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Abstract: We consider broadcasting in multi-hop wireless networks, in which one source transmits information to all the nodes in the networks. We focus on energy efficiency, or minimizing the total number of transmissions.

Our main result is the proof that, from the energy-efficiency perspective, network coding may essentially operate in an optimal way in the core of the network for uniform wireless networks in Euclidean spaces with idealized communication.

In such networks, one corollary is that network coding is expected to outperform routing. We prove that the asymptotic network coding gain is comprised between 1.642 and 1.684 for networks of the plane, and comprised between 1.432 and 2.035 for networks in 3-dimensional space.

Key-words: wireless networks, network coding, broadcasting, multi-hop, min-cut, hypergraph, connected dominating set

Centre de recherche INRIA Paris – Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Diffusion sans-fil par codage réseau: Économie d'énergie, optimalité et gain de codage dans les réseaux sans fil sans perte

Résumé : Nous nous intéressons à la diffusion dans les réseaux ad-hoc multi-sauts, où une source transmet de l'information à tous les nœuds du réseau. Nous nous concentrons sur l'économie d'énergie, c'est-à-dire la minimisation du nombre de transmissions.

Notre résultat principal est la preuve que, du point de vue de l'économie dénergie, le codage réseau peut opérer essentiellement de manière optimale dans le cœur du réseau, pour des réseaux sans fil uniformes dans les espaces euclidiens avec des transmissions idéalisées (sans perte).

Dans de tels réseaux, le corollaire est que le codage réseau peut avoir de meilleures performances que le routage. Nous prouvons que le gain asymptotique du codage réseau est compris entre 1.642 et 1.684 pour des réseaux du plan, et compris entre 1.432 et 2.035 pour des réseaux dans un espace de dimension 3.

Mots-clés : réseaux sans fil, codage de réseau, diffusion, multi-sauts, coupe minimale, hypergraphe, ensemble dominant connecté

1 Introduction

Seminal work from Ahlswede, Cai, Li and Yeung [24] has introduced the idea of *network coding*, whereby intermediate nodes are mixing information from different flows (different bits or different packets), and has illustrated some capacity gains.

In multi-hop wireless networks, one of natural applications of network coding is to reduce the number of transmissions required to transmit some amount of information to the same destinations. This achieves energy efficiency for networks where the cost of wireless communication is a critical design factor. We focus on one specific form of communication: broadcasting information from one source to all the nodes in a wireless multi-hop network. Such communication is commonly used in wireless networks, for instance, for management, information dissemination, multimedia content distribution, or as a simplified form of multicast. Energy efficient broadcast may be formulated as:

- With one broadcast source, minimize the total number of (re)transmissions used to allow all nodes in the network to get the information.

The issue of efficient multicast or broadcast has been studied with network coding; in fact, for single-source multicast or broadcast, and with our network model assumptions (see section 2.1), there exist general *methods* to determine optimal network coding parameters and simultaneously quantify its energy efficiency (see Lun et al. [21] or Wu et al. [31]). Fragouli et al. [14] give insights for all-to-all broadcast and illustrate how gains could be obtained compared to routing.

While the single-source methods may be applied to specific instances of networks and will compute a precise quantification of the energy efficiency of broadcast with optimal parameters, they might not necessarily yield an estimate beforehand without performing the computation. This article seeks and provides some answers for the following questions:

- How efficient is broadcast with network coding, in general? Is it beneficial to use network coding over routing?

For instance, in wired networks, Edmond's Theorem [13] shows that network coding offers no capacity gains for single-source broadcast. But in wireless networks and for energy efficiency, the nature of the problem changes: using a microscopic view, every transmission will reach several neighbors of the emitter. Therefore energy efficiency seeks not only to limit the number of nodes that transmit; the objective also consists of maximizing the number of receivers for which one transmission is useful (*innovative*).

There is not a universal answer to the previous general questions, and we focus on simplified but representative classes of wireless networks: networks of the Euclidian space, where nodes are uniformly distributed. Wireless communication is modeled with hypergraphs.

Energy efficiency of network coding

Starting with the case where nodes are organized on a grid, our main finding is that, asymptotically, network coding may operate in an "optimal" manner in the core of the network for a strong definition of "optimality" that considers the microscopic viewpoint: every transmission is innovative for every receiver (*local* *optimality*). This definition implies a upper bound for energy efficiency. When the area of the network grows indefinitely, the core of the network comprises the majority of nodes and, as a result, we are able to show that network coding achieves this bound asymptotically, hence optimality.

A consequence is for more general networks where the nodes are not distributed regularly on a grid but randomly; on the condition that the density increases, the results of optimality are extended.

Comparison with routing
By observing that, on the other hand, routing cannot achieve optimality in the microscopic view, we answer the question of the gain of network coding: we prove bounds for the gain of network coding of 1.642 and 1.684 when both the area and the density of the network converge toward infinity for networks of the plane ; and bounds of 1.432 and 2.035 for networks of the euclidean space of dimension 3.

Bottleneck around and destination
In extensions of our results, we prove that if the source or if one of the destination are enlarged to set of nodes, the max-flow between there areas will be increased.

The rest of this paper is organized as follows: section 2 provides background material; section 3.4 describes the methods for network coding, as well as main performance results relative to local optimality; section 3.3 discusses energy efficiency in comparison with routing; section 6 indicates how results could be extended; and section 7 concludes the document.

2 Background

2.1 Network Model

2.1.1 Adaptable Source

We assume that one source is present with an infinite number of packets to transmit to the whole network, and that its data rate may be arbitrarily selected.

A practical example of such a source would be a management node in a sensor network that broadcasts a software update: the duration of the broadcast may be arbitrarily selected in order to preserve the battery lifetime of the nodes.

2.1.2 Topology

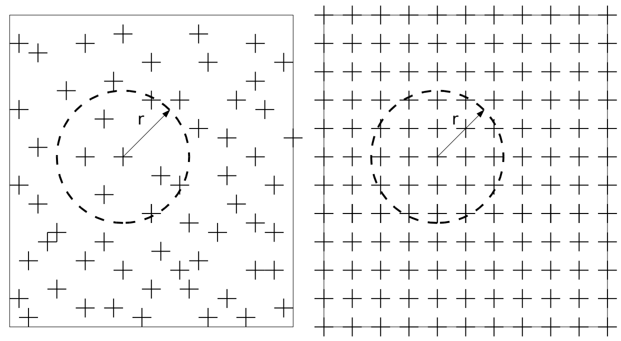
Our assumptions are consistent with some commonly found in the literature of broadcast or connected dominating set (see for instance, those in Fragouli et al. [14] or in Yu and Chong [32]).

We consider multi-hop wireless networks with a number of nodes, without mobility.

Precisely, the considered wireless networks are:

- Random unit disk graphs with nodes uniformly distributed (Fig. 1(a)) on the plane (or, in general, of Euclidean space $\mathbb{R}^n$)
- Unit disk graphs with nodes organized on a lattice (Fig. 1(b))

The primary model for the wireless networks that is considered is the *unit disk graph* model [4], where two nodes are neighbors whenever their distance is lower than a fixed radio range (see Fig. 1(a) for the principle of unit disk graphs).



(a) Random unit disk graph (b) Lattice unit disk graph

Figure 1: Network model and selected rates

The communication model is idealized: every transmission is perfectly received by the neighbors of the emitter, hence wireless transmissions are without loss, collisions, or interferences. Additionally, the network is a packet network and the source is assumed to have identically sized packets.

As discussed in section 2.3.2, because the source is adaptable and because the goal is energy efficiency and not capacity maximization, the ideal communication model does not introduce significant deviations from the behavior of real networks.

2.2 Rate Selection

Several results permit to reformulate the problem of energy efficient multicast of a single source, and they may be found, for instance, in the synthesis of Lun et al. [10, 11]. They can be informally described as follows:

- consider *any* network coding method (deterministic, opportunistic, random, etc.) over a long duration
- compute the average rate of the node (packets transmitted per unit of time).

Then, essentially:

- only the *average rate* of each node has to be considered to have an bound of maximum possible source broadcast rate with the network coding.
- one will achieve asymptotically this bound using the simple method of *random linear coding* of Ho et al. [18, 19], where each node additionally retransmits with a rate equal to the computed average rate (as described in Appendix A).

These results are asymptotic as time converge toward infinity. This is valid for the performance in terms of achieved broadcast capacity, but not, for instance, decoding delay, CPU cost, or other metrics.

It follows that the only relevant characterization of any network coding method is its average rates. Hence, the only issue is to choose a set of rates for the nodes that will yield good performance: a *rate selection*.[1]

[1] rate selection is a specific case of the *subgraph selection* in [21]: in the present case, the nodes have only one hyperarc

Once the rate selection is decided, the maximal performance may be computed by purely graph-theoretic methods. Given any rate selection, one can precisely compute a *maximal achievable broadcast rate* for the source (section 2.5). Essentially, the source may arbitrarily approach this rate and at the same time successfully broadcast all its packets in the long run.

Notice that the rate of one node is not related to the rate of the physical transmission on the wireless medium, as we are below capacity in a packet network (see section 2.3.2); it is simply related to the average delay between two packets' transmissions.

2.3 Energy Efficiency

2.3.1 Metric for Energy Cost

The goal is to operate in an energy efficient way, which is formulated as follows. Consider a network at a given time (t), and then consider the number of source packets that have been successfully broadcast to the entire network $(N_p(t))$ and the number of transmissions that have been made by all nodes in the network $(N_t(t))$. The number of transmissions per broadcast is the ratio between the two (that is: $\frac{N_p(t)}{N_t(t)}$). Energy efficiency corresponds to minimizing this energy cost when the time converge toward infinity.

In the remainder of the article, we will assume that every node has a fixed (average) retransmission rate for reasons detailed later. This defines the *rate selection*. The metric for evaluating energy efficiency is the number of transmissions per broadcast. For one source, we count:

- the number of retransmissions from every node, per unit time, directly given by selected rate.
- the number of packets successfully broadcasted from the source to the entire network per unit time.

By dividing the number of retransmissions by the number of packets successfully broadcasted, the metric for the cost per broadcast is obtained. It is denoted as E_{cost}.

$$E_{\text{cost}} \triangleq \frac{\text{total transmission rate of all nodes}}{\text{broadcast source rate}} \tag{1}$$

The "maximal achievable broadcast rate of the source" (section 2.5) is the rate to use in the previous equation Eq. 1 as "broadcast source rate".

Although it may seem limiting to exclusively consider average rate, it is not, as described in the following section 2.2. Any network coding method may be converted into a method, at least as energy efficient, using *random linear coding* with fixed node rates.

2.3.2 Energy Efficiency and Ideal Communication

The previous expression of the energy efficiency metric Eq. 1 shows that it is invariant when the average rate of the source and of the nodes is scaled by the same amount.

As a result, because we have assumed an adaptable source in a packet network, it is always possible to operate largely below network capacity by decreasing the source rate sufficiently. Then, the limiting characteristics of wireless communication, including half-duplex, limited capacity, and interferences, disappear for the problem of energy efficiency. In practice, once the rates are decided, one may, for instance, scale them appropriately and coordinate the access of the wireless media with a collision-free TDMA system.

2.3.3 Local Optimality for Energy Efficiency

In (linear) network coding terms, *packet* is *innovative* if it will be ultimately decoded into a source packet (otherwise it is not needed or useful at all).

Consider now a microscopic view of the network. When one node u transmits, its transmission will reach its neighbors simultaneously. When using network coding, one coded packet is transmitted by u; this packet will be innovative for a subset of the nodes.

From the local point of view, we can define one metric for the efficiency of the transmission (*local efficiency*) as the proportion of the neighbors for which the transmission is innovative. Then we define as *locally optimal* a transmission that is innovative for all its neighbors (and has maximal local efficiency).

2.3.4 Global Optimality for Energy Efficiency

Turning back to the problem of energy efficiency as it is stated in section 2.3, we define *global optimality* as operating with the most efficient network coding algorithm, the one which minimizes the cost E_{cost} in Eq. 1. With our assumptions, the globally optimal rate selection may be computed for the linear program from Lun et al. [21].

2.3.5 A Bound for Global Efficiency

Global and local optimality are naturally linked, as the globally optimal rate selection will tend to have transmissions that are locally efficient. Generally, even with the globally optimal rate selection, all the individual transmissions are not locally optimal.

Nevertheless, assuming local optimality of every transmission yields a bound for energy efficiency. Indeed, assume that every node has at most $M_{\max}$ neighbors; one single transmission is innovative for at most $M_{\max}$ nodes, assuming local optimality. Hence, in order to broadcast one packet to all N nodes, at least $E_{\text{bound}} = \frac{N}{M_{\max}}$ transmissions are necessary. One metric for energy efficiency can be equivalently written as the ratio to this bound: $\text{Eff}_{\text{bound}} \triangleq \frac{E_{\text{bound}}}{E_{\text{cost}}}$.

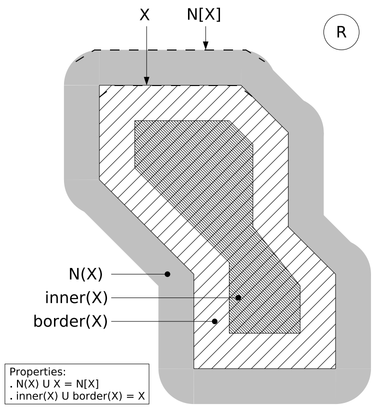
$\text{Eff}_{\text{bound}}$ always verifies $\text{Eff}_{\text{bound}} \leq 1$. A rate selection with $\text{Eff}_{\text{bound}} = 1$ will be locally optimal in every node, hence necessarily globally optimal. In general such a rate selection does not exist.[2].

A common theme is this article is that the proposed rate selections will be asymptotically optimal, in the sense that $\text{Eff}_{\text{bound}} \to 1$ (for size and/or density of the network). As a result, asymptotically, they are on average locally optimal and they must be globally optimal as well.

[2] Indeed, in a multi-hop wireless network, for instance, every retransmission from the neighbors of the source will be noninnovative for the source itself.

2.4 Notations and Definitions

We will use the following general notation in the rest of the article, also illustrated in Fig. 2(a):



(a) Definition

Figure 2: Notations

Some of the notations are more specifically targeted to networks of nodes organized on a lattice. Assume that $\mathcal{V}$ is included in a larger set $\widehat{\mathcal{V}}$ (for a lattice, $\mathcal{V} \subset \widehat{\mathcal{V}} = \mathbb{Z}^n$). We use the following notations for concepts related to neighborhood:

- $\mathcal{N}(X)$: open set of neighbors of $X \in \mathcal{V}$; $\mathcal{N}(X) \subset \widehat{\mathcal{V}}$
- $\mathcal{N}[X]$: closed set of neighbors of $X \in V$, that is nodes and their neighbors $\mathcal{N}[X] \triangleq \mathcal{N}(X) \cup X$
- inner(X): the nodes that have only neighbors in X, that is: inner$(X) \triangleq \{x \mid x \in X \text{ and } \mathcal{N}(x) \subset X\}$
- border(X): nodes that are on the border of X, that is: border$(X) \triangleq X \setminus \text{inner}(X)$
- $|X|$: the number of elements in the set X when it is finite

For a lattice in the unit disk model, the set of neighbors of a node is the same as the neighborhood of the origin node with a translation. And we denote:

- R: the (closed) set of neighbors of the origin node
- M_R: the number of neighbors of the origin node, $M_R \triangleq |R| - 1$
- $\mathcal{L}$: the integer lattice, $\mathcal{L} \triangleq \mathbb{Z}^n$ for n integer > 2
- $B \triangleq$ border$(\mathcal{V})$ will be called the *border of the network*.

We also introduce some requirements for lattice networks:

Requirement 1. *Assumptions on the network:*

$$\mathcal{V} \text{ is a connected network} \tag{2}$$

$$R \text{ is a symmetric set (if } x \in R \text{ then } -x \in R) \tag{3}$$

$$\text{border}(\mathcal{V}) \text{ is a connected area} \tag{4}$$

The requirements are met when $\mathcal{V}$ is a lattice unit graph network (with R as a unit disk neighborhood) and is, for instance, the subset of the infinite lattice inside a rectangle, as represented in Fig. 1(b).

As long as the symmetry of R is verified (Eq. 3 of requirement 1), we have the following properties for any partition X, Y of the integer lattice $\mathcal{L}$:

$$\mathcal{N}(X) = \text{border}(Y) \tag{5}$$

$$\mathcal{N}(\text{inner}(X)) \subset X \tag{6}$$

These properties derive almost immediatly from the definitions, and they are proved with many others in [28] (because in fact $\mathcal{N}(X)$ and inner(X) are two basic operators of mathematical morphology) — for instance, Eq. 6 is identical to a result on the *opening* in (II-23) p52 of [28].

Finally, we define the notion of *dominating set* as a dominating set X of a set Y is a subset of Y, such as $Y \subset \mathcal{N}(X)$. A *connected dominating set (CDS)* X of a set Y is a dominating set X where the subgraph $X \subset \mathcal{V}$ is connected.

2.5 Network Coding: Maximum Achievable Broadcast Rate of the Source

In the network coding literature, several results are for multicast; they apply to the topic of this article, broadcast, since broadcast is a special case of multicast.

A central result for network coding in wireless networks gives the maximum achievable multicast rate for a single source. It is the rate limit for the source, which ensures that every destination may decode the received packets.

This broadcast capacity is given by the min-cut from the source to each individual destination of the network, viewed as a hypergraph for wireless networks [10, 11].

Let us consider the source s and one of the multicast destinations $t \in \mathcal{V}$. The definition of an *s-t cut* is a partition of the set of nodes V in two sets S, T such as $s \in S$ and $t \in T$. Let $Q(s,t)$ be the set of such *s-t cuts*: $(S,T) \in Q(s,t)$.

We denote ΔS, the set of nodes of S that are neighbors of at least one node of T; the *capacity of the cut* $C(S)$ is defined as the maximum rate between the nodes in S and the nodes in T:

$$\begin{array}{c} \Delta S \triangleq \{v \in S : \mathcal{N}(v) \cap T \neq \emptyset\} \\ \text{and} \quad C(S) \triangleq \sum_{v \in \Delta S} C_v \end{array} \tag{7}$$

In other terms, the idea is to cut the network into two parts and check the total rate transmitted from nodes in the part including the source to nodes of the other part.

The *min-cut* between s and t is the cut of $Q(s,t)$ with the minimum capacity. Let us denote $C_{\min}(s,t)$ as its capacity. From [10, 11], the maximum achievable source rate is given by the minimum of capacity of the min-cut of every destination, $C_{\min}(s)$, with:

$$\begin{array}{c} C_{\min}(s,t) \triangleq \min_{(S,T) \in Q(s,t)} C(S) \\ \text{and} \quad C_{\min}(s) \triangleq \min_{t \in \mathcal{V} \setminus \{s\}} C_{\min}(s,t) \end{array} \tag{8}$$

2.6 Related Work

Some results exist about the expected value of the maximum broadcast rate with network coding on some classes of wireline networks (with links between pairs of nodes). For instance Ramamoorthy et al. [1] explored the multicast capacity of networks where a source is two hops from the destinations, through a one network of relay nodes. Aly et al. [27] studied some classes of networks in the plane. From their results [1, 27], one intuition is that most nodes have similar neighborhoods, the approach of setting an identical rate deserves to be explored.

For unit disk graphs, and when every node is a source, Fragouli et al. [14, 15] have previously shown the version of our Th. 1 in the simple case of the torus lattice where nodes have four neighbors and specific topologies such as circular topologies. In these cases their algorithms operate optimally. Their additional theoretic arguments offer pessimistic guarantees of proper functioning with network coding when rates, hence costs, are higher by a factor of three compared to the bound that is reached asymptotically in this article. Hence, they are not sufficient to tightly compare network coding and routing and, indeed, their results of the heuristics, and their analysis, make the case for even better performance than the pessimistic bound.

In previous work, the authors had established Th. 1 in [6] with the method IREN/IRON. The current article is a version derived from [3] that uses the more efficient methods RAUDS and MARAUDS, and gives both lower and upper bounds for the comparison with coding.

3 Our Approach: Efficient Rate Selections

The objective is ultimately to gain insights on the energy efficiency of network coding by considering general classes of networks: networks with nodes organized on a lattice or with nodes randomly distributed.

Consider one instance of such networks: as mentioned in section 2.2, the search for an efficient network coding method on this network is reduced to the search for an efficient a rate selection – deciding the average transmission rate of each node. Then the energy efficiency is computed from the rates and from the topology of the network as shown in Eq. 1 (from the total of the rates and from the maximum broadcast source rate).

Our approach is to construct rate selections, which essentially set the same rate on every node (or a subset of nodes). The rate selections are constructed in such a manner that we are able to give a proof of the energy efficiency of the methods, a cornerstone of this article.

The rate selection methods are the following:

- "IREN/IRON": Increased Rate for Exceptional Nodes/Identical Rate for Other Nodes. This rate selection is for lattice networks.
- "RAUDS": Rate Adjustment Using Dominating Sets. This rate selection is a refinement of the previous method, and is more efficient in the border of the network. It is an intermediate step for the following.

- "MARAUDS": Mapping And Rate Adjustment Using Dominating Sets. This rate selection is the adaptation RAUDS for random unit disk graph networks.

Our starting point is lattice networks such as the one represented in Fig. 1(b) that satisfy the requirement 1 (section 2.4). For a given instance of a lattice network, the rate selections are not the most energy efficient – the optimal one may be computed from the linear program from Lun et al. [21] (without capacity constraints because of ideal communication, see section 2.3.2). Nevertheless, they possess one major property: they imply an energy efficient functioning in the core of the network, which is, in fact, *locally optimal* (defined in section 2.3.3). In each case, we are able to prove that, under some asymptotic conditions, the average efficiency converges to one: global optimality results.

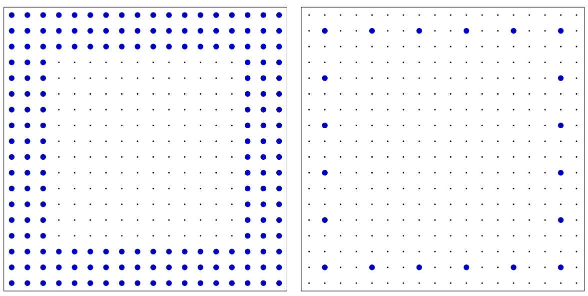
In the next sections, we describe each rate selection method, and formulate and prove theorems that establish their performances as well as local and global optimality. The last one, MARAUDS, will be used to obtain an asymptotic result for random unit disk graphs, and for a comparison with routing.

3.1 The Rate Selection IREN/IRON for Lattice Networks

IREN/IRON is the first rate selection that is proposed. Our approach is to select an identical rate for all nodes, assume local optimality, and improve this rate selection:

1. Assume that every node has an identical retransmission rate, arbitrarily, 1 (e.g., one packet per unit of time)
2. Then every node with M_R neighbors can receive M_R coded packets per unit of time. Assume local optimality: all of them are innovative.
3. Then the source should inject at least M_R packets per unit of time.
4. An issue is the nodes of near the border because they have fewer neighbors, and in general other nodes with less neighbors, therefore some adjustment is required. A rate equal to the source rate (M_R) is chosen.

Precisely, IREN/IRON selects a rate of each node as follows:



(a) Rate selection IREN/IRON (b) Rate selection RAUDS

Figure 3:

- IREN (Increased Rate for Exceptional Nodes): the rate of transmission is set to M_R for the source node and any node v in the border of the network B – defined in section 2.4 (the "exceptional" nodes).

- IRON (Identical Rate for Other Nodes): other nodes transmit coded packets with a rate of 1.

Notice that these rates can be globally scaled by the same amount (see section 2.3.2).

An example is shown on Fig. 3(a), where the larger dots represent the nodes in the border of the network B with rate M_R.

3.2 Maximum Source Broadcast Rate and Local Optimality of IREN/IRON

3.2.1 Maximum Source Broadcast Rate

A central question for the evaluation of the energy efficiency of IREN/IRON, as seen from Eq. 1, is the maximum broadcast rate of the source.

The actual maximum source broadcast rate is also an assessment of whether the assumption of local optimality in section 3.1 is verified. By construction, the rate selection aims at ensuring a fixed maximum broadcast rate equal to M_R. The essence of our main result is the proof that it succeeds:

Theorem 1. *Assume a rate selection IREN/IRON on a network of nodes $\mathcal{V}$ included in the integer lattice $\mathbb{Z}^n$ verifying the requirements (2),(3),(4), then the maximum source broadcast rate is greater or equal to M_R.*
It is exactly M_R when at least one node is not a neighbor of the source nor any node in the border B.

3.2.2 Proof of Th. 1

In this section, we prove the achievable broadcast rate of network coding in Th. 1. The proof relies on fundamental properties of discrete sets of the Euclidean space Eq. 20 and further derived results in appendix B.

We consider the set of nodes $\mathcal{V}$, which is a subset of the integer lattice $\mathcal{L}$. The maximum broadcast rate from the source is computed with graph-theoretic methods as the capacity of min-cut of this network (which is a hypergraph), as indicated in section 2.5. This is done by computing the capacity of the min-cut of the source to any destination. The slight difference with the standard framework is the lack of capacity limitations: the rates of each (hyper)arc are not the maximum capacity of the links, but the rates chosen by the rate selection (IREN/IRON). The results still apply.

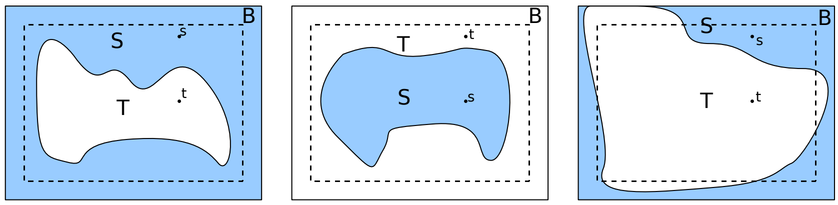
Consider a source s and a destination $t \in \mathcal{V} \setminus \{s\}$. Let (S,T) be a $s-t-$cut of $\mathcal{V}$. The capacity of the cut is $C(S)$, and is given by Eq. 7. The following remark links it with the size of some subsets:

Lemma 1. $C(S) \geq |\Delta S|$

Proof. $C(S)$ is given by Eq. 7: $C(S) \triangleq \sum_{v \in \Delta S} C_v$ which implies $C(S) \geq |\Delta S| \min_{v \in \mathcal{V}} C_v$.

Since with IREN/IRON, $C_v \geq 1$, we have the lemma. □

The next step is to bound the size of ΔS using results from Th. 10. This is the following lemma:



(a) Case $B \subset S$ (b) Case $B \subset T$ (c) Last case

Figure 4:

Lemma 2. *With IREN/IRON, the capacity of one cut $C(S)$ in a lattice graph verifies : $C(S) \geq M_R$.*

Proof. Recall that S and T are a partition of $\mathcal{V}$.

We distinguish three cases, represented on Fig. 4, depending on whether the nodes of the border B are all included in one set S or T or not:

• First case: $B \subset S$ (see Fig. 4(a)).

This is the same as writing: $T \cap \text{border}(\mathcal{V}) = \emptyset$ since $B = \text{border}(\mathcal{V})$. We can apply the Th. 10 by substituting T for X and $\mathcal{V}$ for Y, and we get from Eq. 31: $\mathcal{N}(T) \subset \mathcal{V}$ and $|\mathcal{N}(T)| \geq M_R$.

Since $\mathcal{N}(T)$ is the set of neighbors of T, and thus includes only nodes of S or points of $\mathcal{L}$ outside $\mathcal{V}$, the fact that $\mathcal{N}(T) \subset \mathcal{V}$, implies that $\mathcal{N}(T) \subset S$.

Because of the symmetry of the neighborhood (Eq. 3), all such nodes of S from $\mathcal{N}(T)$ have neighbors in T, hence by definition of ΔS, they are in ΔS. Therefore, here, we have: $\mathcal{N}(T) \subset \Delta S$ (in fact, actually, $\mathcal{N}(T) = \Delta S$)

We can now show the lemma in the current case:

$$
\begin{aligned}
C(S) &\geq |\Delta S| \text{ (due to lemma 1)} \\
C(S) &\geq |\mathcal{N}(T)| \text{ since } \mathcal{N}(T) \subset \Delta S \\
C(S) &\geq M_R \text{ since } |\mathcal{N}(T)| \geq M_R
\end{aligned}
$$

• Second case: $B \subset T$ (see Fig. 4(b)).

We can apply the Th. 10 by substituting S for X and $\mathcal{V}$ for Y; and we get from Eq. 31: $\mathcal{N}(S) \subset \mathcal{V}$. It implies that every node in border(S) has one or several neighbors in $\mathcal{L} \setminus S$, which must be, in fact, in $\mathcal{V}$ hence in T. It follows from the definition of ΔS that border(S) $\subset \Delta S$ (in reality, border(S) $= \Delta S$ even).

We also get from Eq. 30: $|\text{border}(S)| \geq M_R$ or else border(S) $= S$.

• If border(S) $= S$, this implies that the source is in ΔS. We have:

$$
\begin{aligned}
C(S) &= \sum_{v \in \Delta S} C_v \text{ from Eq. 7} \\
&\geq C_s \text{ since } s \in \Delta S \\
&\geq M_R \text{ since } C_s = M_R
\end{aligned}
$$

- Otherwise we have:

$$\begin{aligned} |\text{border}(S)| &\geq M_R \\ |\Delta S| &\geq M_R \text{ because border}(S) \subset \Delta S \\ C(S) &\geq M_R \text{ because of lemma 1} \end{aligned}$$

And the lemma is proven in the second case.

- Third case: both T and S have nodes in B, that is: $T \cap B \neq \emptyset$ and $S \cap B \neq \emptyset$ (see Fig. 4(c)).

Consider one node $u \in S \cap B$ and one node $v \in T \cap B$. From Eq. 4 in requirement 1, B is a connected set, hence there exists one path from u to v consisting of nodes of B. Starting from $u \in S$, and following the path, ultimately one node $v' \in T$ will be found as the immediate successor of one node $u' \in S$. Such $u' \in \Delta S$, and we have:
$C(S) \geq C_{u'}$
$C(S) \geq M_R$ because $u' \in B$ and such nodes have rate $C_{u'} = M_R$

Hence the lemma is proven also in this case. □

Proof of Th. 1

Proof: 1. *From lemma 2, for a fixed source s and destination t, the capacity of cut $s - t-$cut, $C(S)$, verifies $C(S) \geq M_R$, hence the min-cut itself $C_{\min}(s,t) \geq M_R$ (from Eq. 8). It follows the same result for the broadcast min-cut $C_{\min}(s) \geq M_R$, hence the property that the maximum source broadcast rate is greater or equal to M_R.*

For the last statement of Th. 1: in the case where there exists at least one node t_0 which is not neighbor of the source nor of any node in the border B. In this case, consider the cut $S_0 = \mathcal{V} \setminus \{t_0\}$ and $T_0 = \{t_0\}$. Then ΔS_0 is the set of neighbors $\mathcal{N}(t_0)$ of t_0, and verifies $|\Delta S_0| \leq M_R$. Since none of these neighbors is an "exceptional node" (source or border node), with IREN/IRON, they have a rate equal to 1. Hence we have $C(S_0) \leq M_R$, thus $C_{\min}(s) \leq M_R$ and therefore with the opposite inequality that was previously established, we have exactly $C_{\min}(s) = M_R$ which yields the theorem.

3.3 Energy Efficiency of IREN/IRON and Network Coding

3.3.1 Local Optimality of IREN/IRON

To have an understanding of the impact of the previous results with respect to energy efficiency, consider a network organized as a lattice with a rate selection IREN/IRON.

Consider the set of nodes that are sufficiently far from the border and the source, so that all their neighbors will have a rate equal to 1. They have M_R neighbors. Now consider a node u which respects itself the same conditions as well as all its neighbors. Informally, we will call such a node a *core node* [3], and we will refer to their set as the *core of the network*.

For the network, Th. 1 states that the maximum source broadcast rate is M_R. Notice that every rate is an integer, and now if the network also operates

[3] It is at least two-hops away from the border nodes and from the source: formally, it is in the set $\text{inner}(\text{inner}(\text{inner}(\mathcal{V}))) \setminus \mathcal{N}(\mathcal{N}(s))$, non empty if $\mathcal{V}$ is inside a sufficiently large square.

synchronously (as defined in [23]), one can apply the results of Jafarisiavoshani et al. [23] (their Th. 1): the core node u will receive innovative packets from the source at a rate exactly M_R, after a transition phase, provided that linear network coding is done with a sufficiently large field size.

Since in a synchronous operating mode the node u will also receive exactly M_R packets per unit of time (one from each neighbor), it follows that every one of them will be innovative after the transition phase, exactly as hypothesized when constructing IREN/IRON.

Conversely, consider now the perspective of one sender v part of the core of the network ; like u, all its neighbors are receiving exclusively innovative packets after a transition period. Therefore after this transition period, conversely, every of transmissions of v will be innovative for all its receivers, its neighbors. In other words, the transmissions of the node v are locally optimal.

At this point, note that on the lattice, no node may have more than M_R neighbors: in the bound from section 2.3.5, we have $M_{\max} = M_R$, and if the network was including only core nodes, global optimality would result. The non-optimality of IREN/IRON (global optimality) originates from the cost of nodes in the border, the source, and the neighbors of these nodes.

In this section, for simplicity in the explanation, we relied on results of [23], but the presented property of local efficiency is still asymptotically true, in general, for the rate selection IREN/IRON with random linear coding (as in Appendix A) and any field size (see [11]). Indeed, with random linear coding, from the fact that the maximum broadcast rate is M_R, we can deduce that the average value of the local efficiency will converge toward 1 in the core of the network — when the source rate approaches the maximum broadcast source rate and time increases. Is is a different formulation of a similar result of local optimality.

3.3.2 Global Optimality of IREN/IRON and Network Coding

In the previous section, it was shown how the network operates locally optimally in the core of the network. As the size of the network increases, the relative weight of the core increases, and, asymptotically, a global optimality results.

The reasoning is formalized with the following theorem:

Theorem 2. *Consider the sequence of networks* $(\mathcal{V}_L)$ *that are the subsets, of the infinite lattice, included in a square of length* L*, in a unit disk graph model with fixed range* r*.*
We have: when $L \to \infty$*, the relative energy efficiency of the networks converges to* 1, $\text{Eff}_{\text{bound}} \to 1$*. In other words, assymptotically, it is globally optimal.*

Proof: 2. *Consider such a lattice network inside a* $L \times L$ *square. Assume that* $L \geq 2r$*. The number of nodes in the border* B *is* $4r(L-r)$*.*

In that case, the energy efficiency from Eq. 1 is:

$$\begin{aligned} E_{\text{cost}} &= \frac{\text{total transmission rate of all nodes}}{\text{broadcast source rate}} \\ &= \frac{L^2 + (M_R - 1)(4r(L-r))}{M_R} \end{aligned}$$

The bound from local optimality section 2.3.5 is here $E_{\text{bound}} = \frac{L^2}{M_R}$. *Hence the relative energy efficiency:*

$$\text{Eff}_{\text{bound}} = \frac{E_{\text{bound}}}{E_{\text{cost}}} = 1 + O(\frac{1}{L})$$

and the theorem is proven.

3.4 The Rate Selection RAUDS for Lattice Networks

In this section, the rate selection RAUDS (Rate Adjustment Using Dominating Sets) is described. Precisely, it is a specific version of the family of rate selections RAUDS proposed in [3]. It is similar to IREN/IRON except that its energy cost is lower on the border of the network. The motivation is that the rate selection on lattices will be used for random networks in section 3.5, for networks where the density increases, and where M_R increases, the cost of IREN/IRON would prove too high.

Precisely, RAUDS is defined as follows: the nodes in the border B are identified. A connected dominating set D is built to cover these nodes (for instance, with one of the connected dominating set algorithms in [17]).

RAUDS selects a rate of each node as follows:

- The rate of transmission is set to M_R for the source node, and nodes in the connected dominating set D.
- Other nodes have a rate equal to 1.

An example is shown in Fig. 3(b), where the larger dots represent the nodes of the CDS with rate M_R.

3.4.1 Maximum Broadcast Rate of RAUDS

We have the same theorem as for IREN/IRON (which also implies local optimality in the core of the network):

Theorem 3. *Assume a rate selection RAUDS on a network of nodes* $\mathcal{V}$ *included in the integer lattice* $\mathcal{L} = \mathbb{Z}^n$ *verifying the requirements (2),(3),(4). Then the maximum source broadcast rate is greater or equal to* M_R.
It is exactly M_R *when at least one node is not a neighbor of the source or any node in the border* B.

Proof: 3. *The only difference with the proofs of IREN/IRON is for lemma 2, which need to be re-established: consider a source* s *and a destination* $t \in \mathcal{V} \setminus \{s\}$ *and let* (S, T) *be a* $s-t-$*cut of* $\mathcal{V}$.

As in the previous proof of lemma 2, there are three cases represented on Fig. 4, depending on whether or not the nodes in the border B *are all included in one set* S *or* T.

- *First and second cases:* $B \subset S$ *(see Fig. 4(a)) and* $B \subset T$ *(see Fig. 4(b)). The proof in section 3.2.2 still applies in these cases.*
- *Third case: both* T *and* S *have points in* B*, that is:* $T \cap B \neq \emptyset$ *and* $S \cap B \neq \emptyset$ *(see Fig. 4(c)).*

Unlike previously, there are three possibilities, this time, depending on whether or not the nodes in the connected dominating set D are all included in one set S or T.

— *First possibility: $D \subset S$*

Since $T \cap B \neq \emptyset$, consider one node u of this subset: by definition of a (connected) dominating set, it must be neighbor of one node v in D. Since $D \subset S$, $v \in S$ and v has a rate M_R by construction of RAUDS. The transmission $v \to u$ with rate M_R ensures that the capacity of the cut verifies $C(S) \geq M_R$.

— *Second possibility: $D \subset T$*

In this case, consider S as a subset of the infinite lattice $\mathcal{L}$, and denote X its interior, $X \triangleq \mathrm{inner}(S)$, and Y the complementary set of X in $\mathcal{L}$: $Y \triangleq \mathcal{L} \setminus X$. We can apply Th. 9 to X and Y, and we have: $|\mathrm{border}(Y)| \geq M_R$ (from Eq. 31). From Eq. 5, $\mathrm{border}(Y) = \mathcal{N}(X)$, therefore: $|\mathcal{N}(X)| \geq M_R$.

Denote $S_b \triangleq \mathcal{N}(X) = \mathcal{N}(\mathrm{inner}(S))$. The previous result is that $|S_b| \geq M_R$. We will show that in fact all nodes in S_b are in $\mathrm{border}(S)$ and, in turn, in ΔS:

Consider one node $u \in S_b$. Since $u \in \mathcal{N}(\mathrm{inner}(S))$, we have $u \notin \mathrm{inner}(S)$ (by definition of $\mathcal{N}$). Additionally, from Eq. 6, we have $S_b \subset S$, hence $u \in S$. Because $\mathrm{border}(S)$ and $\mathrm{inner}(S)$ are a partition of S, the previous results imply: $u \in \mathrm{border}(S)$. Hence, $S_b \subset \mathrm{border}(S)$.

Consider any node $v \in \mathrm{border}(S)$: by definition of $\mathrm{border}(S)$, the node v must be in S and be neighbor of a node w outside S. $w \notin S$, therefore it is either a node w in T or a point w outside of $\mathcal{V}$.

- *If $w \in T$, then by definition of ΔS, we have $v \in \Delta S$.*
- *If w is a point outside $\mathcal{V}$, by definition of the border of $\mathcal{V}$, we must have $v \in \mathrm{border}(\mathcal{V})$ (that is B); Here we can introduce the hypothesis of the second possibility, that is, $D \subset T$: because v is in B, it must be neighbor of at least one node of the dominating set D, and therefore of a node of T. Again, by definition of ΔS, we have $v \in \Delta S$*

In all cases: $v \in \Delta S$, hence this proves that $\mathrm{border}(S) \subset \Delta S$.

We can combine all the previously established properties:

$$\begin{aligned} C(S) &\geq |\Delta S| \text{ from lemma } \mathit{1} \\ &\geq |\mathrm{border}(S)| \text{ because } \mathrm{border}(S) \subset \Delta S \\ &\geq |S_b| \text{ because } S_b \subset \mathrm{border}(S) \\ &\geq M_R \text{ because } |S_b| \geq M_R \end{aligned}$$

— *Third possibility: D includes both nodes from S and in T. In this case, because D connected, a path can be found between nodes of $S \cap D$ and $T \cap D$, and therefore there exist two neighbors u and v in $S \cap D$ and $T \cap D$ respectively. In that case, the transmissions $u \to v$ contribute to the capacity of the cut with an amount M_R, hence again $C(S) \geq M_R$.*

As a result lemma 2 is restablished, and the theorem follows as for IREN/IRON.

Notice that the maximum broadcast rate of RAUDS is the same as IREN/IRON but its energy cost is always lower (because the CDS is strictly included in the B). Therefore, the results of the local and global optimality of IREN/IRON are valid for RAUDS as well.

3.5 The Rate Selection MARAUDS for Lattice Networks

In this section, we describe the rate selection MARAUDS, an extension of the rate selection RAUDS of section 3.4 for networks in the plane.

Consider an arbitrary network which is a random unit disk graph (see Fig. 1(a)). Denote $\mathcal{V}_{\text{real}}$ the set of the nodes on a random unit disk graph. The rate selection for a random unit disk graph extends RAUDS by considering mapping between a network and a virtual lattice.

3.5.1 Construction of a Virtual Lattice

• Choose some fixed lattice spacing ρ with $0 < \rho < \frac{1}{2}r$ and denote the rescaled lattice as $\mathcal{L}_\rho \triangleq \{\rho\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n\}$
• Choose a quantity δ verifying $\frac{1}{\sqrt{2}}\rho < \delta < \frac{1}{2}r$
• Select a mapping λ from the nodes of the real network $\mathcal{V}_{\text{real}}$ to the points of the virtual lattice (as seen, for example, in Fig. 5); $\lambda : \mathcal{V}_{\text{real}} \to \mathcal{L}_\rho$, which verifies for any $x \in \mathcal{V}_{\text{real}}$:

Property 1. $\|\lambda(x) - x\| \leq \delta$

Such a mapping always exists, for instance, at least the mapping of nodes of $\mathcal{V}_{\text{real}}$ to the closest node of $\mathcal{L}_\rho$.

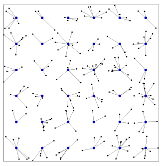


Figure 5: Mapping to a virtual lattice

The *virtual lattice*, denoted $\mathcal{V}$, is then defined as the set of nodes of λ to which a node of $\mathcal{V}_{\text{real}}$ is mapped – formally $\mathcal{V} \triangleq \lambda(\mathcal{V}_{\text{real}})$.

3.5.2 Rate Selection with Virtual Lattice

Define R as a ball, the set of points on the lattice within range r' of the origin, with $r' \triangleq r - 2\delta$ (alternatively, R can be also selected as an arbitrary symmetrical subset of this ball).

It is now possible to apply RAUDS to select rates of nodes in the virtual network $\mathcal{V} \subset \mathcal{L}_\rho$, with the R chosen. Let $C_v^{(\text{lat})}$ be a selected rate for nodes in the virtual lattice. Then, the rate selection for the initial network $\mathcal{V}_{\text{real}}$ is:

- For every point of the virtual lattice $\tau \in \mathcal{V}$, choose one unique $v \in \mathcal{V}_{\text{real}}$ mapped to that point and assign the rate $C_v = C_\tau^{(\text{lat})}$

This is only possible if the following requirement is met:

Requirement 2.
For every point of the virtual lattice $\tau \in \mathcal{V}$, there exists at least one node $v \in \mathcal{V}_{\text{real}}$ mapped to that point.

Notice that the probability that requirement is met increases with the density.

3.5.3 Maximum Broadcast Rate of MARAUDS

The results for the maximum broadcast rate of MARAUDS are extended, and we have the following theorem:

Theorem 4. *If requirement 2 is met, the maximum broadcast rate of the extended MARAUDS for random unit disk graphs* $\geq |R| - 1$.

Proof: 4. *For* $s \in \mathcal{V}_{\text{real}}, t \in \mathcal{V}_{\text{real}}$*, consider a cut* (S, T)*of the graph* $\mathcal{V}_{\text{real}}$*. Denote* $s' = \lambda(s)$ *and* $t' = \lambda(t)$*.*

An induced cut (S', T') *of the* $\mathcal{V} \subset \mathcal{L}_\rho$*, the virtual lattice, is constructed as follows:*

- *For any point of the lattice* $\tau \in \mathcal{V}$*, the rate is* $C_\tau^{(\text{lat})}$*.*
- S' *is the set of the points of* $\mathcal{V}$ *such as only nodes of* S *are mapped to them:*

$$S' \triangleq \{\tau : \lambda^{-1}(\tau) \subset S\} \tag{9}$$

- T' *is the set of the rest of points of* $\mathcal{V}$*.*

Now, if $s' \in T'$*, this implies that the source* s *is mapped to the same node* s' *as at least one other node* $v \in T$*. This implies that the node is neighbor of the source, hence, the capacity of the cut* $C(S)$ *is at least the rate of the source, which is sufficient to establish the theorem in this case.*

Then the only case that need to be considered is the case where $s' \in S'$*. Notice that* $t' \in T'$ *; that all the points of the lattice, to which both nodes from* S *and* T *are mapped, these points are in* T'*. Therefore* S', T' *is indeed a partition and a* $s' - t'$ *cut.*

By Th. 3, we know that the capacity of the cut S', T' *is lower bounded by* $M_R = |R| - 1$*:* $C'(S') \geq |R| - 1$

Consider the expression of the capacity of the cut (S', T')*: From Eq. 7:* $C'(S') = \sum_{v \in \Delta S'} C_v^{(\text{lat})}$
where $\Delta S'$ *are the points of* S'*, neighbors of points of* T' *(from Eq. 7).*

Consider two points in $u' \in \Delta S'$ *and one associated neighbor* $v' \in T'$*. Now consider any two nodes of* $\mathcal{V}_{\text{real}}$ *that are mapped to those points:* $u \in S$ *and* $v \in T$*, such as* $\lambda(u) = u'$ *and* $\lambda(v) = v'$*. Such nodes exist by construction.*

We have:

$$\begin{aligned} \|u - v\| &\leq \|u - u'\| + \|u' - v'\| + \|v' - v\| \\ &\leq \|u - \lambda(u)\| + \|u' - v'\| + \|v - \lambda(v)\| \\ &\leq 2\delta + \|u' - v'\| \text{ (with Property } 1) \leq r \end{aligned} \tag{10}$$

Hence, u *will also be in* ΔS *as defined in Eq. 7 and contributes to the capacity of the cut of* $C(S)$*. Considering all other nodes of* S *mapped to the same* u'*, section 3.5.2 indicates that their total rate is equal to* $C'_{u'}$ *(due to the one node chosen to transmit), and they are also neighbors of* v*. Therefore their total contribution to the capacity of the cut* $C(S)$ *is identical to the contribution of* u' *to the cut* $C'(S')$*.*

Applying the same reasoning to all nodes in S'*, we have:* $C(S) \geq |R| - 1$*. This establishes the theorem.*

3.5.4 Energy Efficiency of MARAUDS for Dense Networks of the Plane

In this section, we provide informal arguments and estimates for the energy efficiency of MARAUDS for general networks of the plane. Formal arguments are presented in the next section 3.5.5.

We focus on networks that are contained in a predefined region of the plane, and we are interested in the energy efficiency of the method MARAUDS when the density increases but the range stays fixed. As the density increases, any point of the space will be arbitrarily close to a node of the network, hence geometric reasonings are used.

Consider networks contained in regions such as the one represented in Fig. 2(a). The networks themselves would be inside the region denoted X. The part denoted inner(X) is the region where the nodes would have a sufficient number of neighbors as the density increases. It is the part where, asymptotically, MARAUDS is locally optimal. The part border(X) is the region where nodes need to be covered by CDS. The idea is that the border(X) can be covered by nodes standing near the border as represented in Fig. 6. In that case, we can informally estimate the number of nodes in the CDS from the perimeter P of the line midway between the edge of X and the edge of inner(X) as $N_{\text{cds}} \approx \frac{P}{r}$. Assume that the area of the network is A and that the virtual lattice of MARAUDS has a step ρ. Then $N_{\text{lattice}} \approx \frac{A}{\rho^2}$, $M \approx \frac{\pi r^2}{\rho^2}$, $E_{\text{cost}} \approx \frac{1}{M}(\frac{P}{r} \times M + N \times 1)$. Finally denoting $\lambda \triangleq \frac{A}{P}$ as the ratio between area and perimeter of region of the network, we can estimate the efficiency bound of section 2.3.5:

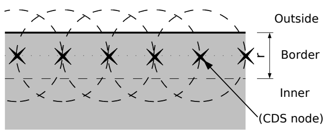


Figure 6: CDS to cover borders in dense networks

$$\text{Eff}^{(\text{est.})}_{\text{bound}} = \frac{1}{1+\frac{\pi r}{\lambda}} + O(\frac{1}{\lambda}) \qquad (11)$$

Notice that when r is kept constant and the entire region of network is scaled by α, the ratio area/perimeter λ is scaled by the same amount α. Hence, $\text{Eff}^{(\text{est.})}_{\text{bound}} \to 1$ when the area of the network grows indefinitely; this is because the area of the border becomes vanishingly small in comparison. This indicates asymptotic local optimality of MARAUDS (as it was the case for RAUDS) and implies, asymptotically, global optimality of MARAUDS as well. For the case of network is included in a square region, a formal proof of Eq. 11 is in the next section.

3.5.5 Energy Efficiency of MARAUDS for Dense Networks in Squares

In this section, we will establish performance bounds of MARAUDS for growing density, but with a fixed network region. Precisely: assume networks contained in the plane, inside a square region $\mathcal{A}$ of width L and with a fixed radio range

$r > 0$. Consider a rate selection MARAUDS as in section 3.5, with the following parameters:

- $\rho = \frac{1}{\mu^{\frac{1}{3}}}$ (μ defined later) and $\delta = \rho$
- the nodes of $\mathcal{V}_{\text{real}}$ are mapped to the closest point on the rescaled lattice $\mathcal{L}_\rho$. Denote $\mathcal{V}(\rho) = \lambda(\mathcal{V}_{\text{real}})$

For an asymptotic result, we consider a sequence of random unit disk graphs where the nodes are defined by spatial Poisson process with rate μ, with $\mu \to \infty$. We are interested in the bound given in section 2.3.5, and the estimate of section 3.5.4 which is proven as the following theorem:

Theorem 5. *The efficiency of MARAUDS is such that* $E[\text{Eff}_{\text{bound}}] \xrightarrow{p} \frac{1}{1+\frac{4\pi r}{L}}$ *for an appropriate rate selection MARAUDS, when the density* $\mu \to \infty$.

Proof: 5. *Consider a lattice with the extension MARAUDS for a random unit disk graph.* $\mathcal{V}(\rho)$ *is the set of points of the lattice* $\mathcal{L}_\rho$ *to which at least one point of* $\mathcal{V}$ *is mapped. When the rate of the spatial Poisson process* μ *is large enough, this sets corresponds to the set of points of the full* $\mathcal{L}_\rho$ *inside the square containing the network. We first select* ρ *to verify this property.*

Precisely: let us denote $\mathcal{E}_0$ *the event, for one point of the lattice:* "there is no point mapped to it". *We have* $\Pr[\mathcal{E}_0] = e^{-\mu\rho^2}$. *If* $\mathcal{E}$ *is the global event* "at least one point of the lattice has no point mapped" *(which is exactly the event* "Requirement 2 is not met"*), an* union bound *on the* $\frac{L^2}{\rho^2}$ *points of the lattice* $\mathcal{L}_\rho$ *yields* $\Pr[\mathcal{E}] \leq \frac{L^2}{\rho^2}\Pr[\mathcal{E}_0]$, *hence:*

$\Pr[\mathcal{E}] \leq \exp(-\mu\rho^2 - 2\log\rho + 2\log L)$

By setting $\rho = \frac{1}{\mu^{\frac{1}{3}}}$ *for instance, we have the desired property* $\Pr[\mathcal{E}] \to 0$ *when* $\mu \to \infty$. *Now consider the efficiency* $\text{Eff}_{\text{bound}}$, *which involves* E_{cost}, *the "transmissions per broadcast.", which in turns requires the maximum broadcast rate, and the total transmission rate* T_{cost}

When $\mathcal{E}_0$ *is verified, requirement 2 is true, and from Th. 4, we have: the maximum broadcast rate is* $|R(\rho)| - 1$. *The number of points in* $|R(\rho)|$ *is the number of lattice points within a circle of radius fixed around the origin (the "*circle problem*"); it is* $|R(\rho)| = \pi(\frac{r}{\rho})^2 + O(\frac{1}{\rho})$ *when* $\rho \to 0$ *([30] p. 133)*

The rate of transmissions T_{cost} *is given by the rate of transmissions of nodes in the network, plus the rate of transmissions of the border nodes, and the source. When the event* $\mathcal{E}$ *is verified, a node in a virtual lattice always has at least one mapping from a node in a network of* $\mathcal{V}$, *and only the nodes on the border of* $\mathcal{V}$ *require rate adjustment from their neighbors.*

We can construct a dominating set composed of nodes of $\mathcal{L}_\rho$ *on four lines parallel to the edges of the square defining the network area (see Fig. 6).*

Their space is chosen as $\approx r$, *i.e.* $r + O(\rho)$, *and from elementary geometry, this is always possible if* ρ *is small enough (from elementary geometry, sufficient condition:* $\rho \leq \frac{r}{8}$*), and the total number of nodes in the dominating set is* $\frac{4L}{r} + O(\rho)$

With the size of dominating set, we can express T_{cost} *as:*

$E[T_{\text{cost}}] = \frac{L^2}{\rho^2} + (|R(\rho)| - 1)(4\frac{L}{r} + O(\rho))$

The cost of transmission per broadcast then:

$$E[E_{\text{cost}}] = \frac{L^2}{\rho^2(|R(\rho)| - 1)} + \frac{4L}{r} + O(\rho)$$

Let us consider the ratio of E_{cost} *with the transmission-level bound* E_{bound} *from section 2.3.1,* $E_{\text{bound}} = \frac{L^2}{\pi r^2}$

Considering the limit of all involved quantities, we get $\frac{1}{\text{Eff}_{\text{bound}}} = \frac{E_{\text{cost}}}{E_{\text{bound}}} \to 1 + \frac{4\pi r}{L}$ *wh en* $\mu \to \infty$ *conditioned to the event* $\mathcal{E}$ *whose probability* $Pr[\mathcal{E}] \to 0$ *(which ensures requirement 2).*

For non-uniform networks, consider a sequence of non-uniform networks indexed by $i \in \mathbb{N}$, $i \to \infty$, and assume that the nodes are given by point processes with of density $\mu_i(x)$ for $x \in \mathcal{A}$. We have the following additional theorem:

Theorem 6. *If the densities verify* $\min_{\mathcal{A}} \mu_i \to \infty$*, the efficiency of MARAUDS is such that:* $E[\text{Eff}_{\text{bound}}] \xrightarrow{p} \frac{1}{1+\frac{4\pi r}{L}}$.

Proof: 6. *We can consider the* $\mu = \min_{x \in \mathcal{A}} \mu_i(x)$ *in the proof of Th. 5, then the proof also applies. What happens is that, with MARAUDS, every region of the plane will contain the same density of nodes of the dominating lattice* $\mathcal{V}$.

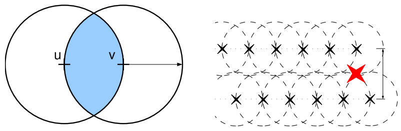
4 Comparison of the Energy Efficiency of Network Coding and Routing in the Plane

In other sections, we described the local optimality of IREN/IRON, RAUDS, and MARAUDS in the parts of the networks that are distant (three hops or more) from the CDS and the source.

One question is: without network coding, can the broadcast also be locally energy efficient (on the lattice and in general)? One answer is negative, from an argument of Fragouli et al. [14]: consider any broadcast with a store-and-forward, non-coding method (including, but not limited to, the use of CDS for broadcast), and consider the sets of nodes which have (re-)transmitted the packet. Apart from the source, any retransmitting node v will have received the packet from another node u. When v retransmits the packet, its transmission will be received again by u itself, and additionally any common neighbors of u and v. For those nodes, the transmission will be redundant (see Fig. 7(a)). Hence, apart from the source, the retransmissions are never locally optimal.

If we consider dense networks of the Euclidean plane and consider areas to evaluate the bound derived from the previous argument, in a unit disk graph with range r, two neighbors share a neighborhood area at least equal to $(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})r^2$ (Fig. 7(a)). Hence a bound for the transmission-level efficiency of broadcast in a dense, uniformly distributed graph, is
$\text{Eff}^{(\text{routing})}_{\text{upper-bound}} = \frac{E^{(\text{routing})}_{\text{bound}}}{E^{(\text{routing})}_{\text{cost}}} \leq \frac{2\pi+3\sqrt{3}}{6\pi} \approx 0.609$ (see also Fragouli et al. [14]).

On the other hand, if the area tends toward infinity, it is at least possible to cover the whole space with alternating rows of aligned nodes such as the border



(a) Inefficiency of routing (b) CDS cover of the plane

Figure 7: Routing – bound and cover

CDS of Fig. 6 (see also Bai et al. [5]).[4] This yields a lower bound of the possible efficiency of routing: $\mathrm{Eff}^{(\mathrm{routing})}_{\mathrm{lower-bound}} = \frac{2+\sqrt{3}}{2\pi} = 0.594\ldots$

Hence, for infinitely large and dense networks and for single-source broadcast, we have the following bounds for the gain of network coding over non-coding:

$$1.642 \approx \frac{6\pi}{2\pi + 3\sqrt{3}} \le \begin{array}{c}\text{coding gain}\\ \text{in dim 2}\end{array} \le \frac{2\pi}{2+\sqrt{3}} \approx 1.684 \qquad (12)$$

4.1 Comparison of MARAUDS and Routing

In the previous section, the comparison of network coding and routing was possible when the area of the network would grow indefinitely, because at the same time, MARAUDS would become asymptotically optimal.

Another question would be to estimate if gains over routing would be expected for a given area of the network (but the density still increases). Our results cannot answer this question, since the efficiency of the optimal rate selection for network coding is not known; however, some insight is possible for the rate selection MARAUDS from Eq. 11 and Th. 5. Comparing Th. 5 and the upper bound of efficiency for routing $\mathrm{Eff}^{(\mathrm{routing})}_{\mathrm{upper-bound}}$ of the previous section, it follows that MARAUDS would be expected to be advantageous over routing, at least for dense square networks where:

$$\frac{r}{L} \le \frac{3}{4\pi+6\sqrt{3}} - \frac{1}{4\pi} \approx 0.0510\ldots$$
$$\text{that is where: } L \ge 20r$$

4.2 Sample Rate Selection with RAUDS

The Fig. 8 represents one example of rate selection RAUDS. The larger (colored) points represent nodes on the CDS with higher rate, whereas other nodes have rate 1. The points with slightly smaller size are nodes on the borders that are covered by 3 CDS. The source may be placed at any node.

- There are $N = 3249$ nodes, the range is $r = 3$, and $M_R = 28$.
- Two neighbors have at most 17 non-common neighbors, and then the lower bound of the size of a CDS is 191

[4] Their spacing is $r(1+\frac{\sqrt{3}}{2})$, and the cost of additional connecting nodes such as the colored one in Fig. 7(a) is $O(\frac{1}{\lambda})$.

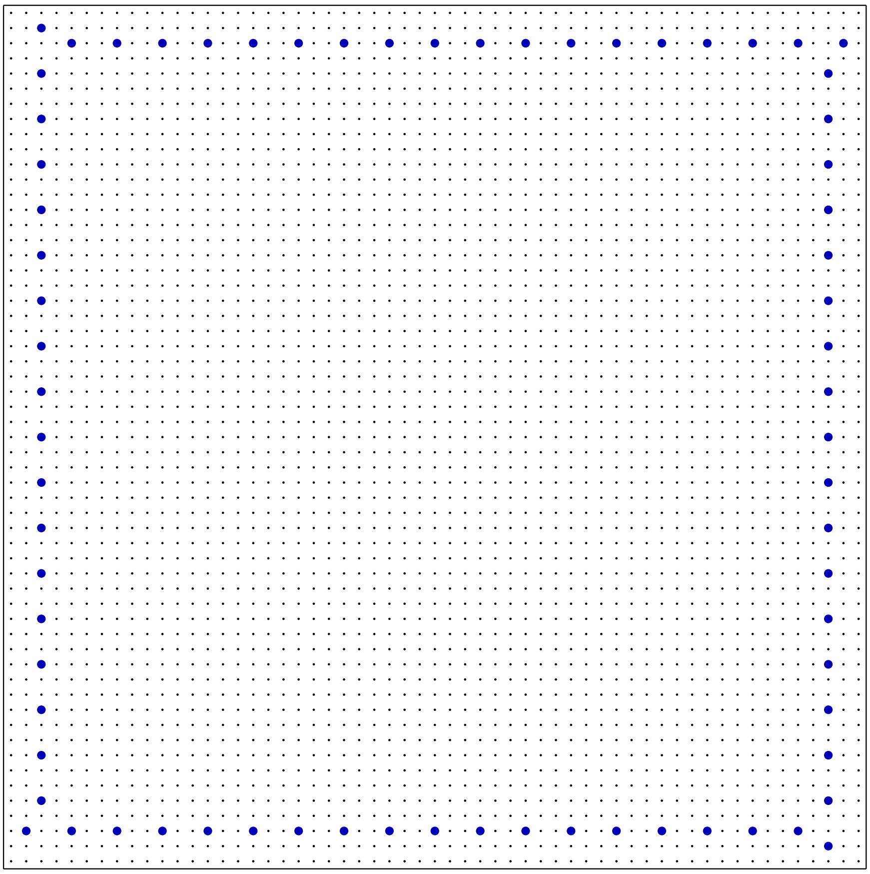


Figure 8: Example of rate selection with RAUDS - in this case it outperforms any method not using coding

- The cost per broadcast of the rate selection RAUDS is $E^{(\mathrm{RAUDS})}_{\mathrm{cost}} = 186.429\ldots$
- Here $\frac{L}{r} = 19.000\ldots$

This example is illustrative because it is one for which RAUDS will outperform any method using routing. The gap with the efficiency of the upper bound of routing and RAUDS is rather small: one reason is that the parameters $\frac{r}{L}$ are slightly lower than those given in section 4.1, beyond which there would be an expectation for RAUDS or MARAUDS to gain advantage over routing.

5 Further Insight on the Max-Flow between Source and Destination

In the previous sections, we were mostly concerned about the energy-efficiency of network coding, computed for specific methods.

In this section, the objective is to explore more closely the available max-flow between the source and the destination, on the integer lattice, when most nodes have the same rate.

Consider the IRON/IREN method (section 3.1). We have shown previously, that the min-cut between the source and the destination is M_R. It is actually the capacity of the cut around the source (for $S = \{s\}; T = \mathcal{V} \setminus \{s\}$) or around the destination ($S = \mathcal{V} \setminus \{t\}; T = \{t\}$), hence two bottlenecks are both around the source and around the destination.

The question that we intend to explore is the following: what is the capacity of the network with IRON/IREN, if these bottlenecks were relaxed?

Our answer will be that the bottlenecks would still be the limited be the size of the region around the source and the destination.

To proceed, we assume that we have a lattice network as in section 3.1 and section 3.2. Since we are only interested by the max-flow available between the source area and the destination area, we assume instant transmission in area that are peripherical to the issues, that: the border, around the source and around the destination.

Precisely:

- We assume that the source s transmits instantlly all information to a set of nodes H_s, and that, in addition, this set of nodes retransmit instantly to their neighbors. This can be modeled with nodes with infinite rate.
- We assume that there is a set of nodes H_t which transmits instantly all information they receive to the destination t. This is modelled with point-to-point links with infinite rate.
- We assume that the nodes on the border (IREN) have also infinite rate.
- By convention, we put $s \in H_s$ and $t \in H_t$

5.1 Generalization of Th. 1

Our objective is a generalization of Th. 1 which could allow larger bounds when H_s and H_t include more than the source s or the destination t respectively.

We start from cuts of $\mathcal{V}$. Consider a source s and a destination $t \in \mathcal{V} \setminus \{s\}$. Let (S,T) be a $s-t-$cut of $\mathcal{V}$. The capacity of the cut is $C(S)$, and is given by Eq. 7.

We first generalize lemma 2:

Lemma 3. *The capacity of one cut $C(S)$ in a lattice graph verifies at least one of the three following properties:*

- *($C(S) \geq b_{\text{out}}(T)$ and $\mathcal{N}(T) \subset \mathcal{V}$)*
- or *($C(S) \geq b_{\text{in}}(S)$ and $\mathcal{N}(S) \subset \mathcal{V}$)*
- or *($C(S)$ is infinite)*

Proof. First, the following the steps of the lemma 2, we distinguish the same three cases, represented on Fig. 4.

• First case: $B \subset S$ (see Fig. 4(a)). We proceed as previously but using Eq. 29 instead of Eq. 31, we get:

$$C(S) \geq b_{\text{out}}(T)$$

and also $\mathcal{N}(T) \subset \mathcal{V}$ from the same Eq. 29.

• Second case: $B \subset T$ (see Fig. 4(b)). We proceed as previously except for using Eq. 28 instead of Eq. 30, and we get either $|border(S)| = S$ (in which case $C(S)$ is infinite) or:

$$C(S) \geq b_{\text{in}}(S)$$

and also $\mathcal{N}(S) \subset \mathcal{V}$ using Eq. 29 with $X = S$ and $Y = \mathcal{V}$.

• Third case: both T and S have nodes in B, that is: $T \cap B \neq \emptyset$ and $S \cap B \neq \emptyset$ (see Fig. 4(c)). As previously, in that case one of the node of T is in range with one of the node of S on the border, hence the capacity of the cut is infinite.

Connecting the results of the three cases yields the lemma. Notice for instance, that even if the partition S/T falls in the first case, the quantity $b_{\text{in}}(S)$ is well defined (as involving sets of $\mathbb{Z}^n$; b_{in} and b_{out} are always defined). □

We start introducing the sets H_t and H_s with the following lemma 4:

Lemma 4. *One cut $(S,T) \in \Omega$ verifies:*

- *If $\mathcal{N}(T) \subset \mathcal{V}$, then:*
  $H_t \subset T$ or else $C(S) = \infty$
- *If $\mathcal{N}(S) \subset \mathcal{V}$, then:*
  $H_s \subset \text{inner}(S)$ or else $C(S) = \infty$

Proof.

• We start with the second, more complex, case: $\mathcal{N}(S) \subset \mathcal{V}$. Then consider what happens if $H_s \not\subset \text{inner}(S)$: there exists at least one node in H_s which is not in inner(S). Denote u this node.

By definition of inner(S), u is neighbor of at least one node which is not in S, denoted v. Because $\mathcal{N}(S) \subset \mathcal{V}$, it follows that $\mathcal{N}(u) \subset \mathcal{V}$, hence $v \in \mathcal{V}$. Since v is not in S, it must be in T.

Since H_s was defined as a set of nodes with infinite rate to their neighbors, it follows that u in S transmits with infinite rate to v in T, hence the capacity of the cut verifies $C(S) = \infty$.

This proves the lemma for this case.

• In the first case $\mathcal{N}(T) \subset \mathcal{V}$. Now if $H_t \not\subset T$, we can find a node $u \in H_t \cap T$, and that node will transmit with infinite rate to t, hence the lemma is proved for this case also.

□

We are now able to combine the two lemmas, lemma 3 and lemma 4, to provide a generalization of Th. 1, in the following Th. 7:

Theorem 7. *Assume a rate selection IREN/IRON on a network of nodes $\mathcal{V}$ included in the integer lattice $\mathbb{Z}^n$ verifying the requirements (2), (3), (4), then the maximum source broadcast rate $C_{\max-\text{flow}}$ verifies:*

$$C_{\max-\text{flow}} \geq \min(M_{\text{out}}, M_{\text{in}})$$

where:

- $M_{\text{out}} \triangleq \min\{b_{\text{out}}(T) : (S,T) \in \Omega_{\text{out}}\}$
- $M_{\text{in}} \triangleq \min\{b_{\text{out}}(\text{inner}(S)) : (S,T) \in \Omega_{\text{in}}\}$
- $\Omega_{\text{in}} = \{(S,T) : (S,T) \in \Omega$ and $\mathcal{N}(S) \subset \mathcal{V}$ and $H_s \subset \text{inner}(S)\}$
- $\Omega_{\text{out}} = \{(S,T) : (S,T) \in \Omega$ and $\mathcal{N}(T) \subset \mathcal{V}$ and $H_t \subset T\}$
- *Ω is the set of partitions of $\mathcal{V}$*
- *b_{out} is defined in Th. 8*
- *M_{in} or M_{out} are set to ∞ when they are a minimum over an empty set.*

Proof. Let (S,T) be a $s-t-$cut of $\mathcal{V}$. It verifies lemma 3, and combining with lemma 4 in the first two cases of lemma 3, we get:

- $(C(S) \geq b_{\text{out}}(T)$ and $\mathcal{N}(T) \subset \mathcal{V}$ and $H_t \subset T)$
- *or* $(C(S) \geq b_{\text{in}}(S)$ and $\mathcal{N}(S) \subset \mathcal{V}$ and $H_s \subset \text{inner}(S))$
- *or* $(C(S)$ is infinite$)$

Hence every cut verifies: $C(S)$ is infinite or $C(S) \geq \min(M_{\text{out}}, M_{\text{in}})$ (notice that by definition, for any set X, $b_{\text{in}}(X) = b_{\text{out}}(\text{inner}(X))$).

As a result the min-cut verifies also the property, and since the maximum source broacast rate is the capacity of the min-cut, then the theorem Th. 7 is established.

Note that from their definition, M_{in} or M_{out} can possibly be minimum over an empty set ; if both are, then it means that the network topology is such that the capacity of the cut is infinite. In that case, the reason is that the topology is also specific, that is, every node is neighbor of the source or of the border, and that we set an infinite rate on the source and on the border in IREN/IRON. IREN/IRON is not specially interesting in this case, but the theorem still holds.

□

5.2 Lower Bounds for Maximum Source Broadcast Rate

We are now able to establish bounds for the maximum source broadcast rate based on Th. 7.

5.3 Generic Bound using Inequalities on Minkowski Sums

Assume that we have the following property for X, Y specific subsets of $\mathbb{Z}^n$:

$$|X \oplus Y| \geq f(|X|, |Y|) + |X| + |Y| \tag{13}$$

where f is a function which is monotonically increasing both in $|X|$ and $|Y|$. Different bound functions f and different conditions of the sets X, Y exists, see in section B.1, the Brunn-Minkowski style inequalities Property 22, Property 23 or Property 21.

Assume that R, and every S and T (for (S, T) in $\Omega_{\text{in}} \cup \Omega_{\text{out}}$) would verify the conditions for Eq. 13 to be true.

Then we have:

$$\begin{aligned} b_{\text{out}}(X) &= |X \oplus R| - |X| \text{ (by def.)} \\ &\geq f(|X|, |R|) + |R| \end{aligned} \tag{14}$$

which depends only on the *size* of X or R.

Consider a network satisfying the conditions of Th. 7. Then starting from the definition of M_{out}:

$$\begin{aligned} M_{\text{out}} &= \min\{b_{\text{out}}(T) : (S, T) \in \Omega_{\text{out}}\} \\ &\overset{(a)}{\geq} |R| + \min\{f(|T|, |R|) : (S, T) \in \Omega_{\text{out}}\} \\ &\overset{(b)}{\geq} |R| + f(\min\{|T| : (S, T) \in \Omega_{\text{out}}\}, |R|) \\ &\overset{(c)}{\geq} |R| + f(|H_t|, |R|) \end{aligned}$$

(a) is because of Eq. 14, (b) is because f is monotonically increasing, (c) is because by definition of Ω_{out}, every T in Ω_{out} must include H_t.

Applying the same reasonning to M_{in}, we have:

$$M_{\text{in}} \geq |R| + f(|H_s|, |R|)$$

Denote $h \triangleq \min(|H_t|, |H_s|)$. Then Th. 7 can be further written as:

$$C_{\text{max-flow}} \geq |R| + f(h, |R|) \tag{15}$$

Notice how the previous results can be recovered from this equation. Using the property Eq. 20, we have a bound function f_0, which is $f_0(|X|, |Y|) = -1$ for all X, Y non-empty. (We also set $H_s = \{s\}$ and $H_t = \{t\}$). This yields: $C_{\text{max-flow}} \geq |R| - 1$, which is Th. 1.

5.4 Bound for small H_s and H_t

The inequality Eq. 21 yields a bound function f_R which is:

$$f_R(|X|, |Y|) \triangleq (n-1)\min(|X|, |Y|) - \frac{n(n+1)}{2}$$

The condition on X, Y is that $\dim(X \oplus Y) = n$. It is always true because, as we use it, one X or Y is R and if the condition were not true, it could be shown that the network is not connected.

For small H_s and H_t, that is, for $h \leq |R|$, we have:

$$C_{\max-\text{flow}} \geq |R| + (n-1)h - \frac{n(n+1)}{2} \tag{16}$$

For dimension $n = 2$, Eq. 16 is never less tight than Th. 1. For dimension $n = 3$, Eq. 16 is tighter than Th. 1 for $h \geq 3$. We observe:

The bound in Eq. 16 increases by $(n-1)$ every time h is increased by 1.

5.5 Bound when H_s and H_t are the neighborhood of s and t

Consider the case where $H_s = \mathcal{N}(s)$ and $H_t = \mathcal{N}(t)$. Then $h = |H_s| = |H_t| = |R|$, and this is the largest neighborhood size for which the results of previous section may be applied. We get from Eq. 16:

$$\boxed{C_{\max-\text{flow}} \geq n|R| - \frac{n(n+1)}{2}} \tag{17}$$

which is noticeably better than Th. 1, essentially by the constant factor n.

5.6 Bound for large H_s and H_t

The inequality Eq. 22 yields a bound function f_{GG} which is, for $|X| \geq |Y|$:

$$f_{GG}(|X|, |Y|) \triangleq (n-2)|Y| - \frac{n(n-1)}{2} + (|X| - n)^{(n-1)/n}(|Y| - n)^{1/n}$$

For large H_s and H_t, that is such that $h \geq |R|$, we have:

$$C_{\max-\text{flow}} \geq (n-1)|R| + (h-n)^{(n-1)/n}(|R| - n)^{1/n} - \frac{n(n-1)}{2}$$

5.7 Consequences on Network Coding Broadcast Protocols

The previous results, and most notably section 5.5, have some implications for the decoding process, and for some protocols. Using IREN/IRON and random linear coding, the source could *asymptotically* approach the maximum broadcast rate Th. 1, and the destination would still be likely to decode.

One issue is that if the rate of the source is set exactly to the maximum broadcast rate, then the exponent of the probability of decoding in the coding delay (established by Lun et al. in [22]) becomes 0 (i.e. no guarantee to decode, no matter how long is the transmission). Hence, in practice, one approach is to set the source rate strictly below the maximum transmission rate M_R.

Another approach is to observe that from Eq. 17, the min-cut between the source s and one destination t is n times larger than the maximum source broadcast rate M_R ; $n = 2$ for the plane. Hence even when setting the transmission rate of the source exactly to M_R, proper decoding could be guaranted provided that 1) the source and its neighbors would exchange sufficiently well coded packets with their neighborhood, 2) the neighbors of the destination would do the same with the destination. This scenario could be achieved for instance by a control protocol, exchanging the state of the nodes within their neighborhood, and a slight increase of the transmission rate around the source and the destination. Then the regions around the source and the destination (i.e. the broadcast process from H_s to H_t), would operate as if it were transmitting at a fraction of the maximum broadcast rate $\frac{1}{n}$, hence with good probabilities of decoding.

To a large extent, this approach can be related to the one chosen in DRAGONCAST [7]: there, the protocol ensures that the information received in one node (rank of received coded packets) is kept in line with the information of neighbors.

Finally, we conclude these observations with one consequence of the results ofsection 5.4. Just setting $h = 2$ (one single neighbor of t transmits instantly with t, another node repeats the information from the source s) even, yields a capacity bound which is $M_R + 1$ when $n = 2$ hence a non-zero decoding error exponent, even when the rate of the source is exactly M_R.

5.8 Consequences on Random Networks

6 Extensions

In the article, results were presented for some families of networks, and with some assumptions; in this section, we informally discuss how the results could be extended in different conditions.

6.1 Generalization to Arbitrary Radio Shapes and Dimension

Some of the previous results were discussed for unit-disk graph networks, where the neighborhood of one node is the set of nodes within a ball.

However many of the results for the integer lattice apply for arbitrary shapes of R provided that the set of requirement 1 (section 2.4) is verified. Specifically, Th. 1 (maximum broadcast rate of IREN/IRON), the property of local optimality in the core (section 3.2), and Th. 3 (maximum broadcast rate of RAUDS) apply directly to any such shape R, and any dimension.

It is also trivial to extend the results of energy efficiency, global optimality (Th. 2, section 3.3.2), by considering the following property: in a space of any dimension n, for a network inside an hypercube of edge length L, the set of nodes in the border of the network will be a fraction $O(\frac{1}{L})$ of the nodes (for any R). As a result their contribution to the energy cost converge towards 0 when $L \to \infty$, and the global optimality derives from local optimality again.

For arbitrary shape of networks, in [3], we actually proposed and proved the results for a more general version of RAUDS, where the network could be within an arbitrary shape (and where there could be "holes" inside the network).

For random unit disk graphs, the results rely on the rate selection MA-RAUDS (section 3.5), which assumes a unit disk graph model where neighbors are nodes within an region R that is a ball (see section 3.5.2). This property is used in the proof at the key step around Eq. 10: it can be replaced by the requirement that the region R may be filled by specific sets of balls of fixed radius (and arbitrarily, closely as the radius converges towards 0, i.e. the region should not be "fractal" for some definition of fractal).

For the energy-efficiency of MARAUDS, it is not immediate to find an equivalent of Eq. 6 for arbitrary dimension or arbitrary neighborhood set R. However the following property of global optimality $E[\text{Eff}_{\text{bound}}] \xrightarrow{p} 1$ is easily derived as for RAUDS.

The bounds of the coding gain of section 4 are however not easily generalized, mostly because the optimal energy-efficiency of routing is not well-known. The coding gain is discussed in section 6.2.

6.2 Coding Gain in Arbitrary Dimension

In previous section 6.1, we surveyed how previous results are generalized. Essentially, for any dimension, it can be easily generalized that network coding is optimal in the sense that $E_{\text{bound}} \rightarrow 1$.

6.3 Lower Bound of Coding Gain from Connexity Constraint

The lower bound based on the connexity constraint of section 4 is easily generalized. It follows from the computation of the intersection of two hyperspheres (see Fig. 7(a)).

In section B.3, we reproduce the computation of volume $V_n(r)$ of an hypersphere, and the volume $W_n(r)$ of intersection of two hyperspheres.

The lower bound for the coding gain G_{lower} is then:

$$G_{\text{n,lower}} = \frac{V_r(n)}{V_r(n) - 2W_r(n)} = \frac{1}{1 - 2\frac{J_n}{I_n}}$$

where I_n, J_n may be computed with the recursive equations:

- $I_0 = \pi$; $I_1 = 2$; $I_n = \frac{n-1}{n} I_{n-2}$
- $J_0 = \frac{\pi}{3}$; $J_1 = \frac{1}{2}$; $J_n = \frac{1}{n}((n-1)J_{n-2} - (\frac{\sqrt{3}}{2})^n)$

6.3.1 Bounds of Coding Gain from Sphere Coverings

We are search for upper bounds of connected dominating sets.

We start from the simpler problem of "dominating sets" in Euclidean space of dimension n. With the unit disk graph model, finding the most efficient dominating set is also know as "optimal sphere covering" in dimension n. An exploration of the topic of (specific) sphere coverings has been proposed, for instance, by F. Valletin in [29] (for low dimensions). He reports that, except for the trivial case of dimension 1, only for dimension 2, the optimality of a sphere is proven at the moment.

Schürmann, Vallentin, and Sikiric give a table of the least dense known lattice coverings [2] (in [29] p.109, a table of the least dense known lattice coverings in dimensions up to dimension 24, noting that *"at the same time this list gives the least dense known sphere coverings in dimensions up to 24 since there is no covering of equal spheres known which is better than the best known lattice covering"*).

The density is defined as a quantity corresponding to the number of spheres that cover one random point on average ; it is equivalent to the coding gain (valid if the sphere covering is connected), since we have shown that network coding is asymptotically optimal.

It is not hard to see that if a sphere covering is known for a bounded subset of a space of dimension n (yielding a dominating set), one may transform it into a *connected* dominated set, by adding iteratively at most one connecting sphere for each sphere of the covering.

Hence the upper bound:

$$G_{\text{n,upper}} = 2\mu_n$$

where μ_n is the density of one known sphere covering (starting from the list in [2]).

For also large dimensions,the bounds have been established:

- Coxeter, Few and Rogers provide a lower bound τ_n in [9], (we used the formulation given by Conway and Sloane [8] for computing numerical results in next section). The asymptotic behavior of τ_n is caracterized by: $\tau_n \sim \frac{n}{e^{3/2}}$
- Dumer proves a upper bound in [12] in dimension $n \geq 3$, for covering of larger spheres by unit spheres. It is proven that the bound on density μ_n^* verifies:

$$\mu_n^* \leq (\frac{1}{2} + \frac{2 \log \log n}{\log n} + \frac{5}{\log n}) n \log n$$

  which is $(\frac{1}{2} + o(1)) n \log n$

Notice that $\tau_n \to \infty$ when $n \to \infty$. τ_n is the number of spheres covering one point in space on average ; hence ideally, one sphere could be used to connect τ_n spheres at the same time (instead of just two). In the most optimistic scenario, the connecting spheres might represent a fraction as low as $\frac{1}{\tau_n}$ which converges towards 0. In reality, one cannot connect arbitrarily different spheres at will, hence some redundancy is expected: as a result, the asymptotic relative cost of connecting spheres appears to be an open question (it is between 0 and 1).

6.3.2 Results of Bounds of Coding Gain for Various Dimensions

The table 1 represents numerical results from the bounds presented in previous dimensions.

When dimension grows, the behavior is the following:

- The connexity constraint is less and less costly ; in fact the lower bound converges towards 1, as it may be seen from the expression of I_n and J_n [5].
- On the other hand, it becomes harder to cover space with spheres (from the lower bound in section 6.3.1).

Hence the coding gain of network coding grows infinitely for the main reason that when the dimension of the space n increases, because space covering with spheres becomes inefficient.

It might be possible to envision the use of several wireless technologies as generally having as similar effect of increasing the dimension n; netherless the

[5] see section B.3: J_n integrates on a smaller interval than I_n - in the range where $\sin^n(\theta)$ that converges faster towards 0

Dimension	Lower Bound from connexity $G_{\text{n,lower}}$	Lower Bound from sphere covering [9]	Upper Bound from known sphere coverings $G_{\text{n,upper}}$	Upper Bound from [12]
2	$6\,\frac{\pi}{2\,\pi+3\,\sqrt{3}} \approx 1.6420$	1.2092	2.4184	-
3	$\frac{16}{11} \approx 1.4545$	1.4323	2.9270	34.4244
4	$12\,\frac{\pi}{4\,\pi+9\,\sqrt{3}} \approx 1.3390$	1.6584	3.5311	50.7713
5	$\frac{256}{203} \approx 1.2611$	1.8852	4.2486	67.5649
6	$30\,\frac{\pi}{10\,\pi+27\,\sqrt{3}} \approx 1.2055$	2.1121	4.9296	84.7473
7	$\frac{2048}{1759} \approx 1.1643$	2.3387	5.8000	102.2618
8	$840\,\frac{\pi}{280\,\pi+837\,\sqrt{3}} \approx 1.1329$	2.5650	6.2844	120.0627
9	$\frac{65536}{59123} \approx 1.1085$	2.7910	8.5372	138.1140
10	$840\,\frac{\pi}{280\,\pi+891\,\sqrt{3}} \approx 1.0892$	3.0168	10.3089	156.3871
11	$\frac{524288}{488293} \approx 1.0737$	3.2424	11.0112	174.8589
12	$660\,\frac{\pi}{220\,\pi+729\,\sqrt{3}} \approx 1.0612$	3.4677	14.9310	193.5102
...	...	...	...	...
n	$1+o(1)$	$\tau_n \sim \frac{n}{e^{3/2}}$	-	$2 \times (\frac{1}{2}+o(1))n \log n$

Table 1: General bounds of coding gain: numerical results in various dimensions. The section 4 and section 6.3.3 present tighter bounds for the practical cases of $n = 2$ and $n = 3$

practically useful results from the table are mostly the results for dimension 3 (dimension 2 was addressed previously).

6.3.3 Further Results for Coding Gain in Dimension 3

In this section, we propose a tighter upper bound for the practical case of dimension 3, and discuss the bounds.

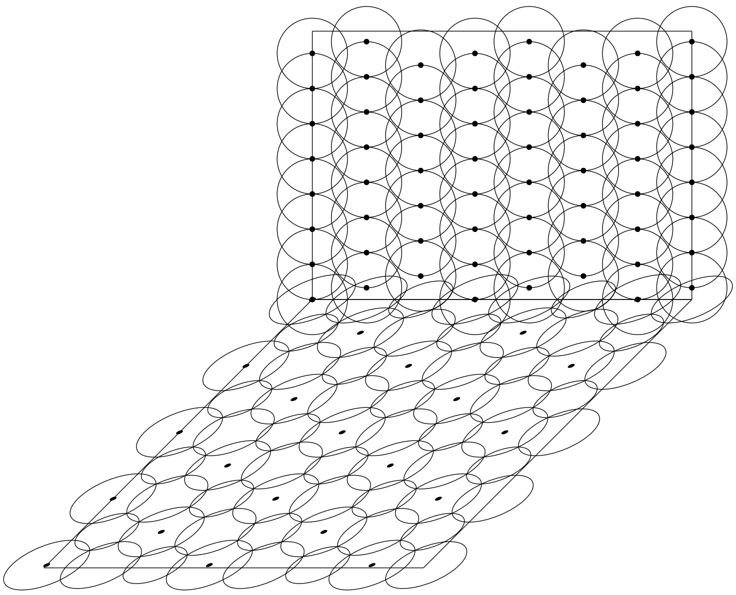


Figure 9: Proposed connected dominating set for our upper bound in dimension 3

Our improved proposed upper bound is based on the following covering of the 3-dimensional Euclidean space:

- The spheres have radius 1
- The centers of the spheres are on the 3-dimensional lattice generated by the 3 vectors

$$v_1 = \begin{pmatrix} \tau \\ 0 \\ \frac{1}{3} \end{pmatrix}, v_2 = \begin{pmatrix} \frac{1}{2}\tau \\ \frac{1}{2}\tau\sqrt{3} \\ \frac{2}{3} \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Precisely the set of the spheres centers is:

$$\left\{ \begin{pmatrix} \tau(i + \frac{1}{2}j) \\ \frac{\sqrt{3}}{2}\tau j \\ \frac{1}{3}((i+2j) \mod 3) + k \end{pmatrix} : (i,j,k) \in \mathbb{Z}^3 \right\}$$

- It can be transformed in a full connected dominated set by adding $O(L^2)$ nodes, when the covered space is a sphere of radius L (or a cube of width L) ; as a result the fraction of the additional spheres necessary to connect the covering $\rightarrow 0$ when $L \rightarrow \infty$.

The Fig. 9 represents two planar cuts and illustrates the construction of the covering. It starts from an hexagonal covering of the plane ($z = 0$, horizontal plane on Fig. 9). That covering is extended on the $z-$axis with strings of connected spheres (as shown on the vertical plane on Fig. 9).

Additionally, in a manner similar to the CDS proposed in section 4, the strings of connected spheres are shifted on the z axis by a given amount (0, 1/3 or 2/3), depending on there position x, y. The underlying idea is the following: consider the hexagons implied in the covering in the horizontal plane ; they are classically colored with 3 colors, so that two neighbors have always different colors ; the amount of shift on the z-axis of the string of spheres associated with one hexagon is based on the color of the hexagon (hence: 0, 1/3 or 2/3). The intent is that any point of the space will be surrounded by 3 spheres with different shiftings ; as for the previously proposed 2-dimensional CDS (on Fig. 7(b)), this allows for a less dense CDS than if no shiftings were present.

We used symbolic computation software to find the maximal value τ, for which we still have a covering ; the value was found to be: $\tau = \frac{1}{3}\sqrt{11 + 6\sqrt{3}}$.

The number of sphere centers per unit volume is:

$$d_c = \frac{2}{\tau^2\sqrt{(3)}}$$

Our proposed upper bound of the coding gain is the gain over this covering and is:

$$G^*_{3,\text{upper}} = \frac{4}{3}\pi d_c = \frac{8\pi\sqrt{3}}{11 + 6\sqrt{3}} \approx 2.0349\ldots$$

With respect to the lower bounds for dimension 3, presented in table 1, we observe that the lower bound from connexity and the lower bound from sphere covering are close from each other. What would be a lower bound when both sphere covering and connexity are considered together (and not independently) is an open question, but one may hypothesize it would be noticeably higher than these bounds taken individually - because both issues tend add to each other.

On the other hand, our proposed 3-dimensional covering uses essentially the same method that yield an efficient CDS in dimension 2 (extend a covering in dimension $n-1$ with strings of connected spheres). For these reasons, it is possible that the coding gain with optimal CDS might be closer to $G^*_{3,\text{upper}}$ than to the previous lower bounds.

The upper bound of this section, combined with the lower bound from connexity yields the following bounds for the coding gain of network coding in dimension 3:

$$1.4323 \approx \frac{16}{11} \leq \begin{array}{c}\text{coding gain}\\ \text{in dim 3}\end{array} \leq \frac{8\pi\sqrt{3}}{11+6\sqrt{3}} \approx 2.0349 \qquad (18)$$

6.4 Impact of Multiple Sources

In general, coding separately several sources may yield an inferior performance than coding the output of the different sources together. However, in our case, since the methods are both locally optimal and, asymptotically, globally optimal, these results directly extend to several sources, even if they are independently coded.

6.5 Impact of Uniformity

In our models, with lattices and random unit disk graphs, a constant underlying theme is that the networks are uniform: most nodes have the same neighborhood, and for those that do not, an adaptation is performed (nodes on the border). For random unit disk graphs with non-uniform density, MARAUDS is using a subset of nodes that is, in fact, uniform (through the mapping of a grid).

One question is: can the result be extended to networks where the neighborhood is not uniform. [26] provides some elements of answers: for random unit disk graphs, and with a simple local adaptation heuristics, the heuristics perform well, but not perfectly. For this reason, we hypothesize that the uniformity of the network is paramount to our results and that deviation from uniformity would imply a deviation from local optimality.

6.6 Maximum Capacity and Scheduling

In this article, energy efficient broadcast was addressed, but not maximum capacity. As a result, with proper scaling of the source rate, the issue of the limited capacity of the wireless medium disappeared.

If we consider the properties of the rate selections (IREN/IRON, RAUDS, and MARAUDS), it appears that they set an identical rate on nodes of the core of the network, but have a higher rate on special nodes: the border nodes and the source. Hence, for maximum capacity, we conjecture that the bottlenecks would be such nodes: At the source for instance, the channel use (counting transmission of the node and of its one-hop neighbors) is approximately twice the channel use of a node in the core of the network; at the border nodes with RAUDS, the channel use is also higher, even if fewer nodes are present in their neighborhood.

Because our methods are not optimal on the border, they cannot answer the problem of maximum capacity of network coding, and of the scheduling of the transmissions in such network.

6.7 Impact of Radio Range

In a network where nodes may emit with different power levels (defining different neighborhoods), one question is, what would be the most energy efficient power level?

In our case, the energy efficiency increases linearly with the number of neighbors, hence the determining factor is, how does the number of neighbors scale with the energy cost of transmissions? Fragouli et al. [15] studied this problem. If we consider a planar network in a three-dimension space, then generally the number of neighbors will scale sub-linearly. Hence, in that case, the radio range should be minimized.

7 Conclusion

In order to characterize the performance of broadcast in wireless networks, we have presented methods of rate selection for network coding in multi-hop wireless networks. The logic behind these rate selections was described. The methods are based on setting the same rate on every node, compensating for the effect of the borders (nonexistent neighbors) and adjusting for non-uniform density of the network.

Proofs were given for their performance and rely on the proof of maximum broadcast rate of the source. We proved that IREN/IRON and RAUDS will operate with local optimality, hence that network coding in general could operate optimally from a local point of view. When the size of the network increases, we have established the global optimality of the methods.

For random unit disk graphs, MARAUDS will asymptotically achieve optimal energy efficiency when the area and density of the network would grow indefinitely. This is used to derive upper and lower bounds of the gain of network coding in general compared to routing under these hypothesis: between 1.642 and 1.684 (in Eq. 12) for the plane, and bounds of 1.432 and 2.035 for networks of the euclidean space of dimension 3.

For dense networks in a fixed square region of the plane, an estimate was given for the condition for expecting the rate selection MARAUDS to outperform any method using routing. It is the case when the radio range is smaller than $\frac{1}{20}$ of the edge length of the square.

We also provide bounds for the theoretical case of larger dimensions.

Finally, in extensions of our results, we prove that if the source or if one of the destination are enlarged to set of nodes, the max-flow between there areas will be increased.

A Practical Operation of Random Linear Coding

In random linear coding [19], the remaining decision is *when* to send packets. In this article, we consider rate selections with fixed rates for every node: C_v is the rate of the node $v \in \mathcal{V}$. Then, random linear coding operates as indicated on algorithm 1.

Algorithm 1: Random Linear Coding with Rate Selection

1.1 **Source scheduling:** the source transmits periodically coded packets (vectors) with rate C_s.
1.2 **Nodes' start and stop conditions:** The nodes start transmitting when they receive the first vector but they continue transmitting until themselves **and their current neighbors** have enough vectors to recover the D source packets.
1.3 **Nodes' scheduling:** every node v retransmits linear combinations of the vectors that it has, and waits for a delay computed from the fixed rate before the next transmission.

With this scheduling of Algorithm 1, the adjustable parameter is the delay between to transmissions, and for fixed rates, it is delay $= 1/C_v$.

Notice that we assume that the source transmits coded packets. In an extreme example, the source could repeatedly transmit the same coded packet, hence another relevant parameter is the innovative packet source rate: in general it should be the same as the source rate C_s in a network without losses ; in any case, the other nodes will be able to decode the broadcast, only if the innovative packet source rate is lower or equal to the maximum broadcast source rate for the network.

B Proof of the Main Discrete Geometry Property on Neighborhood

B.1 Preliminaries: Minkowski sums

The properties are based on the use of the Minkowski addition and specific properties of discrete geometry Eq. 20 below. The Minkowski addition is a classical way to express the neighborhood of one area.

Given two sets A and B of $\mathbb{R}^n$, the Minkowski sum of the two sets $A \oplus B$ is defined as the set of all vector sums generated by all pairs of points in A and B, respectively:

$$A \oplus B \triangleq \{a + b \mid a \in A, b \in B\} \tag{19}$$

Then the closed set of neighbors $\mathcal{N}[t]$ of one point t can be redefined in terms of the Minkowski sum: $\mathcal{N}[t] = \{t\} \oplus R$.
This extends to the neighborhood of subsets: $\mathcal{N}[A] = A \oplus R$

For Minkowski sums on the lattice $\mathcal{L}$, there exist variants of the *Brunn-Minkowsky inequality*, including the following ones found for instance in Gardner and Gronch [16] (which include a survey of some known inequalities, and prove a new one):

Property 2. *Inequality in given [16]:*
For two non-empty subsets A, B of the integer lattice $\mathbb{Z}^n$,

$$|A \oplus B| \geq |A| + |B| - 1 \tag{20}$$

The previous inequality (valid for any non-empty set A, B) is sufficient for most of our results (except section 5).

More refined results may be used, in fact:

Property 3. *Ruzsa proved [25]:*
For two finite subsets A, B of the integer lattice $\mathbb{Z}^n$, with $|B| \leq |A|$ and $dim(A \oplus B) = n$, then

$$|A \oplus B| \geq |A| + n|B| - \frac{n(n+1)}{2} \tag{21}$$

Property 4. *Gartner and Gronch proved (th. 6.5 and th 6.6 of [16]):*

For two non-empty finite subsets A, B of the integer lattice $\mathbb{Z}^n$, with $|B| \leq |A|$ and $dim(B) = n$,

$$|A \oplus B| \geq |A| + (n-1)|B| - \frac{n(n-1)}{2} + (|A| - n)^{(n-1)/n}(|B| - n)^{1/n} \tag{22}$$

For two non-empty finite subsets A, B of the integer lattice $\mathbb{Z}^n$, with $dim(B) = n$,

$$|A \oplus B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}}(|B| - n)^{1/n} \tag{23}$$

B.2 Main Theorem:

In section B.2.1, we prove the following theorem:

Theorem 8. *If X and Y are a partition of the integer lattice $\mathcal{L}$ $(= \mathbb{Z}^n)$, and X is finite, then the following two properties Eq. 24 and Eq. 25 are verified:*

$$\text{border}(X) = X \text{ or } |\text{border}(X)| \geq b_{\text{in}}(X) \tag{24}$$

$$|\text{border}(Y)| \geq b_{\text{out}}(X) \tag{25}$$

where:
- $b_{\text{in}}(X) \triangleq |\text{inner}(X) \oplus R| - |\text{inner}(X)|$,
- $b_{\text{out}}(X) \triangleq |X \oplus R| - |X|$,
- *R is the set of neighbors of the origin point*

Denote $M_R \triangleq |R| - 1$. Notice that the previous theorem implies:
- in all cases, Eq. 20 applied to b_{out} implies that $b_{\text{out}}(Y) \geq M_R$

- in the case where $border(X) \neq X$, which occurs iff $inner(X) \neq \emptyset$ (see proof in next section), Eq. 20 also applies and we get:
$b_{\text{in}}(X) \geq M_R$

This yields the theorem in the following less general form:

Theorem 9. *If X and Y are a partition of the integer lattice $\mathcal{L}$ $(= \mathbb{Z}^n)$, and X is finite, then the following two properties Eq. 26 and Eq. 27 are verified:*

$$\text{border}(X) = X \text{ or } |\text{border}(X)| \geq M_R \tag{26}$$

$$|\text{border}(Y)| \geq M_R \tag{27}$$

where R is the set of neighbors of the origin point, and $M_R = |R| - 1$

It was presented in [3] (Th.5 of [3]), and represented on Fig. 10

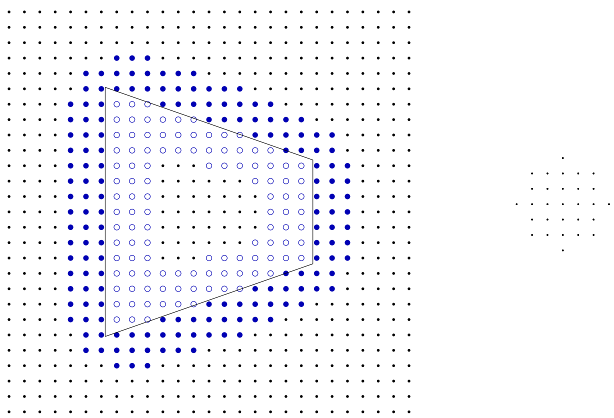


Figure 10: Illustration of Th. 9

An illustration of the Th. 9 is represented on Fig. 10: on the left part, X is the set of points inside the polygon, whereas Y is the rest of the (infinite) integer lattice, only a part of which is represented here. border(X) is the set of clear large points (inside the polygon) ; border(Y) is the set of dark large points (outside the polygon). In this case, we have $|\text{border}(X)| \geq M_R$ and $|\text{border}(Y)| \geq M_R$. For reference, R is represented on the right part ($M_R = 28$).

B.2.1 Proof for Th. 8

Proof. We start by proving Eq. 24 of Th. 8

Consider the set inner(X). If it is empty, by definition of inner(X), all points of X must be neighbors of points of Y: $X = \text{border}(X)$. This implies Eq. 24 of Th. 8 (as a side note, conversely, if $X \neq \text{border}(X)$, at least one point of X is not neighbor of Y, hence inner$(X) \neq \emptyset$).

Otherwise, inner$(X) \neq \emptyset$. Again by definition we have:

$$\begin{aligned} \mathcal{N}[\text{inner}(X)] \setminus \text{inner}(X) &\subset X \setminus \text{inner}(X) \\ &\subset \text{border}(X) \text{ (by def.)} \end{aligned}$$

Therefore:

$$|\text{border}(X)| \geq |\mathcal{N}[\text{inner}(X)] \setminus \text{inner}(X)|$$

For the second part of this inequality, we have:

$$\begin{aligned}|\mathcal{N}[\text{inner}(X)] \setminus \text{inner}(X)| &\geq |\mathcal{N}[\text{inner}(X)]| - |\text{inner}(X)| \\ &\overset{(a)}{\geq} |\text{inner}(X) \oplus R| - |\text{inner}(X)| \\ &\overset{(b)}{\geq} b_{\text{in}}(X)\end{aligned}$$

(a) is by rewriting neighborhood with a Minkowski sum, (b) is from the definition of b_{in}. Eq. 24 follows.

• For Eq. 25 of Th. 8: By definition, border(Y) includes all points $y \in Y$ that are neighbors of points of X. Hence:

$$\begin{aligned}\mathcal{N}(X) &\subset \text{border}(Y) \\ \mathcal{N}[X] \setminus X &\subset \text{border}(Y) \text{ by def. of } \mathcal{N}[X]\end{aligned}$$

Therefore:

$$|\text{border}(Y)| \geq |\mathcal{N}[X] \setminus X|$$

The second part of the equation can be written:

$$\begin{aligned}|\mathcal{N}[X] \setminus X| &\geq |\mathcal{N}[X]| - |X| \\ &\overset{(a)}{\geq} |X \oplus R| - |X| \\ &\overset{(b)}{\geq} b_{\text{out}}(X)\end{aligned}$$

(a) is by rewriting neighborhood with a Minkowski sum, and (b) is by definition. This proves Eq. 25 of Th. 8. □

B.2.2 Corollary for Finite Networks

When considering finite networks, the difficulty is that Th. 8 is proven for partitions of $\mathcal{L}$ the infinite lattice[6], not for partitions of $\mathcal{V}$, which is a finite subset of $\mathcal{L}$. We have the following extension of the results as Th. 10 for the case represented on Fig. 11:

[6]it is false in general for partitions of finite sets ; it is also false for finite torus lattices for pathological R

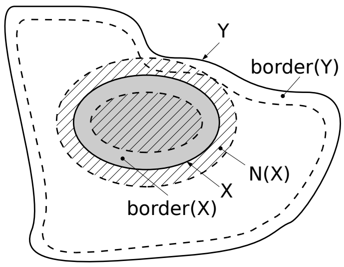


Figure 11: Illustration of Th. 10

Theorem 10. *If X and Y are finite subsets of the integer lattice $\mathcal{L}$, with $X \subset Y$ and $X \cap \text{border}(Y) = \emptyset$, then the following two properties Eq. 28 and Eq. 29 are verified:*

$$\text{border}(X) = X \text{ or } |\text{border}(X)| \geq b_{\text{in}}(X) \tag{28}$$

$$\mathcal{N}(X) \subset Y \text{ and } |\mathcal{N}(X)| \geq b_{\text{out}}(X) \tag{29}$$

where:

- $b_{\text{in}}(X) \triangleq |\text{inner}(X) \oplus R| - |\text{inner}(X)|$,
- $b_{\text{out}}(X) \triangleq |X \oplus R| - |X|$,
- *R is the set of neighbors of the origin point*

In particular, this implies that the following two properties Eq. 30 and Eq. 31 are both true:

$$\text{border}(X) = X \text{ or } |\text{border}(X)| \geq M_R, \tag{30}$$

$$\mathcal{N}(X) \subset Y \text{ and } |\mathcal{N}(X)| \geq M_R \tag{31}$$

where $M_R = |R| - 1$.

Proof: 7. *Denote $\bar{X} \triangleq \mathcal{L} \setminus X$ the complementary of the set X in $\mathcal{L} = \mathbb{Z}^n$: we can apply Th. 8 to the partition X'/Y' of $\mathcal{L}$ with $X' = X$ and $Y' = \bar{X}$.*

We get:

- *Eq. 24 directly yields Eq. 28 (since $X = X'$)*
- *Eq. 25 yields: $|\text{border}(Y')| \geq b_{\text{out}}(X)$*

The latest inequality combined with Eq. 5 yields $|\mathcal{N}(X)| \geq |b_{\text{out}}(X)|$, and the only statement that remains to be verified is: is $\mathcal{N}(X) \subset Y$?

By contradiction: assume that $\mathcal{N}(X) \not\subset Y$, and let u be one of the points $u \in \mathcal{N}(X) \backslash Y$. By definition, it is u is neighbor of at least one point $v \in X$. Since $X \subset Y$, v is also a point of Y. Because v is neighbor of $u \notin Y$, by definition of $\text{border}(Y)$, *$v \in \text{border}(Y)$. v being also in X, we have: $v \in X \cap \text{border}(Y)$. This is impossible because by assumption $X \cap \text{border}(Y) = \emptyset$, hence this contradiction proves that $\mathcal{N}(X) \subset Y$. Hence Eq. 29 is proven.*

Eq. 24 and Eq. 25 are also proven by following the same steps, but using the more specific Th. 9 in place of Th. 8.

The theorem follows.

B.3 Volume of Hyperspheres and of Their Intersection

In [20], A. E. Lawrence describes a recursive method for computing the volume of an hypersphere with elementary calculus, which is summarized here:

- The volume of an hypersphere of radius r in Euclidean space of dimension n has the form $V_n = \alpha_n r^n$
- V_n may be computed recursively from:

$$V_n(r) = r \int_{\theta=0}^{\theta=\pi} (\sin\theta) V_{n-1}(r \sin\theta) d\theta$$

  which implies:
  - $\alpha_n = I_n \alpha_{n-1}$
  - with $I_n = \int_{\theta=0}^{\theta=\pi} \sin^n \theta d\theta$
- Using integration by parts, a relation between I_n and I_{n-2} is found as:

$$I_n = \frac{n-1}{n} I_{n-2}$$

- And by direct computation, $I_0 = \pi$; $I_1 = 2$.

Following the same method as [20], the volume $W_n(r)$ of the intersection of the hypersphere and the region defined by $\{(x_1, \ldots, x_n) | x_1 \geq \frac{r}{2}\}$, may be computed as:

- $W_n(r)$ has the form $V_n(r) = \beta_n r^n$
- V_n may be computed recursively from:

$$W_n(r) = r \int_{\theta=0}^{\theta=\pi/\mathbf{3}} (\sin\theta) V_{n-1}(r \sin\theta) d\theta$$

  (notice the new integration upper limit) which implies:
  - $\beta_n = J_n \alpha_{n-1}$
  - with $J_n = \int_{\theta=0}^{\theta=\pi/3} \sin^n \theta d\theta$
- Using integration by parts, a relation between J_n and J_{n-2} is found as:

$$J_n = \frac{1}{n}((n-1)J_{n-2} - (\frac{\sqrt{3}}{2})^n)$$

- And by direct computation, $J_0 = \frac{\pi}{3}$; $J_1 = \frac{1}{2}$.

References

[1] J. Shi A. Ramamoorthy and R. D. Wesel. On the capacity of network coding for random networks. *IEEE transactions on information theory*, 51:2878–2885, August 2005.

[2] Mathieu Dutour Sikiric Achill Schürmann, Frank Vallentin. A generalization of voronoi's reduction theory and its application. *Duke Math. J.*, 142(1):127–164, Aug 2008.

[3] Cédric Adjih and Song Yean Cho. Wireless Broadcast with Network Coding: A Connected Dominating Sets Approach. Research Report RR-6547, INRIA, June 2008.

[4] C. Colbourn B. Clark and D. Johnson. Unit disk graphs. *Discrete Mathematics*, 86, December 1990.

[5] Xiaole Bai, Santosh Kumar, Dong Xuan, Ziqiu Yun, and Ten H. Lai. Deploying wireless sensors to achieve both coverage and connectivity. In *MobiHoc '06: Proceedings of the 7th ACM international symposium on Mobile ad hoc networking and computing*, pages 131–142, New York, NY, USA, 2006. ACM.

[6] S. Y. Cho C. Adjih and P. Jacquet. Near optimal broadcast with network coding in large sensor networks. In *In Proceedings of the first workshop on Information Theory for Sensor Networks*, June 2007.

[7] S. Y. Cho and C. Adjih. Network coding for wireless broadcast: Rate selection with dynamic heuristics. In *INRIA Research Report RR-6349*, November 2007.

[8] J. Conway and N. Sloane. A lower bound on the average error of vector quantizers (corresp.). *Information Theory, IEEE Transactions on*, 31(1):106–109, Jan 1985.

[9] H. S. M. Coxeter, L. Few, and C. A. Rogers. Covering space with equal spheres. *Mathematika*, 6:147–157, 1959.

[10] R. Koetter D. S. Lun, M. Medard and M. Effros. Further results on coding for reliable communication over packet networks. In *Internatoinal Symposium on Information THeory (ISIT)*, September 2005.

[11] R. Koetter D. S. Lun, M. Medard and M. Effros. On coding for reliable communication over packet networks. In *Technical Report # 2741 MIT LIDS*, January 2007.

[12] Ilya Dumer. Covering spheres with spheres. *Discrete Comput. Geom.*, 38(4):665–679, 2007.

[13] J. Edmonds. *Edge-disjoint branches.* Academic Press, NY, 1973.

[14] C. Fragouli, J. Widmer, and J. Y. Le Boudec. A network coding approach to energy efficient broadcasting: From theory to practice. In *INFOCOM 2006. 25th IEEE International Conference on Computer Communications. Proceedings*, pages 1–11, apr 2006.

[15] C. Fragouli, J. Widmer, and J.-Y. Le Boudec. Efficient broadcasting using network coding. *Networking, IEEE/ACM Transactions on*, 16(2):450–463, April 2008.

[16] R.J. Gardner and P. Gornchi. A brunn-minkowski inequality for the integer lattice. *Transcations of the American Mathematical Society*, 353:3995–4042, 2001.

[17] S. Guha and S. Khuller. Approximation algorithm for connected dominating sets. In *In Proceedings of European Symposium on Algorithms*, pages 179–193, September 1996.

[18] T. Ho, D. Karger R. Koetter, M. Medard, and M. Effros. The benefits of coding over routing in a randomized setting. In *In Proceedings of IEEE International Symposium on Information Theory (ISIT)*, page 442, June 2003.

[19] T. Ho, M. Medard R. Kotter, J. Shi M. Effros, and D. Karger. A random linear network coding approach to multicast. *IEEE Transactions on Information Theory*, 52:4413–4430, October 2006.

[20] A. E. Lawrence. The volume of an n-dimensional hypersphere, 2009.

[21] D. S. Lun, M. Medard R. Rantnakar, D. R. Karger R. Koetter, E. Ahmed T. Ho, and F.Zhao. Minimum-cost multicast over coded packet networks. *IEEE/ACM Transactions on Networking*, 52(6):2608–2623, June 2006.

[22] D.S. Lun, M. Medard, R. Koetter, and Effros M. On coding for reliable communication over packet networks. *Physical Communication*, 1(1):3–20, March 2008.

[23] C. Fragouli M. Jafarisiavoshani and S. Diggavi. On subspace properties for randomized network coding. In *Information Theory Workshop (ITW)*, July 2007.

[24] S.-Y. R. Li R. Ahlswede, N. Cai and R. W. Yeung. Network information flow. *IEEE Transactions on Information Theory*, 46(4):1204–1216, July 2000.

[25] Imre Z. Ruzsa. Sum of sets in several dimensions. *Combinatorica*, 14(4):485–490, 1994.

[26] C. Adjih S. Y. Cho and P. Jacquet. Heuristics for network coding in wireless networks. In *In Proceedings of International Wireless Networks (WICON)*, October 2007.

[27] J. Meng S.A Aly, V. Kapoor and A. Klappenecker. Bounds on the network coding capacity for wireless random networks. In *Third Workshop on Network Coding, Theory and Applications (Netcod)*, January 2007.

[28] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, Inc., 1984.

[29] F. Vallentin. *Sphere Covering, Lattices, and Tilings (in Low Dimensions)*. PhD thesis, Technische Universität München, 2003.

[30] K. Voss. *Discrete Images, Objects and Functions in Z^n*. Springer-Verlag, 1993.

[31] P. A. Chou Y. Wu and S.-Y Kung. Minimum-energy multicast in mobile ad hoc networks using network coding. *IEEE Transactions on Communications*, 53(11):1906–1918, November 2005.

[32] J.Y. Yu and P.H.J. Chong. A survey of clustering schemes for mobile ad hoc networks. *Communications Surveys and Tutorials, IEEE*, 7(1):32–48, Qtr. 2005.

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