathbb{E}^{d+1}$, does not preserve the output size. The complexity of $P$ can exceed that of $S$. Therefore, using output-sensitive algorithms does not help here.
5 Extension to Homothetic Convex Objects

The algorithm for spheres generalizes to a set of homothetic convex objects having the same orientation. The case of non-convex homothetic objects can be reduced to the case of homothetic convex objects by taking the convex hull of each object. More precisely, let us take a convex object $c$ of $\mathbb{E}^d$ and let $c_i (1 \leq i \leq n)$ be a convex object obtained from $c$ by some homothety and some translation. We compute the convex hull $C$ of the set of convex objects $\{c_1, ..., c_n\}$. The main point is that the hypercone $K$ with angle at the apex $\pi/4$ is now replaced by a more general hypercone $G$, which is no longer circular.

Let us associate a half lower hypercone $\lambda(c)$ of $\mathbb{E}^{d+1}$ to $c$ by taking an apex $p(c)$ above the object such that the vertical projection of the apex on $\mathbb{E}^d$ is inside the convex object. $\lambda(c)$ is the half hypercone consisting of the half lines joining $p(c)$ and a point of $c$. Now we may associate to any object homothetic to $c$ a hypercone $\lambda(c_i)$ which is a translated copy of $\lambda(c)$, such that $\{\lambda(c_i) \cap (x_{d+1} = 0)\} = c_i$.

As before $P$ is the convex hull of $\{p(c_1), ..., p(c_n)\}$ and $A$ the convex hull of $\{\lambda(c_1), ..., \lambda(c_n)\}$.

Arguments similar to those of Section 3 can be used. If we replace the half lower hypercone $\lambda_0$ by $\lambda(c)$, we have the following theorem:

**Theorem 2** Any hyperplane of $\mathbb{E}^d$ supporting $C$ is the intersection with $\{x_{d+1} = 0\}$ of a unique hyperplane $H$ of $\mathbb{E}^{d+1}$ satisfying the three properties:

1. $H$ supports $P$
2. $H$ is the translated copy of a hyperplane tangent to $\lambda(c)$ along one of its generatrices.
3. $H$ is above $O'$.

Conversely, let $H$ be a hyperplane of $\mathbb{E}^{d+1}$ satisfying the above three properties. Its intersection with the hyperplane $\{x_{d+1} = 0\}$ is a hyperplane of $\mathbb{E}^d$ supporting $C$.

The dual of the set of hyperplanes $H$ satisfying Condition 2 is now a general hypercone $G$ with apex $O'$, which is no longer circular.

The algorithm of Section 3 can be used if we replace the hypercone $K$ by $G$.

We assume that the convex objects $c$ have constant complexity. Hence the complexity of the hypercone $G$ is constant. The complexity analysis remains the same as for spheres. Replacing $K$ by $G$ does not change the complexity of the algorithm since $G$ has constant complexity. Therefore, the convex hull of $n$ homothetic convex objects of constant complexity in dimension $d$ can be computed in $O(n^{\frac{d}{d+1}} + n \log n)$ time.

For example, the time needed to compute the convex hull of $n$ homothetic ellipsoids in dimension $d$ is $O(n^{\frac{1}{d+1}} + n \log n)$.

Let us notice that, similarly to the case of spheres having the same radii, the convex hull of a set of translates of a given convex object can be easily computed in $O(n^{\frac{1}{d+1}} + n \log n)$. 

6 Conclusion

In this paper, we have reduced by a suitable geometric transform the problem of constructing the convex hull of \( n \) spheres in \( d \)-space to the problem of computing the intersection of a \((d + 1)\)-polytope with \( n \) facets with a hypercone. We have shown that the convex hull of \( n \) spheres in dimension \( d \) can be computed in \( O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n) \) time in the worst case, which is optimal in dimension 3 and in even dimensions. We conjecture that the algorithm is optimal in all dimensions.

We have extended these results to homothetic convex objects: If each object has constant complexity, the time needed in the worst case to compute their convex hull is \( O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n) \). Computing the convex hull of general ellipsoids or convex objects in dimension \( d \geq 3 \) remains an open problem.

References


