



# Variational approximation for detecting points-like target problems

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# *Variational approximation for detecting points-like target problems.*

Daniele Graziani, Gilles Aubert

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## Variational approximation for detecting points-like target problems.

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**Abstract:** The aim of this report is to provide a rigorous variational formulation for the detection of points in 2-d biological images. To this purpose we introduce a new functional of the calculus of variation whose minimizers gives the points we want to detect. Then we build an approximating sequence of functional, for which we prove the  $\Gamma$ -convergence to the initial one..

**Key-words:** points detection, biological images, divergence measure fields,  $p$ -capacity,  $\Gamma$ -convergence.

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Variational approximation for detecting points-like  
target problems.

18 **Résumé :**

**19 Mots-clés :**

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## 1 Introduction

The issue of detecting fine structures, like points or curves in two or three dimensional biological images, is a crucial task in image processing. In particular a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise. Therefore one of the main goals in the application is detecting the spots that the biologists wish to count. This operation is made harder by the presence of the other singular structures like noise, textures or filaments.

In some biological images the image intensity is a function that takes the value 1 on points or other structures like set with Hausdorff dimension  $0 \leq \alpha < 1$ , and it is close to 0 outside. In image processing these concentration sets are called discontinuity without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0.

In the literature there are few variational methods dealing with this problem. In this direction one interesting approach has been proposed in [5]. In that paper the authors consider these kind of pathology as a  $k$ -codimension object, meaning that they should be regarded as a singularity of a map  $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$ , with  $k \geq 2$  and  $m \geq 0$  (see [8] for a complete survey on this subject). So that the detecting point case corresponds to the case  $k = 2$  and  $m = 0$ .

This point of view makes possible a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map  $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ , where  $\mathbb{S}^1$  is a unit sphere of  $\mathbb{R}^2$ . This strategy makes crucial the construction, starting from the initial image  $I : \mathbb{R}^2 \rightarrow \mathbb{R}$ , of an initial vector field  $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$  with a topological singularity of degree 1, and how to build such a vector field in a rigorous way, is a subject of a current investigation.

Our first purpose here is finding a most natural variational framework in which a rigorous definition of discontinuity without jump can be given.

In our model the image  $I$  is a Radon measure, that is an element of  $(C_0(\Omega))^*$ : the dual space of  $C_0(\Omega)$  (the closure with respect to the sup-norm of the space of continuous functions with compact support). It is crucial for detecting points that this Radon measure be capable of charging points. The preliminary step is finding a space whose elements are capable of producing these kind of measures. This space is given by  $\mathcal{DM}^p(\Omega)$ : the space of the  $L^p$ -vector fields whose distributional divergence measure is a Radon measure, with  $1 < p < 2$ . The restriction on  $p$  is due to the fact that when  $p \geq 2$  the distributional divergence  $\text{Div}U$  of  $U$  cannot be a measure concentrated on points. (see section 3.1 below). Then we have to construct from the original image  $I$  a data vector field  $U_0 \in \mathcal{DM}^p(\Omega)$ . Clearly there are, at least in principle, many ways to do this. The one we propose here seems to be the most natural. We consider the classical elliptic problem with measure data  $I$ :

$$\begin{cases} \Delta u_0 = I & \text{on } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Then by setting  $U_0 = \nabla u_0$  we have  $U_0 \in \mathcal{DM}^p(\Omega)$  with  $\text{Div}U_0 = I$ . Nevertheless the support of the measure  $\text{Div}U_0$  is too large and could contains several structures like curves

or fractal sets, while the singularities, we are interested in, are contained in the atomic part of the measure  $\text{Div}U_0$  and therefore we have to isolate it. To do this the notion of  $p$ -capacity of a set plays a key role. Indeed when  $p < 2$  the  $p$ -capacity of a point in  $\Omega$  is zero and besides every Radon measure can be decomposed (see [17]) in two mutually singular measures: the first one is absolutely continuous with respect the  $p$ -capacity and the second one is singular with respect to the  $p$ -capacity, meaning that it is a measure concentrated on sets with 0  $p$ -capacity.

Moreover for a compact subset  $E \subset \Omega$  a necessary and sufficient condition to have 0  $p$ -capacity is the existence of a sequence of test functions  $\{\phi_k\}$  converging to 0 strongly in  $W_0^{1,p}$  and equals to 1 on  $E$  for every  $k$ , and hence one can say, in this sense, that a set of 0  $p$ -capacity is a discontinuity with no jump.

As well known in dimension 2 the sets with 0  $p$ -capacity, and hence discontinuity without jump, can be isolated points, countable set of points, fractal sets with Hausdorff dimension  $0 \leq \alpha < 1$  (see subsection 2.3 for the definition of  $p$ -capacity and related properties).

By the way the goal here is keeping nothing else but points in the image. The achievement of such a purpose makes necessary the minimization of a suitable energy that must remove all the discontinuity which are not discontinuities without jump, and remove all the discontinuities without jump which are no isolated point.

From one hand we have to force the concentration set of the divergence measure of  $U$  to contain only the points we want to catch, and on the other hand we have to regularize the initial data  $U_0$  outside the points of singularities. To this end we introduce the auxiliary space  $SDM^p(\Omega)$  of the vector field belonging to  $\mathcal{DM}^p(\Omega)$  whose divergence measure has no absolutely continuous part with respect to the  $p$ -capacity. Then, by taking into account that the initial vector field is a gradient of a Sobolev function, our goal is to minimize the following energy, involving a competition between a divergence term and the counting Hausdorff measure  $\mathcal{H}^0$ ,

$$\mathcal{F}(U) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\text{supp}(\text{div}^s \nabla u)_0)$$

where  $u \in W_0^{1,p}(\Omega)$  with  $\nabla u \in SDM^p(\Omega)$  and  $p < 2$ . The gradient of a minimizer of the energy  $\mathcal{F}$  is the vector field we are looking for, that is a vector field whose divergence measure can be decomposed in an absolutely continuous (with respect to the Lebesgue's measure) term plus an atomic measure concentrated on the points we want to catch in the image.

Even if a pointwise characterization of discontinuity without jump is not available, thanks to our definition the singular set of points can be linked to the vector field  $\nabla u$ , in the spirit of classical SBV formulation of the Mumford-Shah's Functional. (we refer to [1] for a complete survey on Mumford Shah's functional).

Our next task is to provide an approximation in the sense of the  $\Gamma$ -convergence introduced by [19, 20]. Our approach is close in the spirit to the one used to approximate the Mumford Shah functionals by a families of depending curvature functionals in [11]. Indeed as in their



work (see also [10]) we replace the atomic measure  $\mathcal{H}^0$  by the term

$$G_\varepsilon(D) = \frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

where  $D$  is a proper regular set containing the atomic set  $P$ ,  $\kappa$  is the curvature of its boundary, the constant  $\frac{1}{4\pi}$  is a normalization factor. Roughly speaking the minimum of these functionals are achieved on the union of balls of small radius, so that when  $\varepsilon \rightarrow 0$  the sequence  $G_\varepsilon$  shrinks to the atomic measure  $\mathcal{H}^0(P)$ .

This leads to an intermediate approximation given by

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (1)$$

This strategy permits to work with the perimeter measure  $H^1[\partial D]$ , that can be approximated, according to the Modica-Mortola's approach ( see [24, 25]), by the measure:

$$\mu_\varepsilon(w, \nabla w) dx = \left( \varepsilon |\nabla w|^2 + \frac{V(w)}{\varepsilon} \right) dx,$$

where  $V(w) = w^2(1 - w)^2$  is a double well functional.

Besides by using the Sard Theorem and the Coarea Formula (see also [6] for a similar approach) one can formally replace in the second integral on the boundary of  $D$  the curvature  $\kappa$  by the integral of the  $\operatorname{div} \frac{\nabla w}{|\nabla w|}$  and the integral is computed over the level sets of  $w$ . So that one can formally write the complete approximating sequence:

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_{\Omega} w^2 |\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left( \operatorname{div} \left( \frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_{\Omega} (1 - w_\varepsilon)^2 dx \end{aligned}$$

where, as usual,  $C = \int_0^1 \sqrt{W(t)} dt$ ,  $\beta_\varepsilon$  and  $\gamma_\varepsilon$  are infinitesimal as  $\varepsilon \rightarrow 0$ . The last integral is a penalization term that forces  $w$  to tend to 1 as  $\varepsilon \rightarrow 0$ .

The goal of the second part of this work is then to show that the family of energies  $\mathcal{F}_\varepsilon$   $\Gamma$ -converges to the functional  $\mathcal{F}$  when the parameters are related in a suitable way.

As in [11] we deal with the Hausdorff convergence for sets of points and with a suitable convergence of functions involving the convergence of a sub-level sets

Despites this approach is inspired by some ideas contained in [10, 11], we point out that in our case the regularization term involves a second order differential operator, due to the fact that our goal is to detect points and not segment curves. This deep difference requires a non trivial adaption of the arguments used in those papers.

122     The paper is organized as follows. Section 2 is devoted to notation, preliminaries, defi-  
123     nitions and result. In section 3 we illustrate the new variational model and we present the  
124     functional we deal with. Section 4 and 5 are devoted to the  $\Gamma$ -convergence result Finally in  
125     the last section we conclude the paper by comparing this approach with the celebrate con-  
126     jecture by De Giorgi concerning the approximation of the curvature depending functionals.  
127     We do not give here experimental result illustrating our approach. We refer the reader  
128     for that to [22].

## 2 Preliminaries

### 2.1 Notation

In all the paper  $\Omega \subset \mathbb{R}^2$  is an open bounded set with  $C^2$ -boundary. The operation  $\cdot$  stands for the usual scalar product. The Euclidean norm will be denoted by  $|\cdot|$ , while the symbol  $\|\cdot\|$  indicates the norm of some functional space. The brackets  $\langle, \rangle$  denotes the duality product in some distributional spaces.  $\mathcal{L}^d$  or  $dx$  is the  $d$ -dimensional Lebesgue measure and  $\mathcal{H}^k$  is the  $k$ -dimensional Hausdorff measure.  $B_\rho(x_0)$  is the ball centered at  $x_0$  with radius  $\rho$ . We say that a set  $D \in \Omega$  is a regular set if it can be written as  $\{F < 0\}$  with  $F \in C_0^\infty(\Omega)$ . In the following we will denote by  $R(\Omega)$  the set of the all regular sets in  $\Omega$ .

### 2.2 Distributional divergence and classical spaces

In this subsection we recall the definition of the distributional space  $L^{p,q}(\text{div}; \Omega)$  and  $\mathcal{DM}^p(\Omega)$  (see [4, 14]). We start by recalling the distributional definition of the divergence operator. Let  $\Omega \subset \mathbb{R}^2$  be an open set and let  $U : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a regular vector field. If  $U = (u, v)$  then classically  $\text{div}U = u_{x_1} + v_{x_2}$ . In the sense of distribution  $\text{Div}U$  is defined

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} (u, v) \cdot \nabla \varphi$$

for all  $\varphi \in C_0^\infty(\Omega)$ .

**Definition 2.1.** We say that  $U \in L^{p,q}(\text{div}; \Omega)$  if  $U \in L^p(\Omega; \mathbb{R}^2)$  and if its distributional divergence  $\text{Div}U = \text{div} \in L^q(\Omega)$ . If  $p = q$  the space  $L^{p,q}(\text{div}; \Omega)$  will be denoted by  $L^p(\text{div}; \Omega)$ .

We say that  $U \in L_{loc}^{p,q}(\text{div}; \Omega)$  if  $U \in L^{p,q}(\text{div}; A)$  for every open set  $A \subset \subset \Omega$ .

We say that a function  $u \in W^{1,p}(\Omega)$  belongs to  $W^{1,p,q}(\text{div}; \Omega)$  if  $\nabla u \in L^{p,q}(\text{div}; \Omega)$ . We say that a function  $u \in W_0^{1,p}(\Omega)$  belongs to  $W_0^{1,p,q}(\text{div}; \Omega)$  if  $\nabla u \in L^{p,q}(\text{div}; \Omega)$ .

**Definition 2.2.** For  $U \in L^p(\Omega; \mathbb{R}^2)$ ,  $1 \leq p \leq +\infty$ , set

$$|\text{Div}U|(\Omega) := \sup\{\langle U, \nabla \varphi \rangle : \varphi \in C_0^\infty(\Omega), |\varphi| \leq 1\}.$$

We say that  $U$  is an  $L^p$ -divergence measure field, i.e.  $U \in \mathcal{DM}^p(\Omega)$  if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\text{Div}U|(\Omega) < +\infty.$$

We say that  $U \in \mathcal{DM}_{loc}^p(\Omega)$  if  $U \in \mathcal{DM}^p(A)$  for every open set  $A \subset \subset \Omega$ .

Let us recall the classical result (see [15] Proposition 3.1)

**Theorem 2.1.** Let  $\{U_k\}_k \subset \mathcal{DM}^p(\Omega)$  such that

$$U_k \rightharpoonup U \quad L^p(\Omega), \text{ for } 1 \leq p < +\infty \quad (2)$$

Then

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_{h \rightarrow \infty} \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \quad |\text{Div}U|(\Omega) \leq \liminf_{h \rightarrow \infty} |\text{Div}U_h|(\Omega)$$

Moreover for a vector field  $U \in L^{p,q}(\text{div}; \Omega)$  it is possible to define a trace operator as stated by the next proposition (see [28], Proposition 3.1)

**Proposition 2.1.** *Let  $\Omega$  be a bounded,  $C^2$  domain. There is a continuous linear operator  $\gamma_\nu : U \in L^{p,q}(\text{div}; \Omega) \rightarrow (C^1(\partial\Omega))^*$  such that*

(i) *for each  $\varphi \in C^1(\partial\Omega; \mathbb{R})$  and each  $U \in L^{p,q}(\text{div}; \Omega)$ ,*

$$\langle \gamma_\nu(U), \varphi \rangle = \int_\Omega U \cdot \nabla \psi + \int_\Omega \psi \text{div} U \quad (3)$$

whenever  $\psi \in C^1(\overline{\Omega})$  and  $\psi|_{\partial\Omega} = \varphi$ .

(ii) *for  $U \in C^1(\overline{\Omega}; \mathbb{R}^2)$ ,  $\gamma_\nu(U) = U \cdot \nu|_{\partial\Omega}$ .*

## 2.3 $p$ -capacity

The  $p$ -capacity will be crucial to find a convenient functional framework to deal with. If  $K \subset \mathbb{R}^2$  is a compact set and  $\chi_K$  denotes its characteristic function, we define:

$$\text{Cap}_p(K, \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p dx \mid f \in C_0^\infty(\Omega), f \geq \chi_K \right\}$$

If  $U \subset \Omega$  is an open set, the  $p$ -capacity is given by

$$\text{Cap}_p(U, \Omega) = \sup_{K \subset U} \text{Cap}_p(K, \Omega).$$

Finally if  $A \subset \Omega$  is a Borel set

$$\text{Cap}_p(A, \Omega) = \inf_{A \subset U \subset \Omega} \text{Cap}_p(U, \Omega).$$

We recall the following result (see for instance [23], Theorem 2.27) theorem that explains the relationship between  $p$ -capacity and Hausdorff measure. Such a result is crucial to have geometric information on the null  $p$ -capacity sets.

**Theorem 2.2.** *Assume  $1 \leq p < 2$ . If  $\mathcal{H}^{2-p}(A) < \infty$  then  $\text{Cap}_p(A, \Omega) = 0$*

Another useful tool to manage the sets of  $p$ -capacity 0 is provided by the following characterization.

**Theorem 2.3.** *Let  $E$  be a compact subset of  $\Omega$ . Then  $\text{Cap}_p(E, \Omega) = 0$  if and only if there exists a sequence  $\{\phi_k\} \subset C_0^\infty(\Omega)$ , converging to 0 strongly in  $W_0^{1,p}(\Omega)$ , such that  $0 \leq \phi_k \leq 1$  and  $\phi_k = 1$  on  $E$  for every  $k$ .*

For a general survey we refer the reader to [21, 23, 29].

### 170 3 The Variational Model

171 This section is devoted to illustrate the functional framework we will work with.

Roughly speaking in biological images the image is a function that could be very high on points or other structures like set with Hausdorff dimension  $0 \leq \alpha < 1$ , and it is close to 0 outside. In image processing these concentration sets are called discontinuity without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0. From a mathematical point of view it seems to be much more appropriate to think of the image as a Radon measure, that is  $I = \mu \in (C_0(\Omega))^*$ . The next step is finding a space whose elements are capable of producing these discontinuity without jump. This space is given by  $\mathcal{DM}^p(\Omega)$ , with  $1 < p < 2$ . The restriction on  $p$  is due to the fact that when  $p \geq 2$  the distributional divergence of  $U$  cannot be a measure concentrated on points. Set  $p \geq 2$ . Looking at the definition we have

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} U \cdot \nabla \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

172 Since  $p \geq 2$  this distribution is a well defined even for any test  $\varphi \in W_0^{1,p'}(\Omega)$  where  $p' \leq 2$   
 173 is the dual exponent of  $p$ . In particular  $\text{Div}U$  belongs to the dual space  $W^{-1,p'}(\Omega)$  of the  
 174 Sobolev space  $W_0^{1,p}(\Omega)$ . Then in this case, the distributional divergence of  $U$  cannot be  
 175 an atomic measure, whereas  $\delta_0 \notin W^{-1,p'}(\Omega)$ . To see this, one can consider as  $\Omega$  the disk  
 176  $B_0(1)$  and the function  $\tilde{\varphi}(x) = \log(\log(1 + |x|)) - \log(\log(2))$ . This function is in the space  
 177  $W_0^{1,p'}(\Omega)$  for every  $p' \leq 2$  and therefore it is an admissible test function and it easy to check  
 178 that  $\langle \delta_0, \varphi \rangle = +\infty$ .

179 When  $1 < p < 2$  we have that  $\text{Div}U \in W^{-1,p'}(\Omega)$ , but in this case since  $p < 2$ , we have  
 180  $p' > 2$  and hence the function  $\tilde{\varphi}$  is no longer an admissible test function. One can check  
 181 that the distribution  $\text{Div}U$  is an element of  $(C_0(\Omega))^*$  capable of charging the points. Take  
 182 for instance the map  $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ .

183 The next step is to transform the initial image  $I$  as the divergence measure of a suitable  
 184 vector field. We consider the elliptic problem with measure data  $I$ :

$$\begin{cases} \Delta u = I & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

Classical results (see [27]) ensures the existence of a unique weak solution  $u \in W_0^{1,p}(\Omega)$  with  $p < 2$ . Then it easy to see that the distributional divergence of  $\nabla u$  is given by  $I$ . In particular by setting  $U = \nabla u$ , we have  $U \in \mathcal{DM}^p(\Omega)$ . According to the Radon Nykodem decomposition of the measure  $\text{Div}U$  we have

$$\text{Div}U = \text{div}U + \text{div}^s U,$$

185 where  $\text{div}U \in L^1(\Omega)$ . For our purpose the support of the singular measure  $\text{div}^s U$  is too large.

186 In particular the measure  $\text{div}^s U$  could charge sets with Hausdorff dimension  $0 \leq \alpha < 2$ ,

187 hence it can be concentrated also on discontinuity with jump. So that in order to isolate  
 188 the singularities we are interested in, we need a further decomposition of the measure  $\text{Div}U$ .  
 189 This can be done by using the capacitary decomposition of the Radon measure  $\text{div}^s U$ . It is  
 190 known (see [17]) that given a Radon measure  $\mu$  the following decomposition holds

$$\mu = \mu_a + \mu_0 \quad (5)$$

191 where the measure  $\mu_a$  is absolutely continuous with respect to the  $p$ -capacity and  $\mu_0$  is  
 192 singular with respect to the  $p$ -capacity, that is concentrated on sets with 0  $p$ -capacity. Besides  
 193 it is also known (see [17]) that every measure which is absolutely continuous with respect  
 194 to the  $p$ -capacity can be characterized as an element of  $L^1 + W^{-1,p'}$ , leading to the finer  
 195 decomposition:

$$\mu = f - \text{Div}G + \mu_0 \quad (6)$$

196 where  $G \in L^p(\Omega; \mathbb{R}^2)$ ,  $f \in L^1(\Omega)$ .

197 By applying this decomposition to the measure  $\text{div}^s U$  we obtain the following decom-  
 198 position of the measure  $\text{Div}U$

$$\text{Div}U = \text{div}U + f - \text{Div}G + (\text{div}^s U)_0 \quad (7)$$

199 with  $G \in L^p(\Omega)$ .

200 According to this decomposition and taking into account Theorem 2.3 we give the defi-  
 201 nition of discontinuity without and with jump.

202 **Definition 3.1.** *We say that a point  $x \in \Omega \subset \mathbb{R}^2$  is a point of discontinuity without jump*  
 203 *of  $U$  if  $x \in \text{supp}(\text{div}^s U)_0$ .*

204 The other singularities where there is a jump, are contained in the second term of the  
 205 decomposition (7). Indeed the space  $W^{-1,p'}(\Omega)$  contains Hausdorff measures restricted to  
 206 sub-manifolds of dimension greater or equal to 1 (see [29] chapter 4, section 7 for a detailed  
 207 discussion on the space  $W^{-1,p'}(\Omega)$ ), like for instance Hausdorff measures concentrated on  
 208 regular closed curves, that is contours, that are a classical example of discontinuity with  
 209 jump. More precisely a contour of a regular set is the jump set of the characteristic  
 210 function of the set  $D$  and its  $p$ -capacity is strictly positive. This is of course in agreement  
 211 with Theorem 2.3. Indeed, since the characteristic function of the set  $D$  is  $BV$ -function, it  
 212 cannot be approximated by regular functions in the  $W^{1,p}$  strong norm.

213 **Definition 3.2.** *We say that a point  $x \in \Omega \subset \mathbb{R}^2$  is a point of discontinuity with jump of  $U$*   
 214 *if  $x \in \text{supp}(f - \text{Div}G)$ .*

### 215 3.1 The variational framework

216 We shall build an energy whose minimizer will be a vector field whose singular part of its  
 217 divergence measure will be given by nothing else but points.

218 This minimizer must be an  $L^p$  (with  $p < 2$ ) vector field with the following properties:

- 219 1. It must be close to the initial data  $U_0$  which is, in general, an  $L^p$  vector field  $U_0$  with  
 220  $1 < p < 2$ .
- 221 2. The absolutely continuous part with respect to the Lebesgue measure of  $\text{Div}U$  is an  
 222  $L^2$  function.
- 223 3. The support of the measure  $\text{div}^s U$  must be given by set of points  $P_U$  with  $\mathcal{H}^0(P_U) <$   
 224  $+\infty$

225 According to these considerations it seems to be natural to introduce the space

$$SDM^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \text{Div}G = 0\} \quad (8)$$

226 so that, as a consequence, the decomposition (7) yields for any  $U \in SDM^p(\Omega)$

$$\text{Div}U = \text{div}U + (\text{div}^s U)_0. \quad (9)$$

227 For our purposes the following result concerning the features of the elements of the space  
 228  $SDM^p(\Omega)$  will play a crucial role.

229 **Proposition 3.1.** *Let  $U \in L^{p,q}(\text{div}; \Omega \setminus P)$  and  $P \subset \Omega$  be a set of finite number of points*  
 230 *then  $U \in SDM^p(\Omega)$ .*

231 **Proof.** We set  $P = \{x_1, \dots, x_n\}$ . Let  $B_\rho(x_i)$   $i=1, \dots, n$ . We can find a sequence  $\rho(h) \rightarrow 0$  when  
 232  $h \rightarrow +\infty$  such that  $B_{\rho(h)}(x_i) \cap B_{\rho(h)}(x_j) = \emptyset$  for  $h$  large and  $i \neq j$ . Then we define the  
 233 following sequence  $U_h$ .

$$U_h = \begin{cases} 0 & \text{on } \Omega \cap \bigcup_i^n B_{\rho(h)}(x_i) \\ U & \text{on } \Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i) \end{cases} \quad (10)$$

234 The sequence  $U_h \in \mathcal{DM}^p(\Omega)$  for every  $h$  fixed. If  $\varphi \in C_0^\infty(\Omega)$ , by applying the Proposition  
 235 2.1 to the vector field  $U \in L^{p,q}(\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i))$ . we get:

$$\begin{aligned} |\text{Div}U_h|(\Omega) &\leq \sup_{0 \leq \varphi \leq 1} \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} U \nabla \varphi \right| \leq \sup_{0 \leq \varphi \leq 1} \{ |\langle \gamma_\nu(U), \varphi|_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} \text{div}U \varphi \right| \} \leq \sup_{\varphi} |\langle \gamma_\nu(U), \varphi|_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \|\varphi\|_{L^\infty(\Omega)} \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C \|U\|_p + (1 + C) \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C_1 \end{aligned} \quad (11)$$

$$(12)$$

236 where  $C_1$  does not depend on  $h$ .

Now since  $\mathcal{L}(\bigcup_i^n B_{\rho(h)}(x_i)) \rightarrow 0$  we have  $U_h \rightarrow U$  in  $L^p(\Omega)$  the by Theorem 2.1

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_h \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \leq C$$

and by theorem 2.1

$$|\operatorname{Div} U|(\Omega) < \liminf_{h \rightarrow \infty} |\operatorname{Div} U_h| \leq C.$$

237 Therefore  $U \in \mathcal{DM}^p(\Omega)$ . Finally we know that  $U \in L^{p,q}(\operatorname{div}; \Omega \setminus P)$  and the support  
 238 of the measure  $\operatorname{div}^s U$  is given by the set  $P$ , then, according to decomposition (7), the  
 239 measure  $f - \operatorname{Div} G$  vanishes on sets with 0  $p$ -capacity, we deduce  $f - \operatorname{Div} G = 0$ , that is  
 240  $U \in \mathcal{SDM}^p(\Omega)$ .  $\square$

### 241 3.2 The Functional

According to our purpose the natural energy to deal with is the following  $F : \mathcal{SDM}^p(\Omega) \rightarrow [0, \infty]$  given by

$$F(U) = \int_{\Omega} |\operatorname{div} U|^2 dx + \int_{\Omega} |U - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s U)_0).$$

242 We note that  $\inf F(U) > 0$  on  $\mathcal{SDM}^p(\Omega)$ . Indeed if we had  $\inf_{\mathcal{SDM}^p(\Omega)} F(U) = 0$  then it  
 243 would be possible exhibiting a minimizing sequence  $U_n \rightarrow U_0$  in  $L^p$  such that  $\operatorname{Div} U_n \rightarrow 0$   
 244 in  $\mathcal{D}'(\Omega)$ . On the other hand, the  $L^p$ -distance between  $U_n$  and  $U_0$  can be arbitrary small  
 245 only if  $\operatorname{Div} U_0 = 0$  as well, because the constraint  $\operatorname{Div} U = 0$  is stable under  $L^p$ -convergence.  
 246 Moreover to mimic the construction of  $U_0$  we restrict ourselves to vector fields  $U$  which are  
 247 the gradient of a function  $u \in W_0^{1,p}(\Omega)$ .

248

$$\Delta \mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \nabla u \in \mathcal{SDM}^p\} \quad (13)$$

249 So that the new energy  $\mathcal{F} : \Delta \mathcal{M}^p(\Omega) \rightarrow (0, \infty]$  is given by

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s \nabla u)_0). \quad (14)$$



## 250 4 $\Gamma$ -convergence: The intermediate approximation

251 The functional  $\mathcal{F}$  is finite on the class of the functions  $u \in \Delta\mathcal{M}^p(\Omega)$  whose support of the  
 252 measure  $(\operatorname{div}^s \nabla u)_0$  is given by a finite set and the absolutely continuous part of the measure  
 253  $\operatorname{Div} \nabla u$  is given by  $\Delta u \in L^2(\Omega)$ . Consequently it is convenient to introduce the following  
 254 space:

$$\Delta\mathcal{AM}^{p,2}(\Omega) = \{u \in \Delta\mathcal{M}^p(\Omega) : \operatorname{supp}(\operatorname{div}^s \nabla u)_0 = P_{\nabla u} \text{ with } \mathcal{H}^0(P_{\nabla u}) < +\infty, \Delta u \in L^2(\Omega)\} \quad (15)$$

In the spirit [11] we introduce an intermediate approximation, in the sense of  $\Gamma$ -convergence, of the functional  $\mathcal{F}$ . We define a sequence of functionals where the counting measure  $\mathcal{H}^0(P_{\nabla u})$  is replaced by a functional defined on regular sets  $D$  and which involves the curvature of the boundary  $\partial D$ . The approximating sequence is given by:

$$\begin{aligned} F_\varepsilon(u, D) &= \int_\Omega (1 - \chi_D) |\Delta u|^2 + \int_\Omega |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (16)$$

255 Where  $u \in W_0^{1,p,2}(\operatorname{div}; \Omega)$ ,  $D$  is a regular set, and  $\kappa$  denotes the curvature of its boundary .

256 In order to guarantee that the measure of the sets  $D$  is small we define a new functional  
 257 still denoted by  $F_\varepsilon(u, D)$  given by

$$F_\varepsilon(u, D) = \begin{cases} \int_{\Omega \setminus D} |\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{4\pi} \int_{\partial D} \left( \frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) & \text{if } \mathcal{L}(D) \leq a(\varepsilon) \\ +\infty & \text{otherwise} \end{cases} \quad (17)$$

258 while the limit functional is

$$\mathcal{F}(u) = \int_\Omega |\Delta u|^2 dx + \int_\Omega |\nabla u - U_0|^p dx + \mathcal{H}^0(P_{\nabla u}).$$

259 where  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ ,  $U_0 \in L^p(\Omega; \mathbb{R}^2)$ .

260 To simplify the notation let us set  $Y(\Omega) = \{(u, D) \mid u \in W_0^{1,p,2}(\operatorname{div}; \Omega), D \in R(\Omega)\}$ ,  
 261 where  $R(\Omega)$  is a regular set.

262 By adapting to our case the convergence introduced in [11], we endow the set  $Y(\Omega)$  with  
 263 the following convergence.

264 **Definition 4.1.** We say that  $\{(u_h, D_h)\} \subset Y(\Omega)$  *A-converges* to  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  if the  
 265 following conditions hold

- 266 1.  $\mathcal{L}(D_h) \rightarrow 0$
- 267 2.  $\{\partial D\}_h \rightarrow P$  in the Hausdorff metric, where  $P$  is a finite set of points.
- 268 3.  $u_h \rightarrow u$  in  $L^p(\Omega)$  and  $P_{\nabla u} \subseteq P$

269 The definition of  $\Gamma$ -convergence with respect to the convergence above is the following.

270 **Definition 4.2.** We say that a sequence  $G_\varepsilon$   $\Gamma$ -converges to  $G$  if for every sequence of positive  
271 numbers  $\{\varepsilon_h\} \rightarrow 0$  and for every  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  we have:

1. for every sequence  $\{(u_h, D_h)\} \subset Y(\Omega)$   $A$ -convergent to  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \geq G(u)$$

2. there exists a sequence  $\{(u_h, D_h)\} \subset Y(\Omega)$   $A$ -convergent to  $u$  such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \leq G(u)$$

272 By following this approach the fundamental theorem of the  $\Gamma$ -convergence can be stated  
273 as follows.

**Theorem 4.1.** Let  $\mathcal{F}_\varepsilon$  be a sequence of functional  $\Gamma$ -convergent to  $\mathcal{F}$ . If  $\{\varepsilon_h\}$  is a sequence  
of positive numbers converging to zero and  $\{(u_h, D_h)\} \subset Y(\Omega)$  such that

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} \mathcal{F}_{\varepsilon_h}) = 0$$

274 then there exists a subsequence  $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$  and a minimizer  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  of  
275  $\mathcal{F}(u)$ , such that  $\{(u_{h_k}, D_{h_k})\}$   $A$ -converges to  $u$ .

276 By following the classical pattern (see for instance [12]), we will split the proof in three  
277 steps. Theorem 4.1 will follow from: the compactness of the minimizing sequence of  $F_\varepsilon$ ,  
278 the  $\Gamma \liminf$  inequality (1) and the  $\Gamma$ -limsup inequality (2) in Definition 4.2.

## 279 4.1 Compactness

280 We state and prove the following compactness result.

281 **Theorem 4.2.** Let  $h$  goes to  $+\infty$  and  $\varepsilon_h \rightarrow 0$

$$F_{\varepsilon_h}(u_h, D_h) \leq M \tag{18}$$

282 then there exist a subsequence  $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$  and a function  $u \in \Delta\mathcal{AM}^p(\Omega)$  such  
283 that  $(u_{h_k}, D_{h_k})$   $A$ -converges to  $u$ .

**Proof.** We adapt an argument of [11]. We have  $\mathcal{L}(D_h) \rightarrow 0$  otherwise  $F_{\varepsilon_h} = +\infty$ . Let  
 $D_h$  be a sequence of regular sets. Then we can parameterize every  $C_h = \partial D_h$  by a finite  
and disjoint union of Jordan curves. Let us set for every  $h$   $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$ , then we have  
according to the 2-dimensional version of Gauss-Bonnet's Theorem and Young's inequality

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left( \frac{1}{\varepsilon_h} + \varepsilon_h \kappa_h^2 \right) d\mathcal{H}^1 \geq \frac{1}{4\pi} \int_{\partial D_h} 2\kappa_h d\mathcal{H}^1 = \frac{1}{4\pi} \int_{\bigcup_h C_h} 2\kappa_h d\mathcal{H}^1 = m(h)$$

Note that the number  $m(h) \leq M$ , with  $M \geq 0$ , is independent of  $h$ . Then it is possible to extract a subsequence  $C_{h_k}$  with the number of curves in  $C_{h_k}$  equal to  $n$  for every  $k$ . Then we set  $C_{h_k} = \{\gamma_{h_k}^1, \dots, \gamma_{h_k}^n\}$  for any  $k$ . From (18) we also have for any  $\gamma \in C_{h_k}$  that  $\mathcal{H}^1(\gamma) \leq 4\pi M \varepsilon_{h_k}$  and consequently  $\max\{\mathcal{H}^1(\gamma) : \gamma \in C_{h_k}\} \rightarrow 0$ . Then there exists a finite set of point  $P = \{x_1, \dots, x_n\} \subset \overline{\Omega}$  such that for any radius  $\rho$  there is an index  $k_\rho$  with

$$\gamma_{h_k}^i \subset B_\rho(x_i) \text{ for all } k > k_\rho \text{ and } i \in \{1, \dots, n\},$$

so that if we set  $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$ , then the Hausdorff distance  $d_H(\partial D_{h_k}, P) \rightarrow 0$  since  $\mathcal{L}^1(D_{h_k}) \rightarrow 0$  and therefore  $\rho \rightarrow 0$  as well.

Now we prove the compactness property for  $u_h$ . First of all from the estimate

$$\|\nabla u_h\|_{L^p(\Omega)}^p \leq 2^p (\|\nabla u_h - U_0\|_{L^p(\Omega)} + \|U_0\|_{L^p(\Omega)}) \quad (19)$$

and (18) we may extract a subsequence  $\{u_{h_k}\} \subset W_0^{1,p}(\Omega)$  weakly convergent to  $u \in W_0^{1,p}(\Omega)$ .

Let  $\Omega_j$  be a sequence of open sets  $\Omega_j \subset \subset \Omega \setminus P$  invading  $\Omega \setminus P$ . We claim that that it is possible to extract a sequence of  $D_{h_k}$  such that  $\Omega_j \cap \partial D_{h_k} = \emptyset$ . Indeed since the distance between  $\Omega_j$  and  $P$  is positive for any  $j$  there exists  $\eta_j$  such that  $\Omega_j \cap (\bigcup_i B_{\eta_j}(x_i)) = \emptyset$ . On the other hand we know from before that for every  $\rho$  we can find  $k_\rho$  such that  $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$ . Then in particular if  $\rho = \eta_j$  there exists  $k_j$  such that for all  $k \geq k_j$

$$\Omega_j \cap \partial D_{h_k} = \emptyset.$$

Therefore for any  $x \in \Omega_j$  there exists  $\delta > 0$  such that either  $B_\delta(x) \subset D_{h_k}$  or  $B_\delta(x) \subset \Omega \setminus D_{h_k}$ .

Finally by taking into account that  $\mathcal{L}^1(D_{h_k}) \rightarrow 0$  we conclude  $\Omega_j \cap \partial D_{h_k} = \emptyset$  for  $k \geq k_j$ .

Then for every  $k \geq k_j$  we have that  $u_{h_k} \in W^{1,p,2}(\text{div}; \Omega_j)$  and by (18) we get

$$\int_{\Omega_j} |\Delta u_h|^2 \leq \int_{\Omega \setminus D_{h_k}} |\Delta u_{h_k}|^2 dx \leq M \quad (20)$$

Then we can extract a further subsequence still denoted by  $\{u_{h_k}\} \subset W^{1,p,2}(\text{div}; \Omega_j)$  such that

$$\begin{cases} u_{h_k} \rightharpoonup u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \text{ and a.e} \\ \nabla u_{h_k} \rightharpoonup \nabla u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \\ \Delta u_{h_k} \rightharpoonup \Delta u & \text{in } L^2(\Omega_j). \end{cases}$$

Let now  $x \in \Omega' \subset \subset \Omega \setminus P$ . Then there exists a sequence  $x_j \rightarrow x$  with  $j \in \mathbb{N}$ . By applying the diagonal argument to the sequence  $u_{h_{k_l}}(x_j)$  we obtain a subsequence  $u_l = u_{h_{k_l}}(x_l)$  such that  $\Delta u_l$  converges weakly in  $L^{p'}(\Omega')$  to  $\Delta u$  for any  $\Omega' \subset \subset \Omega$ . Then by the semicontinuity of the  $L^2$ -norm we have

$$\sup_j \int_{\Omega_j} |\Delta u|^2 dx \leq \sup_j \liminf_{l \rightarrow +\infty} \int_{\Omega_j} |\Delta u_l|^2 dx \leq M.$$

If we set  $\tilde{P} = P \setminus \partial\Omega$ , then we deduce  $u \in W_0^{1,p,2}(\text{div}; \Omega \setminus \tilde{P})$  and therefore  $\nabla u \in \mathcal{SDM}^p(\Omega)$  with  $P_{\nabla u} \subseteq P$ , by Proposition 3.1. So we conclude that  $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$   $\square$ .

## 297 4.2 Lower Bound

298 According to the convergence we deal with, we have to provide the following lower bound.

**Theorem 4.3.** *Let  $\{\varepsilon_h\}_h$  be a sequence of positive numbers converging to zero. For every  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  and for every sequence  $\{(u_h, D_h)\}_h \subset Y(\Omega)$   $A$ -converging to  $u$  we have*

$$\liminf_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u).$$

**Proof.** First of all we may assume that  $\liminf F_{\varepsilon_h}(u_h, D_h) < M$  otherwise the result is trivial. Moreover up to a subsequence we may assume that the  $\liminf$  is actually a limit. Let  $\{\Omega_j\}_j$  be a sequence of open sets  $\Omega_j \subset \subset \Omega \setminus P$  invading  $\Omega \setminus P$ . As in the proof of the Theorem 4.2 we may assume up to a subsequence, that  $\Delta u_h \rightarrow \Delta u$  weakly in  $L^2(\Omega_j)$ . Furthermore, since in this case all the sequence  $D_h$  satisfies the condition (1) and (2) in Definition 4.1 we have  $\Omega_j \subset \Omega \setminus D_h$  for  $h$  large and for any  $j$ . We also have  $\Delta u_h$  converges weakly in  $L^2(\Omega_j)$  to  $\Delta u$ . Consequently

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 \geq \liminf_h \int_{\Omega_j} |\Delta u_h|^2 dx \geq \int_{\Omega_j} |\Delta u|^2 dx.$$

299 On the other hand we know that the limit  $u$  of the subsequence  $u_h$  belongs to  $\Delta\mathcal{AM}^p(\Omega)$ ,  
300 hence  $\Delta u \in L^2(\Omega)$ . So that by monotone convergence

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \geq \int_{\Omega \setminus P} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 \quad (21)$$

301 As in the proof of the theorem 4.2 we have the inequality (19). Then we easily get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx. \quad (22)$$

This conclude the first part of the proof. Let now  $D_h$  be a sequence a regular sets. Then we can find a parameterization of every  $C_h = \partial D_h$  by a finite and disjoint union of Jordan curves. As in the in the proof of Theorem 4.2 if we set for every  $h$ ,  $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left( \frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 = m(h)$$

302 and up to subsequences we have  $m(h) = n$  and there exist a set  $P_1$  of  $n$  points such that  $\partial D_h$   
303 converges in the Hausdorff metric to  $P_1$ . On the other hand we have that  $\partial D_h$  converges in  
304 the Hausdorff metric to  $P$  and, since the limit is unique  $P = P_1$  and therefore.

$$\frac{1}{4\pi} \int_{\partial D_h} \left( \frac{1}{\varepsilon} + \varepsilon_h k^2 \right) d\mathcal{H}^1 \geq n = \mathcal{H}^0(P) \geq \mathcal{H}^0(P_{\nabla u}) \quad (23)$$

305 Eventually by (21),(22) (23) and by the superlinearity property of the  $\liminf$  operator we  
306 achieve the result.  $\square$

### 307 4.3 Upper bound

308 In [11] for the construction of the optimal sequence it is crucial to use a result due to  
 309 Chambolle and Doveri (see [13]). This result states that it is possible to approximate, in  
 310 the  $H^1$  norm, a function  $u \in W^{1,2}(\Omega \setminus C)$  (where  $C$  is a closed set), by means of a sequence  
 311 of functions  $u_h \in W^{1,2}(\Omega \setminus C_h)$  with  $C_h$  convergent to  $C$  in the Hausdorff metric. In our  
 312 case this argument does not apply due to presence of a second order differential operator.  
 313 Nevertheless since we work only with set of points it is possible to build an optimal sequence  
 314 in a more direct way.

315 According to the convergence we deal with the theorem, we have to prove the following  
 316 theorem.

317 **Theorem 4.4.** *Let  $\varepsilon_h$  be a sequence of positive numbers with  $\varepsilon_h \rightarrow 0$ . For every  $u \in$   
 318  $\Delta\mathcal{AM}^{p,2}(\Omega)$  there exists a sequence  $\{(u_h, D_h)\} \subset Y(\Omega)$  A-converging to  $u$  such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u). \quad (24)$$

319 **Proof.** We start by the sequence  $D_h$ . Let  $n$  be the number of points  $x_i$  in  $P_{\nabla u}$ . Then  
 320 we take  $D_h = \bigcup_1^n B_{\varepsilon_h}(x_i)$ . So that  $\mathcal{L}(D_h) \rightarrow 0$  and  $\partial D_h$  goes to  $P$  with respect to the  
 321 Hausdorff distance. Moreover for  $h$  large we may assume  $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$  for  $i \neq j$ .  
 322 Now we build  $u_h$ . Let  $\{\rho_h\} \subset \mathbb{R}$  be such that  $\rho_h \geq 0$  and  $\rho_h \rightarrow 0$  when  $h \rightarrow \infty$ . Let  
 323  $\theta_h \in C^\infty(\Omega)$  with the following property:

$$\begin{cases} \theta_h = 1 & \text{on } B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ 0 \leq \theta_h \leq 1 & \text{on } B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ \theta = 0 & \text{on } \Omega \setminus B_{\rho_h}(x_i) \text{ for any } i = 1, \dots, n \\ \|\nabla \theta_h\|_\infty \leq \frac{1}{\rho_h} \end{cases} \quad (25)$$

We set  $u_h = (1 - \theta_h)u$  and we claim that the pair  $(u_h, D_h)$  realizes the inequality (24). By making the computation we have

$$\nabla u_h = (1 - \theta_h)\nabla u - u\nabla \theta_h$$

Then we have

$$\int_\Omega |\nabla u_h - U_0|^p dx = \int_\Omega |\nabla u - U_0 - \theta_h \nabla u - u \nabla \theta_h|^p dx$$

324 So that

$$\limsup_{h \rightarrow +\infty} \int_\Omega |\nabla u_h - U_0|^p dx \leq \limsup_{h \rightarrow +\infty} \left( \left( \int_\Omega |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left( \int_\Omega |\theta_h \nabla u|^p \right)^{\frac{1}{p}} + \left( \int_\Omega |u \nabla \theta_h|^p \right)^{\frac{1}{p}} \right)^p \quad (26)$$

325 Since  $\nabla u \in L^1(\Omega)$ , we have by applying the dominated convergence theorem  $\int_{\Omega} |\theta_h \nabla u|^p \rightarrow 0$ .  
 326 Let us focus on the term  $\int_{\Omega} |\nabla \theta_h u|^p$ . By the Sobolev embedding we have  $u \in L^{p^*}(\Omega)$  with  
 327  $p^* = \frac{2p}{2-p}$  and hence  $|u|^p \in L^{\frac{p^*}{p}}(\Omega)$ , with  $\frac{p^*}{p} = \frac{2}{2-p}$ .

By (25), using the Holder's inequality with dual exponents  $\frac{2}{2-p}$  and  $\frac{2}{p}$  and taking into account that  $p < 2$

$$\begin{aligned} \int_{\Omega} |\nabla \theta_h u|^p &\leq \sum_{i=1}^n \int_{B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx = \sum_{i=1}^n \left( \int_{B_{\rho_h}(x_i)} |\nabla \theta_h u|^p dx - \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx \right) \\ &\leq n \left( \left( \int_{B_{\rho_h}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} + \left( \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \right) \\ &\leq n \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \left( \frac{\pi^2 \rho_h^2}{\rho_h^p} + \frac{\pi^2 \rho_h^2}{4 \rho_h^p} \right) \rightarrow 0. \end{aligned} \quad (27)$$

From (26) it follows that

$$\begin{aligned} \limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx &\leq \lim_{h \rightarrow +\infty} \left( \left( \int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\theta_h \nabla u|^p \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla \theta_h u|^p \right)^{\frac{1}{p}} \right)^p \\ &= \left( \left( \int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} \right)^p = \int_{\Omega} |\nabla u - U_0|^p dx \end{aligned} \quad (28)$$

Now we compute  $\Delta \theta_h$ . The identity  $\operatorname{div}(fA) = f \operatorname{div} A + \nabla f \cdot A$  yields

$$\Delta u_h = (1 - \theta_h) \Delta u - 2 \nabla \theta_h \nabla u - \Delta \theta_h u.$$

328 Then by choosing  $\rho_h$  small enough we have from (25)

$$\limsup_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 \leq \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u|^2 \rightarrow \int_{\Omega} |\Delta u|^2 dx. \quad (29)$$

329 Moreover it is not difficult to check that  $\{(u_h, D_h)\} \subset Y(\Omega)$  and A-converges to  $u$ .

330 Finally since for  $h$  large we have  $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$  for  $i \neq j$  we get

$$\lim_h \frac{1}{4\pi} \int_{\partial D_h} \left( \varepsilon_h + \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = \lim_h \sum_1^n \int_{\partial B_{\varepsilon_h}} \left( \varepsilon_h \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = n = \mathcal{H}^0(P_{\nabla u}). \quad (30)$$

331 By recalling that the limsup is sublinear operation and by (28),(29),(30), we achieve the  
 332 result.  $\square$

## 333 5 Approximation by smooth function

334 By following the Braides-March's approach in [11] we approximate the measure  $\mathcal{H}^1$  measure  
 335 by the Modica-Mortola's energy density given by  $(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx$  where  $W(w) =$   
 336  $w^2(1-w)^2$  and  $w \in C^\infty(\Omega)$ . The next step is to replace the regular set  $D$  with the level  
 337 set of  $w$ . Let us set  $H = \{\nabla w(x) = 0\}$ . By Sard's Lemma we have that  $\mathcal{L}^1(w(H)) = 0$ . In  
 338 particular, if  $w$  takes values into the interval  $[0, 1]$ , we infer that for almost every  $t \in (0, 1)$   
 339 the set  $H \cap w^{-1}(t)$  is empty. Consequently for almost every  $t \in (0, 1)$  the  $t$ -level set  $\{w < t\}$   
 340 is a  $C^\infty$  set with boundary  $\{w = t\}$ . Now since we want to replace the set  $D$  we need that  
 341  $\{w < t\} \subset\subset \Omega$  and then we require that  $w$  is such that  $1 - w \in C_0^\infty(\Omega; [0, 1])$ . Furthermore  
 342 for almost every  $t$  we have  $k(\{w = t\}) = \operatorname{div}(\frac{\nabla w}{|\nabla w|})$ . From all of this we are led to define the  
 343 following space:

$$S(\Omega) = \{(u, w); u \in W_0^{1,p,2}(\operatorname{div}; \Omega); 1 - w \in C_0^\infty(\Omega; [0, 1])\} \quad (31)$$

and having in mind the coarea formula, the following sequence of functionals defined on  $S(\Omega)$

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_\Omega w|\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w=0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)^2\right) (\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx \\ &+ \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_\Omega (1 - w_\varepsilon)^2 dx \end{aligned}$$

344 whith  $C = \int_0^1 \sqrt{W(t)}dt$ .

345 The last term forces  $w_\varepsilon$  be 1 almost everywhere in the limit and hence plays the role of  
 346 the condition  $\mathcal{L}(D) \leq a(\varepsilon)$ . From now on the parameters  $\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$  will be related as follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 0 \quad (32)$$

347

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0 \quad (33)$$

348 The convergence that plays the role of the  $A$ -convergence is the following.

349 **Definition 5.1.** Let  $\{(u_h, w_h)\}$  be a sequence in  $S(\Omega)$  and  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ . We say that  
 350 that  $(u_h, w_h)$   $S$ -converges to  $u$  if for every  $t \in (0, 1)$  the pair  $(u_h, D_h^t)$   $A$ -converges to  $u$ .

351 We define the  $\Gamma$ -convergence with respect to this convergence.

352 **Definition 5.2.** We say that a sequence  $G_\varepsilon$   $\Gamma$ -converges to  $G$  if for every sequence of positive  
 353 numbers  $\varepsilon_h \rightarrow 0$  and for every  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  we have:

1. for every sequence  $(u_h, w_h)$   $S$ -convergent to  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \geq G(u)$$

2. there exists a sequence  $(u_h, w_h)$   $S$ -convergent to  $u$  such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \leq G(u)$$

354 As in the previous section we state a suitable version of the fundamental theorem of the  
355  $\Gamma$ -convergence.

**Theorem 5.1.** *Let  $\mathcal{F}_\varepsilon$  a sequence of functional  $\Gamma$ -convergent to  $\mathcal{F}$ . If  $\{\varepsilon_h\}$  is a sequence of positive numbers converging to zero and  $\{(u_h, w_h)\} \subset S(\Omega)$  such that*

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf \mathcal{F}_{\varepsilon_h}) = 0$$

356 *then there exists a subsequence  $\{(u_{h_k}, w_{h_k})\}$  and a minimizer  $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$  of  $\mathcal{F}$  such*  
357 *that  $\{(u_{h_k}, w_{h_k})\}$   $S$ -converges to  $u$ .*

## 358 5.1 Compactness

359 **Theorem 5.2.** *Let  $h$  goes to  $+\infty$  and  $\varepsilon_h \rightarrow 0$*

$$F_{\varepsilon_h}(u_h, w_h) \leq M \tag{34}$$

360 *there exists a subsequence  $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$  and  $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$  such that  $(u_{h_k}, w_{h_k})$*   
361  *$S$ -converges to  $u$ .*

362 **Proof.** The first part of proof is as in [11]. For the convenience of the reader we give the  
363 complete proof.

364 For any  $t \in (0, 1)$  since  $0 \leq w_h \leq 1$ , we have  $w_h > \frac{1}{(1-t)^2}$  then by Chebyshev's inequality  
365 and by (34)

$$\mathcal{L}(\{w_h < t\}) \leq \frac{M \gamma_{\varepsilon_h}}{(1-t)^2}. \tag{35}$$

366 The inequality ensures that the conditon (1) of Definition 4.1 holds for every sub level sets  
367  $\{w_h < t\}$ .

By Young'inequality and by (34) we get

$$M \geq 2 \int_{\Omega \setminus \{|\nabla w_h|=0\}} |\nabla w_h| \sqrt{W(w_h)} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left( \operatorname{div} \left( \frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) dx$$

368 Now by coarea formula and taking into account that  $|\nabla w_h|$  coincides with the norm of the  
369 tangent vector to the boundary of the  $t$ -level set we obtain

$$M \geq \int_0^1 \sqrt{W(t)} \int_{\{w_h=t\} \cap \{|\nabla w| \neq 0\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left( \operatorname{div} \left( \frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) d\mathcal{H}^1 dt \tag{36}$$



Thanks to the Sard's Lemma for any  $h$  there exists a  $\mathcal{L}$ -negligible set  $\mathcal{N}_{w_h} \subseteq (0, 1)$  such that

$$\{w_h = t\} = \partial\{w_h < t\}, \quad \{w_h < t\} \in R(\Omega), \quad \text{for } t \in (0, 1) \setminus \mathcal{N}_{w_h}$$

and on  $\{w_h = t\}$  for  $t \in (0, 1) \setminus \mathcal{N}_{w_h}$  we have

$$|\nabla w_h| \neq 0 \text{ and } \kappa(\{w_h = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right).$$

Now since the union  $\mathcal{N}_w$  of the sets  $\mathcal{N}_h$  is  $\mathcal{L}$ -negligible (almost countable) from (36) we have

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt$$

370 By applying Fatou's Lemma and taking into account that the set  $\mathcal{N}_w$  does not depend on  $h$   
371 we get

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt \quad (37)$$

372 Hence we deduce the existence of a  $\mathcal{L}$ -negligible set  $Q_w$ , with  $\mathcal{N}_w \subseteq Q_w$ , such that

$$\liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 \leq M_t \quad (38)$$

373 where the constant  $M_t$  does not depend on  $h$ .

Then for every  $t \in (0, 1) \setminus Q_w$  we can extract a sequence  $w_{h,t}$  such that  $\partial\{w_{h,t} < t\}$  converges with respect to the Hausdorff metric to a set  $P^t \subset \overline{\Omega}$ . Let  $\mathcal{N} = \{t_i\}$  in  $(0, 1)$  be a dense countable set. Up to a diagonal argument we can find a subsequence  $\{w_{h_k}\}$  such that, for every  $t^i \in \mathcal{N}$ ,  $\partial\{w_{h_k} < t_i\}$  converges to  $P^{t_i}$ . Let  $t_i \in \mathcal{N}$  such that  $t_i > t$  and consequently  $\{w_{h,k} < t\} \subseteq \{w_{h,k} < t_i\}$ . From the definition of Hausdorff convergence it follows that for every  $\rho > 0$  there exists  $k_0(\rho)$  such that for any  $k > k_0$  we have  $\{w_{h,k} < t_i\} \cap B_\rho(x) \neq \emptyset$  for every  $x \in P^{t_i}$ . Since the  $t_i$ -level set is open for every  $\rho$  and for every  $x \in P^{t_i}$  such that  $\{w_{h,k} < t_i\} \cap B_\rho(x) \neq \emptyset$ , we may choose  $t_n \in \mathcal{N}$  with  $t_n < t$  and so obtain for  $k$  large enough  $\{w_{h,k} < t_n\} \cap B_\rho(x) \neq \emptyset$ . By choosing  $t_{max} = \max_{x \in P^{t_i}} t_n(x)$  for every  $x \in P^{t_i}$ , the inclusion  $\{w_{h,k} < t_{max}\} \subset \{w_{h,k} < t\}$  gives

$$\{w_{h,k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h,k} < t\} \cap B_\rho(x) \subseteq \{w_{h,k} < t_n\} \cap B_\rho(x)$$

374 with  $t_{max}, t_n \in \mathcal{N}$ . Then by taking the limit  $\delta \rightarrow 0^+$  we infer  $\partial\{w_{h,t} < t\}$  converges with  
375 respect to the Hausdorff metric to a set  $P^t \subset \overline{\Omega}$  for every  $t \in (0, 1)$ .

376 Finally as in the proof of Theorem 4.2 we can extract a subsequence  $\{u_{h_k}\}$  which con-  
377 verges strongly in  $L^p(\Omega)$  to a function  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  with  $P_{\nabla u} \subset P^t$  for every  $t \in (0, 1)$ .  
378 Therefore we have that for every  $t \in (0, 1)$  the pair  $(u_h, \{w_{h,t} < t\})$  S-converges to  $u$  and  
379 the proof is achieved.  $\square$

## 5.2 Lower bound

**Theorem 5.3.** *Let  $\{\varepsilon_h\}$  be a sequence of positive numbers converging to 0. For every  $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$  and for every sequence  $\{u_h, w_h\} \subset S(\Omega)$   $S$ -converging to  $u$  the following inequality holds*

$$\liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}(u, w) \geq F(u).$$

**Proof.** Let  $u \in \Delta\mathcal{AM}^p(\Omega)$  and  $\{u_h, w_h\} \subset S(\Omega)$   $S$ -converging to  $u$ . As usual up to subsequences we can assume

$$+\infty > \liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}.$$

For any  $t \in (0, 1)$  we have (see also [11] for a similar argument)

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 = \int_{\{w_h < t\} \cap \Omega} w_h^2 |\Delta u_h|^2 + \int_{\{w_h \geq t\} \cap \Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \quad (39)$$

Let  $\{\Omega_j\}_j$  be a sequence of open sets  $\Omega_j \subset \subset \Omega \setminus P^t$  invading  $\Omega \setminus P^t$ . Then we may assume that  $u_h \rightarrow u$  weakly in  $W_0^{1,p}(\Omega)$  and  $\Delta u_h$  converges weakly in  $L^2(\Omega_j)$  to  $\Delta u$ . Therefore as in the proof of Theorem 4.3 we get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx \quad (40)$$

and

$$\liminf_{h \rightarrow +\infty} t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx$$

for any  $j$ .

Then by (39) and, by taking into account that  $|\Delta u|$  is an  $L^2(\Omega)$  function, it follows that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} |\Delta u|^2 dx$$

And eventually by taking the limit  $t \rightarrow 1$

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq \int_{\Omega} |\Delta u|^2 dx \quad (41)$$

By definition we know that for every  $t \in (0, 1)$   $\mathcal{L}^d(\{w_h < t\}) \rightarrow 0$  and  $\partial\{w_h < t\} \rightarrow P^t$  in the Hausdorff distance. So that, as in the proof of Theorem 4.3 (inequality 23) we have

$$\liminf_{h \rightarrow +\infty} \frac{1}{4\pi} \int_{\partial\{w_h < t\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}) \quad (42)$$

Now arguing as in the proof of Theorem 5.2 and by taking into account (42), we get

$$\begin{aligned}
& \liminf_{h \rightarrow \infty} \int_{\Omega \setminus \{\nabla w = 0\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left( \operatorname{div} \left( \frac{\nabla w}{|\nabla w|} \right) \right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \right. \\
& \geq 2 \liminf_{h \rightarrow +\infty} \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 dt \\
& \geq 2 \int_{(0,1)} \mathcal{H}^0(P^t) \sqrt{W(t)} dt \geq 8\pi C \mathcal{H}^0(P_{\nabla u})
\end{aligned} \tag{43}$$

By collecting (40) (41) and (43) we achieve the thesis.  $\square$

### 5.3 Upper bound

As in [11] to build  $w_h$  we use the construction used by Bellettini in [6], while the optimal sequence  $u_k$  is given as in Theorem 4.4.

**Theorem 5.4.** *Let  $\varepsilon_h$  be a sequence of positive numbers with  $\varepsilon_h \rightarrow 0$ . For every  $u \in \Delta\mathcal{M}^{p,2}(\Omega)$  there exists a sequence  $(u_h, D_h) \in S(\Omega)$   $S$ -converging to  $u$  such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{44}$$

**Proof.** If  $A \subset \mathbb{R}^2$  we set

$$\delta_A(x) = d(x, A) - d(x, \mathbb{R}^2 \setminus A).$$

We start by the construction of  $w_h$ .

As in the proof of Theorem 4.4 we set  $P_{\nabla u} = \{x_1, \dots, x_n\}$  and we define

$$D_h = \bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

Since  $D_h$  is a regular set by taking into account the condition (33) for  $h$  large enough we have

$$\{x \in \Omega : d(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset \subset \Omega. \tag{45}$$

Let  $\eta$  be the optimal profile for the Modica Mortola energy, that is the solution of

$$\begin{cases} \eta'(t) = \sqrt{W(t)} & \text{on } \mathbb{R} \\ \eta(-\infty) = 0, \\ \eta(+\infty) = 1 \end{cases}$$

given by  $\eta(t) = \frac{1}{2}(1 + \tanh \frac{t}{2})$ .

For every  $h$  let  $\psi_h : [0, +\infty) \rightarrow [0, 1]$  be a  $C^\infty$  function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h|, +\infty] \\ \psi_h' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|\psi_h'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O\left(\frac{1}{|\log \varepsilon_h|}\right) \end{cases}$$

As in [6] and in [11] we define

$$\eta_h(t) = \begin{cases} \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) + 1 - \psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t \geq 0 \\ \psi\left(\frac{t}{\varepsilon_h}\right) - \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t < 0 \end{cases}$$

Then we set  $w_h(x) = \eta_h(\delta_{D_h}(x))$ . We claim that  $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$  for  $h$  large. Indeed for any  $x \in \Omega \setminus D_h$  we have  $\delta_{D_h}(x) \geq 0$ , hence

$$1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$$

400 and then  $0 \leq 1 - w_h \leq 1$  by the properties of  $\psi_h$  and  $\eta$ . The case  $x \in D_h$  is similar. Let now  
 401  $x \in \partial\Omega$  then  $\delta_{D_h}(x) \geq 0$  and  $1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$ . From (45), it follows  
 402  $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$  for  $h$  large enough, hence the claim follows. Then we take  $\{(u_h, w_h)\}$  as  
 403 an optimal sequence, where  $u_h$  is given as in Theorem 4.4

404 First all we have to check that  $(u_h, w_h(x))$  S-converges to  $u$ . For any  $x \in \Omega \setminus P_{\nabla u}$  we have  
 405 that for  $h$  large enough  $\delta_{D_h}(x) \geq 0$  and one can check that  $w_h(x) \rightarrow 1$  for every  $x \in \Omega \setminus P_{\nabla u}$ .  
 406 This implies that  $\mathcal{L}(\{w_h < t\}) \rightarrow 0$  for every  $t \in (0, 1)$ . Now for every  $t \in (0, 1)$  we write

$$\{w_h = t\} = (\{w_h = t\} \cap D_h) \cup (\{w_h = t\} \cap \Omega \setminus D_h). \quad (46)$$

407 Hence, since  $w_h(x) \rightarrow 1$  for  $x \in \Omega \setminus D_h$ , for any  $t \in (0, 1)$  there exists  $h(t)$  such that  
 408  $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$  for every  $h \geq h(t)$ . So that from (46) it follows that for every  
 409  $t \in (0, 1)$ ,  $\{w_h = t\} \rightarrow P_{\nabla u}$  when  $h \rightarrow +\infty$ . So we can conclude that  $(u_h, w_h)$  S-converges  
 410 to  $U$ .

As in [6] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h |\log \varepsilon_h|\}, \quad D_h^2 = \{x \in \Omega : \varepsilon_h |\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h| |\log \varepsilon_h|\}$$

and we split

$$\begin{aligned} & \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &= \int_{D_h^1} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx + \\ & \quad \int_{D_h^2} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx = \\ & \quad = I_h + II_h \end{aligned} \quad (47)$$

For  $x \in D_h^1$  we have  $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$  therefore  $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$ . By taking into account the definition of  $\eta$  we have  $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$ . Moreover it is easy to check that

$$\eta'_h(t) = \frac{1}{\varepsilon_h} \eta'(t) = \frac{1}{\varepsilon_h} \sqrt{W(t)}; \quad |\nabla w_h| = |\eta'_k(\delta_{D_h})|$$

411

This, together with the coarea formula yields

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h=t\}} \left( \frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt$$

Now we have that  $\partial D_h = \{w_h = \frac{1}{2}\}$  then

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} \left( \frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)}$$

412 Then

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt \quad (48)$$

413 Moreover with the same argument and by using the definition of  $w_h$  one can check that

$$\lim_{h \rightarrow \infty} II_h = 0 \quad (49)$$

414 By (48) and (49) we have

415

$$\lim_{h \rightarrow \infty} \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w=0\}} \left( \frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left( \operatorname{div} \left( \frac{\nabla w}{|\nabla w|} \right) \right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) \right) dx = \mathcal{H}^0(P_{\nabla u}) \quad (50)$$

416 Now we take care of the terms involving  $u_h$ . As in the proof of Theorem 4.4 we have

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p \leq \int_{\Omega} |\nabla u - U_0|^p \quad (51)$$

Furthermore by taking into account that  $w_h(x) = 1$  if  $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$  and  $w_h(x) = 0$  if  $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$  we have

$$\int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \int_{\Omega \setminus D_h^0} |\Delta u_h|^2 dx$$

417 where  $D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$ .418 By choosing  $\rho_h$  in such a way that  $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$  we obtain

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega \setminus P_{\nabla u}} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx \quad (52)$$

419 Finally from the definition of  $w_h$ , it follows that  $w_h = 1$  outside the disk  $(D_h)_{2 \log \varepsilon_h}$  and  
 420 hence thanks to (32) and (33)

$$\limsup_{h \rightarrow +\infty} \frac{1}{\gamma_h} \int_{\Omega} (1 - w_h)^2 dx \leq \lim_{h \rightarrow +\infty} \mathcal{L}(D_{h 2 \log \varepsilon_h}) \frac{1}{\gamma_h} = 0. \quad (53)$$

421 The thesis follows by collecting (50), (51), (52) and (53)  $\square$

## 422 6 De Giorgi's Conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

423 where  $D$  is a regular set and  $\kappa$  is a curvature of its boundary  $\partial D$ .

424 Since  $\partial D$  can be represented as the discontinuity set of the function  $w_0 = 1 - \chi_D$ ,  
425 by Modica-Mortola's Theorem it follows that there is a sequence of non constant local  
426 minimizers such that  $w_\varepsilon \rightarrow w_0$  with respect to the  $L^1$ -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(w_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

427 Furthermore looking at the Euler-Lagrange equation associated to a contour length term,  
428 yields a contour curvature term  $\kappa$ , while the Euler-Lagrange equations for the functional  
429  $F_\varepsilon^1(w)$  contains a term  $2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon}$ .

430 Then De Giorgi suggested to approximate by  $\Gamma$ -convergence the functional  $F^2$  by adding  
431 to the Modica-Mortola approximating functionals the following term

$$F_\varepsilon^2(w) = \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx.$$

432 In ([7]) Bellettini and Paolini have proven the  $\limsup$  inequality, while the validity of  
433  $\liminf$  for a modified version of the original conjecture has been proven by Röger and Shätzle  
434 (see [26]).

Inspired by the De Giorgi's conjecture it appears natural investigate, in the spirit of [11],  
the possibility of approximating the functional  $\mathcal{F}$  by a means of a sequence  $\mathcal{F}_\varepsilon$  much more  
convenient from a numerical point view (see [22]):

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &= \int_{\Omega} w^2 |\Delta u|^2 dx + \frac{1}{8\pi C} \left( \frac{\beta_\varepsilon}{2\varepsilon} \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \right) \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} (1 - w^2) dx. \end{aligned}$$

435 The presence of the term  $\frac{1}{2\varepsilon}$  will be clear in the proof.

436 By the way we are able to prove only the  $\Gamma$ -limsup inequality.

437 **Theorem 6.1.** *Let  $\varepsilon_h$  be a sequence of positive numbers with  $\varepsilon_h \rightarrow 0$ . For every  $u \in$*   
438  *$\Delta \mathcal{M}^{p,2}(\Omega)$  there exists a sequence  $(u_h, D_h) \in S(\Omega)$   $S$ -converging to  $u$  such that*

$$\limsup \mathcal{F}_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{54}$$

439 **Proof.** Let  $(u_h, w_h)$  the optimal sequence of Theorem 5.4.

It is not difficult to see that for every  $x \in D_h^1$  the function  $\delta_h(x)$  is regular and using the definition of  $w_h$  and taking into account that  $\eta' = \sqrt{W(\eta)}$  the following identity holds

$$2\varepsilon_h \Delta w_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h \Delta \delta_{D_h}(x) + 2\varepsilon_h \eta''_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h (\delta_{D_h}(x)).$$

For  $h$  large we also have  $\Delta \delta_{D_h}(x) = \kappa(\{\delta_{D_h}(x) = t\})$ . Besides on  $D_h^1$  we have  $w_h(x) = \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})$  and hence the level set  $\{\delta_{D_h}(x) = t\}$  corresponds to the level set  $\{w_h(x) = \eta(\frac{t}{\varepsilon_h})\}$  with  $0 \leq \eta \leq 1$ , so that we infer

$$\kappa(\{\delta_{D_h}(x) = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)$$

By proceeding as in the proof of Theorem 5.4 and taking into account the equality  $2\varepsilon_h |\eta'_h(\delta_{D_h}(x))| = 2\varepsilon_h |\nabla w_h|$  we have

$$I_h = \int_{D_h^1} \frac{\beta_{\varepsilon_h}}{2\varepsilon_h} \left(2\varepsilon_h \Delta w - \frac{W'(w)}{\varepsilon_h}\right)^2 + \frac{1}{\beta_{\varepsilon_h}} (\varepsilon_h |\nabla w_{\varepsilon_h}|^2 + \frac{1}{\varepsilon_h} W(w_{\varepsilon_h})) dx = 2 \int_{D_h^1} \left(\beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)^2 + \frac{1}{\beta_{\varepsilon_h}}\right) \sqrt{W(w_h)} |\nabla w_h| dx$$

then as in the proof of Theorem 5.4 we conclude

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt.$$

440 By the same calculation on  $D_h^2$  one can check that the integral over  $D_h^2$  vanishes as in the  
 441 proof of Theorem 5.4.





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