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Variational approximation for detecting points-like target problems.

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Thème COG



*Rapport
de recherche*

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1 Introduction

The issue of detecting fine structures, like points or curves in two or three dimensional biological images, is a crucial task in image processing. In particular a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise. Therefore one of the main goals in the application is detecting the spots that the biologists wish to count. This operation is made harder by the presence of the other singular structures like noise, textures or filaments.

In some biological images the image intensity is a function that takes the value 1 on points or other structures like set with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. In image processing these concentration sets are called discontinuity without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0.

In the literature there are few variational methods dealing with this problem. In this direction one interesting approach has been proposed in [5]. In that paper the authors consider these kind of pathology as a k -codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [8] for a complete survey on this subject). So that the detecting point case corresponds to the case $k = 2$ and $m = 0$.

This point of view makes possible a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$, where \mathbb{S}^1 is a unit sphere of \mathbb{R}^2 . This strategy makes crucial the construction, starting from the initial image $I : \mathbb{R}^2 \rightarrow \mathbb{R}$, of an initial vector field $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ with a topological singularity of degree 1, and how to build such a vector field in a rigorous way, is a subject of a current investigation.

Our first purpose here is finding a most natural variational framework in which a rigorous definition of discontinuity without jump can be given.

In our model the image I is a Radon measure, that is an element of $(C_0(\Omega))^*$: the dual space of $C_0(\Omega)$ (the closure with respect to the sup-norm of the space of continuous functions with compact support). It is crucial for detecting points that this Radon measure be capable of charging points. The preliminary step is finding a space whose elements are capable of producing these kind of measures. This space is given by $\mathcal{DM}^p(\Omega)$: the space of the L^p -vector fields whose distributional divergence measure is a Radon measure, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence $\text{Div}U$ of U cannot be a measure concentrated on points. (see section 3.1 below). Then we have to construct from the original image I a data vector field $U_0 \in \mathcal{DM}^p(\Omega)$. Clearly there are, at least in principle, many ways to do this. The one we propose here seems to be the most natural. We consider the classical elliptic problem with measure data I :

$$\begin{cases} \Delta u_0 = I & \text{on } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Then by setting $U_0 = \nabla u_0$ we have $U_0 \in \mathcal{DM}^p(\Omega)$ with $\text{Div}U_0 = I$. Nevertheless the support of the measure $\text{Div}U_0$ is too large and could contains several structures like curves

72 or fractal sets, while the singularities, we are interested in, are contained in the atomic part
 73 of the measure $\text{Div}U_0$ and therefore we have to isolate it. To do this the notion of p -capacity
 74 of a set plays a key role. Indeed when $p < 2$ the p -capacity of a point in Ω is zero and besides
 75 every Radon measure can be decomposed (see [17]) in two mutually singular measures: the
 76 first one is absolutely continuous with respect the p -capacity and the second one is singular
 77 with respect to the p -capacity, meaning that it is a measure concentrated on sets with 0
 78 p -capacity.

79 Moreover for a compact subset $E \subset \Omega$ a necessary and sufficient condition to have 0
 80 p -capacity is the existence of a sequence of test functions $\{\phi_k\}$ converging to 0 strongly in
 81 $W_0^{1,p}$ and equals to 1 on E for every k , and hence one can say, in this sense, that a set of 0
 82 p -capacity is a discontinuity with no jump.

83 As well known in dimension 2 the sets with 0 p -capacity, and hence discontinuity without
 84 jump, can be isolated points, countable set of points, fractal sets with Hausdorff dimension
 85 $0 \leq \alpha < 1$ (see subsection 2.3 for the definition of p -capacity and related properties).

86 By the way the goal here is keeping nothing else but points in the image. The achieve-
 87 ment of such a purpose makes necessary the minimization of a suitable energy that must
 88 remove all the discontinuity which are not discontinuities without jump, and remove all the
 89 discontinuities without jump which are no isolated point.

90 From one hand we have to force the concentration set of the divergence measure of U
 91 to contain only the points we want to catch, and on the other hand we have to regularize
 92 the initial data U_0 outside the points of singularities. To this end we introduce the auxiliar
 93 space $SDM^p(\Omega)$ of the vector field belonging to $\mathcal{DM}^p(\Omega)$ whose divergence measure has
 94 no absolutely continuous part with respect to the p -capacity. Then, by taking into account
 95 that the initial vector field is a gradient of a Sobolev function, our goal is to minimize
 96 the following energy, involving a competition between a divergence term and the counting
 97 Hausdorff measure \mathcal{H}^0 ,

$$\mathcal{F}(U) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\text{supp}(\text{div}^s \nabla u)_0)$$

98 where $u \in W_0^{1,p}(\Omega)$ with $\nabla u \in SDM^p(\Omega)$ and $p < 2$. The gradient of a minimizer of the
 99 energy \mathcal{F} is the vector field we are looking for, that is a vector field whose divergence measure
 100 can be decomposed in an absolutely continuous (with respect to the Lebesgue's measure)
 101 term plus an atomic measure concentrated on the points we want to catch in the image.

102 Even if a pointwise characterization of discontinuity without jump is not available, thanks
 103 to our definition the singular set of points can be linked to the vector field ∇u , in the spirit of
 104 classical SBV formulation of the Mumford-Shah's Functional. (we refer to [1] for a complete
 105 survey on Mumford Shah's functional).

Our next task is to provide an approximation in the sense of the Γ -convergence introduced
 by [19, 20]. Our approach is close in the spirit to the one used to approximate the Mumford
 Shah functionals by a families of depending curvature functionals in [11]. Indeed as in their

work (see also [10]) we replace the atomic measure \mathcal{H}^0 by the term

$$G_\varepsilon(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

106 where D is a proper regular set containing the atomic set P , κ is the curvature of its
 107 boundary, the constant $\frac{1}{4\pi}$ is a normalization factor. Roughly speaking the minimum of
 108 these functionals are achieved on the union of balls of small radius, so that when $\varepsilon \rightarrow 0$ the
 109 sequence G_ε shrinks to the atomic measure $\mathcal{H}^0(P)$.

110 This leads to an intermediate approximation given by

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (1)$$

111 This strategy permits to work with the perimeter measure $H^1[\partial D]$, that can be approx-
 112 imated, according to the Modica-Mortola's approach (see [24, 25]), by the measure:

$$\mu_\varepsilon(w, \nabla w) dx = \left(\varepsilon |\nabla w|^2 + \frac{V(w)}{\varepsilon} \right) dx,$$

113 where $V(w) = w^2(1 - w)^2$ is a double well functional.

114 Besides by using the Sard Theorem and the Coarea Formula (see also [6] for a similar
 115 approach) one can formally replace in the second integral on the boundary of D the curvature
 116 κ by the integral of the $\operatorname{div} \frac{\nabla w}{|\nabla w|}$ and the integral is computed over the level sets of w . So
 117 that one can formally write the complete approximating sequence:

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_{\Omega} w^2 |\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_{\Omega} (1 - w_\varepsilon)^2 dx \end{aligned}$$

118 where, as usual, $C = \int_0^1 \sqrt{W(t)} dt$, β_ε and γ_ε are infinitesimal as $\varepsilon \rightarrow 0$. The last integral is
 119 a penalization term that forces w to tend to 1 as $\varepsilon \rightarrow 0$.

120 The goal of the second part of this work is then to show that the family of energies \mathcal{F}_ε
 121 Γ -converges to the functional \mathcal{F} when the parameters are related in a suitable way.

122 As in [11] we deal with the Hausdorff convergence for sets of points and with a suitable
 123 convergence of functions involving the convergence of a sub-level sets

124 Despitse this approach is inspired by some ideas contained in [10, 11], we point out that
 125 in our case the regularization term involves a second order differential operator, due to the
 126 fact that our goal is to detect points and not segment curves. This deep difference requires
 127 a non trivial adaption of the arguments used in those papers.

128 The paper is organized as follows. Section 2 is devoted to notation, preliminaries, defi-
129 nitions and result. In section 3 we illustrate the new variational model and we present the
130 functional we deal with. Section 4 and 5 are devoted to the Γ -convergence result Finally in
131 the last section we conclude the paper by comparing this approach with the celebrate con-
132 jecture by De Giorgi concerning the approximation of the curvature depending functionals.
133 We do not give here experimental result illustrating our approach. We refer the reader
134 for that to [22].

135 2 Preliminaries

136 2.1 Notation

137 In all the paper $\Omega \subset \mathbb{R}^2$ is an open bounded set with C^2 -boundary. The operation \cdot stands
 138 for the usual scalar product. The Euclidean norm will be denoted by $|\cdot|$, while the symbol
 139 $\|\cdot\|$ indicates the norm of some functional space. The brackets $\langle \cdot, \cdot \rangle$ denotes the duality
 140 product in some distributional spaces. \mathcal{L}^d or dx is the d -dimensional Lebesgue measure and
 141 \mathcal{H}^k is the k -dimensional Hausdorff measure. $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ .
 142 We say that a set $D \in \Omega$ is a regular set if it can be written as $\{F < 0\}$ with $F \in C_0^\infty(\Omega)$.
 143 In the following we will denote by $R(\Omega)$ the set of the all regular sets in Ω .

144 2.2 Distributional divergence and classical spaces

145 In this subsection we recall the definition of the distributional space $L^{p,q}(\text{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$
 146 (see [4, 14]). We start by recalling the distributional definition of the divergence operator.
 147 Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a regular vector field. If $U = (u, v)$
 148 then classically $\text{div}U = u_{x_1} + v_{x_2}$. In the sense of distribution $\text{Div}U$ is defined

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} (u, v) \cdot \nabla \varphi$$

149 for all $\varphi \in C_0^\infty(\Omega)$.

150 **Definition 2.1.** We say that $U \in L^{p,q}(\text{div}; \Omega)$ if $U \in L^p(\Omega; \mathbb{R}^2)$ and if its distributional
 151 divergence $\text{Div}U = \text{div} \in L^q(\Omega)$. If $p = q$ the space $L^{p,q}(\text{div}; \Omega)$ will be denoted by $L^p(\text{div}; \Omega)$.

152 We say that $U \in L_{loc}^{p,q}(\text{div}; \Omega)$ if $U \in L^{p,q}(\text{div}; A)$ for every open set $A \subset \subset \Omega$.

153 We say that a function $u \in W^{1,p}(\Omega)$ belongs to $W^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$. We
 154 say that a function $u \in W_0^{1,p}(\Omega)$ belongs to $W_0^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$.

Definition 2.2. For $U \in L^p(\Omega; \mathbb{R}^2)$, $1 \leq p \leq +\infty$, set

$$|\text{Div}U|(\Omega) := \sup\{\langle U, \nabla \varphi \rangle : \varphi \in C_0^\infty(\Omega), |\varphi| \leq 1\}.$$

We say that U is an L^p -divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$ if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\text{Div}U|(\Omega) < +\infty.$$

155 We say that $U \in \mathcal{DM}_{loc}^p(\Omega)$ if $U \in \mathcal{DM}^p(A)$ for every open set $A \subset \subset \Omega$.

156 Let us recall the classical result (see [15] Proposition 3.1)

157 **Theorem 2.1.** Let $\{U_k\}_k \subset \mathcal{DM}^p(\Omega)$ such that

$$U_k \rightharpoonup U \quad L^p(\Omega), \text{ for } 1 \leq p < +\infty \quad (2)$$

Then

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_{h \rightarrow \infty} \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \quad |\text{Div}U|(\Omega) \leq \liminf_{h \rightarrow \infty} |\text{Div}U_h|(\Omega)$$

158 Moreover for a vector field $U \in L^{p,q}(\text{div}; \Omega)$ it is possible to define a trace operator as
 159 stated by the next proposition (see [28], Proposition 3.1)

160 **Proposition 2.1.** *Let Ω be a bounded, C^2 domain. There is a continuous linear operator*
 161 $\gamma_\nu : U \in L^{p,q}(\text{div}; \Omega) \rightarrow (C^1(\partial\Omega))^*$ such that

162 (i) for each $\varphi \in C^1(\partial\Omega; \mathbb{R})$ and each $U \in L^{p,q}(\text{div}; \Omega)$,

$$\langle \gamma_\nu(U), \varphi \rangle = \int_\Omega U \cdot \nabla \psi + \int_\Omega \psi \text{div} U \quad (3)$$

163 whenever $\psi \in C^1(\overline{\Omega})$ and $\psi_{\partial\Omega} = \varphi$.

164 (ii) for $U \in C^1(\overline{\Omega}; \mathbb{R}^2)$, $\gamma_\nu(U) = U \cdot \nu|_{\partial\Omega}$.

165 2.3 p -capacity

The p -capacity will be crucial to find a convenient functional framework to deal with. If $K \subset \mathbb{R}^2$ is a compact set and χ_K denotes its characteristic function, we define:

$$\text{Cap}_p(K, \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p dx \mid f \in C_0^\infty(\Omega), f \geq \chi_K \right\}$$

If $U \subset \Omega$ is an open set, the p -capacity is given by

$$\text{Cap}_p(U, \Omega) = \sup_{K \subset U} \text{Cap}_p(K, \Omega).$$

Finally if $A \subset \Omega$ is a Borel set

$$\text{Cap}_p(A, \Omega) = \inf_{A \subset U \subset \Omega} \text{Cap}_p(U, \Omega).$$

166 We recall the following result (see for instance [23], Theorem 2.27) theorem that explains
 167 the relationship between p -capacity and Hausdorff measure. Such a result is crucial to have
 168 geometric information on the null p -capacity sets.

169 **Theorem 2.2.** *Assume $1 \leq p < 2$. If $\mathcal{H}^{2-p}(A) < \infty$ then $\text{Cap}_p(A, \Omega) = 0$*

170 Another useful tool to manage the sets of p -capacity 0 is provided by the following
 171 characterization.

172 **Theorem 2.3.** *Let E be a compact subset of Ω . Then $\text{Cap}_p(E, \Omega) = 0$ if and only if there*
 173 *exists a sequence $\{\phi_k\} \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \leq \phi_k \leq 1$*
 174 *and $\phi_k = 1$ on E for every k .*

175 For a general survey we refer the reader to [21, 23, 29].

176 3 The Variational Model

177 This section is devoted to illustrate the functional framework we will work with.

Roughly speaking in biological images the image is a function that could be very high on points or other structures like set with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. In image processing these concentration sets are called discontinuity without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0. From a mathematical point of view it seems to be much more appropriate to think of the image as a Radon measure, that is $I = \mu \in (C_0(\Omega))^*$. The next step is finding a space whose elements are capable of producing these discontinuity without jump. This space is given by $\mathcal{DM}^p(\Omega)$, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence of U cannot be a measure concentrated on points. Set $p \geq 2$. Looking at the definition we have

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} U \cdot \nabla \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega)$$

178 Since $p \geq 2$ this distribution is a well defined even for any test $\varphi \in W_0^{1,p'}(\Omega)$ where $p' \leq 2$
 179 is the dual exponent of p . In particular $\text{Div}U$ belongs to the dual space $W^{-1,p'}(\Omega)$ of the
 180 Sobolev space $W_0^{1,p}(\Omega)$. Then in this case, the distributional divergence of U cannot be
 181 an atomic measure, whereas $\delta_0 \notin W^{-1,p'}(\Omega)$. To see this, one can consider as Ω the disk
 182 $B_0(1)$ and the function $\tilde{\varphi}(x) = \log(\log(1 + |x|)) - \log(\log(2))$. This function is in the space
 183 $W_0^{1,p'}(\Omega)$ for every $p' \leq 2$ and therefore it is an admissible test function and it easy to check
 184 that $\langle \delta_0, \varphi \rangle = +\infty$.

185 When $1 < p < 2$ we have that $\text{Div}U \in W^{-1,p'}(\Omega)$, but in this case since $p < 2$, we have
 186 $p' > 2$ and hence the function $\tilde{\varphi}$ is no longer an admissible test function. One can check
 187 that the distribution $\text{Div}U$ is an element of $(C_0(\Omega))^*$ capable of charging the points. Take
 188 for instance the map $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$.

189 The next step is to transform the initial image I as the divergence measure of a suitable
 190 vector field. We consider the elliptic problem with measure data I :

$$\begin{cases} \Delta u = I & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

Classical results (see [27]) ensures the existence of a unique weak solution $u \in W_0^{1,p}(\Omega)$ with $p < 2$. Then it easy to see that the distributional divergence of ∇u is given by I . In particular by setting $U = \nabla u$, we have $U \in \mathcal{DM}^p(\Omega)$. According to the Radon Nykodim decomposition of the measure $\text{Div}U$ we have

$$\text{Div}U = \text{div}U + \text{div}^s U,$$

191 where $\text{div}U \in L^1(\Omega)$. For our purpose the support of the singular measure $\text{div}^s U$ is too large.
 192 In particular the measure $\text{div}^s U$ could charge sets with Hausdorff dimension $0 \leq \alpha < 2$,

193 hence it can be concentrated also on discontinuity with jump. So that in order to isolate
 194 the singularities we are interested in, we need a further decomposition of the measure $\text{Div}U$.
 195 This can be done by using the capacity decomposition of the Radon measure $\text{div}^s U$. It is
 196 known (see [17]) that given a Radon measure μ the following decomposition holds

$$\mu = \mu_a + \mu_0 \quad (5)$$

197 where the measure μ_a is absolutely continuous with respect to the p -capacity and μ_0 is
 198 singular with respect to the p -capacity, that is concentrated on sets with 0 p -capacity. Besides
 199 it is also known (see [17]) that every measure which is absolutely continuous with respect
 200 to the p -capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer
 201 decomposition:

$$\mu = f - \text{Div}G + \mu_0 \quad (6)$$

202 where $G \in L^p(\Omega; \mathbb{R}^2)$, $f \in L^1(\Omega)$.

203 By applying this decomposition to the measure $\text{div}^s U$ we obtain the following decom-
 204 position of the measure $\text{Div}U$

$$\text{Div}U = \text{div}U + f - \text{Div}G + (\text{div}^s U)_0 \quad (7)$$

205 with $G \in L^p(\Omega)$.

206 According to this decomposition and taking into account Theorem 2.3 we give the defi-
 207 nition of discontinuity without and with jump.

208 **Definition 3.1.** *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity without jump*
 209 *of U if $x \in \text{supp}(\text{div}^s U)_0$.*

210 The other singularities where there is a jump, are contained in the second term of the
 211 decomposition (7). Indeed the space $W^{-1,p'}(\Omega)$ contains Hausdorff measures restricted to
 212 sub-manifolds of dimension greater or equal to 1 (see [29] chapter 4, section 7 for a detailed
 213 discussion on the space $W^{-1,p'}(\Omega)$), like for instance Hausdorff measures concentrated on
 214 regular closed curves, that is contours, that are a classical example of discontinuity with
 215 jump. More precisely a contour of a regular set is the jump set of the characteristic
 216 function of the set D and its p -capacity is strictly positive. This is of course in agreement
 217 with Theorem 2.3. Indeed, since the characteristic function of the set D is BV -function, it
 218 cannot be approximated by regular functions in the $W^{1,p}$ strong norm.

219 **Definition 3.2.** *We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity with jump of U*
 220 *if $x \in \text{supp}(f - \text{Div}G)$.*

221 3.1 The variational framework

222 We shall build an energy whose minimizer will be a vector field whose singular part of its
 223 divergence measure will be given by nothing else but points.

224 This minimizer must be an L^p (with $p < 2$) vector field with the following properties:

- 225 1. It must be close to the initial data U_0 which is, in general, an L^p vector field U_0 with
 226 $1 < p < 2$.
- 227 2. The absolutely continuous part with respect to the Lebesgue measure of $\text{Div}U$ is an
 228 L^2 function.
- 229 3. The support of the measure $\text{div}^s U$ must be given by set of points P_U with $\mathcal{H}^0(P_U) <$
 230 $+\infty$

231 According to these considerations it seems to be natural to introduce the space

$$SDM^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \text{Div}G = 0\} \quad (8)$$

232 so that, as a consequence, the decomposition (7) yields for any $U \in SDM^p(\Omega)$

$$\text{Div}U = \text{div}U + (\text{div}^s U)_0. \quad (9)$$

233 For our purposes the following result concerning the features of the elements of the space
 234 $SDM^p(\Omega)$ will play a crucial role.

235 **Proposition 3.1.** *Let $U \in L^{p,q}(\text{div}; \Omega \setminus P)$ and $P \subset \Omega$ be a set of finite number of points
 236 then $U \in SDM^p(\Omega)$.*

237 **Proof.** We set $P = \{x_1, \dots, x_n\}$. Let $B_\rho(x_i)$ $i=1, \dots, n$. We can find a sequence $\rho(h) \rightarrow 0$ when
 238 $h \rightarrow +\infty$ such that $B_{\rho(h)}(x_i) \cap B_{\rho(h)}(x_j) = \emptyset$ for h large and $i \neq j$. Then we define the
 239 following sequence U_h .

$$U_h = \begin{cases} 0 & \text{on } \Omega \cap \bigcup_i^n B_{\rho(h)}(x_i) \\ U & \text{on } \Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i) \end{cases} \quad (10)$$

240 The sequence $U_h \in \mathcal{DM}^p(\Omega)$ for every h fixed. If $\varphi \in C_0^\infty(\Omega)$, by applying the Proposition
 241 2.1 to the vector field $U \in L^{p,q}(\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i))$. we get:

$$\begin{aligned} |\text{Div}U_h|(\Omega) &\leq \sup_{0 \leq \varphi \leq 1} \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} U \nabla \varphi \right| \leq \sup_{0 \leq \varphi \leq 1} \{ |\langle \gamma_\nu(U), \varphi |_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \left| \int_{\Omega \setminus \bigcup_i^n B_{\rho(h)}(x_i)} \text{div}U \varphi \right| \} \leq \sup_{\varphi} |\langle \gamma_\nu(U), \varphi |_{\partial \bigcup_i^n B_{\rho(h)}(x_i)} \rangle| \\ &+ \|\varphi\|_{L^\infty(\Omega)} \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C \|U\|_p + (1 + C) \|\text{div}U\|_{L^q(\Omega \setminus P)} \leq C_1 \end{aligned} \quad (11)$$

$$(12)$$

242 where C_1 does not depend on h .

Now since $\mathcal{L}(\bigcup_i^n B_{\rho(h)}(x_i)) \rightarrow 0$ we have $U_h \rightarrow U$ in $L^p(\Omega)$ the by Theorem 2.1

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_h \|U_h\|_{L^p(\Omega; \mathbb{R}^2)} \leq C$$

and by theorem 2.1

$$|\operatorname{Div}U|(\Omega) < \liminf_{h \rightarrow \infty} |\operatorname{Div}U_h| \leq C.$$

243 Therefore $U \in \mathcal{DM}^p(\Omega)$. Finally we know that $U \in L^{p,q}(\operatorname{div}; \Omega \setminus P)$ and the support
 244 of the measure $\operatorname{div}^s U$ is given by the set P , then, according to decomposition (7), the
 245 measure $f - \operatorname{Div}G$ vanishes on sets with 0 p -capacity, we deduce $f - \operatorname{Div}G = 0$, that is
 246 $U \in \mathcal{SDM}^p(\Omega)$. \square

247 3.2 The Functional

According to our purpose the natural energy to deal with is the following $F : \mathcal{SDM}^p(\Omega) \rightarrow [0, \infty]$ given by

$$F(U) = \int_{\Omega} |\operatorname{div}U|^2 dx + \int_{\Omega} |U - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s U)_0).$$

248 We note that $\inf F(U) > 0$ on $\mathcal{SDM}^p(\Omega)$. Indeed if we had $\inf_{\mathcal{SDM}^p(\Omega)} F(U) = 0$ then it
 249 would be possible exhibiting a minimizing sequence $U_n \rightarrow U_0$ in L^p such that $\operatorname{Div}U_n \rightarrow 0$
 250 in $\mathcal{D}'(\Omega)$. On the other hand, the L^p -distance between U_n and U_0 can be arbitrary small
 251 only if $\operatorname{Div}U_0 = 0$ as well, because the constraint $\operatorname{Div}U = 0$ is stable under L^p -convergence.
 252 Moreover to mimic the construction of U_0 we restrict ourselves to vector fields U which are
 253 the gradient of a function $u \in W_0^{1,p}(\Omega)$.

254

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \nabla u \in \mathcal{SDM}^p\} \quad (13)$$

255 So that the new energy $\mathcal{F} : \Delta\mathcal{M}^p(\Omega) \rightarrow (0, \infty]$ is given by

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s \nabla u)_0). \quad (14)$$

256 4 Γ -convergence: The intermediate approximation

257 The functional \mathcal{F} is finite on the class of the functions $u \in \Delta\mathcal{M}^p(\Omega)$ whose support of the
 258 measure $(\operatorname{div}^s \nabla u)_0$ is given by a finite set and the absolutely continuous part of the measure
 259 $\operatorname{Div} \nabla u$ is given by $\Delta u \in L^2(\Omega)$. Consequently it is convenient to introduce the following
 260 space:

$$\Delta\mathcal{AM}^{p,2}(\Omega) = \{u \in \Delta\mathcal{M}^p(\Omega) : \operatorname{supp}(\operatorname{div}^s \nabla u)_0 = P_{\nabla u} \text{ with } \mathcal{H}^0(P_{\nabla u}) < +\infty, \Delta u \in L^2(\Omega)\} \quad (15)$$

In the spirit [11] we introduce an intermediate approximation, in the sense of Γ -convergence, of the functional \mathcal{F} . We define a sequence of functionals where the counting measure $\mathcal{H}^0(P_{\nabla u})$ is replaced by a functional defined on regular sets D and which involves the curvature of the boundary ∂D . The approximating sequence is given by:

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) \end{aligned} \quad (16)$$

261 Where $u \in W_0^{1,p,2}(\operatorname{div}; \Omega)$, D is a regular set, and κ denotes the curvature of its boundary .

262 In order to guarantee that the measure of the sets D is small we define a new functional
 263 still denoted by $F_\varepsilon(u, D)$ given by

$$F_\varepsilon(u, D) = \begin{cases} \int_{\Omega \setminus D} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x) & \text{if } \mathcal{L}(D) \leq a(\varepsilon) \\ +\infty & \text{otherwise} \end{cases} \quad (17)$$

264 while the limit functional is

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(P_{\nabla u}).$$

265 where $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, $U_0 \in L^p(\Omega; \mathbb{R}^2)$.

266 To simplify the notation let us set $Y(\Omega) = \{(u, D) \mid u \in W_0^{1,p,2}(\operatorname{div}; \Omega), D \in R(\Omega)\}$,
 267 where $R(\Omega)$ is a regular set.

268 By adapting to our case the convergence introduced in [11], we endow the set $Y(\Omega)$ with
 269 the following convergence.

270 **Definition 4.1.** We say that $\{(u_h, D_h)\} \subset Y(\Omega)$ *A-converges* to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ if the
 271 following conditions hold

- 272 1. $\mathcal{L}(D_h) \rightarrow 0$
- 273 2. $\{\partial D\}_h \rightarrow P$ in the Hausdorff metric, where P is a finite set of points.
- 274 3. $u_h \rightarrow u$ in $L^p(\Omega)$ and $P_{\nabla u} \subseteq P$

275 The definition of Γ -convergence with respect to the convergence above is the following.

276 **Definition 4.2.** We say that a sequence G_ε Γ -converges to G if for every sequence of positive
277 numbers $\{\varepsilon_h\} \rightarrow 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -convergent to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \geq G(u)$$

2. there exists a sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -convergent to u such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, D_h) \leq G(u)$$

278 By following this approach the fundamental theorem of the Γ -convergence can be stated
279 as follows.

Theorem 4.1. Let \mathcal{F}_ε be a sequence of functional Γ -convergent to \mathcal{F} . If $\{\varepsilon_h\}$ is a sequence
of positive numbers converging to zero and $\{(u_h, D_h)\} \subset Y(\Omega)$ such that

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} \mathcal{F}_{\varepsilon_h}) = 0$$

280 then there exists a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ and a minimizer $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ of
281 $\mathcal{F}(u)$, such that $\{(u_{h_k}, D_{h_k})\}$ A -converges to u .

282 By following the classical pattern (see for instance [12]), we will split the proof in three
283 steps. Theorem 4.1 will follow from: the compactness of the minimizing sequence of F_ε ,
284 the Γ lim inf inequality (1) and the Γ -limsup inequality (2) in Definition 4.2.

285 4.1 Compactness

286 We state and prove the following compactness result.

287 **Theorem 4.2.** Let h goes to $+\infty$ and $\varepsilon_h \rightarrow 0$

$$F_{\varepsilon_h}(u_h, D_h) \leq M \tag{18}$$

288 then there exist a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ and a function $u \in \Delta\mathcal{AM}^p(\Omega)$ such
289 that (u_{h_k}, D_{h_k}) A -converges to u .

Proof. We adapt an argument of [11]. We have $\mathcal{L}(D_h) \rightarrow 0$ otherwise $F_{\varepsilon_h} = +\infty$. Let
 D_h be a sequence of regular sets. Then we can parameterize every $C_h = \partial D_h$ by a finite
and disjoint union of Jordan curves. Let us set for every h $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$, then we have
according to the 2-dimensional version of Gauss-Bonnet's Theorem and Young's inequality

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h \kappa_h^2 \right) d\mathcal{H}^1 \geq \frac{1}{4\pi} \int_{\partial D_h} 2\kappa_h d\mathcal{H}^1 = \frac{1}{4\pi} \int_{\bigcup_h C_h} 2\kappa_h d\mathcal{H}^1 = m(h)$$

Note that the number $m(h) \leq M$, with $M \geq 0$, is independent of h . Then it is possible to extract a subsequence C_{h_k} with the number of curves in C_{h_k} equal to n for every k . Then we set $C_{h_k} = \{\gamma_{h_k}^1, \dots, \gamma_{h_k}^n\}$ for any k . From (18) we also have for any $\gamma \in C_{h_k}$ that $\mathcal{H}^1(\gamma) \leq 4\pi M \varepsilon_{h_k}$ and consequently $\max\{\mathcal{H}^1(\gamma) : \gamma \in C_{h_k}\} \rightarrow 0$. Then there exists a finite set of point $P = \{x_1, \dots, x_n\} \subset \overline{\Omega}$ such that for any radius ρ there is an index k_ρ with

$$\gamma_{h_k}^i \subset B_\rho(x_i) \text{ for all } k > k_\rho \text{ and } i \in \{1, \dots, n\},$$

290 so that if we set $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$, then the Hausdorff distance $d_H(\partial D_{h_k}, P) \rightarrow$
 291 0 since $\mathcal{L}^1(D_{h_k}) \rightarrow 0$ and therefore $\rho \rightarrow 0$ as well.

292 Now we prove the compactness property for u_h . First of all from the estimate

$$\|\nabla u_h\|_{L^p(\Omega)}^p \leq 2^p (\|\nabla u_h - U_0\|_{L^p(\Omega)} + \|U_0\|_{L^p(\Omega)}) \quad (19)$$

293 and (18) we may extract a subsequence $\{u_{h_k}\} \subset W_0^{1,p}(\Omega)$ weakly convergent to $u \in W_0^{1,p}(\Omega)$.

Let Ω_j be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P$ invading $\Omega \setminus P$. We claim that that it is possible to extract a sequence of D_{h_k} such that $\Omega_j \cap \partial D_{h_k} = \emptyset$. Indeed since the distance between Ω_j and P is positive for any j there exists η_j such that $\Omega_j \cap (\bigcup_i^n B_{\eta_j}(x_i)) = \emptyset$. On the other hand we know from before that for every ρ we can find k_ρ such that $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$. Then in particular if $\rho = \eta_j$ there exists k_j such that for all $k \geq k_j$

$$\Omega_j \cap \partial D_{h_k} = \emptyset.$$

294 Therefore for any $x \in \Omega_j$ there exists $\delta > 0$ such that either $B_\delta(x) \subset D_{h_k}$ or $B_\delta(x) \subset \Omega \setminus D_{h_k}$.

295 Finally by taking into account that $\mathcal{L}^1(D_{h_k}) \rightarrow 0$ we conclude $\Omega_j \cap \partial D_{h_k} = \emptyset$ for $k \geq k_j$.

296 Then for every $k \geq k_j$ we have that $u_{h_k} \in W^{1,p,2}(\text{div}; \Omega_j)$ and by (18) we get

$$\int_{\Omega_j} |\Delta u_h|^2 \leq \int_{\Omega \setminus D_{h_k}} |\Delta u_{h_k}|^2 dx \leq M \quad (20)$$

Then we can extract a further subsequence still denoted by $\{u_{h_k}\} \subset W^{1,p,2}(\text{div}; \Omega_j)$ such that

$$\begin{cases} u_{h_k} \rightarrow u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \text{ and a.e} \\ \nabla u_{h_k} \rightharpoonup \nabla u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \\ \Delta u_{h_k} \rightharpoonup \Delta u & \text{in } L^2(\Omega_j). \end{cases}$$

297 Let now $x \in \Omega' \subset \subset \Omega \setminus P$. Then there exists a sequence $x_j \rightarrow x$ with $j \in \mathbb{N}$. By applying
 298 the diagonal argument to the sequence $u_{h_{k_l}}(x_j)$ we obtain a subsequence $u_l = u_{h_{k_l}}(x_l)$ such
 299 that Δu_l converges weakly in $L^{p'}(\Omega')$ to Δu for any $\Omega' \subset \subset \Omega$. Then by the semicontinuity
 300 of the L^2 -norm we have

$$\sup_j \int_{\Omega_j} |\Delta u|^2 dx \leq \sup_j \liminf_{l \rightarrow +\infty} \int_{\Omega_j} |\Delta u_l|^2 dx \leq M.$$

301 If we set $\tilde{P} = P \setminus \partial\Omega$, then we deduce $u \in W_0^{1,p,2}(\text{div}; \Omega \setminus \tilde{P})$ and therefore $\nabla u \in S\mathcal{DM}^p(\Omega)$
 302 with $P_{\nabla u} \subseteq P$, by Proposition 3.1. So we conclude that $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ \square .

303 4.2 Lower Bound

304 According to the convergence we deal with, we have to provide the following lower bound.

Theorem 4.3. *Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and for every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ A -converging to u we have*

$$\liminf_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u).$$

Proof. First of all we may assume that $\liminf F_{\varepsilon_h}(u_h, D_h) < M$ otherwise the result is trivial. Moreover up to a subsequence we may assume that the \liminf is actually a limit. Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P$ invading $\Omega \setminus P$. As in the proof of the Theorem 4.2 we may assume up to a subsequence, that $\Delta u_h \rightarrow \Delta u$ weakly in $L^2(\Omega_j)$. Furthermore, since in this case all the sequence D_h satisfies the condition (1) and (2) in Definition 4.1 we have $\Omega_j \subset \subset \Omega \setminus D_h$ for h large and for any j . We also have Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Consequently

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 \geq \liminf_h \int_{\Omega_j} |\Delta u_h|^2 dx \geq \int_{\Omega_j} |\Delta u|^2 dx.$$

305 On the other hand we know that the limit u of the subsequence u_h belongs to $\Delta\mathcal{AM}^p(\Omega)$,
306 hence $\Delta u \in L^2(\Omega)$. So that by monotone convergence

$$\liminf_h \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \geq \int_{\Omega \setminus P} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 \quad (21)$$

307 As in the proof of the theorem 4.2 we have the inequality (19). Then we easily get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx. \quad (22)$$

This concludes the first part of the proof. Let now D_h be a sequence of regular sets. Then we can find a parameterization of every $C_h = \partial D_h$ by a finite and disjoint union of Jordan curves. As in the proof of Theorem 4.2 if we set for every h , $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 = m(h)$$

308 and up to subsequences we have $m(h) = n$ and there exist a set P_1 of n points such that ∂D_h
309 converges in the Hausdorff metric to P_1 . On the other hand we have that ∂D_h converges in
310 the Hausdorff metric to P and, since the limit is unique $P = P_1$ and therefore.

$$\frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon} + \varepsilon_h k^2 \right) d\mathcal{H}^1 \geq n = \mathcal{H}^0(P) \geq \mathcal{H}^0(P_{\nabla u}) \quad (23)$$

311 Eventually by (21), (22) (23) and by the superlinearity property of the \liminf operator we
312 achieve the result. \square

313 4.3 Upper bound

314 In [11] for the construction of the optimal sequence it is crucial to use a result due to
 315 Chambolle and Doveri (see [13]). This result states that it is possible to approximate, in
 316 the H^1 norm, a function $u \in W^{1,2}(\Omega \setminus C)$ (where C is a closed set), by means of a sequence
 317 of functions $u_h \in W^{1,2}(\Omega \setminus C_h)$ with C_h convergent to C in the Hausdorff metric. In our
 318 case this argument does not apply due to presence of a second order differential operator.
 319 Nevertheless since we work only with set of points it is possible to build an optimal sequence
 320 in a more direct way.

321 According to the convergence we deal with the theorem, we have to prove the following
 322 theorem.

323 **Theorem 4.4.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
 324 $\Delta AM^{p,2}(\Omega)$ there exists a sequence $\{(u_h, D_h)\} \subset Y(\Omega)$ A -converging to u such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u). \quad (24)$$

325 **Proof.** We start by the sequence D_h . Let n be the number of points x_i in $P_{\nabla u}$. Then
 326 we take $D_h = \bigcup_1^n B_{\varepsilon_h}(x_i)$. So that $\mathcal{L}(D_h) \rightarrow 0$ and ∂D_h goes to P with respect to the
 327 Hausdorff distance. Moreover for h large we may assume $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$.
 328 Now we build u_h . Let $\{\rho_h\} \subset \mathbb{R}$ be such that $\rho_h \geq 0$ and $\rho_h \rightarrow 0$ when $h \rightarrow \infty$. Let
 329 $\theta_h \in C^\infty(\Omega)$ with the following property:

$$\begin{cases} \theta_h = 1 & \text{on } B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ 0 \leq \theta_h \leq 1 & \text{on } B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ \theta = 0 & \text{on } \Omega \setminus B_{\rho_h}(x_i) \text{ for any } i = 1, \dots, n \\ \|\nabla \theta_h\|_\infty \leq \frac{1}{\rho_h} \end{cases} \quad (25)$$

We set $u_h = (1 - \theta_h)u$ and we claim that the pair (u_h, D_h) realizes the inequality (24). By making the computation we have

$$\nabla u_h = (1 - \theta_h)\nabla u - u\nabla \theta_h$$

Then we have

$$\int_\Omega |\nabla u_h - U_0|^p dx = \int_\Omega |\nabla u - U_0 - \theta_h \nabla u - u \nabla \theta_h|^p dx$$

330 So that

$$\limsup_{h \rightarrow +\infty} \int_\Omega |\nabla u_h - U_0|^p dx \leq \limsup_{h \rightarrow +\infty} \left(\left(\int_\Omega |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_\Omega |\theta_h \nabla u|^p \right)^{\frac{1}{p}} + \left(\int_\Omega |u \nabla \theta_h|^p \right)^{\frac{1}{p}} \right)^p \quad (26)$$

331 Since $\nabla u \in L^1(\Omega)$, we have by applying the dominated convergence theorem $\int_{\Omega} |\theta_h \nabla u|^p \rightarrow 0$.
 332 Let us focus on the term $\int_{\Omega} |\nabla \theta_h u|^p$. By the Sobolev embedding we have $u \in L^{p^*}(\Omega)$ with
 333 $p^* = \frac{2p}{2-p}$ and hence $|u|^p \in L^{\frac{p^*}{p}}(\Omega)$, with $\frac{p^*}{p} = \frac{2}{2-p}$.

By (25), using the Holder's inequality with dual exponents $\frac{2}{2-p}$ and $\frac{2}{p}$ and taking into account that $p < 2$

$$\begin{aligned} \int_{\Omega} |\nabla \theta_h u|^p &\leq \sum_{i=1}^n \int_{B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx = \sum_{i=1}^n \left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h u|^p dx - \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx \right) \\ &\leq n \left(\left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} + \left(\int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \right) \\ &\leq n \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \left(\frac{\pi^2 \rho_h^2}{\rho_h^p} + \frac{\pi^2 \rho_h^2}{4\rho_h^p} \right) \rightarrow 0. \end{aligned} \quad (27)$$

From (26) it follows that

$$\begin{aligned} \limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx &\leq \lim_{h \rightarrow +\infty} \left(\left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\theta_h \nabla u|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\nabla \theta_h u|^p dx \right)^{\frac{1}{p}} \right)^p \\ &= \left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} = \int_{\Omega} |\nabla u - U_0|^p dx \end{aligned} \quad (28)$$

Now we compute $\Delta \theta_h$. The identity $\operatorname{div}(fA) = f \operatorname{div} A + \nabla f \cdot A$ yields

$$\Delta u_h = (1 - \theta_h) \Delta u - 2 \nabla \theta_h \nabla u - \Delta \theta_h u.$$

334 Then by choosing ρ_h small enough we have from (25)

$$\limsup_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 \leq \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u|^2 \rightarrow \int_{\Omega} |\Delta u|^2 dx. \quad (29)$$

335 Moreover it is not difficult to check that $\{(u_h, D_h)\} \subset Y(\Omega)$ and A-converges to u .

336 Finally since for h large we have $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$ we get

$$\lim_h \frac{1}{4\pi} \int_{\partial D_h} \left(\varepsilon_h + \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = \lim_h \sum_1^n \int_{\partial B_{\varepsilon_h}} \left(\varepsilon_h \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = n = \mathcal{H}^0(P_{\nabla u}). \quad (30)$$

337 By recalling that the limsup is sublinear operation and by (28),(29),(30), we achieve the
 338 result. \square

339 5 Approximation by smooth function

340 By following the Braides-March's approach in [11] we approximate the measure \mathcal{H}^1 measure
 341 by the Modica-Mortola's energy density given by $(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx$ where $W(w) =$
 342 $w^2(1-w)^2$ and $w \in C^\infty(\Omega)$. The next step is to replace the regular set D with the level
 343 set of w . Let us set $H = \{\nabla w(x) = 0\}$. By Sard's Lemma we have that $\mathcal{L}^1(w(H)) = 0$. In
 344 particular, if w takes values into the interval $[0, 1]$, we infer that for almost every $t \in (0, 1)$
 345 the set $H \cap w^{-1}(t)$ is empty. Consequently for almost every $t \in (0, 1)$ the t -level set $\{w < t\}$
 346 is a C^∞ set with boundary $\{w = t\}$. Now since we want to replace the set D we need that
 347 $\{w < t\} \subset\subset \Omega$ and then we require that w is such that $1 - w \in C_0^\infty(\Omega; [0, 1])$. Furthermore
 348 for almost every t we have $k(\{w = t\}) = \operatorname{div}(\frac{\nabla w}{|\nabla w|})$. From all of this we are led to define the
 349 following space:

$$S(\Omega) = \{(u, w); u \in W_0^{1,p,2}(\operatorname{div}; \Omega); 1 - w \in C_0^\infty(\Omega; [0, 1])\} \quad (31)$$

and having in mind the coarea formula, the following sequence of functionals defined on $S(\Omega)$

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_\Omega w|\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)^2\right) (\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx \\ &+ \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_\Omega (1 - w_\varepsilon)^2 dx \end{aligned}$$

350 whit $C = \int_0^1 \sqrt{W(t)} dt$.

351 The last term forces w_ε be 1 almost everywhere in the limit and hence plays the role of
 352 the condition $\mathcal{L}(D) \leq a(\varepsilon)$. From now on the parameters $\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ will be related as follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 0 \quad (32)$$

353

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0 \quad (33)$$

354 The convergence that plays the role of the A -convergence is the following.

355 **Definition 5.1.** Let $\{(u_h, w_h)\}$ be a sequence in $S(\Omega)$ and $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$. We say that
 356 that (u_h, w_h) S -converges to u if for every $t \in (0, 1)$ the pair (u_h, D_h^t) A -converges to u .

357 We define the Γ -convergence with respect to this convergence.

358 **Definition 5.2.** We say that a sequence G_ε Γ -converges to G if for every sequence of positive
 359 numbers $\varepsilon_h \rightarrow 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence (u_h, w_h) S -convergent to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \geq G(u)$$

2. there exists a sequence (u_h, w_h) S -convergent to u such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \leq G(u)$$

360 As in the previous section we state a suitable version of the fundamental theorem of the
361 Γ -convergence.

Theorem 5.1. *Let \mathcal{F}_ε a sequence of functional Γ -convergent to \mathcal{F} . If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, w_h)\} \subset S(\Omega)$ such that*

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf \mathcal{F}_{\varepsilon_h}) = 0$$

362 then there exists a subsequence $\{(u_{h_k}, w_{h_k})\}$ and a minimizer $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ of \mathcal{F} such
363 that $\{(u_{h_k}, w_{h_k})\}$ S -converges u .

364 5.1 Compactness

365 **Theorem 5.2.** *Let h goes to $+\infty$ and $\varepsilon_h \rightarrow 0$*

$$F_{\varepsilon_h}(u_h, w_h) \leq M \tag{34}$$

366 there exists a subsequence $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$ and $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ such that (u_{h_k}, w_{h_k})
367 S -converges to u .

368 **Proof.** The first part of proof is as in [11]. For the convenience of the reader we give the
369 complete proof.

370 For any $t \in (0, 1)$ since $0 \leq w_h \leq 1$, we have $w_h > \frac{1}{(1-t)^2}$ then by Chebyshev's inequality
371 and by (57)

$$\mathcal{L}(\{w_h < t\}) \leq \frac{M\gamma_{\varepsilon_h}}{(1-t)^2}. \tag{35}$$

372 The inequality ensures that the condition (1) of Definition 4.1 holds for every sub level sets
373 $\{w_h < t\}$.

By Young's inequality and by (57) we get

$$M \geq 2 \int_{\Omega \setminus \{|\nabla w_h|=0\}} |\nabla w_h| \sqrt{W(w_h)} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) dx$$

374 Now by coarea formula and taking into account that $|\nabla w_h|$ coincides with the norm of the
375 tangent vector to the boundary of the t -level set we obtain

$$M \geq \int_0^1 \sqrt{W(t)} \int_{\{w_h=t\} \cap \{|\nabla w| \neq 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) d\mathcal{H}^1 dt \tag{36}$$

Thanks to the Sard's Lemma for any h there exists a \mathcal{L} -negligible set $\mathcal{N}_{w_h} \subseteq (0, 1)$ such that

$$\{w_h = t\} = \partial\{w_h < t\}, \quad \{w_h < t\} \in R(\Omega), \quad \text{for } t \in (0, 1) \setminus \mathcal{N}_{w_h}$$

and on $\{w_h = t\}$ for $t \in (0, 1) \setminus \mathcal{N}_{w_h}$ we have

$$|\nabla w_h| \neq 0 \text{ and } \kappa(\{w_h = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right).$$

Now since the union \mathcal{N}_w of the sets \mathcal{N}_h is \mathcal{L} -negligible (almost countable) from (59) we have

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt$$

376 By applying Fatou's Lemma and taking into account that the set \mathcal{N}_w does not depend on h
377 we get

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt \quad (37)$$

378 Hence we deduce the existence of a \mathcal{L} -negligible set Q_w , with $\mathcal{N}_w \subseteq Q_w$, such that

$$\liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 \leq M_t \quad (38)$$

379 where the constant M_t does not depend on h .

Then for every $t \in (0, 1) \setminus Q_w$ we can extract a sequence $w_{h,t}$ such that $\partial\{w_{h,t} < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$. Let $\mathcal{N} = \{t_i\}$ in $(0, 1)$ be a dense countable set. Up to a diagonal argument we can find a subsequence $\{w_{h_k}\}$ such that, for every $t^i \in \mathcal{N}$, $\partial\{w_{h_k} < t_i\}$ converges to P^{t_i} . Let $t_i \in \mathcal{N}$ such that $t_i > t$ and consequently $\{w_{h_k} < t\} \subseteq \{w_{h_k} < t_i\}$. From the definition of Hausdorff convergence it follows that for every $\rho > 0$ there exists $k_0(\rho)$ such that for any $k > k_0$ we have $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$ for every $x \in P^{t_i}$. Since the t_i -level set is open for every ρ and for every $x \in P^{t_i}$ such that $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$, we may choose $t_n \in \mathcal{N}$ with $t_n < t$ and so obtain for k large enough $\{w_{h_k} < t_n\} \cap B_\rho(x) \neq \emptyset$. By choosing $t_{max} = \max_{x \in P^{t_i}} t_n(x)$ for every $x \in P^{t_i}$, the inclusion $\{w_{h_k} < t_{max}\} \subset \{w_{h_k} < t\}$ gives

$$\{w_{h_k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h_k} < t\} \cap B_\rho(x) \subseteq \{w_{h_k} < t_n\} \cap B_\rho(x)$$

380 with $t_{max}, t_n \in \mathcal{N}$. Then by taking the limit $\delta \rightarrow 0^+$ we infer $\partial\{w_{h,t} < t\}$ converges with
381 respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$ for every $t \in (0, 1)$.

382 Finally as in the proof of Theorem 4.2 we can extract a subsequence $\{u_{h_k}\}$ which con-
383 verges strongly in $L^p(\Omega)$ to a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ with $P_{\nabla u} \subset P^t$ for every $t \in (0, 1)$.
384 Therefore we have that for every $t \in (0, 1)$ the pair $(u_h, \{w_{h,t} < t\})$ S-converges to u and
385 the proof is achieved. \square

386 **5.2 Lower bound**

Theorem 5.3. *Let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and for every sequence $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u the following inequality holds*

$$\liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}(u, w) \geq F(u).$$

Proof. Let $u \in \Delta\mathcal{AM}^p(\Omega)$ and $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u . As usual up to subsequences we can assume

$$+\infty > \liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}.$$

387 For any $t \in (0, 1)$ we have (see also [11] for a similar argument)

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 = \int_{\{w_h < t\} \cap \Omega} w_h^2 |\Delta u_h|^2 + \int_{\{w_h \geq t\} \cap \Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \quad (39)$$

388 Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P^t$ invading $\Omega \setminus P^t$. Then we may assume
 389 that $u_h \rightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Therefore as
 390 in the proof of Theorem 4.3 we get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx \quad (40)$$

and

$$\liminf_{h \rightarrow +\infty} t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx$$

391 for any j .

Then by (62) and, by taking into account that $|\Delta u|$ is an $L^2(\Omega)$ function, it follows that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} |\Delta u|^2 dx$$

392 And eventually by taking the limit $t \rightarrow 1$

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq \int_{\Omega} |\Delta u|^2 dx \quad (41)$$

393 By definition we know that for every $t \in (0, 1)$ $\mathcal{L}^d(\{w_h < t\}) \rightarrow 0$ and $\partial\{w_h < t\} \rightarrow P^t$ in
 394 the Hausdorff distance. So that, as in the proof of Theorem 4.3 (inequality 23) we have

395

$$\liminf_{h \rightarrow +\infty} \frac{1}{4\pi} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}) \quad (42)$$

Now arguing as in the proof of Theorem 6.2 and by taking into account (65), we get

$$\begin{aligned}
& \liminf_{h \rightarrow \infty} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\
& \geq 2 \liminf_{h \rightarrow +\infty} \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 dt \\
& \geq 2 \int_{(0,1)} \mathcal{H}^0(P^t) \sqrt{W(t)} dt \geq 8\pi C \mathcal{H}^0(P_{\nabla u})
\end{aligned} \tag{43}$$

396 By collecting (63) (64) and (66) we achieve the thesis. \square

397 5.3 Upper bound

398 As in [11] to build w_h we use the construction used by Bellettini in [6], while the optimal
399 sequence u_k is given as in Theorem 4.4.

400 **Theorem 5.4.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
401 $\Delta\mathcal{M}^{p,2}(\Omega)$ there exists a sequence $(u_h, D_h) \in S(\Omega)$ S -converging to u such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{44}$$

Proof. If $A \subset \mathbb{R}^2$ we set

$$\delta_A(x) = d(x, A) - d(x, \mathbb{R}^2 \setminus A).$$

402 We start by the construction of w_h .

As in the proof of Theorem 4.4 we set $P_{\nabla u} = \{x_1, \dots, x_n\}$ and we define

$$D_h = \bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

403 Since D_h is a regular set by taking into account the condition (56) for h large enough we
404 have

$$\{x \in \Omega : d(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset \subset \Omega. \tag{45}$$

Let η be the optimal profile for the Modica Mortola energy, that is the solution of

$$\begin{cases} \eta'(t) = \sqrt{W(t)} & \text{on } \mathbb{R} \\ \eta(-\infty) = 0, \\ \eta(+\infty) = 1 \end{cases}$$

405 given by $\eta(t) = \frac{1}{2}(1 + \tanh \frac{t}{2})$.

For every h let $\psi_h : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h, +\infty] \\ \psi_h' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|\psi_h'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O\left(\frac{1}{|\log \varepsilon_h|}\right) \end{cases}$$

As in [6] and in [11] we define

$$\eta_h(t) = \begin{cases} \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) + 1 - \psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t \geq 0 \\ \psi\left(\frac{t}{\varepsilon_h}\right) - \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t < 0 \end{cases}$$

Then we set $w_h(x) = \eta_h(\delta_{D_h}(x))$. We claim that $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$ for h large. Indeed for any $x \in \Omega \setminus D_h$ we have $\delta_{D_h}(x) \geq 0$, hence

$$1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$$

406 and then $0 \leq 1 - w_h \leq 1$ by the properties of ψ_h and η . The case $x \in D_h$ is similar. Let now
 407 $x \in \partial\Omega$ then $\delta_{D_h}(x) \geq 0$ and $1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$. From (68), it follows
 408 $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$ for h large enough, hence the claim follows. Then we take $\{(u_h, w_h)\}$ as
 409 an optimal sequence, where u_h is given as in Theorem 4.4

410 First all we have to check that $(u_h, w_h(x))$ S-converges to u . For any $x \in \Omega \setminus P_{\nabla u}$ we have
 411 that for h large enough $\delta_{D_h}(x) \geq 0$ and one can check that $w_h(x) \rightarrow 1$ for every $x \in \Omega \setminus P_{\nabla u}$.
 412 This implies that $\mathcal{L}(\{w_h < t\}) \rightarrow 0$ for every $t \in (0, 1)$. Now for every $t \in (0, 1)$ we write

$$\{w_h = t\} = (\{w_h = t\} \cap D_h) \cup (\{w_h = t\} \cap \Omega \setminus D_h). \quad (46)$$

413 Hence, since $w_h(x) \rightarrow 1$ for $x \in \Omega \setminus D_h$, for any $t \in (0, 1)$ there exists $h(t)$ such that
 414 $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$ for every $h \geq h(t)$. So that from (69) it follows that for every
 415 $t \in (0, 1)$, $\{w_h = t\} \rightarrow P_{\nabla u}$ when $h \rightarrow +\infty$. So we can conclude that (u_h, w_h) S-converges
 416 to U .

As in [6] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h |\log \varepsilon_h|\}, \quad D_h^2 = \{x \in \Omega : \varepsilon_h |\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h |\log \varepsilon_h|\}$$

and we split

$$\begin{aligned} & \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &= \int_{D_h^1} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx + \\ & \int_{D_h^2} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx = \\ &= I_h + II_h \end{aligned} \quad (47)$$

For $x \in D_h^1$ we have $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$ therefore $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$. By taking into account the definition of η we have $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$. Moreover it is easy to check that

$$\eta'_h(t) = \frac{1}{\varepsilon_h} \eta'(t) = \frac{1}{\varepsilon_h} \sqrt{W(t)}; \quad |\nabla w_h| = |\eta'_k(\delta_{D_h})|$$

417

This, together with the coarea formula yields

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h=t\}} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt$$

Now we have that $\partial D_h = \{w_h = \frac{1}{2}\}$ then

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)}$$

418 Then

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt \quad (48)$$

419 Moreover with the same argument and by using the definition of w_h one can check that

$$\lim_{h \rightarrow \infty} II_h = 0 \quad (49)$$

420 By (71) and (72) we have

421

$$\lim_{h \rightarrow \infty} \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) \right) dx = \mathcal{H}^0(P_{\nabla u}) \quad (50)$$

422 Now we take care of the terms involving u_h . As in the proof of Theorem 4.4 we have

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p \leq \int_{\Omega} |\nabla u - U_0|^p \quad (51)$$

Furthermore by taking into account that $w_h(x) = 1$ if $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$ and $w_h(x) = 0$ if $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$ we have

$$\int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \int_{\Omega \setminus D_h^0} |\Delta u_h|^2 dx$$

423 where $D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$.424 By choosing ρ_h in such a way that $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$ we obtain

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega \setminus P_{\nabla u}} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx \quad (52)$$

425 Finally from the definition of w_h , it follows that $w_h = 1$ outside the disk $(D_h)_{2 \log \varepsilon_h}$ and
426 hence thanks to (55) and (56)

$$\limsup_{h \rightarrow +\infty} \frac{1}{\gamma_h} \int_{\Omega} (1 - w_h)^2 dx \leq \lim_{h \rightarrow +\infty} \mathcal{L}(D_{h 2 \log \varepsilon_h}) \frac{1}{\gamma_h} = 0. \quad (53)$$

427 The thesis follows by collecting (73), (74), (75) and (76) \square

6 Approximation by smooth function

By following the Braides-March's approach in [11] we approximate the measure \mathcal{H}^1 measure by the Modica-Mortola's energy density given by $(\varepsilon|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon))dx$ where $W(w) = w^2(1-w)^2$ and $w \in C^\infty(\Omega)$. The next step is to replace the regular set D with the level set of w . Let us set $H = \{\nabla w(x) = 0\}$. By Sard's Lemma we have that $\mathcal{L}^1(w(H)) = 0$. In particular, if w takes values into the interval $[0, 1]$, we infer that for almost every $t \in (0, 1)$ the set $H \cap w^{-1}(t)$ is empty. Consequently for almost every $t \in (0, 1)$ the t -level set $\{w < t\}$ is a C^∞ set with boundary $\{w = t\}$. Now since we want to replace the set D we need that $\{w < t\} \subset\subset \Omega$ and then we require that w is such that $1 - w \in C_0^\infty(\Omega; [0, 1])$. Furthermore for almost every t we have $k(\{w = t\}) = \operatorname{div}(\frac{\nabla w}{|\nabla w|})$. From all of this we are led to define the following space:

$$S(\Omega) = \{(u, w); u \in W_0^{1,p,2}(\operatorname{div}; \Omega); 1 - w \in C_0^\infty(\Omega; [0, 1])\} \quad (54)$$

and having in mind the coarea formula, the following sequence of functionals defined on $S(\Omega)$

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_\Omega w |\Delta u|^2 + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &+ \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_\Omega (1 - w_\varepsilon)^2 dx \end{aligned}$$

whit $C = \int_0^1 \sqrt{W(t)} dt$.

The last term forces w_ε be 1 almost everywhere in the limit and hence plays the role of the condition $\mathcal{L}(D) \leq a(\varepsilon)$. From now on the parameters $\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ will be related as follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 0 \quad (55)$$

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0 \quad (56)$$

The convergence that plays the role of the A -convergence is the following.

Definition 6.1. Let $\{(u_h, w_h)\}$ be a sequence in $S(\Omega)$ and $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$. We say that (u_h, w_h) S -converges to u if for every $t \in (0, 1)$ the pair (u_h, D_h^t) A -converges to u .

We define the Γ -convergence with respect to this convergence.

Definition 6.2. We say that a sequence G_ε Γ -converges to G if for every sequence of positive numbers $\varepsilon_h \rightarrow 0$ and for every $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence (u_h, w_h) S -convergent to $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \geq G(u)$$

2. there exists a sequence (u_h, w_h) S -convergent to u such that

$$\limsup_{h \rightarrow \infty} G_{\varepsilon_h}(u_h, w_h) \leq G(u)$$

449 As in the previous section we state a suitable version of the fundamental theorem of the
450 Γ -convergence.

Theorem 6.1. *Let \mathcal{F}_ε a sequence of functional Γ -convergent to \mathcal{F} . If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, w_h)\} \subset S(\Omega)$ such that*

$$\lim_{h \rightarrow \infty} (\mathcal{F}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf \mathcal{F}_{\varepsilon_h}) = 0$$

451 then there exists a subsequence $\{(u_{h_k}, w_{h_k})\}$ and a minimizer $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ of \mathcal{F} such
452 that $\{(u_{h_k}, w_{h_k})\}$ S -converges to u .

453 6.1 Compactness

454 **Theorem 6.2.** *Let h goes to $+\infty$ and $\varepsilon_h \rightarrow 0$*

$$F_{\varepsilon_h}(u_h, w_h) \leq M \tag{57}$$

455 there exists a subsequence $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$ and $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ such that (u_{h_k}, w_{h_k})
456 S -converges to u .

457 **Proof.** The first part of proof is as in [11]. For the convenience of the reader we give the
458 complete proof.

459 For any $t \in (0, 1)$ since $0 \leq w_h \leq 1$, we have $w_h > \frac{1}{(1-t)^2}$ then by Chebyshev's inequality
460 and by (57)

$$\mathcal{L}(\{w_h < t\}) \leq \frac{M\gamma_{\varepsilon_h}}{(1-t)^2}. \tag{58}$$

461 The inequality ensures that the condition (1) of Definition 4.1 holds for every sub level sets
462 $\{w_h < t\}$.

By Young's inequality and by (57) we get

$$M \geq 2 \int_{\Omega \setminus \{|\nabla w_h|=0\}} |\nabla w_h| \sqrt{W(w_h)} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) dx$$

463 Now by coarea formula and taking into account that $|\nabla w_h|$ coincides with the norm of the
464 tangent vector to the boundary of the t -level set we obtain

$$M \geq \int_0^1 \sqrt{W(t)} \int_{\{w_h=t\} \cap \{|\nabla w| \neq 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) d\mathcal{H}^1 dt \tag{59}$$

Thanks to the Sard's Lemma for any h there exists a \mathcal{L} -negligible set $\mathcal{N}_{w_h} \subseteq (0, 1)$ such that

$$\{w_h = t\} = \partial\{w_h < t\}, \quad \{w_h < t\} \in R(\Omega), \quad \text{for } t \in (0, 1) \setminus \mathcal{N}_{w_h}$$

and on $\{w_h = t\}$ for $t \in (0, 1) \setminus \mathcal{N}_{w_h}$ we have

$$|\nabla w_h| \neq 0 \quad \text{and} \quad \kappa(\{w_h = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right).$$

Now since the union \mathcal{N}_w of the sets \mathcal{N}_h is \mathcal{L} -negligible (almost countable) from (59) we have

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt$$

465 By applying Fatou's Lemma and taking into account that the set \mathcal{N}_w does not depend on h
466 we get

$$M \geq 2 \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 dt \quad (60)$$

467 Hence we deduce the existence of a \mathcal{L} -negligible set Q_w , with $\mathcal{N}_w \subseteq Q_w$, such that

$$\liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2\right) d\mathcal{H}^1 \leq M_t \quad (61)$$

468 where the constant M_t does not depend on h .

Then for every $t \in (0, 1) \setminus Q_w$ we can extract a sequence $w_{h,t}$ such that $\partial\{w_{h,t} < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$. Let $\mathcal{N} = \{t_i\}$ in $(0, 1)$ be a dense countable set. Up to a diagonal argument we can find a subsequence $\{w_{h_k}\}$ such that, for every $t^i \in \mathcal{N}$, $\partial\{w_{h_k} < t^i\}$ converges to P^{t^i} . Let $t_i \in \mathcal{N}$ such that $t_i > t$ and consequently $\{w_{h_k} < t\} \subseteq \{w_{h_k} < t^i\}$. From the definition of Hausdorff convergence it follows that for every $\rho > 0$ there exists $k_0(\rho)$ such that for any $k > k_0$ we have $\{w_{h_k} < t^i\} \cap B_\rho(x) \neq \emptyset$ for every $x \in P^{t^i}$. Since the t_i -level set is open for every ρ and for every $x \in P^{t^i}$ such that $\{w_{h_k} < t^i\} \cap B_\rho(x) \neq \emptyset$, we may choose $t_n \in \mathcal{N}$ with $t_n < t$ and so obtain for k large enough $\{w_{h_k} < t_n\} \cap B_\rho(x) \neq \emptyset$. By choosing $t_{max} = \max_{x \in P^{t^i}} t_n(x)$ for every $x \in P^{t^i}$, the inclusion $\{w_{h_k} < t_{max}\} \subset \{w_{h_k} < t\}$ gives

$$\{w_{h_k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h_k} < t\} \cap B_\rho(x) \subseteq \{w_{h_k} < t_n\} \cap B_\rho(x)$$

469 with $t_{max}, t_n \in \mathcal{N}$. Then by taking the limit $\delta \rightarrow 0^+$ we infer $\partial\{w_{h,t} < t\}$ converges with
470 respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$ for every $t \in (0, 1)$.

471 Finally as in the proof of Theorem 4.2 we can extract a subsequence $\{u_{h_k}\}$ which con-
472 verges strongly in $L^p(\Omega)$ to a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ with $P_{\nabla u} \subset P^t$ for every $t \in (0, 1)$.
473 Therefore we have that for every $t \in (0, 1)$ the pair $(u_h, \{w_{h,t} < t\})$ S-converges to u and
474 the proof is achieved. \square

475 6.2 Lower bound

Theorem 6.3. *Let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0. For every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and for every sequence $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u the following inequality holds*

$$\liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}(u, w) \geq F(u).$$

Proof. Let $u \in \Delta\mathcal{AM}^p(\Omega)$ and $\{u_h, w_h\} \subset S(\Omega)$ S -converging to u . As usual up to subsequences we can assume

$$+\infty > \liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}.$$

476 For any $t \in (0, 1)$ we have (see also [11] for a similar argument)

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 = \int_{\{w_h < t\} \cap \Omega} w_h^2 |\Delta u_h|^2 + \int_{\{w_h \geq t\} \cap \Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \quad (62)$$

477 Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P^t$ invading $\Omega \setminus P^t$. Then we may assume
 478 that $u_h \rightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Therefore as
 479 in the proof of Theorem 4.3 we get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx \quad (63)$$

and

$$\liminf_{h \rightarrow +\infty} t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx$$

480 for any j .

Then by (62) and, by taking into account that $|\Delta u|$ is an $L^2(\Omega)$ function, it follows that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq t^2 \int_{\Omega} |\Delta u|^2 dx$$

481 And eventually by taking the limit $t \rightarrow 1$

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 \geq \int_{\Omega} |\Delta u|^2 dx \quad (64)$$

482 By definition we know that for every $t \in (0, 1)$ $\mathcal{L}^d(\{w_h < t\}) \rightarrow 0$ and $\partial\{w_h < t\} \rightarrow P^t$ in
 483 the Hausdorff distance. So that, as in the proof of Theorem 4.3 (inequality 23) we have

484

$$\liminf_{h \rightarrow +\infty} \frac{1}{4\pi} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}) \quad (65)$$

Now arguing as in the proof of Theorem 6.2 and by taking into account (65), we get

$$\begin{aligned}
& \liminf_{h \rightarrow \infty} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\
& \geq 2 \liminf_{h \rightarrow +\infty} \int_{(0,1) \setminus \mathcal{N}_w} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon k^2 \right) d\mathcal{H}^1 dt \\
& \geq 2 \int_{(0,1)} \mathcal{H}^0(P^t) \sqrt{W(t)} dt \geq 8\pi C \mathcal{H}^0(P_{\nabla u})
\end{aligned} \tag{66}$$

485 By collecting (63) (64) and (66) we achieve the thesis. \square

486 6.3 Upper bound

487 As in [11] to build w_h we use the construction used by Bellettini in [6], while the optimal
488 sequence u_k is given as in Theorem 4.4.

489 **Theorem 6.4.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$
490 $\Delta\mathcal{M}^{p,2}(\Omega)$ there exists a sequence $(u_h, D_h) \in S(\Omega)$ S -converging to u such that*

$$\limsup F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{67}$$

Proof. If $A \subset \mathbb{R}^2$ we set

$$\delta_A(x) = d(x, A) - d(x, \mathbb{R}^2 \setminus A).$$

491 We start by the construction of w_h .

As in the proof of Theorem 4.4 we set $P_{\nabla u} = \{x_1, \dots, x_n\}$ and we define

$$D_h = \bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

492 Since D_h is a regular set by taking into account the condition (56) for h large enough we
493 have

$$\{x \in \Omega : d(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset\subset \Omega. \tag{68}$$

Let η be the optimal profile for the Modica Mortola energy, that is the solution of

$$\begin{cases} \eta'(t) = \sqrt{W(t)} & \text{on } \mathbb{R} \\ \eta(-\infty) = 0, \\ \eta(+\infty) = 1 \end{cases}$$

494 given by $\eta(t) = \frac{1}{2}(1 + \tanh \frac{t}{2})$.

For every h let $\psi_h : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h, +\infty] \\ \psi_h' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|\psi_h'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O\left(\frac{1}{|\log \varepsilon_h|}\right) \end{cases}$$

As in [6] and in [11] we define

$$\eta_h(t) = \begin{cases} \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) + 1 - \psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t \geq 0 \\ \psi\left(\frac{t}{\varepsilon_h}\right) - \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t < 0 \end{cases}$$

Then we set $w_h(x) = \eta_h(\delta_{D_h}(x))$. We claim that $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$ for h large. Indeed for any $x \in \Omega \setminus D_h$ we have $\delta_{D_h}(x) \geq 0$, hence

$$1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$$

495 and then $0 \leq 1 - w_h \leq 1$ by the properties of ψ_h and η . The case $x \in D_h$ is similar. Let now
 496 $x \in \partial\Omega$ then $\delta_{D_h}(x) \geq 0$ and $1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$. From (68), it follows
 497 $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$ for h large enough, hence the claim follows. Then we take $\{(u_h, w_h)\}$ as
 498 an optimal sequence, where u_h is given as in Theorem 4.4

499 First all we have to check that $(u_h, w_h(x))$ S-converges to u . For any $x \in \Omega \setminus P_{\nabla u}$ we have
 500 that for h large enough $\delta_{D_h}(x) \geq 0$ and one can check that $w_h(x) \rightarrow 1$ for every $x \in \Omega \setminus P_{\nabla u}$.
 501 This implies that $\mathcal{L}(\{w_h < t\}) \rightarrow 0$ for every $t \in (0, 1)$. Now for every $t \in (0, 1)$ we write

$$\{w_h = t\} = (\{w_h = t\} \cap D_h) \cup (\{w_h = t\} \cap \Omega \setminus D_h). \tag{69}$$

502 Hence, since $w_h(x) \rightarrow 1$ for $x \in \Omega \setminus D_h$, for any $t \in (0, 1)$ there exists $h(t)$ such that
 503 $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$ for every $h \geq h(t)$. So that from (69) it follows that for every
 504 $t \in (0, 1)$, $\{w_h = t\} \rightarrow P_{\nabla u}$ when $h \rightarrow +\infty$. So we can conclude that (u_h, w_h) S-converges
 505 to U .

As in [6] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h |\log \varepsilon_h|\}, \quad D_h^2 = \{x \in \Omega : \varepsilon_h |\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h |\log \varepsilon_h|\}$$

and we split

$$\begin{aligned} & \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \\ &= \int_{D_h^1} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx + \\ & \int_{D_h^2} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)\right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx = \\ &= I_h + II_h \end{aligned} \tag{70}$$

For $x \in D_h^1$ we have $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$ therefore $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$. By taking into account the definition of η we have $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$. Moreover it is easy to check that

$$\eta'_h(t) = \frac{1}{\varepsilon_h} \eta'(t) = \frac{1}{\varepsilon_h} \sqrt{W(t)}; \quad |\nabla w_h| = |\eta'_k(\delta_{D_h})|$$

506 .

This, together with the coarea formula yields

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h=t\}} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt$$

Now we have that $\partial D_h = \{w_h = \frac{1}{2}\}$ then

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} \left(\frac{1}{\beta_h} + \beta_h k^2 \right) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)}$$

507 Then

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt \quad (71)$$

508 Moreover with the same argument and by using the definition of w_h one can check that

$$\lim_{h \rightarrow \infty} II_h = 0 \quad (72)$$

509 By (71) and (72) we have

510

$$\lim_{h \rightarrow \infty} \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) \right) dx = \mathcal{H}^0(P_{\nabla u}) \quad (73)$$

511 Now we take care of the terms involving u_h . As in the proof of Theorem 4.4 we have

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p \leq \int_{\Omega} |\nabla u - U_0|^p \quad (74)$$

Furthermore by taking into account that $w_h(x) = 1$ if $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$ and $w_h(x) = 0$ if $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$ we have

$$\int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \int_{\Omega \setminus D_h^0} |\Delta u_h|^2 dx$$

512 where $D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$.513 By choosing ρ_h in such a way that $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$ we obtain

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |w_h|^2 |\Delta u_h|^2 dx = \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega \setminus P_{\nabla u}} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx \quad (75)$$

514 Finally from the definition of w_h , it follows that $w_h = 1$ outside the disk $(D_h)_{2 \log \varepsilon_h}$ and
515 hence thanks to (55) and (56)

$$\limsup_{h \rightarrow +\infty} \frac{1}{\gamma_h} \int_{\Omega} (1 - w_h)^2 dx \leq \lim_{h \rightarrow +\infty} \mathcal{L}(D_{h 2 \log \varepsilon_h}) \frac{1}{\gamma_h} = 0. \quad (76)$$

516 The thesis follows by collecting (73), (74), (75) and (76) \square

517 7 De Giorgi's Conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

518 where D is a regular set and κ is a curvature of its boundary ∂D .

519 Since ∂D can be represented as the discontinuity set of the function $w_0 = 1 - \chi_D$,
520 by Modica-Mortola's Theorem it follows that there is a sequence of non constant local
521 minimizers such that $w_\varepsilon \rightarrow w_0$ with respect to the L^1 -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(w_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

522 Furthermore looking at the Euler-Lagrange equation associated to a contour length term,
523 yields a contour curvature term κ , while the Euler-Lagrange equations for the functional
524 $F_\varepsilon^1(w)$ contains a term $2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon}$.

525 Then De Giorgi suggested to approximate by Γ -convergence the functional F^2 by adding
526 to the Modica-Mortola approximating functionals the following term

$$F_\varepsilon^2(w) = \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx.$$

527 In ([7]) Bellettini and Paolini have proven the lim sup inequality, while the validity of
528 lim inf for a modified version of the original conjecture has been proven by Röger and Shätzle
529 (see [26]).

Inspired by the De Giorgi's conjecture it appears natural investigate, in the spirit of [11],
the possibility of approximating the functional \mathcal{F} by a means of a sequence \mathcal{F}_ε much more
convenient from a numerical point view (see [22]):

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &= \int_{\Omega} w^2 |\Delta u|^2 dx + \frac{1}{8\pi C} \left(\frac{\beta_\varepsilon}{2\varepsilon} \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} (\varepsilon |\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)) dx \right) \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} (1 - w^2) dx. \end{aligned}$$

530 The presence of the term $\frac{1}{2\varepsilon}$ will be clear in the proof.

531 By the way we are able to prove only the Γ -limsup inequality.

532 **Theorem 7.1.** *Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in$*
533 *$\Delta \mathcal{M}^{p,2}(\Omega)$ there exists a sequence $(u_h, D_h) \in S(\Omega)$ S -converging to u such that*

$$\limsup \mathcal{F}_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u) \tag{77}$$

534 **Proof.** Let (u_h, w_h) the optimal sequence of Theorem 6.4.

It is not difficult to see that for every $x \in D_h^1$ the function $\delta_h(x)$ is regular and using the definition of w_h and taking into account that $\eta' = \sqrt{W(\eta)}$ the following identity holds

$$2\varepsilon_h \Delta w_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h \Delta \delta_{D_h}(x) + 2\varepsilon_h \eta''_h - \frac{W'(w)}{\varepsilon_h} = 2\varepsilon_h \eta'_h (\delta_{D_h}(x)).$$

For h large we also have $\Delta \delta_{D_h}(x) = \kappa(\{\delta_{D_h}(x) = t\})$. Besides on D_h^1 we have $w_h(x) = \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})$ and hence the level set $\{\delta_{D_h}(x) = t\}$ corresponds to the level set $\{w_h(x) = \eta(\frac{t}{\varepsilon_h})\}$ with $0 \leq \eta \leq 1$, so that we infer

$$\kappa(\{\delta_{D_h}(x) = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)$$

By proceeding as in the proof of Theorem 6.4 and taking into account the equality $2\varepsilon_h |\eta'_h(\delta_{D_h}(x))| = 2\varepsilon_h |\nabla w_h|$ we have

$$I_h = \int_{D_h^1} \frac{\beta_{\varepsilon_h}}{2\varepsilon_h} \left(2\varepsilon_h \Delta w - \frac{W'(w)}{\varepsilon_h}\right)^2 + \frac{1}{\beta_{\varepsilon_h}} (\varepsilon_h |\nabla w_{\varepsilon_h}|^2 + \frac{1}{\varepsilon_h} W(w_{\varepsilon_h})) dx = 2 \int_{D_h^1} \left(\beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)^2 + \frac{1}{\beta_{\varepsilon_h}}\right) \sqrt{W(w_h)} |\nabla w_h| dx$$

then as in the proof of Theorem 6.4 we conclude

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt.$$

535 By the same calculation on D_h^2 one can check that the integral over D_h^2 vanishes as in the
536 proof of Theorem 6.4.

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