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***Variational approximation for detecting points-like
target problems.***

Daniele Graziani, Gilles Aubert

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Thème COG



R *apport
de recherche*

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39 1 Introduction

40 The issue of detecting fine structures, like points or curves in two or three dimensional
 41 biological images, is a crucial task in image processing. In particular a point may represent
 42 a viral particle whose visibility is compromised by the presence of other structures like cell
 43 membranes or some noise. Therefore one of the main goals in the application is detecting
 44 the spots that the biologists wish to count. This operation is made harder by the presence
 45 of the other singular structures like noise, textures or filaments.

46 In some biological images the image intensity is a function that takes the value 1 on
 47 points or other structures like set with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0
 48 outside. In image processing these concentration sets are called discontinuity without jump,
 49 meaning that there is no jump across the set and therefore the gradient of the image is 0.

50 In the literature there are few variational methods dealing with this problem. In this
 51 direction one interesting approach has been proposed in [5]. In that paper the authors
 52 consider these kind of pathology as a k -codimension object, meaning that they should be
 53 regarded as a singularity of a map $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [8] for a
 54 complete survey on this subject). So that the detecting point case corresponds to the case
 55 $k = 2$ and $m = 0$.

56 This point of view makes possible a variational approach based on the theory of Ginzburg-
 57 Landau systems. In their work the isolated points in 2-D images are regarded as the topo-
 58 logical singularities of a map $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$, where \mathbb{S}^1 is a unit sphere of \mathbb{R}^2 . This strategy
 59 makes crucial the construction, starting from the initial image $I : \mathbb{R}^2 \rightarrow \mathbb{R}$, of an initial
 60 vector field $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ with a topological singularity of degree 1, and how to build such
 61 a vector field in a rigorous way, is a subject of a current investigation.

62 Our first purpose here is finding a most natural variational framework in which a rigorous
 63 definition of discontinuity without jump can be given.

In our model the image I is a Radon measure, that is an element of $(C_0(\Omega))^*$: the dual
 space of $C_0(\Omega)$ (the closure with respect to the sup-norm of the space of continuous functions
 with compact support). It is crucial for detecting points that this Radon measure be capable
 of charging points. The preliminary step is finding a space whose elements are capable of
 producing these kind of measures. This space is given by $\mathcal{DM}^p(\Omega)$: the space of the L^p -
 vector fields whose distributional divergence measure is a Radon measure, with $1 < p < 2$.
 The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence $\text{Div}U$
 of U cannot be a measure concentrated on points. (see section 3.1 below). Then we have
 to construct from the original image I a data vector field $U_0 \in \mathcal{DM}^p(\Omega)$. Clearly there are,
 at least in principle, many ways to do this. The one we propose here seems to be the most
 natural. We consider the classical elliptic problem with measure data I :

$$\begin{cases} \Delta u_0 = I & \text{on } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

64 Then by setting $U_0 = \nabla u_0$ we have $U_0 \in \mathcal{DM}^p(\Omega)$ with $\text{Div}U_0 = I$. Nevertheless the
 65 support of the measure $\text{Div}U_0$ is too large and could contains several structures like curves

66 or fractal sets, while the singularities, we are interested in, are contained in the atomic part
 67 of the measure $\text{Div}U_0$ and therefore we have to isolate it. To do this the notion of p -capacity
 68 of a set plays a key role. Indeed when $p < 2$ the p -capacity of a point in Ω is zero and besides
 69 every Radon measure can be decomposed (see [17]) in two mutually singular measures: the
 70 first one is absolutely continuous with respect the p -capacity and the second one is singular
 71 with respect to the p -capacity, meaning that it is a measure concentrated on sets with 0
 72 p -capacity.

73 Moreover for a compact subset $E \subset \Omega$ a necessary and sufficient condition to have 0
 74 p -capacity is the existence of a sequence of test functions $\{\phi_k\}$ converging to 0 strongly in
 75 $W_0^{1,p}$ and equals to 1 on E for every k , and hence one can say, in this sense, that a set of 0
 76 p -capacity is a discontinuity with no jump.

77 As well known in dimension 2 the sets with 0 p -capacity, and hence discontinuity without
 78 jump, can be isolated points, countable set of points, fractal sets with Hausdorff dimension
 79 $0 \leq \alpha < 1$ (see subsection 2.3 for the definition of p -capacity and related properties).

80 By the way the goal here is keeping nothing else but points in the image. The achieve-
 81 ment of such a purpose makes necessary the minimization of a suitable energy that must
 82 remove all the discontinuity which are not discontinuities without jump, and remove all the
 83 discontinuities without jump which are no isolated point.

84 From one hand we have to force the concentration set of the divergence measure of U
 85 to contain only the points we want to catch, and on the other hand we have to regularize
 86 the initial data U_0 outside the points of singularities. To this end we introduce the auxiliar
 87 space $SDM^p(\Omega)$ of the vector field belonging to $\mathcal{DM}^p(\Omega)$ whose divergence measure has
 88 no absolutely continuous part with respect to the p -capacity. Then, by taking into account
 89 that the initial vector field is a gradient of a Sobolev function, our goal is to minimize
 90 the following energy, involving a competition between a divergence term and the counting
 91 Hausdorff measure \mathcal{H}^0 ,

$$\mathcal{F}(U) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\text{supp}(\text{div}^s \nabla u)_0)$$

92 where $u \in W_0^{1,p}(\Omega)$ with $\nabla u \in SDM^p(\Omega)$ and $p < 2$. The gradient of a minimizer of the
 93 energy \mathcal{F} is the vector field we are looking for, that is a vector field whose divergence measure
 94 can be decomposed in an absolutely continuous (with respect to the Lebesgue's measure)
 95 term plus an atomic measure concentrated on the points we want to catch in the image.

96 Even if a pointwise characterization of discontinuity without jump is not available, thanks
 97 to our definition the singular set of points can be linked to the vector field ∇u , in the spirit of
 98 classical SBV formulation of the Mumford-Shah's Functional. (we refer to [1] for a complete
 99 survey on Mumford Shah's functional).

Our next task is to provide an approximation in the sense of the Γ -convergence introduced
 by [19, 20]. Our approach is close in the spirit to the one used to approximate the Mumford
 Shah functionals by a families of depending curvature functionals in [11]. Indeed as in their

work (see also [10]) we replace the atomic measure \mathcal{H}^0 by the term

$$G_\varepsilon(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

100 where D is a proper regular set containing the atomic set P , κ is the curvature of its
 101 boundary, the constant $\frac{1}{4\pi}$ is a normalization factor. Roughly speaking the minimum of
 102 these functionals are achieved on the union of balls of small radius, so that when $\varepsilon \rightarrow 0$ the
 103 sequence G_ε shrinks to the atomic measure $\mathcal{H}^0(P)$.

104 This leads to an intermediate approximation given by

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (1)$$

105 This strategy permits to work with the perimeter measure $H^1[\partial D]$, that can be approx-
 106 imated, according to the Modica-Mortola's approach (see [24, 25]), by the measure:

$$\mu_\varepsilon(w, \nabla w) dx = \left(\varepsilon |\nabla w|^2 + \frac{V(w)}{\varepsilon} \right) dx,$$

107 where $V(w) = w^2(1 - w)^2$ is a double well functional.

108 Besides by using the Sard Theorem and the Coarea Formula (see also [6] for a similar
 109 approach) one can formally replace in the second integral on the boundary of D the curvature
 110 κ by the integral of the $\operatorname{div} \frac{\nabla w}{|\nabla w|}$ and the integral is computed over the level sets of w . So
 111 that one can formally write the complete approximating sequence:

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_{\Omega} w^2 |\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_{\Omega} (1 - w_\varepsilon)^2 dx \end{aligned}$$

112 where, as usual, $C = \int_0^1 \sqrt{W(t)} dt$, β_ε and γ_ε are infinitesimal as $\varepsilon \rightarrow 0$. The last integral is
 113 a penalization term that forces w to tend to 1 as $\varepsilon \rightarrow 0$.

114 The goal of the second part of this work is then to show that the family of energies \mathcal{F}_ε
 115 Γ -converges to the functional \mathcal{F} when the parameters are related in a suitable way.

116 As in [11] we deal with the Hausdorff convergence for sets of points and with a suitable
 117 convergence of functions involving the convergence of a sub-level sets

118 Despite this approach is inspired by some ideas contained in [10, 11], we point out that
 119 in our case the regularization term involves a second order differential operator, due to the
 120 fact that our goal is to detect points and not segment curves. This deep difference requires
 121 a non trivial adaption of the arguments used in those papers.

122 The paper is organized as follows. Section 2 is devoted to notation, preliminaries, defi-
123 nitions and result. In section 3 we illustrate the new variational model and we present the
124 functional we deal with. Section 4 and 5 are devoted to the Γ -convergence result Finally in
125 the last section we conclude the paper by comparing this approach with the celebrate con-
126 jecture by De Giorgi concerning the approximation of the curvature depending functionals.
127 We do not give here experimental result illustrating our approach. We refer the reader
128 for that to [22].

129 2 Preliminaries

130 2.1 Notation

131 In all the paper $\Omega \subset \mathbb{R}^2$ is an open bounded set with C^2 -boundary. The operation \cdot stands
 132 for the usual scalar product. The Euclidean norm will be denoted by $|\cdot|$, while the symbol
 133 $\|\cdot\|$ indicates the norm of some functional space. The brackets $\langle \cdot, \cdot \rangle$ denotes the duality
 134 product in some distributional spaces. \mathcal{L}^d or dx is the d -dimensional Lebesgue measure and
 135 \mathcal{H}^k is the k -dimensional Hausdorff measure. $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ .
 136 We say that a set $D \in \Omega$ is a regular set if it can be written as $\{F < 0\}$ with $F \in C_0^\infty(\Omega)$.
 137 In the following we will denote by $R(\Omega)$ the set of the all regular sets in Ω .

138 2.2 Distributional divergence and classical spaces

139 In this subsection we recall the definition of the distributional space $L^{p,q}(\text{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$
 140 (see [4, 14]). We start by recalling the distributional definition of the divergence operator.
 141 Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a regular vector field. If $U = (u, v)$
 142 then classically $\text{div}U = u_{x_1} + v_{x_2}$. In the sense of distribution $\text{Div}U$ is defined

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} (u, v) \cdot \nabla \varphi$$

143 for all $\varphi \in C_0^\infty(\Omega)$.

144 **Definition 2.1.** We say that $U \in L^{p,q}(\text{div}; \Omega)$ if $U \in L^p(\Omega; \mathbb{R}^2)$ and if its distributional
 145 divergence $\text{Div}U = \text{div} \in L^q(\Omega)$. If $p = q$ the space $L^{p,q}(\text{div}; \Omega)$ will be denoted by $L^p(\text{div}; \Omega)$.

146 We say that $U \in L_{loc}^{p,q}(\text{div}; \Omega)$ if $U \in L^{p,q}(\text{div}; A)$ for every open set $A \subset \subset \Omega$.

147 We say that a function $u \in W^{1,p}(\Omega)$ belongs to $W^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$. We
 148 say that a function $u \in W_0^{1,p}(\Omega)$ belongs to $W_0^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$.

Definition 2.2. For $U \in L^p(\Omega; \mathbb{R}^2)$, $1 \leq p \leq +\infty$, set

$$|\text{Div}U|(\Omega) := \sup\{\langle U, \nabla \varphi \rangle : \varphi \in C_0^\infty(\Omega), |\varphi| \leq 1\}.$$

We say that U is an L^p -divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$ if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\text{Div}U|(\Omega) < +\infty.$$

149 We say that $U \in \mathcal{DM}_{loc}^p(\Omega)$ if $U \in \mathcal{DM}^p(A)$ for every open set $A \subset \subset \Omega$.

150 Let us recall the classical result (see [15] Proposition 3.1)

151 **Theorem 2.1.** Let $\{U_k\}_k \subset \mathcal{DM}^p(\Omega)$ such that

$$U_k \rightharpoonup U \quad L^p(\Omega), \text{ for } 1 \leq p < +\infty \quad (2)$$

Then

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_{h \rightarrow \infty} \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \quad |\text{Div}U|(\Omega) \leq \liminf_{h \rightarrow \infty} |\text{Div}U_h|(\Omega)$$

152 Moreover for a vector field $U \in L^{p,q}(\text{div}; \Omega)$ it is possible to define a trace operator as
 153 stated by the next proposition (see [28], Proposition 3.1)

154 **Proposition 2.1.** *Let Ω be a bounded, C^2 domain. There is a continuous linear operator*
 155 $\gamma_\nu : U \in L^{p,q}(\text{div}; \Omega) \rightarrow (C^1(\partial\Omega))^*$ such that

156 (i) for each $\varphi \in C^1(\partial\Omega; \mathbb{R})$ and each $U \in L^{p,q}(\text{div}; \Omega)$,

$$\langle \gamma_\nu(U), \varphi \rangle = \int_\Omega U \cdot \nabla \psi + \int_\Omega \psi \text{div} U \quad (3)$$

157 whenever $\psi \in C^1(\overline{\Omega})$ and $\psi_{\partial\Omega} = \varphi$.

158 (ii) for $U \in C^1(\overline{\Omega}; \mathbb{R}^2)$, $\gamma_\nu(U) = U \cdot \nu_{|\partial\Omega}$.

159 2.3 p -capacity

The p -capacity will be crucial to find a convenient functional framework to deal with. If $K \subset \mathbb{R}^2$ is a compact set and χ_K denotes its characteristic function, we define:

$$\text{Cap}_p(K, \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p dx \mid f \in C_0^\infty(\Omega), f \geq \chi_K \right\}$$

If $U \subset \Omega$ is an open set, the p -capacity is given by

$$\text{Cap}_p(U, \Omega) = \sup_{K \subset U} \text{Cap}_p(K, \Omega).$$

Finally if $A \subset \Omega$ is a Borel set

$$\text{Cap}_p(A, \Omega) = \inf_{A \subset U \subset \Omega} \text{Cap}_p(U, \Omega).$$

160 We recall the following result (see for instance [23], Theorem 2.27) theorem that explains
 161 the relationship between p -capacity and Hausdorff measure. Such a result is crucial to have
 162 geometric information on the null p -capacity sets.

163 **Theorem 2.2.** *Assume $1 \leq p < 2$. If $\mathcal{H}^{2-p}(A) < \infty$ then $\text{Cap}_p(A, \Omega) = 0$*

164 Another useful tool to manage the sets of p -capacity 0 is provided by the following
 165 characterization.

166 **Theorem 2.3.** *Let E be a compact subset of Ω . Then $\text{Cap}_p(E, \Omega) = 0$ if and only if there*
 167 *exists a sequence $\{\phi_k\} \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \leq \phi_k \leq 1$*
 168 *and $\phi_k = 1$ on E for every k .*

169 For a general survey we refer the reader to [21, 23, 29].

170 3 The Variational Model

171 This section is devoted to illustrate the functional framework we will work with.

Roughly speaking in biological images the image is a function that could be very high on points or other structures like set with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. In image processing these concentration sets are called discontinuity without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0. From a mathematical point of view it seems to be much more appropriate to think of the image as a Radon measure, that is $I = \mu \in (C_0(\Omega))^*$. The next step is finding a space whose elements are capable of producing these discontinuity without jump. This space is given by $\mathcal{DM}^p(\Omega)$, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence of U cannot be a measure concentrated on points. Set $p \geq 2$. Looking at the definition we have

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} U \cdot \nabla \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

172 Since $p \geq 2$ this distribution is a well defined even for any test $\varphi \in W_0^{1,p'}(\Omega)$ where $p' \leq 2$
 173 is the dual exponent of p . In particular $\text{Div}U$ belongs to the dual space $W^{-1,p'}(\Omega)$ of the
 174 Sobolev space $W_0^{1,p}(\Omega)$. Then in this case, the distributional divergence of U cannot be
 175 an atomic measure, whereas $\delta_0 \notin W^{-1,p'}(\Omega)$. To see this, one can consider as Ω the disk
 176 $B_0(1)$ and the function $\tilde{\varphi}(x) = \log(\log(1 + |x|)) - \log(\log(2))$. This function is in the space
 177 $W_0^{1,p'}(\Omega)$ for every $p' \leq 2$ and therefore it is an admissible test function and it easy to check
 178 that $\langle \delta_0, \varphi \rangle = +\infty$.

179 When $1 < p < 2$ we have that $\text{Div}U \in W^{-1,p'}(\Omega)$, but in this case since $p < 2$, we have
 180 $p' > 2$ and hence the function $\tilde{\varphi}$ is no longer an admissible test function. One can check
 181 that the distribution $\text{Div}U$ is an element of $(C_0(\Omega))^*$ capable of charging the points. Take
 182 for instance the map $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$.

183 The next step is to transform the initial image I as the divergence measure of a suitable
 184 vector field. We consider the elliptic problem with measure data I :

$$\begin{cases} \Delta u = I & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

Classical results (see [27]) ensures the existence of a unique weak solution $u \in W_0^{1,p}(\Omega)$ with $p < 2$. Then it easy to see that the distributional divergence of ∇u is given by I . In particular by setting $U = \nabla u$, we have $U \in \mathcal{DM}^p(\Omega)$. According to the Radon Nykodim decomposition of the measure $\text{Div}U$ we have

$$\text{Div}U = \text{div}U + \text{div}^s U,$$

185 where $\text{div}U \in L^1(\Omega)$. For our purpose the support of the singular measure $\text{div}^s U$ is too large.
 186 In particular the measure $\text{div}^s U$ could charge sets with Hausdorff dimension $0 \leq \alpha < 2$,

187 hence it can be concentrated also on discontinuity with jump. So that in order to isolate
 188 the singularities we are interested in, we need a further decomposition of the measure $\text{Div}U$.
 189 This can be done by using the capacity decomposition of the Radon measure $\text{div}^s U$. It is
 190 known (see [17]) that given a Radon measure μ the following decomposition holds

$$\mu = \mu_a + \mu_0 \quad (5)$$

191 where the measure μ_a is absolutely continuous with respect to the p -capacity and μ_0 is
 192 singular with respect to the p -capacity, that is concentrated on sets with 0 p -capacity. Besides
 193 it is also known (see [17]) that every measure which is absolutely continuous with respect
 194 to the p -capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer
 195 decomposition:

$$\mu = f - \text{Div}G + \mu_0 \quad (6)$$

196 where $G \in L^p(\Omega; \mathbb{R}^2)$, $f \in L^1(\Omega)$.

197 By applying this decomposition to the measure $\text{div}^s U$ we obtain the following decom-
 198 position of the measure $\text{Div}U$

$$\text{Div}U = \text{div}U + f - \text{Div}G + (\text{div}^s U)_0 \quad (7)$$

199 with $G \in L^p(\Omega)$.

200 According to this decomposition and taking into account Theorem 2.3 we give the defi-
 201 nition of discontinuity without and with jump.

202 **Definition 3.1.** *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity without jump*
 203 *of U if $x \in \text{supp}(\text{div}^s U)_0$.*

204 The other singularities where there is a jump, are contained in the second term of the
 205 decomposition (7). Indeed the space $W^{-1,p'}(\Omega)$ contains Hausdorff measures restricted to
 206 sub-manifolds of dimension greater or equal to 1 (see [29] chapter 4, section 7 for a detailed
 207 discussion on the space $W^{-1,p'}(\Omega)$), like for instance Hausdorff measures concentrated on
 208 regular closed curves, that is contours, that are a classical example of discontinuity with
 209 jump. More precisely a contour of a regular set is the jump set of the characteristic
 210 function of the set D and its p -capacity is strictly positive. This is of course in agreement
 211 with Theorem 2.3. Indeed, since the characteristic function of the set D is BV -function, it
 212 cannot be approximated by regular functions in the $W^{1,p}$ strong norm.

213 **Definition 3.2.** *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity with jump of U*
 214 *if $x \in \text{supp}(f - \text{Div}G)$.*

215 3.1 The variational framework

216 We shall build an energy whose minimizer will be a vector field whose singular part of its
 217 divergence measure will be given by nothing else but points.

218 This minimizer must be an L^p (with $p < 2$) vector field with the following properties:

- 219 1. It must be close to the initial data U_0 which is, in general, an L^p vector field U_0 with
 220 $1 < p < 2$.
- 221 2. The absolutely continuous part with respect to the Lebesgue measure of $\text{Div}U$ is an
 222 L^2 function.
- 223 3. The support of the measure $\text{div}^s U$ must be given by set of points P_U with $\mathcal{H}^0(P_U) <$
 224 $+\infty$

225 According to these considerations it seems to be natural to introduce the space

$$SDM^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \text{Div}G = 0\} \quad (8)$$

226 so that, as a consequence, the decomposition (7) yields for any $U \in SDM^p(\Omega)$

$$\text{Div}U = \text{div}U + (\text{div}^s U)_0. \quad (9)$$

227 For our purposes the following result concerning the features of the elements of the space
 228 $SDM^p(\Omega)$ will play a crucial role.

229 **Proposition 3.1.** *Let $U \in L^{p,q}(\text{div}; \Omega \setminus P)$ and $P \subset \Omega$ be a set of finite number of points
 230 then $U \in SDM^p(\Omega)$.*

231 **Proof.** We set $P = \{x_1, \dots, x_n\}$. Let $B_\rho(x_i)$ $i=1, \dots, n$. We can find a sequence $\rho(h) \rightarrow 0$ when
 232 $h \rightarrow +\infty$ such that $B_{\rho(h)}(x_i) \cap B_{\rho(h)}(x_j) = \emptyset$ for h large and $i \neq j$. Then we define the
 233 following sequence U_h .

$$U_h = \begin{cases} 0 & \text{on } \Omega \cap \bigcup_i^n B_{\rho(h)}(x_i) \\ U & \text{on } \Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i) \end{cases} \quad (10)$$

234 The sequence $U_h \in \mathcal{DM}^p(\Omega)$ for every h fixed. If $\varphi \in C_0^\infty(\Omega)$, by applying the Proposition
 235 2.1 to the vector field $U \in L^{p,q}(\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i))$. we get:

$$\begin{aligned} |\text{Div}U_h|(\Omega) &\leq \sup_{0 \leq \varphi \leq 1} \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} U \nabla \varphi \right| \leq \sup_{0 \leq \varphi \leq 1} \{ |\langle \gamma_\nu(U), \varphi |_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} \text{div}U \varphi \right| \} \leq \sup_{\varphi} |\langle \gamma_\nu(U), \varphi |_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \|\varphi\|_{L^\infty(\Omega)} \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C \|U\|_p + (1 + C) \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C_1 \end{aligned} \quad (11)$$

$$(12)$$

236 where C_1 does not depend on h .

Now since $\mathcal{L}(\bigcup_i^n B_{\rho(h)}(x_i)) \rightarrow 0$ we have $U_h \rightarrow U$ in $L^p(\Omega)$ the by Theorem 2.1

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_h \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \leq C$$

and by theorem 2.1

$$|\operatorname{Div}U|(\Omega) < \liminf_{h \rightarrow \infty} |\operatorname{Div}U_h| \leq C.$$

237 Therefore $U \in \mathcal{DM}^p(\Omega)$. Finally we know that $U \in L^{p,q}(\operatorname{div}; \Omega \setminus P)$ and the support
 238 of the measure $\operatorname{div}^s U$ is given by the set P , then, according to decomposition (7), the
 239 measure $f - \operatorname{Div}G$ vanishes on sets with 0 p -capacity, we deduce $f - \operatorname{Div}G = 0$, that is
 240 $U \in \mathcal{SDM}^p(\Omega)$. \square

241 3.2 The Functional

According to our purpose the natural energy to deal with is the following $F : \mathcal{SDM}^p(\Omega) \rightarrow [0, \infty]$ given by

$$F(U) = \int_{\Omega} |\operatorname{div}U|^2 dx + \int_{\Omega} |U - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s U)_0).$$

242 We note that $\inf F(U) > 0$ on $\mathcal{SDM}^p(\Omega)$. Indeed if we had $\inf_{\mathcal{SDM}^p(\Omega)} F(U) = 0$ then it
 243 would be possible exhibiting a minimizing sequence $U_n \rightarrow U_0$ in L^p such that $\operatorname{Div}U_n \rightarrow 0$
 244 in $\mathcal{D}'(\Omega)$. On the other hand, the L^p -distance between U_n and U_0 can be arbitrary small
 245 only if $\operatorname{Div}U_0 = 0$ as well, because the constraint $\operatorname{Div}U = 0$ is stable under L^p -convergence.
 246 Moreover to mimic the construction of U_0 we restrict ourselves to vector fields U which are
 247 the gradient of a function $u \in W_0^{1,p}(\Omega)$.

248

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \nabla u \in \mathcal{SDM}^p\} \quad (13)$$

249 So that the new energy $\mathcal{F} : \Delta\mathcal{M}^p(\Omega) \rightarrow (0, \infty]$ is given by

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s \nabla u)_0). \quad (14)$$

250 4 Γ -convergence: The intermediate approximation

251 The functional \mathcal{F} is finite on the class of the functions $u \in \Delta\mathcal{M}^p(\Omega)$ whose support of the
 252 measure $(\operatorname{div}^s \nabla u)_0$ is given by a finite set and the absolutely continuous part of the measure
 253 $\operatorname{Div} \nabla u$ is given by $\Delta u \in L^2(\Omega)$. Consequently it is convenient to introduce the following
 254 space:

$$\Delta\mathcal{AM}^{p,2}(\Omega) = \{u \in \Delta\mathcal{M}^p(\Omega) : \operatorname{supp}(\operatorname{div}^s \nabla u)_0 = P_{\nabla u} \text{ with } \mathcal{H}^0(P_{\nabla u}) < +\infty, \Delta u \in L^2(\Omega)\} \quad (15)$$

In the spirit [11] we introduce an intermediate approximation, in the sense of Γ -convergence, of the functional \mathcal{F} . We define a sequence of functionals where the counting measure $\mathcal{H}^0(P_{\nabla u})$ is replaced by a functional defined on regular sets D and which involves the curvature of the boundary ∂D . The approximating sequence is given by:

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (16)$$

255 Where $u \in W_0^{1,p,2}(\operatorname{div}; \Omega)$, D is a regular set, and κ denotes the curvature of its boundary .

256 In order to guarantee that the measure of the sets D is small we define a new functional
 257 still denoted by $F_\varepsilon(u, D)$ given by

$$F_\varepsilon(u, D) = \begin{cases} \int_{\Omega \setminus D} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) & \text{if } \mathcal{L}(D) \leq a(\varepsilon) \\ +\infty & \text{otherwise} \end{cases} \quad (17)$$

258 while the limit functional is

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(P_{\nabla u}).$$

259 where $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, $U_0 \in L^p(\Omega; \mathbb{R}^2)$.

260 To simplify the notation let us set $Y(\Omega) = \{(u, D) \mid u \in W_0^{1,p,2}(\operatorname{div}; \Omega), D \in R(\Omega)\}$,
 261 where $R(\Omega)$ is a regular set.

262 By adapting to our case the convergence introduced in [11], we endow the set $Y(\Omega)$ with
 263 the following convergence.

264 **Definition 4.1.** We say that $\{(u_h, D_h)\} \subset Y(\Omega)$ *A-converges* to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ if the
 265 following conditions hold

- 266 1. $\mathcal{L}(D_h) \rightarrow 0$
- 267 2. $\{\partial D\}_h \rightarrow P$ in the Hausdorff metric, where P is a finite set of points.
- 268 3. $u_h \rightarrow u$ in $L^p(\Omega)$ and $P_{\nabla u} \subseteq P$

269 The definition of Γ -convergence with respect to the convergence above is the following.

270 **Definition 4.2.** We say that a sequence G_ε Γ -converges to G if for every sequence of positive
271 numbers $\{\varepsilon_h\} \rightarrow 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -convergent to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \geq G(u)$$

2. there exists a sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -convergent to u such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \leq G(u)$$

272 By following this approach the fundamental theorem of the Γ -convergence can be stated
273 as follows.

Theorem 4.1. Let \mathcal{F}_ε be a sequence of functional Γ -convergent to \mathcal{F} . If $\{\varepsilon_h\}$ is a sequence
of positive numbers converging to zero and $\{(u_h, D_h)\} \subset Y(\Omega)$ such that

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} \mathcal{F}_{\varepsilon_h}) = 0$$

274 then there exists a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ and a minimizer $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ of
275 $\mathcal{F}(u)$, such that $\{(u_{h_k}, D_{h_k})\}$ A -converges to u .

276 By following the classical pattern (see for instance [12]), we will split the proof in three
277 steps. Theorem 4.1 will follow from: the compactness of the minimizing sequence of F_ε ,
278 the Γ lim inf inequality (1) and the Γ -limsup inequality (2) in Definition 4.2.

279 4.1 Compactness

280 We state and prove the following compactness result.

281 **Theorem 4.2.** Let h goes to $+\infty$ and $\varepsilon_h \rightarrow 0$

$$F_{\varepsilon_h}(u_h, D_h) \leq M \tag{18}$$

282 then there exist a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ and a function $u \in \Delta\mathcal{AM}^p(\Omega)$ such
283 that (u_{h_k}, D_{h_k}) A -converges to u .

Proof. We adapt an argument of [11]. We have $\mathcal{L}(D_h) \rightarrow 0$ otherwise $F_{\varepsilon_h} = +\infty$. Let
 D_h be a sequence of regular sets. Then we can parameterize every $C_h = \partial D_h$ by a finite
and disjoint union of Jordan curves. Let us set for every h $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$, then we have
according to the 2-dimensional version of Gauss-Bonnet's Theorem and Young's inequality

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h \kappa_h^2 \right) d\mathcal{H}^1 \geq \frac{1}{4\pi} \int_{\partial D_h} 2\kappa_h d\mathcal{H}^1 = \frac{1}{4\pi} \int_{\bigcup_h C_h} 2\kappa_h d\mathcal{H}^1 = m(h)$$

Note that the number $m(h) \leq M$, with $M \geq 0$, is independent of h . Then it is possible to extract a subsequence C_{h_k} with the number of curves in C_{h_k} equal to n for every k . Then we set $C_{h_k} = \{\gamma_{h_k}^1, \dots, \gamma_{h_k}^n\}$ for any k . From (18) we also have for any $\gamma \in C_{h_k}$ that $\mathcal{H}^1(\gamma) \leq 4\pi M \varepsilon_{h_k}$ and consequently $\max\{\mathcal{H}^1(\gamma) : \gamma \in C_{h_k}\} \rightarrow 0$. Then there exists a finite set of point $P = \{x_1, \dots, x_n\} \subset \overline{\Omega}$ such that for any radius ρ there is an index k_ρ with

$$\gamma_{h_k}^i \subset B_\rho(x_i) \text{ for all } k > k_\rho \text{ and } i \in \{1, \dots, n\},$$

284 so that if we set $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$, then the Hausdorff distance $d_H(\partial D_{h_k}, P) \rightarrow$
 285 0 since $\mathcal{L}^1(D_{h_k}) \rightarrow 0$ and therefore $\rho \rightarrow 0$ as well.

286 Now we prove the compactness property for u_h . First of all from the estimate

$$\|\nabla u_h\|_{L^p(\Omega)}^p \leq 2^p (\|\nabla u_h - U_0\|_{L^p(\Omega)} + \|U_0\|_{L^p(\Omega)}) \quad (19)$$

287 and (18) we may extract a subsequence $\{u_{h_k}\} \subset W_0^{1,p}(\Omega)$ weakly convergent to $u \in W_0^{1,p}(\Omega)$.

Let Ω_j be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P$ invading $\Omega \setminus P$. We claim that that it is possible to extract a sequence of D_{h_k} such that $\Omega_j \cap \partial D_{h_k} = \emptyset$. Indeed since the distance between Ω_j and P is positive for any j there exists η_j such that $\Omega_j \cap (\bigcup_i^n B_{\eta_j}(x_i)) = \emptyset$. On the other hand we know from before that for every ρ we can find k_ρ such that $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$. Then in particular if $\rho = \eta_j$ there exists k_j such that for all $k \geq k_j$

$$\Omega_j \cap \partial D_{h_k} = \emptyset.$$

288 Therefore for any $x \in \Omega_j$ there exists $\delta > 0$ such that either $B_\delta(x) \subset D_{h_k}$ or $B_\delta(x) \subset \Omega \setminus D_{h_k}$.

289 Finally by taking into account that $\mathcal{L}^1(D_{h_k}) \rightarrow 0$ we conclude $\Omega_j \cap \partial D_{h_k} = \emptyset$ for $k \geq k_j$.

290 Then for every $k \geq k_j$ we have that $u_{h_k} \in W^{1,p,2}(\text{div}; \Omega_j)$ and by (18) we get

$$\int_{\Omega_j} |\Delta u_h|^2 \leq \int_{\Omega \setminus D_{h_k}} |\Delta u_{h_k}|^2 dx \leq M \quad (20)$$

Then we can extract a further subsequence still denoted by $\{u_{h_k}\} \subset W^{1,p,2}(\text{div}; \Omega_j)$ such that

$$\begin{cases} u_{h_k} \rightarrow u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \text{ and a.e} \\ \nabla u_{h_k} \rightharpoonup \nabla u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \\ \Delta u_{h_k} \rightharpoonup \Delta u & \text{in } L^2(\Omega_j). \end{cases}$$

291 Let now $x \in \Omega' \subset \subset \Omega \setminus P$. Then there exists a sequence $x_j \rightarrow x$ with $j \in \mathbb{N}$. By applying
 292 the diagonal argument to the sequence $u_{h_{k_l}}(x_j)$ we obtain a subsequence $u_l = u_{h_{k_l}}(x_l)$ such
 293 that Δu_l converges weakly in $L^{p'}(\Omega')$ to Δu for any $\Omega' \subset \subset \Omega$. Then by the semicontinuity
 294 of the L^2 -norm we have

$$\sup_j \int_{\Omega_j} |\Delta u|^2 dx \leq \sup_j \liminf_{l \rightarrow +\infty} \int_{\Omega_j} |\Delta u_l|^2 dx \leq M.$$

295 If we set $\tilde{P} = P \setminus \partial\Omega$, then we deduce $u \in W_0^{1,p,2}(\text{div}; \Omega \setminus \tilde{P})$ and therefore $\nabla u \in S\mathcal{DM}^p(\Omega)$
 296 with $P_{\nabla u} \subseteq P$, by Proposition 3.1. So we conclude that $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ \square .

297 **4.2 Lower Bound**

298 According to the convergence we deal with, we have to provide the following lower bound.

Theorem 4.3. *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and for every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ A -converging to u we have*

$$\liminf_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u).$$

Proof. First of all we may assume that $\liminf F_{\varepsilon_h}(u_h, D_h) < M$ otherwise the result is trivial. Moreover up to a subsequence we may assume that the \liminf is actually a limit. Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P$ invading $\Omega \setminus P$. As in the proof of the Theorem 4.2 we may assume up to a subsequence, that $\Delta u_h \rightarrow \Delta u$ weakly in $L^2(\Omega_j)$. Furthermore, since in this case all the sequence D_h satisfies the condition (1) and (2) in Definition 4.1 we have $\Omega_j \subset \subset \Omega \setminus D_h$ for h large and for any j . We also have Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Consequently

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 \geq \liminf_h \int_{\Omega_j} |\Delta u_h|^2 dx \geq \int_{\Omega_j} |\Delta u|^2 dx.$$

299 On the other hand we know that the limit u of the subsequence u_h belongs to $\Delta\mathcal{AM}^p(\Omega)$,
300 hence $\Delta u \in L^2(\Omega)$. So that by monotone convergence

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \geq \int_{\Omega \setminus P} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 \quad (21)$$

301 As in the proof of the theorem 4.2 we have the inequality (19). Then we easily get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx. \quad (22)$$

This conclude the first part of the proof. Let now D_h be a sequence a regular sets. Then we can find a parameterization of every $C_h = \partial D_h$ by a finite and disjoint union of Jordan curves. As in the in the proof of Theorem 4.2 if we set for every h , $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 = m(h)$$

302 and up to subsequences we have $m(h) = n$ and there exist a set P_1 of n points such that ∂D_h
303 converges in the Hausdorff metric to P_1 . On the other hand we have that ∂D_h converges in
304 the Hausdorff metric to P and, since the limit is unique $P = P_1$ and therefore.

$$\frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon} + \varepsilon_h k^2 \right) d\mathcal{H}^1 \geq n = \mathcal{H}^0(P) \geq \mathcal{H}^0(P_{\nabla u}) \quad (23)$$

305 Eventually by (21),(22) (23) and by the superlinearity property of the \liminf operator we
306 achieve the result. \square

307 4.3 Upper bound

308 In [11] for the construction of the optimal sequence it is crucial to use a result due to
 309 Chambolle and Doveri (see [13]). This result states that it is possible to approximate, in
 310 the H^1 norm, a function $u \in W^{1,2}(\Omega \setminus C)$ (where C is a closed set), by means of a sequence
 311 of functions $u_h \in W^{1,2}(\Omega \setminus C_h)$ with C_h convergent to C in the Hausdorff metric. In our
 312 case this argument does not apply due to presence of a second order differential operator.
 313 Nevertheless since we work only with set of points it is possible to build an optimal sequence
 314 in a more direct way.

315 According to the convergence we deal with the theorem, we have to prove the following
 316 theorem.

317 **Theorem 4.4.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
 318 $\Delta AM^{p,2}(\Omega)$ there exists a sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -converging to u such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u). \quad (24)$$

319 **Proof.** We start by the sequence D_h . Let n be the number of points x_i in $P_{\nabla u}$. Then
 320 we take $D_h = \bigcup_1^n B_{\varepsilon_h}(x_i)$. So that $\mathcal{L}(D_h) \rightarrow 0$ and ∂D_h goes to P with respect to the
 321 Hausdorff distance. Moreover for h large we may assume $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$.
 322 Now we build u_h . Let $\{\rho_h\} \subset \mathbb{R}$ be such that $\rho_h \geq 0$ and $\rho_h \rightarrow 0$ when $h \rightarrow \infty$. Let
 323 $\theta_h \in C^\infty(\Omega)$ with the following property:

$$\begin{cases} \theta_h = 1 & \text{on } B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ 0 \leq \theta_h \leq 1 & \text{on } B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ \theta = 0 & \text{on } \Omega \setminus B_{\rho_h}(x_i) \text{ for any } i = 1, \dots, n \\ \|\nabla \theta_h\|_\infty \leq \frac{1}{\rho_h} \end{cases} \quad (25)$$

We set $u_h = (1 - \theta_h)u$ and we claim that the pair (u_h, D_h) realizes the inequality (24). By making the computation we have

$$\nabla u_h = (1 - \theta_h)\nabla u - u\nabla \theta_h$$

Then we have

$$\int_\Omega |\nabla u_h - U_0|^p dx = \int_\Omega |\nabla u - U_0 - \theta_h \nabla u - u \nabla \theta_h|^p dx$$

324 So that

$$\limsup_{h \rightarrow +\infty} \int_\Omega |\nabla u_h - U_0|^p dx \leq \limsup_{h \rightarrow +\infty} \left(\left(\int_\Omega |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_\Omega |\theta_h \nabla u|^p \right)^{\frac{1}{p}} + \left(\int_\Omega |u \nabla \theta_h|^p \right)^{\frac{1}{p}} \right)^p \quad (26)$$

325 Since $\nabla u \in L^1(\Omega)$, we have by applying the dominated convergence theorem $\int_{\Omega} |\theta_h \nabla u|^p \rightarrow 0$.
 326 Let us focus on the term $\int_{\Omega} |\nabla \theta_h u|^p$. By the Sobolev embedding we have $u \in L^{p^*}(\Omega)$ with
 327 $p^* = \frac{2p}{2-p}$ and hence $|u|^p \in L^{\frac{p^*}{p}}(\Omega)$, with $\frac{p^*}{p} = \frac{2}{2-p}$.

By (25), using the Holder's inequality with dual exponents $\frac{2}{2-p}$ and $\frac{2}{p}$ and taking into account that $p < 2$

$$\begin{aligned}
 \int_{\Omega} |\nabla \theta_h u|^p &\leq \sum_{i=1}^n \int_{B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx = \sum_{i=1}^n \left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h u|^p dx - \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx \right) \\
 &\leq n \left(\left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} + \left(\int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \right) \\
 &\leq n \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \left(\frac{\pi^2 \rho_h^2}{\rho_h^p} + \frac{\pi^2 \rho_h^2}{4\rho_h^p} \right) \rightarrow 0.
 \end{aligned} \tag{27}$$

From (26) it follows that

$$\begin{aligned}
 \limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx &\leq \lim_{h \rightarrow +\infty} \left(\left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\theta_h \nabla u|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\nabla \theta_h u|^p dx \right)^{\frac{1}{p}} \right)^p \\
 &= \left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} = \int_{\Omega} |\nabla u - U_0|^p dx
 \end{aligned} \tag{28}$$

Now we compute $\Delta \theta_h$. The identity $\operatorname{div}(fA) = f \operatorname{div} A + \nabla f \cdot A$ yields

$$\Delta u_h = (1 - \theta_h) \Delta u - 2 \nabla \theta_h \nabla u - \Delta \theta_h u.$$

328 Then by choosing ρ_h small enough we have from (25)

$$\limsup_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 \leq \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u|^2 \rightarrow \int_{\Omega} |\Delta u|^2 dx. \tag{29}$$

329 Moreover it is not difficult to check that $\{(u_h, D_h)\} \subset Y(\Omega)$ and A-converges to u .

330 Finally since for h large we have $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$ we get

$$\lim_h \frac{1}{4\pi} \int_{\partial D_h} \left(\varepsilon_h + \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = \lim_h \sum_1^n \int_{\partial B_{\varepsilon_h}} \left(\varepsilon_h \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = n = \mathcal{H}^0(P_{\nabla u}). \tag{30}$$

331 By recalling that the limsup is sublinear operation and by (28),(29),(30), we achieve the
 332 result. \square

333 5 Approximation by smooth function

334 By following the Braides-March's approach in [11] we approximate the measure \mathcal{H}^1 measure
 335 by the Modica-Mortola's energy density given by $(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx$ where $W(w) =$
 336 $w^2(1-w)^2$ and $w \in C^\infty(\Omega)$. The next step is to replace the regular set D with the level
 337 set of w . Let us set $H = \{\nabla w(x) = 0\}$. By Sard's Lemma we have that $\mathcal{L}^1(w(H)) = 0$. In
 338 particular, if w takes values into the interval $[0, 1]$, we infer that for almost every $t \in (0, 1)$
 339 the set $H \cap w^{-1}(t)$ is empty. Consequently for almost every $t \in (0, 1)$ the t -level set $\{w < t\}$
 340 is a C^∞ set with boundary $\{w = t\}$. Now since we want to replace the set D we need that
 341 $\{w < t\} \subset\subset \Omega$ and then we require that w is such that $1 - w \in C_0^\infty(\Omega; [0, 1])$. Furthermore
 342 for almost every t we have $k(\{w = t\}) = \operatorname{div}(\frac{\nabla w}{|\nabla w|})$. From all of this we are led to define the
 343 following space:

$$S(\Omega) = \{(u, w); u \in W_0^{1,p,2}(\operatorname{div}; \Omega); 1 - w \in C_0^\infty(\Omega; [0, 1])\} \quad (31)$$

and having in mind the coarea formula, the following sequence of functionals defined on $S(\Omega)$

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_\Omega w|\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)^2\right) (\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx \\ &+ \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_\Omega (1 - w_\varepsilon)^2 dx \end{aligned}$$

344 whith $C = \int_0^1 \sqrt{W(t)} dt$.

345 The last term forces w_ε be 1 almost everywhere in the limit and hence plays the role of
 346 the condition $\mathcal{L}(D) \leq a(\varepsilon)$. From now on the parameters $\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ will be related as follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 0 \quad (32)$$

347

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0 \quad (33)$$

348 The convergence that plays the role of the A -convergence is the following.

349 **Definition 5.1.** Let $\{(u_h, w_h)\}$ be a sequence in $S(\Omega)$ and $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$. We say that
 350 that (u_h, w_h) S -converges to u if for every $t \in (0, 1)$ the pair (u_h, D_h^t) A -converges to u .

351 We define the Γ -convergence with respect to this convergence.

352 **Definition 5.2.** We say that a sequence G_ε Γ -converges to G if for every sequence of positive
 353 numbers $\varepsilon_h \rightarrow 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence (u_h, w_h) S -convergent to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \geq G(u)$$

2. there exists a sequence (u_h, w_h) S -convergent to u such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \leq G(u)$$

354 As in the previous section we state a suitable version of the fundamental theorem of the
355 Γ -convergence.

Theorem 5.1. *Let \mathcal{F}_ε a sequence of functional Γ -convergent to \mathcal{F} . If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, w_h)\} \subset S(\Omega)$ such that*

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf \mathcal{F}_{\varepsilon_h}) = 0$$

356 then there exists a subsequence $\{(u_{h_k}, w_{h_k})\}$ and a minimizer $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ of \mathcal{F} such
357 that $\{(u_{h_k}, w_{h_k})\}$ S -converges u .

358 5.1 Compactness

359 **Theorem 5.2.** *Let h goes to $+\infty$ and $\varepsilon_h \rightarrow 0$*

$$F_{\varepsilon_h}(u_h, w_h) \leq M \tag{34}$$

360 there exists a subsequence $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$ and $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ such that (u_{h_k}, w_{h_k})
361 S -converges to u .

362 **Proof.** The first part of proof is as in [11]. For the convenience of the reader we give the
363 complete proof.

364 For any $t \in (0, 1)$ since $0 \leq w_h \leq 1$, we have $w_h > \frac{1}{(1-t)^2}$ then by Chebyshev's inequality
365 and by (34)

$$\mathcal{L}(\{w_h < t\}) \leq \frac{M\gamma_{\varepsilon_h}}{(1-t)^2}. \tag{35}$$

366 The inequality ensures that the condition (1) of Definition 4.1 holds for every sub level sets
367 $\{w_h < t\}$.

By Young's inequality and by (34) we get

$$M \geq 2 \int_{\Omega \setminus \{|\nabla w_h|=0\}} |\nabla w_h| \sqrt{W(w_h)} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) dx$$

368 Now by coarea formula and taking into account that $|\nabla w_h|$ coincides with the norm of the
369 tangent vector to the boundary of the t -level set we obtain

$$M \geq \int_0^1 \sqrt{W(t)} \int_{\{w_h=t\} \cap \{|\nabla w| \neq 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) d\mathcal{H}^1 dt \tag{36}$$

Thanks to the Sard's Lemma for any h there exists a \mathcal{L} -negligible set $\mathcal{N}_{w_h} \subseteq (0, 1)$ such that

$$\{w_h = t\} = \partial\{w_h < t\}, \quad \{w_h < t\} \in R(\Omega), \quad \text{for } t \in (0, 1) \setminus \mathcal{N}_{w_h}$$

and on $\{w_h = t\}$ for $t \in (0, 1) \setminus \mathcal{N}_{w_h}$ we have

$$|\nabla w_h| \neq 0 \text{ and } \kappa(\{w_h = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right).$$

Now since the union \mathcal{N}_w of the sets \mathcal{N}_h is \mathcal{L} -negligible (almost countable) from (36) we have

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt$$

370 By applying Fatou's Lemma and taking into account that the set \mathcal{N}_w does not depend on h
371 we get

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt \quad (37)$$

372 Hence we deduce the existence of a \mathcal{L} -negligible set Q_w , with $\mathcal{N}_w \subseteq Q_w$, such that

$$\liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 \leq M_t \quad (38)$$

373 where the constant M_t does not depend on h .

Then for every $t \in (0, 1) \setminus Q_w$ we can extract a sequence $w_{h,t}$ such that $\partial\{w_{h,t} < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$. Let $\mathcal{N} = \{t_i\}$ in $(0, 1)$ be a dense countable set. Up to a diagonal argument we can find a subsequence $\{w_{h_k}\}$ such that, for every $t^i \in \mathcal{N}$, $\partial\{w_{h_k} < t_i\}$ converges to P^{t_i} . Let $t_i \in \mathcal{N}$ such that $t_i > t$ and consequently $\{w_{h_k} < t\} \subseteq \{w_{h_k} < t_i\}$. From the definition of Hausdorff convergence it follows that for every $\rho > 0$ there exists $k_0(\rho)$ such that for any $k > k_0$ we have $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$ for every $x \in P^{t_i}$. Since the t_i -level set is open for every ρ and for every $x \in P^{t_i}$ such that $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$, we may choose $t_n \in \mathcal{N}$ with $t_n < t$ and so obtain for k large enough $\{w_{h_k} < t_n\} \cap B_\rho(x) \neq \emptyset$. By choosing $t_{max} = \max_{x \in P^{t_i}} t_n(x)$ for every $x \in P^{t_i}$, the inclusion $\{w_{h_k} < t_{max}\} \subset \{w_{h_k} < t\}$ gives

$$\{w_{h_k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h_k} < t\} \cap B_\rho(x) \subseteq \{w_{h_k} < t_n\} \cap B_\rho(x)$$

374 with $t_{max}, t_n \in \mathcal{N}$. Then by taking the limit $\delta \rightarrow 0^+$ we infer $\partial\{w_{h,t} < t\}$ converges with
375 respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$ for every $t \in (0, 1)$.

376 Finally as in the proof of Theorem 4.2 we can extract a subsequence $\{u_{h_k}\}$ which con-
377 verges strongly in $L^p(\Omega)$ to a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ with $P_{\nabla u} \subset P^t$ for every $t \in (0, 1)$.
378 Therefore we have that for every $t \in (0, 1)$ the pair $(u_h, \{w_{h,t} < t\})$ S-converges to u and
379 the proof is achieved. \square

380 5.2 Lower bound

Theorem 5.3. *Let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and for every sequence $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u the following inequality holds*

$$\liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}(u, w) \geq F(u).$$

Proof. Let $u \in \Delta\mathcal{AM}^p(\Omega)$ and $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u . As usual up to subsequences we can assume

$$+\infty > \liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}.$$

381 For any $t \in (0, 1)$ we have (see also [11] for a similar argument)

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 = \int_{\{w_h < t\} \cap \Omega} w_h^2 |\Delta u_h|^2 + \int_{\{w_h \geq t\} \cap \Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \quad (39)$$

382 Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P^t$ invading $\Omega \setminus P^t$. Then we may assume
 383 that $u_h \rightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Therefore as
 384 in the proof of Theorem 4.3 we get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx \quad (40)$$

and

$$\liminf_{h \rightarrow +\infty} t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx$$

385 for any j .

Then by (39) and, by taking into account that $|\Delta u|$ is an $L^2(\Omega)$ function, it follows that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} |\Delta u|^2 dx$$

386 And eventually by taking the limit $t \rightarrow 1$

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq \int_{\Omega} |\Delta u|^2 dx \quad (41)$$

387 By definition we know that for every $t \in (0, 1)$ $\mathcal{L}^d(\{w_h < t\}) \rightarrow 0$ and $\partial\{w_h < t\} \rightarrow P^t$ in
 388 the Hausdorff distance. So that, as in the proof of Theorem 4.3 (inequality 23) we have

389

$$\liminf_{h \rightarrow +\infty} \frac{1}{4\pi} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}) \quad (42)$$

Now arguing as in the proof of Theorem 5.2 and by taking into account (42), we get

$$\begin{aligned}
& \liminf_{h \rightarrow \infty} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\
& \geq 2 \liminf_{h \rightarrow +\infty} \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 dt \\
& \geq 2 \int_{(0,1)} \mathcal{H}^0(P^t) \sqrt{W(t)} dt \geq 8\pi C \mathcal{H}^0(P_{\nabla u})
\end{aligned} \tag{43}$$

390 By collecting (40) (41) and (43) we achieve the thesis. \square

391 5.3 Upper bound

392 As in [11] to build w_h we use the construction used by Bellettini in [6], while the optimal
393 sequence u_k is given as in Theorem 4.4.

394 **Theorem 5.4.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
395 $\Delta\mathcal{M}^{p,2}(\Omega)$ there exists a sequence $(u_h, D_h) \in S(\Omega)$ S -converging to u such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{44}$$

Proof. If $A \subset \mathbb{R}^2$ we set

$$\delta_A(x) = d(x, A) - d(x, \mathbb{R}^2 \setminus A).$$

396 We start by the construction of w_h .

As in the proof of Theorem 4.4 we set $P_{\nabla u} = \{x_1, \dots, x_n\}$ and we define

$$D_h = \bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

397 Since D_h is a regular set by taking into account the condition (33) for h large enough we
398 have

$$\{x \in \Omega : d(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset \subset \Omega. \tag{45}$$

Let η be the optimal profile for the Modica Mortola energy, that is the solution of

$$\begin{cases} \eta'(t) = \sqrt{W(t)} & \text{on } \mathbb{R} \\ \eta(-\infty) = 0, \\ \eta(+\infty) = 1 \end{cases}$$

399 given by $\eta(t) = \frac{1}{2}(1 + \tanh \frac{t}{2})$.

For every h let $\psi_h : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h, +\infty] \\ \psi_h' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|\psi_h'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O\left(\frac{1}{|\log \varepsilon_h|}\right) \end{cases}$$

As in [6] and in [11] we define

$$\eta_h(t) = \begin{cases} \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) + 1 - \psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t \geq 0 \\ \psi\left(\frac{t}{\varepsilon_h}\right) - \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t < 0 \end{cases}$$

Then we set $w_h(x) = \eta_h(\delta_{D_h}(x))$. We claim that $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$ for h large. Indeed for any $x \in \Omega \setminus D_h$ we have $\delta_{D_h}(x) \geq 0$, hence

$$1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$$

400 and then $0 \leq 1 - w_h \leq 1$ by the properties of ψ_h and η . The case $x \in D_h$ is similar. Let now
 401 $x \in \partial\Omega$ then $\delta_{D_h}(x) \geq 0$ and $1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$. From (45), it follows
 402 $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$ for h large enough, hence the claim follows. Then we take $\{(u_h, w_h)\}$ as
 403 an optimal sequence, where u_h is given as in Theorem 4.4

404 First all we have to check that $(u_h, w_h(x))$ S-converges to u . For any $x \in \Omega \setminus P_{\nabla u}$ we have
 405 that for h large enough $\delta_{D_h}(x) \geq 0$ and one can check that $w_h(x) \rightarrow 1$ for every $x \in \Omega \setminus P_{\nabla u}$.
 406 This implies that $\mathcal{L}(\{w_h < t\}) \rightarrow 0$ for every $t \in (0, 1)$. Now for every $t \in (0, 1)$ we write

$$\{w_h = t\} = (\{w_h = t\} \cap D_h) \cup (\{w_h = t\} \cap \Omega \setminus D_h). \quad (46)$$

407 Hence, since $w_h(x) \rightarrow 1$ for $x \in \Omega \setminus D_h$, for any $t \in (0, 1)$ there exists $h(t)$ such that
 408 $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$ for every $h \geq h(t)$. So that from (46) it follows that for every
 409 $t \in (0, 1)$, $\{w_h = t\} \rightarrow P_{\nabla u}$ when $h \rightarrow +\infty$. So we can conclude that (u_h, w_h) S-converges
 410 to U .

As in [6] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h |\log \varepsilon_h|\}, \quad D_h^2 = \{x \in \Omega : \varepsilon_h |\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h |\log \varepsilon_h|\}$$

and we split

$$\begin{aligned} & \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &= \int_{D_h^1} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx + \\ & \int_{D_h^2} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx = \\ &= I_h + II_h \end{aligned} \quad (47)$$

For $x \in D_h^1$ we have $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$ therefore $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$. By taking into account the definition of η we have $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$. Moreover it is easy to check that

$$\eta'_h(t) = \frac{1}{\varepsilon_h} \eta'(t) = \frac{1}{\varepsilon_h} \sqrt{W(t)}; \quad |\nabla w_h| = |\eta'_k(\delta_{D_h})|$$

411

This, together with the coarea formula yields

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h=t\}} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt$$

Now we have that $\partial D_h = \{w_h = \frac{1}{2}\}$ then

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)}$$

412 Then

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt \quad (48)$$

413 Moreover with the same argument and by using the definition of w_h one can check that

$$\lim_{h \rightarrow \infty} II_h = 0 \quad (49)$$

414 By (48) and (49) we have

415

$$\lim_{h \rightarrow \infty} \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon (\operatorname{div}(\frac{\nabla w}{|\nabla w|}))^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) \right) dx = \mathcal{H}^0(P_{\nabla u}) \quad (50)$$

416 Now we take care of the terms involving u_h . As in the proof of Theorem 4.4 we have

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p \leq \int_{\Omega} |\nabla u - U_0|^p \quad (51)$$

Furthermore by taking into account that $w_h(x) = 1$ if $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$ and $w_h(x) = 0$ if $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$ we have

$$\int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \int_{\Omega \setminus D_h^0} |\Delta u_h|^2 dx$$

417 where $D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$.418 By choosing ρ_h in such a way that $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$ we obtain

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega \setminus P_{\nabla u}} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx \quad (52)$$

419 Finally from the definition of w_h , it follows that $w_h = 1$ outside the disk $(D_h)_{2 \log \varepsilon_h}$ and
420 hence thanks to (32) and (33)

$$\limsup_{h \rightarrow +\infty} \frac{1}{\gamma_h} \int_{\Omega} (1 - w_h)^2 dx \leq \lim_{h \rightarrow +\infty} \mathcal{L}(D_{h 2 \log \varepsilon_h}) \frac{1}{\gamma_h} = 0. \quad (53)$$

421 The thesis follows by collecting (50), (51), (52) and (53) \square

422 6 De Giorgi's Conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

423 where D is a regular set and κ is a curvature of its boundary ∂D .

424 Since ∂D can be represented as the discontinuity set of the function $w_0 = 1 - \chi_D$,
425 by Modica-Mortola's Theorem it follows that there is a sequence of non constant local
426 minimizers such that $w_\varepsilon \rightarrow w_0$ with respect to the L^1 -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(w_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

427 Furthermore looking at the Euler-Lagrange equation associated to a contour length term,
428 yields a contour curvature term κ , while the Euler-Lagrange equations for the functional
429 $F_\varepsilon^1(w)$ contains a term $2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon}$.

430 Then De Giorgi suggested to approximate by Γ -convergence the functional F^2 by adding
431 to the Modica-Mortola approximating functionals the following term

$$F_\varepsilon^2(w) = \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx.$$

432 In ([7]) Bellettini and Paolini have proven the lim sup inequality, while the validity of
433 lim inf for a modified version of the original conjecture has been proven by Röger and Shätzle
434 (see [26]).

Inspired by the De Giorgi's conjecture it appears natural investigate, in the spirit of [11],
the possibility of approximating the functional \mathcal{F} by a means of a sequence \mathcal{F}_ε much more
convenient from a numerical point view (see [22]):

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &= \int_{\Omega} w^2 |\Delta u|^2 dx + \frac{1}{8\pi C} \left(\frac{\beta_\varepsilon}{2\varepsilon} \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \right) \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} (1 - w^2) dx. \end{aligned}$$

435 The presence of the term $\frac{1}{2\varepsilon}$ will be clear in the proof.

436 By the way we are able to prove only the Γ -limsup inequality.

437 **Theorem 6.1.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
438 $\Delta \mathcal{M}^{p,2}(\Omega)$ there exists a sequence $(u_h, D_h) \in S(\Omega)$ S -converging to u such that*

$$\limsup \mathcal{F}_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{54}$$

439 **Proof.** Let (u_h, w_h) the optimal sequence of Theorem 5.4.

It is not difficult to see that for every $x \in D_h^1$ the function $\delta_h(x)$ is regular and using the definition of w_h and taking into account that $\eta' = \sqrt{W(\eta)}$ the following identity holds

$$2\varepsilon_h \Delta w_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h \Delta \delta_{D_h}(x) + 2\varepsilon_h \eta''_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h (\delta_{D_h}(x)).$$

For h large we also have $\Delta \delta_{D_h}(x) = \kappa(\{\delta_{D_h}(x) = t\})$. Besides on D_h^1 we have $w_h(x) = \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})$ and hence the level set $\{\delta_{D_h}(x) = t\}$ corresponds to the level set $\{w_h(x) = \eta(\frac{t}{\varepsilon_h})\}$ with $0 \leq \eta \leq 1$, so that we infer

$$\kappa(\{\delta_{D_h}(x) = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)$$

By proceeding as in the proof of Theorem 5.4 and taking into account the equality $2\varepsilon_h |\eta'_h(\delta_{D_h}(x))| = 2\varepsilon_h |\nabla w_h|$ we have

$$I_h = \int_{D_h^1} \frac{\beta_{\varepsilon_h}}{2\varepsilon_h} \left(2\varepsilon_h \Delta w - \frac{W'(w)}{\varepsilon_h}\right)^2 + \frac{1}{\beta_{\varepsilon_h}} (\varepsilon_h |\nabla w_{\varepsilon_h}|^2 + \frac{1}{\varepsilon_h} W(w_{\varepsilon_h})) dx = 2 \int_{D_h^1} \left(\beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)^2 + \frac{1}{\beta_{\varepsilon_h}}\right) \sqrt{W(w_h)} |\nabla w_h| dx$$

then as in the proof of Theorem 5.4 we conclude

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt.$$

440 By the same calculation on D_h^2 one can check that the integral over D_h^2 vanishes as in the
441 proof of Theorem 5.4.

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