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1 INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Variational approximation of a functional governing point-like singularities.

Daniele Graziani, Gilles Aubert

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Thème COG

 *Rapport
de recherche*

15 **Variational approximation of a functional governing**
16 **point-like singularities.**

17 **Résumé :** Nous proposons une nouvelle méthode variationnelle pour isoler des points dans
18 une image 2-D. Dans ce but nous introduisons une énergie dont les points de minimum sont
19 donnés par l'ensemble des points que l'on veut détecter. En suite on approche cette énergie
20 par une suite de fonctionnelles plus régulières, pour laquelle on montre la Γ -convergence vers
21 la fonctionnelle initiale.

22 **Mots-clés :** détection de points, champs avec divergence mesurée, p -capacité, Γ -convergence.

23	Contents	
24	1 Introduction	ii
25	2 Preliminaries	v
26	2.1 Notation	v
27	2.2 Distributional divergence and classical spaces	v
28	2.3 p -capacity	vi
29	3 The Variational Model	vii
30	3.1 The variational framework	ix
31	3.2 The Functional	xi
32	4 Γ-convergence: The intermediate approximation	xii
33	4.1 Compactness	xiii
34	4.2 Lower bound	xv
35	4.3 Upper bound	xvi
36	4.4 Variational property	xvii
37	5 Approximation by smooth function	xix
38	5.1 Compactness	xx
39	5.2 Lower bound	xxii
40	5.3 Upper bound	xxiii
41	5.4 Variational property	xxvi
42	6 De Giorgi's Conjecture	xxvii
43	References	xxix

1 Introduction

The issue of detecting fine structures, like points or curves in two or three dimensional biological images, is a crucial task in image processing. In particular a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise. Therefore one of the main goals is detecting the spots that the biologists wish to count. This operation is made harder by the presence of other singular structures.

In some biological images the image intensity is a function that takes the value 1 on points or other structures like sets with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. In image processing these concentration sets are called discontinuities without jump, meaning that there is no jump across the set and therefore the gradient of the image is 0.

In the literature there are few variational methods dealing with this problem. In this direction one interesting approach has been proposed in [3]. In that paper the authors consider this kind of pathology as a k -codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [6] for a complete survey on this subject). In particular the detecting point case corresponds to the case $k = 2$ and $m = 0$.

This point of view makes possible a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$, where \mathbb{S}^1 is the unit sphere of \mathbb{R}^2 . Starting from the initial image $I : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, this strategy makes crucial the construction of an initial vector field $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ with a topological singularity of degree 1. How to build such a vector field in a rigorous way, is a subject of a current investigation.

Our first purpose here is finding a most natural variational framework in which a rigorous definition of discontinuity without jump can be given. In our model the image I is a Radon measure. It is crucial for detecting points that this Radon measure be able of charging points. The preliminary step is finding a space whose elements are able of producing this kind of measures. This space is given by $\mathcal{DM}^p(\Omega)$: the space of L^p -vector fields whose distributional divergence is a Radon measure, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence $\text{Div}U$ of U cannot be a measure concentrated on points (see Section 3.1 below). Then we have to construct, from the original image I , a data vector $U_0 \in \mathcal{DM}^p(\Omega)$. Clearly there are, at least in principle, many ways to do this. The one we propose here seems to be the most natural. We consider the classical elliptic problem with measure data I :

$$\begin{cases} -\Delta u_0 = I & \text{on } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then by setting $U_0 = \nabla u_0$ we have $U_0 \in \mathcal{DM}^p(\Omega)$ with $\text{Div}U_0 = I$. However the support of the measure $\text{Div}U_0$ is too large and could contains several structures like curves or fractals, while the singularities, we are interested in, are contained in the atomic part of the measure

71 $\text{Div}U_0$ and therefore we have to isolate it. To do this the notion of p -capacity of a set
 72 plays a key role. Indeed when $p < 2$ the p -capacity of a point in Ω is zero and one can
 73 say, in this sense, that it is a discontinuity with no jump. Besides every Radon measure
 74 can be decomposed (see [14]) in two mutually singular measures: the first one is absolutely
 75 continuous with respect to the p -capacity and the second one is singular with respect to the
 76 p -capacity, meaning that it is a measure concentrated on sets with 0 p -capacity.

77 As it is known in dimension 2, sets with 0 p -capacity, and hence discontinuities without
 78 jump, can be isolated points, countable set of points or fractals with Hausdorff dimension
 79 $0 \leq \alpha < 1$ (see Subsection 2.3 for the definition of p -capacity and related properties).

80 Our goal here is keeping nothing else but points in the image. The achievement of such
 81 a purpose makes necessary the minimization of a suitable energy that must remove all the
 82 discontinuities which are not discontinuities without jump, and remove all the discontinuities
 83 without jump which are not isolated point.

84 From one hand we have to force the concentration set of the divergence measure of U to
 85 contain only the points we want to catch, and on the other hand we have to regularize the
 86 initial data U_0 outside the points of singularities. To this end we introduce the auxiliary
 87 space $SDM^p(\Omega)$ of vector fields belonging to $\mathcal{DM}^p(\Omega)$ whose divergence measure has no
 88 absolutely continuous part with respect to the p -capacity. Then, by taking into account
 89 that the initial vector field is a gradient of a Sobolev function, our goal is to minimize the
 90 following energy:

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \lambda \int_{\Omega} |\nabla u - U_0|^p dx + \mu \mathcal{H}^0(\text{supp}(\text{div}^s \nabla u)_0),$$

91 where $u \in W_0^{1,p}(\Omega)$ with $\nabla u \in SDM^p(\Omega)$, $1 < p < 2$ and λ, μ are positive weights. The
 92 gradient of a minimizer of the energy \mathcal{F} is the vector field we are looking for, that is a
 93 vector field whose divergence measure can be decomposed in an absolutely continuous (with
 94 respect to the Lebesgue's measure) term plus an atomic measure concentrated on the points
 95 we want to isolate in the image.

96 Even if a pointwise characterization of discontinuity without jump is not available, thanks
 97 to our definition the singular set of points can be linked to the vector field ∇u , in the spirit
 98 of the classical SBV formulation of the Mumford-Shah's functional (we refer to [1] for a
 99 complete survey on the Mumford Shah's functional).

For future computational purposes, the next task is to provide an approximation in the
 sense of Γ -convergence introduced in [16, 17]. Our approach is close in the spirit to the
 one used to approximate the Mumford Shah functional by a family of depending curvature
 functionals as in [9]. Indeed, as in their work (see also [8]), we replace the atomic measure
 \mathcal{H}^0 by the term

$$G_{\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2 \right) d\mathcal{H}^1;$$

100 where D is a proper regular set containing the atomic set P , κ is the curvature of its
 101 boundary, and the constant $\frac{1}{4\pi}$ is a normalization factor. Roughly speaking the minima of

102 these functionals are achieved on the union of balls of small radius, so that when $\varepsilon \rightarrow 0$ the
 103 sequence G_ε shrinks to the atomic measure $\mathcal{H}^0(P)$.

This leads to an intermediate approximation given by

$$\begin{aligned} F_\varepsilon(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2 \right) d\mathcal{H}^1. \end{aligned} \quad (1)$$

This strategy permits to work with the perimeter measure $\mathcal{H}^1|_{\partial D}$, that can be approximated, according to the Modica-Mortola's approach (see [21, 22]), by the measure:

$$\mu_\varepsilon(w, \nabla w) dx = \left(\varepsilon |\nabla w|^2 + \frac{W(w)}{\varepsilon} \right) dx,$$

104 where $W(w) = w^2(1-w)^2$ is a double well function.

Besides by using Sard's Theorem and coarea formula (see also [4] for a similar approach) one can formally replace the integral on ∂D by an integral computed over the level sets of w , whose curvature κ becomes $\operatorname{div} \frac{\nabla w}{|\nabla w|}$ and the integral is computed over the level sets of w . So that one can formally write the complete approximating sequence:

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_{\Omega} w^2 |\Delta u|^2 dx + \mu \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w=0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) \left(\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} W(w) \right) dx \\ &+ \lambda \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_{\Omega} (1-w)^2 dx, \end{aligned}$$

105 where, as usual, $C = \int_0^1 \sqrt{W(t)} dt$, β_ε and γ_ε are infinitesimal as $\varepsilon \rightarrow 0$. The last integral is
 106 a penalization term that forces w to tend to 1 as $\varepsilon \rightarrow 0$.

107 The goal of the second part of this work is then to show that the family of energies \mathcal{F}_ε
 108 Γ -converges to the functional \mathcal{F} when the parameters are related in a suitable way.

109 As in [9] we deal with a suitable convergence of functions involving the Hausdorff con-
 110 vergence of a sub-level sets. This strategy requires a careful statement of the Γ -convergence
 111 definitions and results, in order to have that sequences asymptotically minimizing \mathcal{F}_ε con-
 112 verges to a minimum of \mathcal{F} .

113 Despite this approach is inspired by some ideas contained in [8, 9], we point out that
 114 in our case the regularization term involves a second order differential operator, due to the
 115 fact that our goal is to detect points and not segment curves. This deep difference requires
 116 a non trivial adaptation of the arguments used in those papers.

117 The paper is organized as follows. Section 2 is devoted to notations, preliminary defini-
 118 tions and results. In Section 3 we illustrate the new variational model and we present the
 119 functional we deal with. Section 4 and 5 are devoted to the Γ -convergence result. Finally in
 120 the last Section we conclude the paper by comparing this approach with the celebrated con-
 121 jecture by De Giorgi, concerning the approximation of the curvature depending functionals.

122 We do not give here experimental result illustrating our approach. We refer the reader
 123 for that to [19].

124 2 Preliminaries

125 2.1 Notation

126 In all the paper $\Omega \subset \mathbb{R}^2$ is an open bounded set with lipschitz boundary. The Euclidean
 127 norm will be denoted by $|\cdot|$, while the symbol $\|\cdot\|$ indicates the norm of some functional
 128 spaces. The brackets \langle, \rangle denotes the duality product in some distributional spaces. \mathcal{L}^d or
 129 dx is the d -dimensional Lebesgue measure and \mathcal{H}^k is the k -dimensional Hausdorff measure.
 130 $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ . We say that a set $D \subset \Omega$ is a regular set
 131 if it can be written as $\{F < 0\}$ with $F \in C_0^\infty(\Omega)$. In the following we will denote by $R(\Omega)$
 132 the family of all regular sets in Ω . Finally we will use the symbol \rightharpoonup for denoting a weak
 133 convergence.

134 2.2 Distributional divergence and classical spaces

135 In this Subsection we recall the definition of the distributional space $L^{p,q}(\text{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$,
 136 $1 \leq p, q \leq +\infty$, (see [2, 12]).

137 **Definition 2.1.** We say that $U \in L^{p,q}(\text{div}; \Omega)$ if $U \in L^p(\Omega; \mathbb{R}^2)$ and if its distributional
 138 divergence $\text{Div}U = \text{div}U \in L^q(\Omega)$. If $p = q$ the space $L^{p,q}(\text{div}; \Omega)$ will be denoted by
 139 $L^p(\text{div}; \Omega)$.

140 We say that a function $u \in W^{1,p}(\Omega)$ belongs to $W^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$. We
 141 say that a function $u \in W_0^{1,p}(\Omega)$ belongs to $W_0^{1,p,q}(\text{div}; \Omega)$ if $\nabla u \in L^{p,q}(\text{div}; \Omega)$.

Definition 2.2. For $U \in L^p(\Omega; \mathbb{R}^2)$, $1 \leq p \leq +\infty$, set

$$|\text{Div}U|(\Omega) := \sup\{\langle U, \nabla\varphi \rangle : \varphi \in C_0^\infty(\Omega), |\varphi| \leq 1\}.$$

We say that U is an L^p -divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$, if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\text{Div}U|(\Omega) < +\infty.$$

142 Let us recall the following classical result (see [13] Proposition 3.1).

143 **Theorem 2.1.** Let $\{U_k\}_k \subset \mathcal{DM}^p(\Omega)$ be such that

$$U_k \rightharpoonup U \quad \text{in } L^p(\Omega; \mathbb{R}^2), \quad \text{as } k \rightarrow +\infty \text{ for } 1 \leq p < +\infty. \quad (2)$$

Then

$$\|U\|_{L^p(\Omega; \mathbb{R}^2)} \leq \liminf_{k \rightarrow +\infty} \|U_k\|_{L^p(\Omega; \mathbb{R}^2)}, \quad |\text{Div}U|(\Omega) \leq \liminf_{k \rightarrow +\infty} |\text{Div}U_k|(\Omega).$$

144 2.3 p -capacity

The p -capacity will be crucial to find a convenient functional framework to deal with. If $K \subset \mathbb{R}^2$ is a compact set and χ_K denotes its characteristic function, we define:

$$Cap_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla f|^p dx, f \in C_0^\infty(\Omega), f \geq \chi_K \right\}.$$

If $U \subset \Omega$ is an open set, the p -capacity is given by

$$Cap_p(U, \Omega) = \sup_{K \subset U} Cap_p(K, \Omega).$$

Finally if $A \subset \Omega$ is a Borel set

$$Cap_p(A, \Omega) = \inf_{A \subset U \subset \Omega} Cap_p(U, \Omega).$$

145 We recall the following result (see for instance [20], Theorem 2.27) that explains the relation-
 146 ship between p -capacity and Hausdorff measures. Such a result is crucial to have geometric
 147 informations on null p -capacity sets.

148 **Theorem 2.2.** *Assume $1 < p < 2$. If $\mathcal{H}^{2-p}(A) < \infty$ then $Cap_p(A, \Omega) = 0$.*

149 Another useful tool to manage sets of p -capacity 0 is provided by the following charac-
 150 terization.

151 **Theorem 2.3.** *Let E be a compact subset of Ω . Then $Cap_p(E, \Omega) = 0$ if and only if there
 152 exists a sequence $\{\phi_k\}_k \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \leq \phi_k \leq 1$
 153 and $\phi_k = 1$ on E for every k .*

154 For a general survey we refer the reader to [18, 20, 25].

155 3 The Variational Model

156 In this section we set the functional framework and the functional to be minimized.

Roughly speaking in biological images the image is a function that could be very high on points or other structures like sets with Hausdorff dimension $0 \leq \alpha < 1$, and it is close to 0 outside. From a mathematical point of view it seems to be much more appropriate to think of the image as a Radon measure, that is $I = \mu \in (C_0(\Omega))^*$. The next step is finding a space whose elements are able of producing this kind of discontinuities: the space $\mathcal{DM}^p(\Omega)$, with $1 < p < 2$. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence of U cannot be a measure concentrated on points. Set $p \geq 2$, according to the definition, we have

$$\langle \text{Div}U, \varphi \rangle = - \int_{\Omega} U \cdot \nabla \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

157 Since $p \geq 2$ this distribution is well-defined for any test $\varphi \in W_0^{1,p'}(\Omega)$, where $p' \leq 2$ is the
 158 dual exponent of p . In particular $\text{Div}U$ belongs to the dual space $W^{-1,p'}(\Omega)$ of the Sobolev
 159 space $W_0^{1,p}(\Omega)$. Then in this case, the distributional divergence of U cannot be an atomic
 160 measure, since $\delta_0 \notin W^{-1,p'}(\Omega)$. To see this, one can consider as Ω the disk $B_1(0)$ and the
 161 function $\tilde{\varphi}(x) = \log(\log(1 + |x|)) - \log(\log(2))$. This function is in the space $W_0^{1,p'}(\Omega)$ for
 162 every $p' \leq 2$ and therefore it is an admissible test function, however it easy to check that
 163 $\langle \delta_0, \tilde{\varphi} \rangle = +\infty$.

164 When $1 < p < 2$ we have that $\text{Div}U \in W^{-1,p'}(\Omega)$, but in this case since $p < 2$, we have
 165 $p' > 2$ and hence the function $\tilde{\varphi}$ is no longer an admissible test function. One can check
 166 that the distribution $\text{Div}U$ is an element of $(C_0(\Omega))^*$ able of charging the points. Take for
 167 instance the map $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$.

168 The next step is to transform the initial image I as the divergence measure of a suitable
 169 vector field. We consider the elliptic problem with measure data I :

$$\begin{cases} -\Delta u = I & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

Classical results (see [24]) ensures the existence of a unique weak solution $u \in W_0^{1,p}(\Omega)$ with $p < 2$. Then it easy to see that the distributional divergence of ∇u is given by I . In particular by setting $U = \nabla u$, we have $U \in \mathcal{DM}^p(\Omega)$. According to the Radon-Nikodym decomposition of the measure $\text{Div}U$ we have

$$\text{Div}U = \text{div}U + \text{div}^s U,$$

170 where $\text{div}U \in L^1(\Omega)$ and $\text{div}^s U$ is a singular measure with respect to \mathcal{L}^2 . For our purpose
 171 the support of the singular measure $\text{div}^s U$ is too large. In particular the measure $\text{div}^s U$
 172 could charge sets with Hausdorff dimension $0 \leq \alpha < 2$. So that in order to isolate the
 173 singularities we are interested in, we need a further decomposition of the measure $\text{Div}U$.

174 This can be done by using the capacity decomposition of the Radon measure $\operatorname{div}^s U$. It is
 175 known (see [14]) that given a Radon measure μ the following decomposition holds

$$\mu = \mu_a + \mu_0, \quad (4)$$

176 where the measure μ_a is absolutely continuous with respect to the p -capacity and μ_0 is
 177 singular with respect to the p -capacity, that is concentrated on sets with 0 p -capacity. Besides
 178 it is also known (see [14]) that every measure which is absolutely continuous with respect
 179 to the p -capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer
 180 decomposition:

$$\mu = f - \operatorname{Div}G + \mu_0, \quad (5)$$

181 where $G \in L^{p'}(\Omega; \mathbb{R}^2)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^1(\Omega)$.

182 By applying this decomposition to the measure $\operatorname{div}^s U$ we obtain the following decompo-
 183 sition of the measure $\operatorname{Div}U$

$$\operatorname{Div}U = \operatorname{div}U + f - \operatorname{Div}G + (\operatorname{div}^s U)_0, \quad (6)$$

184 with $G \in L^{p'}(\Omega; \mathbb{R}^2)$, $f \in L^1(\Omega)$, $\operatorname{div}U \in L^1(\Omega)$, and $(\operatorname{div}^s U)_0$ is a measure concentrated on
 185 a set with 0 p -capacity.

186 According to this decomposition and taking into account Theorem 2.3 we give the defi-
 187 nition of discontinuity without and with jump.

188 **Definition 3.1.** We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity without jump
 189 of U if $x \in \operatorname{supp}(\operatorname{div}^s U)_0$.

Remark 3.1. The other singularities, where there is a jump, are contained in the second term of decomposition (6). Indeed the space $W^{-1,p'}(\Omega)$ contains Hausdorff measures restricted to sub-manifolds of dimension greater than or equal to 1. (We refer to [25] Section 4.7 for a detailed discussion on the space $W^{-1,p'}(\Omega)$), like for instance Hausdorff measures concentrated on regular closed curves, which are classical examples of discontinuities with jump. More precisely a contour of a regular set D is the jump set of the characteristic function of D and its p -capacity is strictly positive. This is of course in agreement with Theorem 2.3. Indeed if there were a sequence $\{\phi_k\}_k \subset C_0^\infty(\Omega)$, converging to 0 strongly in $W_0^{1,p}(\Omega)$, such that $0 \leq \phi_k \leq 1$ and $\phi_k = 1$ on ∂D for every k , it would be possible to define the sequence

$$\tilde{\phi}_k = \begin{cases} \phi_k & \text{on } D \\ 1 & \text{on } \Omega \setminus D, \end{cases}$$

190 which converges, in the $W^{1,p}$ -norm, to the BV-function $1 - \chi_D$, which cannot be approxi-
 191 mated by regular functions in the $W^{1,p}$ -norm.

192 **Definition 3.2.** We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity with jump of U
 193 if $x \in \operatorname{supp}(f - \operatorname{Div}G)$.

194 3.1 The variational framework

195 We shall build an energy whose minimizers will be vector fields whose divergence measure's
196 singular part will be given by nothing else but points.

197 Each minimizer must be an L^p (with $p < 2$) vector field with the following properties:

- 198 1. It must be close to the initial data U_0 which is, in general, an L^p vector field U_0 with
199 $1 < p < 2$.
- 200 2. The absolutely continuous part with respect to the Lebesgue measure of $\text{Div}U$ is an
201 L^2 function.
- 202 3. The support of the measure $(\text{div}^s U)_0$ must be given by set of points P_U with $\mathcal{H}^0(P_U) <$
203 $+\infty$.

204 According to these considerations it is natural to introduce the space

$$SDM^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \text{Div}G = 0\}, \quad (7)$$

205 so that, as a consequence, decomposition (6) yields for any $U \in SDM^p(\Omega)$

$$\text{Div}U = \text{div}U + (\text{div}^s U)_0. \quad (8)$$

206 For our purposes the following result concerning the features of elements of the space
207 $SDM^p(\Omega)$ will play a crucial role.

208 **Proposition 3.1.** *Let $u \in W_0^{1,p,2}(\text{div}; \Omega \setminus P)$, with $1 < p < 2$. Let $P \subset \Omega$ be a set of finite
209 number of points. Then $\nabla u \in SDM^p(\Omega)$, with $(\text{div}^s \nabla u)_0 = P$.*

210 **Proof.** We set $P = \{x_1, \dots, x_n\}$. Let $\rho(h) \rightarrow 0$ as $h \rightarrow +\infty$ be such that $B_{\rho_h}(x_i) \cap B_{\rho_h}(x_j) =$
211 \emptyset for h large enough and $i \neq j$. We set $\Omega_h = \bigcup_{i=1}^n B_{\rho_h}(x_i)$ and we define the following
212 sequence $\{U_h\} \subset L^p(\Omega; \mathbb{R}^2)$.

$$\begin{cases} U_h = \nabla u & \text{on } \Omega \setminus \Omega_h, \\ 0 & \text{on } \Omega_h. \end{cases} \quad (9)$$

213 Since $\Delta u \in L^2(\Omega \setminus P)$, by standard elliptic regularity we deduce that $u \in W_{loc}^{2,p}(\Omega \setminus P)$. In
214 particular the exterior trace $\gamma_0^{ext}(u) \in W^{\frac{3}{2},p}(\partial\Omega_h)$. Therefore we infer that $u \in W^{2,p}(\Omega \setminus \Omega_h)$.
215 For every $i = 1, \dots, n$ and h small enough we can find an open set A_i such that $B_{\rho_h}(x_i) \subset$
216 $A_i \subset \Omega \setminus \bigcup_{j \neq i} B_{\rho_h}(x_j)$ and A_i does not depend on h . Let θ_i be a cutoff function associated
217 to A_i such that

$$\begin{cases} \theta_i = 1 & \text{on } B_{\rho_h}(x_i) \text{ for any } i = 1, \dots, n, \\ 0 \leq \theta_i \leq 1 & \text{for any } i = 1, \dots, n, \\ \theta_i = 0 & \text{on } \Omega \setminus A_i \text{ for any } i = 1, \dots, n, \\ \|\nabla \theta_h\|_\infty \leq \frac{M_i}{d(\partial A_i, \partial B_{\rho_h}(x_i))} & \text{for any } i = 1, \dots, n. \end{cases} \quad (10)$$

Then, if $\varphi \in C_0^1(\Omega)$ with $|\varphi| \leq 1$, by applying Gauss-Green's formula we obtain:

$$\begin{aligned}
\int_{\Omega} U_h \cdot \nabla \varphi dx &= \int_{\Omega \setminus \Omega_h} \nabla u \cdot \nabla \varphi dx = - \int_{\Omega \setminus \Omega_h} \Delta u \varphi dx + \int_{\partial(\Omega \setminus \Omega_h)} \nabla u \cdot \nu \varphi d\mathcal{H}^1 \\
&= - \int_{\Omega \setminus \Omega_h} \Delta u \varphi dx + \sum_{i=1}^n \int_{\partial(\Omega \setminus B_{\rho_h}(x_i))} \nabla u \cdot \nu (\varphi - \theta_i \varphi(x_i)) d\mathcal{H}^1 \\
&\quad + \sum_{i=1}^n \varphi(x_i) \int_{\partial(\Omega \setminus B_{\rho_h}(x_i))} \theta_i \nabla u \cdot \nu d\mathcal{H}^1 \\
&= - \int_{\Omega \setminus \Omega_h} \Delta u \varphi dx + \sum_{i=1}^n \int_{\partial \Omega} \nabla u \cdot \nu (\varphi - \theta_i \varphi(x_i)) \\
&\quad + \sum_{i=1}^n \int_{\partial B_{\rho_h}(x_i)} \nabla u \cdot \nu (\varphi - \varphi(x_i)) d\mathcal{H}^1 \\
&\quad + \sum_{i=1}^n \left\{ \varphi(x_i) \int_{A_i \setminus B_{\rho_h}(x_i)} \Delta u \theta_i dx + \int_{A_i \setminus B_{\rho_h}(x_i)} \nabla u \nabla \theta_i dx \right\}. \quad (11)
\end{aligned}$$

218 where in the last equality we have applied again the Gauss-Green's formula and the definition
219 of θ_i .

Now for every i we have that $\{\partial B_{\rho_h}(x_i)\}$ converges in the Hausdorff metric to the singleton $\{x_i\}$. Then, since the support of the function $\psi = \varphi - \varphi(x_i)$ is contained in $\Omega \setminus \{x_i\}$, we have that $\text{supp} \psi \cap \partial \{B_h(x_i)\} = \emptyset$ for h large enough, by standard properties of the Hausdorff convergence. Therefore the third term in (11) is equal to 0. Moreover for h large enough we can find a proper open regular set A , that does not depend on h , such that $u \in W^{2,p}(\Omega \setminus A)$. Hence we infer $\frac{\partial u}{\partial \nu} \in W^{\frac{1}{2},p}(\partial \Omega)$. Therefore, from (11) it follows that

$$\begin{aligned}
|\text{Div} U_h|(\Omega) &\leq \sup_{0 \leq \varphi \leq 1} \int_{\Omega} |\nabla u \cdot \nabla \varphi| dx \leq (n+1)C_1(\Omega) \|\Delta u\|_{L^2(\Omega \setminus P)} + 2n \left\| \frac{\partial u}{\partial \nu} \right\|_{W^{\frac{1}{2},p}(\partial \Omega)} \\
&\quad + \|\nabla u\|_{L^p(\Omega; \mathbb{R}^2)} \sum_{i=1}^n \frac{M_i}{d(\partial A_i, \partial B_{\rho_h}(x_i))} := C(n, \Omega),
\end{aligned}$$

for h large enough. Since $U_h \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^2)$, by Theorem 2.1

$$|\text{Div} \nabla u|(\Omega) \leq \liminf_{h \rightarrow \infty} |\text{Div} \nabla u_h| \leq C.$$

220 Therefore $\nabla u \in \mathcal{DM}^p(\Omega)$. Finally we know that $u \in W^{1,p,2}(\text{div}; \Omega \setminus P)$ and thus the
221 support of the measure $\text{div}^s \nabla u$ is given by the set P . Since $\text{Cap}_p(P, \Omega) = 0$, according to
222 decomposition (6) the measure $f - \text{Div} G$ vanishes on sets with 0 p -capacity, and we deduce
223 $f - \text{Div} G = 0$, that is $\nabla u \in \mathcal{SDM}^p(\Omega)$, with $(\text{div}^s \nabla u)_0 = P$. \square

224 3.2 The Functional

According to our purpose the natural energy to deal with is the following $F : \mathcal{SDM}^p(\Omega) \rightarrow [0, \infty]$, $1 < p < 2$, given by

$$F(U) = \int_{\Omega} |\operatorname{div} U|^2 dx + \lambda \int_{\Omega} |U - U_0|^p dx + \mu \mathcal{H}^0(\operatorname{supp}(\operatorname{div}^s U)_0).$$

225 From now on we assume without loosing generality that the weights λ and μ are equal to 1.

226 We note that, if $\operatorname{Div} U_0 \neq 0$ in the sense of distributions, then $\inf F(U) > 0$ on $\mathcal{SDM}^p(\Omega)$.

227 Indeed if we had $\inf_{\mathcal{SDM}^p(\Omega)} F(U) = 0$ then, it would be possible exhibiting a minimizing

228 sequence $\{U_n\}$, such that $F(U_n) \rightarrow 0$. This would imply $U_n \rightarrow U_0$ in L^p and $\operatorname{Div} U_n \rightarrow 0$ in

229 $\mathcal{D}'(\Omega)$. On the other hand, the L^p -distance between U_n and U_0 can be arbitrary small only

230 if $\operatorname{Div} U_0 = 0$ as well, because the constraint $\operatorname{Div} U = 0$ is stable under L^p -convergence.

231 4 Γ -convergence: The intermediate approximation

232 By analogy with the construction of U_0 we restrict ourselves to vector fields U which are
233 the gradient of a function $u \in W_0^{1,p}(\Omega)$.

234 Thus the functional \mathcal{F} is finite on the class of functions whose support of the measure
235 $(\operatorname{div}^s \nabla u)_0$ is given by a finite set. Consequently it is convenient to introduce the following
236 spaces:

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \nabla u \in S\mathcal{DM}^p(\Omega)\}, \quad (12)$$

237 and

$$\Delta\mathcal{AM}^{p,2}(\Omega) = \{u \in \Delta\mathcal{M}^p(\Omega) : \Delta u \in L^2(\Omega), \operatorname{supp}(\operatorname{div}^s \nabla u)_0 = P_{\nabla u} \text{ with } \mathcal{H}^0(P_{\nabla u}) < +\infty\}. \quad (13)$$

238 So that the target-limit energy $\mathcal{F} : \Delta\mathcal{AM}^{p,2}(\Omega) \rightarrow (0, \infty)$ is given by

$$\mathcal{F}(u) = \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \mathcal{H}^0(P_{\nabla u}). \quad (14)$$

In the spirit [9] we introduce an intermediate variational approximation of the functional \mathcal{F} . We define a sequence of functionals where the counting measure $\mathcal{H}^0(P_{\nabla u})$ is replaced by a functional defined on regular sets D and which involves the curvature of the boundary ∂D . The approximating sequence is given by:

$$\begin{aligned} F_{\varepsilon}(u, D) &= \int_{\Omega} (1 - \chi_D) |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx \\ &+ \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2 \right) d\mathcal{H}^1. \end{aligned}$$

239 Where $u \in W_0^{1,p,2}(\operatorname{div}; \Omega)$, D is a regular set, and κ denotes the curvature of its boundary.

240 In order to guarantee that the measure of the sets D is small we define a new functional
241 still denoted by $F_{\varepsilon}(u, D)$ given by

$$F_{\varepsilon}(u, D) = \int_{\Omega} (1 - \chi_D) |\Delta u|^2 dx + \int_{\Omega} |\nabla u - U_0|^p dx + \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\varepsilon} + \varepsilon \kappa^2 \right) d\mathcal{H}^1 + \frac{1}{\varepsilon} \mathcal{L}^2(D) \quad \text{on } Y(\Omega), \quad (15)$$

242 where $Y(\Omega) = \{(u, D) : u \in W_0^{1,p,2}(\operatorname{div}; \Omega), D \in R(\Omega)\}$. We endow the set $Y(\Omega)$ with the
243 following convergence.

244 **Definition 4.1.** Let $h \in \mathbb{N}$ go to $+\infty$. We say that a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H -
245 converges to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ if the following conditions hold

- 246 1. $\mathcal{L}^2(D_h) \rightarrow 0$;
- 247 2. $\{\partial D_h\}_h \rightarrow P \subset \Omega$ in the Hausdorff metric, where P is a finite set of points;
- 248 3. $u_h \rightarrow u$ in $L^p(\Omega)$ and $P_{\nabla u} \subseteq P$.

249 As in [9] we adopte the following ad hoc definition of Γ -convergence.

250 **Definition 4.2.** Let $h \in \mathbb{N}$ go to $+\infty$. We say that F_ε Γ -converges to \mathcal{F} if for every sequence
251 of positive numbers $\{\varepsilon_h\} \rightarrow 0$ and for every $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ we have:

1. for every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H -converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$

$$\liminf_{h \rightarrow +\infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u);$$

2. there exists a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H -converging to u such that

$$\limsup_{h \rightarrow +\infty} F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u).$$

252 We point out that with this approach, the fundamental theorem of the Γ -convergence
253 cannot be applied directly, since we do not deal with a metric space (for a complete survey
254 on Γ -convergence we refer to [7, 10]). However it is still possible to prove that a sequence
255 $\{(u_h, D_h)\}_h$ asymptotically minimizing $F_\varepsilon(u, D)$ admits a subsequence H -converging to a
256 minimizer of $\mathcal{F}(u)$. Indeed we will show at the end of the Section (see Theorem 4.4) that
257 the convergence of the minimum problems can still obtained as a consequence of compactness
258 of the minimizing sequence of F_ε , Γ -lim inf inequality (1) and Γ -lim sup inequality (2).

259 4.1 Compactness

260 We state and prove the following compactness result.

261 **Theorem 4.1.** Let $h \in \mathbb{N}$ go to $+\infty$ and $\varepsilon_h \rightarrow 0$ such that

$$F_{\varepsilon_h}(u_h, D_h) \leq M, \tag{16}$$

262 then there exist a subsequence $\{(u_{h_k}, D_{h_k})\}_k \subset Y(\Omega)$, a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ and a set
263 $P \subset \bar{\Omega}$ of finite number of points, such that $\{(u_{h_k}, D_{h_k})\}_k$ H -converges to u .

Proof. We adapt an argument of [9]. From (16) we have immediately $\{D_h\} \subset R(\Omega)$ with
 $\mathcal{L}^2(D_h) \rightarrow 0$. Then we can parametrize every $C_h = \partial D_h$ by a finite and disjoint union
of Jordan curves. Let us set for every h , $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$. Then we have according to the
2-dimensional version of Gauss-Bonnet's Theorem and Young's inequality

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h \kappa_h^2 \right) d\mathcal{H}^1 \geq \frac{1}{4\pi} \int_{\partial D_h} 2\kappa_h d\mathcal{H}^1 = \frac{1}{4\pi} \int_{\bigcup_h C_h} 2\kappa_h d\mathcal{H}^1 = m(h).$$

Note that the number $m(h) \leq M$, with $M \geq 0$, is independent of h . Then it is possible to
extract a subsequence C_{h_k} with the number of curves in C_{h_k} equal to some n for every k .
Then we set $C_{h_k} = \{\gamma_{h_k}^1, \dots, \gamma_{h_k}^n\}$ for any k . From (16) we also have for any $\gamma \in C_{h_k}$ that

$\mathcal{H}^1(\gamma) \leq 4\pi M \varepsilon_{h_k}$ and consequently $\max\{\mathcal{H}^1(\gamma) : \gamma \in C_{h_k}\} \rightarrow 0$. Then there exists a finite set of point $P = \{x_1, \dots, x_n\} \subset \bar{\Omega}$ such that for any radius ρ there is an index k_ρ with

$$\gamma_{h_k}^i \subset B_\rho(x_i) \text{ for all } k > k_\rho \text{ and } i \in \{1, \dots, n\},$$

264 so that if we set $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$, then the Hausdorff distance $d_H(\partial D_{h_k}, P) \rightarrow$
 265 0 since $\mathcal{L}^2(D_{h_k}) \rightarrow 0$ and therefore $\rho \rightarrow 0$ as well.

266 Now we prove the compactness property for u_h . First of all from the estimate

$$\|\nabla u_h\|_{L^p(\Omega)}^p \leq 2^p (\|\nabla u_h - U_0\|_{L^p(\Omega)}^p + \|U_0\|_{L^p(\Omega)}^p), \quad (17)$$

267 and (16), we may extract a subsequence $\{u_{h_k}\} \subset W_0^{1,p}(\Omega)$ weakly convergent to $u \in W_0^{1,p}(\Omega)$.

Let Ω_j be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P$ invading $\Omega \setminus P$. We claim that it is possible to extract a sequence of D_{h_k} such that $\Omega_j \cap \partial D_{h_k} = \emptyset$. Indeed since the distance between Ω_j and P is positive for any j there exists η_j such that $\Omega_j \cap (\bigcup_{i=1}^n B_{\eta_j}(x^i)) = \emptyset$. On the other hand we know that for every ρ we can find k_ρ such that $\partial D_{h_k} = \bigcup_{i=1}^n \gamma_{h_k}^i \subset \bigcup_{i=1}^n B_\rho(x_i)$. Then in particular if $\rho = \eta_j$ there exists k_j such that for all $k \geq k_j$

$$\Omega_j \cap \partial D_{h_k} = \emptyset.$$

268 Therefore for any $x \in \Omega_j$ there exists $\delta > 0$ such that either $B_\delta(x) \subset D_{h_k}$ or $B_\delta(x) \subset \Omega \setminus D_{h_k}$.

269 Finally by taking into account that $\mathcal{L}^2(D_{h_k}) \rightarrow 0$ we conclude $\Omega_j \cap \partial D_{h_k} = \emptyset$ for $k \geq k_j$.

270 Then for every $k \geq k_j$ we have that $u_{h_k} \in W^{1,p,2}(\text{div}; \Omega_j)$ and by (16) we get

$$\int_{\Omega_j} |\Delta u_h|^2 dx \leq \int_{\Omega \setminus D_{h_k}} |\Delta u_{h_k}|^2 dx \leq M. \quad (18)$$

Then we can extract a further subsequence still denoted by $\{u_{h_k}\} \subset W^{1,p,2}(\text{div}; \Omega_j)$ such that

$$\begin{cases} u_{h_k} \rightharpoonup u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \text{ and a.e.} \\ \nabla u_{h_k} \rightharpoonup \nabla u & \text{in } L^p(\Omega_j; \mathbb{R}^2) \\ \Delta u_{h_k} \rightharpoonup \Delta u & \text{in } L^2(\Omega_j). \end{cases}$$

271 Let now $x \in \Omega' \subset \subset \Omega \setminus P$. Then there exists a sequence $x_j \rightarrow x$ with $j \in \mathbb{N}$. By applying
 272 the diagonal argument to the sequence $u_{h_{k_l}}(x_j)$ we obtain a subsequence $u_l = u_{h_{k_l}}(x_l)$ such
 273 that Δu_l converges weakly in $L^2(\Omega')$ to Δu for any $\Omega' \subset \subset \Omega$. Then by the semicontinuity
 274 of the L^2 -norm we have

$$\sup_j \int_{\Omega_j} |\Delta u|^2 dx \leq \sup_j \liminf_{l \rightarrow +\infty} \int_{\Omega_j} |\Delta u_l|^2 dx \leq M.$$

275 If we set $\tilde{P} = P \setminus \partial\Omega$, then we deduce $u \in W_0^{1,p,2}(\text{div}; \Omega \setminus \tilde{P})$ and therefore $\nabla u \in S\mathcal{DM}^p(\Omega)$
 276 with $P_{\nabla u} \subseteq P$, by Proposition 3.1. So we conclude that $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$. \square

277 4.2 Lower bound

278 We provide the lower bound (1) in Definition 4.2.

Theorem 4.2. *Let $h \in \mathbb{N}$ go to $+\infty$. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$, H -converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, we have*

$$\liminf_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, D_h) \geq \mathcal{F}(u).$$

Proof. Up to a subsequence we may assume that the \liminf is actually a limit. As in the proof of Theorem 4.1, by setting for every h , $C_h = \bigcup_{i=1}^{m(h)} \gamma^i$, we get

$$M \geq \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 = m(h).$$

279 Up to subsequences we have $m(h) = n$ for some natural number n . Hence there exists a set
280 P_1 of n points such that ∂D_h converges in the Hausdorff metric to P_1 . On the other hand
281 we have that ∂D_h converges in the Hausdorff metric to P with $P_{\nabla u} \subseteq P$. Then, since the
282 limit is unique, we have $P = P_1$.

Let now $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset \subset \Omega \setminus P_1$ invading $\Omega \setminus P_1$. As in the proof of Theorem 4.1 we may assume up to a subsequence, that $\Delta u_h \rightharpoonup \Delta u$ in $L^2(\Omega_j)$. Furthermore, since in this case all the sequence D_h converges to the set P_1 we have, by the same argument used in the proof of Theorem 4.1, $\Omega_j \subset \Omega \setminus D_h$ for h large and for any j . Consequently

$$\liminf_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \geq \liminf_{h \rightarrow +\infty} \int_{\Omega_j} |\Delta u_h|^2 dx \geq \int_{\Omega_j} |\Delta u|^2 dx.$$

283 On the other hand, arguing as in Theorem 4.1, we infer that the limit u of the subsequence
284 u_h belongs to $\Delta\mathcal{AM}^{p,2}(\Omega)$, with $\Delta u \in L^2(\Omega \setminus P_1)$ and $P_{\nabla u} \subseteq P_1$. So that by monotone
285 convergence

$$\liminf_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \geq \int_{\Omega \setminus P_1} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx. \quad (19)$$

286 As in the proof of Theorem 4.1, inequality (17) holds. Then we easily get

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx. \quad (20)$$

287 Finally we have

$$\frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 \geq n = \mathcal{H}^0(P_1) \geq \mathcal{H}^0(P_{\nabla u}). \quad (21)$$

288 Eventually by (19),(20) (21) and by the superlinearity property of the \liminf operator we
289 achieve the result. \square

290 4.3 Upper bound

291 In [9] for the construction of the optimal sequence it is crucial to use a result due to Cham-
 292 bolle and Doveri (see [11]). This result states that it is possible to approximate, in the
 293 H^1 -norm, a function $u \in W^{1,2}(\Omega \setminus C)$ (where C is a closed set), by means of a sequence
 294 of functions $u_h \in W^{1,2}(\Omega \setminus C_h)$ with C_h convergent to C in the Hausdorff metric. In our
 295 case this argument does not apply due to presence of a second order differential operator.
 296 Nevertheless since we work only with set of points it is possible to build an optimal sequence
 297 in a more direct way.

298 **Theorem 4.3.** *Let $h \in \mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive converging to 0. For*
 299 *every $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$ there exists a sequence $\{(u_h, D_h)\}_h \subset Y(\Omega)$ H-converging to u such*
 300 *that*

$$\limsup_{h \rightarrow +\infty} F_{\varepsilon_h}(u_h, D_h) \leq \mathcal{F}(u). \quad (22)$$

301 **Proof.** We start by the construction of the sequence D_h . Let n be the number of points x_i
 302 in $P_{\nabla u}$. Then we take $D_h = \bigcup_{i=1}^n B_{\varepsilon_h}(x_i)$. So that $\mathcal{L}^2(D_h) \rightarrow 0$, $\frac{1}{\varepsilon_h} \mathcal{L}^2(D_h) \rightarrow 0$ and ∂D_h
 303 converges with respect to the Hausdorff distance to $P_{\nabla u}$. Moreover for h large enough we
 304 may assume $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$. Now we build u_h . Let $\{\rho_h\} \subset \mathbb{R}$ be such that
 305 $\rho_h \geq 0$ and $\rho_h \rightarrow 0$ when $h \rightarrow \infty$. Let $\theta_h \in C^\infty(\Omega)$ with the following property:

$$\begin{cases} \theta_h = 1 & \text{on } B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ 0 \leq \theta_h \leq 1 & \text{on } B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i) \text{ for any } i = 1, \dots, n \\ \theta_h = 0 & \text{on } \Omega \setminus B_{\rho_h}(x_i) \text{ for any } i = 1, \dots, n \\ \|\nabla \theta_h\|_\infty \leq \frac{1}{\rho_h}. \end{cases} \quad (23)$$

We set $u_h = (1 - \theta_h)u$. It is not difficult to check that $\{(u_h, D_h)\}_h \subset Y(\Omega)$ and H-converges
 to u . We claim that the pair (u_h, D_h) realizes the inequality (22) for a suitable choice of the
 sequence ρ_h . By making the computation we have

$$\nabla u_h = (1 - \theta_h)\nabla u - u\nabla \theta_h.$$

Then

$$\int_{\Omega} |\nabla u_h - U_0|^p dx = \int_{\Omega} |\nabla u - U_0 - \theta_h \nabla u - u \nabla \theta_h|^p dx,$$

306 so that

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \leq \limsup_{h \rightarrow +\infty} \left(\left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\theta_h \nabla u|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\nabla \theta_h u|^p dx \right)^{\frac{1}{p}} \right)^p. \quad (24)$$

307 Since $|\nabla u|^p \in L^1(\Omega)$, we have by applying the dominated convergence theorem $\int_{\Omega} |\theta_h \nabla u|^p dx \rightarrow$
 308 0. Let us focus on the term $\int_{\Omega} |\nabla \theta_h u|^p$. By the Sobolev embedding we have $u \in L^{p^*}(\Omega)$
 309 with $p^* = \frac{2p}{2-p}$ and hence $|u|^p \in L^{\frac{p^*}{p}}(\Omega)$, with $\frac{p^*}{p} = \frac{2}{2-p}$.

By (23), using Holder's inequality with dual exponents $\frac{2}{2-p}$ and $\frac{2}{p}$, and taking into account that $p < 2$

$$\begin{aligned} \int_{\Omega} |\nabla \theta_h u|^p dx &\leq \sum_{i=1}^n \int_{B_{\rho_h}(x_i) \setminus B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx = \sum_{i=1}^n \left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h u|^p dx - \int_{B_{\frac{\rho_h}{2}}(x_i)} |\nabla \theta_h u|^p dx \right) \\ &\leq \sum_{i=1}^n \left(\int_{B_{\rho_h}(x_i)} |\nabla \theta_h|^2 dx \right)^{\frac{p}{2}} \|u\|_{L^{\frac{2}{2-p}}(\Omega)} \leq \sum_{i=1}^n \|u\|_{L^{p^*}(\Omega)} \left(\frac{\pi^2 \rho_h^2}{\rho_h^p} \right) \rightarrow 0. \end{aligned} \quad (25)$$

From (24) it follows that

$$\begin{aligned} \limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx &\leq \lim_{h \rightarrow +\infty} \left(\left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\theta_h \nabla u|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |\nabla \theta_h u|^p dx \right)^{\frac{1}{p}} \right)^p \\ &= \left(\left(\int_{\Omega} |\nabla u - U_0|^p dx \right)^{\frac{1}{p}} \right)^p = \int_{\Omega} |\nabla u - U_0|^p dx. \end{aligned} \quad (26)$$

Now we compute Δu_h . The identity $\operatorname{div}(fA) = f \operatorname{div} A + \nabla f \cdot A$ yields

$$\Delta u_h = (1 - \theta_h) \Delta u - 2 \nabla \theta_h \nabla u - \Delta \theta_h u.$$

310 Then by choosing ρ_h small enough we have from (23)

$$\limsup_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u_h|^2 dx \leq \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h} |\Delta u|^2 dx \rightarrow \int_{\Omega} |\Delta u|^2 dx. \quad (27)$$

311 Finally since for h large we have $B_{\varepsilon_h}(x_i) \cap B_{\varepsilon_h}(x_j) = \emptyset$ for $i \neq j$ we get

$$\lim_h \frac{1}{4\pi} \int_{\partial D_h} \left(\frac{1}{\varepsilon_h} + \varepsilon_h k^2 \right) d\mathcal{H}^1 = \lim_h \sum_{i=1}^n \frac{1}{4\pi} \int_{\partial B_{\varepsilon_h}} \left(\varepsilon_h \frac{1}{\varepsilon_h} k^2 \right) d\mathcal{H}^1 = n = \mathcal{H}^0(P_{\nabla u}). \quad (28)$$

312 By recalling that the limsup is sublinear operation and by (26),(27),(28), we achieve the
313 result. \square

314 4.4 Variational property

315 We conclude this section by properly stating and proving the particular version of funda-
316 mental Theorem, which is, in this case, a direct consequence of Theorems 4.1, 4.2, 4.3. The
317 proof can be achieved by a classical argument (see [7], Section 1.5). However we prefer to
318 give the proof in order to make clear that the classical variational setting is not directly
319 available, and therefore the variational property has to be proven.

Theorem 4.4. *Let $h \in \mathbb{N}$ go to $+\infty$. Let F_{ε} and \mathcal{F} be given respectively by (15) and (14). If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, D_h)\} \subset Y(\Omega)$ such that*

$$\lim_{h \rightarrow +\infty} (F_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} F_{\varepsilon_h}(u, D)) = 0,$$

320 then there exists a subsequence $\{(u_{h_k}, D_{h_k})\} \subset Y(\Omega)$ and a minimizer \bar{u} of $\mathcal{F}(u)$ with $\bar{u} \in$
 321 $\Delta\mathcal{AM}^{p,2}(\Omega)$, such that $\{(u_{h_k}, D_{h_k})\}$ H-converges to \bar{u} .

Proof. We know from Theorems 4.2 and 4.3 that F_ε Γ -converges to \mathcal{F} . Let $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ be such that

$$\mathcal{F}(u) \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}(u) + \delta.$$

From Theorem 4.3 there exists a sequence $\{(\tilde{u}_h, \tilde{D}_h)\} \subset Y(\Omega)$, such that

$$\inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F} + \delta \geq \mathcal{F}(u) \geq \limsup_{h \rightarrow +\infty} F_{\varepsilon_h}(\tilde{u}_h, \tilde{D}_h).$$

322 Then since δ is arbitrary it follows that

$$\limsup_{h \rightarrow +\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \limsup_{h \rightarrow +\infty} F_{\varepsilon_h}(\tilde{u}_h, \tilde{D}_h) \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}. \quad (29)$$

Let now $\{(u_h, D_h)\} \subset Y(\Omega)$ be such that $\lim_{h \rightarrow +\infty} (F_{\varepsilon_h}(u_h, D_h) - \inf_{Y(\Omega)} F_{\varepsilon_h}(u, D)) = 0$. Then from Theorem 4.1, up to subsequences, the sequence $\{(u_h, D_h)\}_h$ H-converges to some $\bar{u} \in \Delta\mathcal{AM}^{p,2}(\Omega)$. Then by Theorem 4.2 and taking into account (29) we deduce

$$\inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F} \leq \mathcal{F}(\bar{u}) \leq \liminf_{h \rightarrow +\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \limsup_{h \rightarrow +\infty} \inf_{Y(\Omega)} F_{\varepsilon_h} \leq \inf_{\Delta\mathcal{AM}^{p,2}(\Omega)} \mathcal{F}.$$

323 Then we easily get the thesis. \square

324 5 Approximation by smooth function

325 By following the Braides-March's approach in [9] we approximate the measure $\mathcal{H}^1 \llcorner \partial D$ by the
 326 Modica-Mortola's energy density given by $(\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} W(w)) dx$ where $W(w) = w^2(1-w)^2$
 327 and $w \in C^\infty(\Omega)$. The next step is to replace the regular set D with the level set of w . Let
 328 us set $Z = \{\nabla w(x) = 0\}$. By Sard's Lemma we have that $\mathcal{L}^1(w(Z)) = 0$. In particular,
 329 if w takes values into the interval $[0, 1]$, we infer that for almost every $t \in (0, 1)$ the set
 330 $Z \cap w^{-1}(t)$ is empty. Consequently for almost every $t \in (0, 1)$ the t -level set $\{w < t\}$ is a
 331 regular set with boundary $\{w = t\}$. Now, since we want to replace the set D , we need that
 332 $\{w < t\} \subset\subset \Omega$. Then we require $1 - w \in C_0^\infty(\Omega; [0, 1])$. Furthermore for almost every t , we
 333 have $k(\{w = t\}) = \operatorname{div}(\frac{\nabla w}{|\nabla w|})$. From all of this we are led to define the following space:

$$S(\Omega) = \{(u, w); u \in W_0^{1,p,2}(\operatorname{div}; \Omega); 1 - w \in C_0^\infty(\Omega; [0, 1])\} \quad (30)$$

and having in mind the coarea formula, the following sequence of functionals defined on $S(\Omega)$

$$\begin{aligned} \mathcal{G}_\varepsilon(u, w) &= \int_\Omega w^2 |\Delta u|^2 dx + \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w = 0\}} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \left(\operatorname{div} \left(\frac{\nabla w}{|\nabla w|} \right) \right)^2 \right) (\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} W(w)) dx \\ &+ \int_\Omega |\nabla u - U_0|^p dx + \frac{1}{\gamma_\varepsilon} \int_\Omega (1 - w)^2 dx, \end{aligned} \quad (31)$$

334 with $C = \int_0^1 \sqrt{W(t)} dt$. The last term forces w_ε be 1 almost everywhere in the limit. From
 335 now on the parameters $\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ will be related as follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 0, \quad (32)$$

336

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon |\log(\varepsilon)|}{\beta_\varepsilon} = 0. \quad (33)$$

337 The convergence that plays the role of the H-convergence is the following. With a slight
 338 abuse of notation this convergence will be still denoted by H.

339 **Definition 5.1.** Let $h \in \mathbb{N}$ goto $+\infty$ and $\{(u_h, w_h)\}_h$ be a sequence $S(\Omega)$. Set $D_h^t = \{w_h <$
 340 $t\}$. We say that $\{(u_h, w_h)\}_h$ H-converges to $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$, if for every $t \in (0, 1)$ the
 341 sequence $\{(u_h, D_h^t)\}_h$ in $Y(\Omega)$ H-converges to u .

342 As in the previous Section, we adopte the ad hoc definition of Γ -convergence with respect
 343 to the convergence above.

344 **Definition 5.2.** Let $h \in \mathbb{N}$ go to $+\infty$. We say that \mathcal{G}_ε Γ -converges to \mathcal{F} if, for every
 345 sequence of positive numbers $\varepsilon_h \rightarrow 0$ and for every $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$, we have:

1. for every sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H -converging to u

$$\liminf_{h \rightarrow +\infty} \mathcal{G}_{\varepsilon_h}(u_h, w_h) \geq \mathcal{F}(u);$$

2. there exists a sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H -converging to u such that

$$\limsup_{h \rightarrow +\infty} \mathcal{G}_{\varepsilon_h}(u_h, w_h) \leq \mathcal{F}(u).$$

346 As in the previous Section, we remark that the convergence of the minimum problems
 347 must be proved, since we cannot apply the fundamental Theorem of Γ -convergence. We will
 348 state the analogous of Theorem 4.4 at the end of the Section.

349 5.1 Compactness

350 The compactness result goes as follows.

351 **Theorem 5.1.** *Let $h \in \mathbb{N}$ goes to $+\infty$ and $\varepsilon_h \rightarrow 0$ such that*

$$F_{\varepsilon_h}(u_h, w_h) \leq M. \quad (34)$$

352 *Then there exists a subsequence $\{(u_{h_k}, w_{h_k})\}_k \subset S(\Omega)$, $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$ such that $\{(u_{h_k}, w_{h_k})\}_k$
 353 H -converges to u .*

354 **Proof.** The first part of proof is as in [9]. For the convenience of the reader we give the
 355 complete proof.

By Young's inequality and by (34) we get

$$M \geq 2 \int_{\Omega \setminus \{|\nabla w_h|=0\}} |\nabla w_h| \sqrt{W(w_h)} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div} \left(\frac{\nabla w_h}{|\nabla w_h|} \right) \right)^2 \right) dx.$$

356 Now by coarea formula, we obtain

$$M \geq 2 \int_0^1 \sqrt{W(t)} \int_{\{w_h=t\} \cap \{|\nabla w_h| \neq 0\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div} \left(\frac{\nabla w_h}{|\nabla w_h|} \right) \right)^2 \right) d\mathcal{H}^1 dt. \quad (35)$$

Thanks to Sard's Lemma, for any h there exists a \mathcal{L}^1 -negligible set $\mathcal{N}_{w_h} \subseteq (0, 1)$ such that

$$\{w_h = t\} = \partial\{w_h < t\}, \quad \{w_h < t\} \in R(\Omega), \quad \text{for } t \in (0, 1) \setminus \mathcal{N}_{w_h}.$$

On $\{w_h = t\}$ for $t \in (0, 1) \setminus \mathcal{N}_{w_h}$ we have

$$|\nabla w_h| \neq 0 \quad \text{and} \quad \kappa(\{w_h = t\}) = \operatorname{div} \left(\frac{\nabla w_h}{|\nabla w_h|} \right).$$

Now since the union $\bigcup_h \mathcal{N}_{w_h}$ of the sets \mathcal{N}_{w_h} is \mathcal{L}^2 -negligible (almost countable) from (35) we have

$$M \geq 2 \int_{(0,1) \setminus \bigcup_h \mathcal{N}_{w_h}} \sqrt{W(t)} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \kappa^2 \right) d\mathcal{H}^1 dt.$$

357 By applying Fatou's Lemma and taking into account that the set $\bigcup_h \mathcal{N}_{w_h}$ does not depend
358 on h we get

$$M \geq 2 \int_{(0,1) \setminus \bigcup_h \mathcal{N}_{w_h}} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \kappa^2 \right) d\mathcal{H}^1 dt. \quad (36)$$

359 Hence we deduce the existence of a \mathcal{L}^2 -negligible set Q , with $\bigcup_h \mathcal{N}_{w_h} \subseteq Q$, such that

$$\liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \kappa^2 \right) d\mathcal{H}^1 \leq M_t, \quad (37)$$

360 where the constant M_t does not depend on h .

Then for every $t \in (0,1) \setminus Q$ we can extract a sequence $\{w_h^t\}_h$ such that $\partial\{w_h^t < t\}$ converges with respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$. Let $\mathcal{N} = \{t_i\}$ in $(0,1)$ be a dense countable set. Up to a diagonal argument we can find a subsequence $\{w_{h_k}\}_k$ such that, for every $t_i \in \mathcal{N}$, $\partial\{w_{h_k} < t_i\}$ converges to P^{t_i} . Let $t_i \in \mathcal{N}$ such that $t_i > t$ and consequently $\{w_{h_k} < t\} \subseteq \{w_{h_k} < t_i\}$. From the definition of Hausdorff convergence it follows that for every $\rho > 0$ there exists $k_0(\rho)$ such that for any $k > k_0$ we have $\{w_{h_k} < t_i\} \cap B_\rho(x) \neq \emptyset$ for every $x \in P^{t_i}$. Since the t -level set is open for every ρ and for every $x \in P^{t_i}$ such that $\{w_{h_k} < t\} \cap B_\rho(x) \neq \emptyset$, we may choose $t_n \in \mathcal{N}$ with $t_n < t$ and so obtain for k large enough $\{w_{h_k} < t_n\} \cap B_\rho(x) \neq \emptyset$. By choosing $t_{max} = \max_{x \in P^{t_i}} t_n(x)$ for every $x \in P^{t_i}$, the inclusion $\{w_{h_k} < t_{max}\} \subset \{w_{h_k} < t\}$ gives

$$\{w_{h_k} < t_{max}\} \cap B_\rho(x) \subseteq \{w_{h_k} < t\} \cap B_\rho(x) \subseteq \{w_{h_k} < t_n\} \cap B_\rho(x),$$

361 with $t_{max}, t_i \in \mathcal{N}$. Then by taking the limit $\rho \rightarrow 0^+$ we infer $\partial\{w_{h_k} < t\}$ converges with
362 respect to the Hausdorff metric to a set $P^t \subset \overline{\Omega}$ for every $t \in (0,1)$.

363 Finally for any $t \in (0,1)$ since $0 \leq w_h \leq 1$, we have $\mathcal{L}^2(\{w_h < t\}) = \mathcal{L}^2(\{1 - w_h >$
364 $1 - t\}) \leq \mathcal{L}^2(\{1 - w_h > (1 - t)^2\})$, then by Chebyshev's inequality and by (34)

$$\mathcal{L}^2(\{w_h < t\}) \leq \frac{M\gamma_{\varepsilon_h}}{(1-t)^2}. \quad (38)$$

365 Therefore, as in the proof of Theorem 4.1, we can extract a subsequence $\{u_{h_k}\}_k$ which
366 converges strongly in $L^p(\Omega)$ to a function $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$ with $P_{\nabla u} \subseteq P^t$ for every $t \in$
367 $(0,1)$. Hence we have that for every $t \in (0,1)$ the sequence $\{(u_{h_k}, D_{h_k}^t)\}_k$ H-converges to u
368 and the proof is achieved. \square

369 5.2 Lower bound

370 We give the proof of the lower bound (1) in Definition 5.2. In the proof it will be crucial
371 having the convergence of the t -level set for every $t \in (0, 1)$.

Theorem 5.2. *Let $h \in \mathbb{N}$ go to $+\infty$. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. For every sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H -converging to $u \in \Delta\mathcal{AM}^{p,2}(\Omega)$, we have*

$$\liminf_{h \rightarrow +\infty} F_{\varepsilon_h}(u_h, w_h) \geq \mathcal{F}(u).$$

Proof. Without loss of generality we assume, up to subsequences,

$$+\infty > \liminf_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h} = \lim_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}.$$

372 As in the proof of Theorem 5.1 we get that for every $t \in (0, 1)$ $\mathcal{L}^2(\{w_h < t\}) \rightarrow 0$ and
373 $\partial\{w_h < t\} \rightarrow P^t$ in the Hausdorff distance. For any $t \in (0, 1)$ we have (see also [9] for a
374 similar argument)

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 dx = \int_{\{w_h < t\}} w_h^2 |\Delta u_h|^2 dx + \int_{\{w_h \geq t\}} w_h^2 |\Delta u_h|^2 dx \geq t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx. \quad (39)$$

375 Let $\{\Omega_j\}_j$ be a sequence of open sets $\Omega_j \subset\subset \Omega \setminus P^t$ invading $\Omega \setminus P^t$. Then we may assume
376 that $u_h \rightharpoonup$ weakly in $W_0^{1,p}(\Omega)$ and Δu_h converges weakly in $L^2(\Omega_j)$ to Δu . Therefore as in
377 the proof of Theorem 4.2 we get

$$\lim_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \geq \int_{\Omega} |\nabla u - U_0|^p dx, \quad (40)$$

and

$$\liminf_{h \rightarrow +\infty} t^2 \int_{\Omega} (1 - \chi_{\{w_h < t\}}) |\Delta u_h|^2 dx \geq t^2 \int_{\Omega_j} |\Delta u|^2 dx,$$

378 for any j .

Then by (39) and, by taking into account that $|\Delta u|$ is in $L^2(\Omega \setminus P^t)$ with $P_{\nabla u} \subseteq P^t$, it follows that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 dx \geq t^2 \int_{\Omega} |\Delta u|^2 dx.$$

379 And eventually by taking the limit $t \rightarrow 1$

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 dx \geq \int_{\Omega} |\Delta u|^2 dx. \quad (41)$$

380 Finally, as in the proof of Theorem 4.2 (inequality 21) we have

$$\liminf_{h \rightarrow +\infty} \frac{1}{4\pi} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} k^2 \right) d\mathcal{H}^1 \geq \mathcal{H}^0(P^t) \geq \mathcal{H}^0(P_{\nabla u}). \quad (42)$$

Now arguing as in the proof of Theorem 5.1 and by taking into account (42), we get

$$\begin{aligned}
& \liminf_{h \rightarrow +\infty} \int_{\Omega \setminus \{\nabla w_h = 0\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div} \left(\frac{\nabla w_h}{|\nabla w_h|} \right) \right)^2 (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) \right) dx \\
& \geq 2 \liminf_{h \rightarrow +\infty} \int_{(0,1) \setminus \bigcup_h \mathcal{N}_{w_h}} \sqrt{W(t)} \liminf_{h \rightarrow +\infty} \int_{\partial\{w_h < t\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} k^2 \right) d\mathcal{H}^1 dt \\
& \geq 2 \int_{(0,1)} \mathcal{H}^0(P^t) \sqrt{W(t)} dt \geq 8\pi C \mathcal{H}^0(P_{\nabla u}). \tag{43}
\end{aligned}$$

381 By collecting (40) (41) and (43) we achieve the thesis. \square

382 5.3 Upper bound

383 As in [9] to build w_h we use the construction given in [4], while the optimal sequence u_k is
 384 chosen as in Theorem 4.3.

385 **Theorem 5.3.** *Let $h \in \mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$.
 386 For every $u \in \Delta \mathcal{AM}^{p,2}$ there exists a sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$, H -converging to u , such
 387 that*

$$\limsup_{h \rightarrow \infty} F_{\varepsilon_h}(u_h, w_h) \leq \mathcal{F}(u). \tag{44}$$

Proof. If $A \subset \mathbb{R}^2$ we set

$$\delta_A(x) = d(x, A) - d(x, \mathbb{R}^2 \setminus A).$$

388 We start with the construction of w_h .

As in the proof of Theorem 4.4 we set $P_{\nabla u} = \{x_1, \dots, x_n\}$ and we define

$$D_h = \bigcup_{i=1}^n B_{\beta_{\varepsilon_h}}(x_i).$$

389 Since D_h is a regular set by taking into account the condition (33) for h large enough we
 390 have

$$\{x \in \Omega : d(x, D_h) < 2\varepsilon_h |\log \varepsilon_h|\} \subset\subset \Omega. \tag{45}$$

Let η be the optimal profile for Modica-Mortola's energy, that is the solution of the ODE

$$\begin{cases} \eta'(t) = \sqrt{W(t)} & \text{on } \mathbb{R} \\ \eta(-\infty) = 0, \\ \eta(+\infty) = 1, \end{cases}$$

391 given by $\eta(t) = \frac{1}{2}(1 + \tanh \frac{t}{2})$.

For every h let $\psi_h : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ -function such that

$$\begin{cases} \psi_h = 1 & \text{on } [0, |\log \varepsilon_h|] \\ \psi_h = 0 & \text{on } [2|\log \varepsilon_h|, +\infty] \\ \psi_h' < 0 & \text{on } [|\log \varepsilon_h|, 2|\log \varepsilon_h|] \\ \|\psi_h'\|_{L^\infty(|\log \varepsilon_h|, 2|\log \varepsilon_h|)} = O\left(\frac{1}{|\log \varepsilon_h|}\right). \end{cases}$$

As in [4] and in [9] we define

$$\eta_h(t) = \begin{cases} \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) + 1 - \psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t \geq 0 \\ \psi\left(\frac{t}{\varepsilon_h}\right) - \eta\left(\frac{t}{\varepsilon_h}\right)\psi\left(\frac{t}{\varepsilon_h}\right) & \text{if } t < 0. \end{cases}$$

Then we set $w_h(x) = \eta_h(\delta_{D_h}(x))$. We claim that $1 - w_h(x) \in C_0^\infty(\Omega; [0, 1])$ for h large enough. Indeed for any $x \in \Omega \setminus D_h$ we have $\delta_{D_h}(x) \geq 0$, hence

$$1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right).$$

392 Then $0 \leq 1 - w_h \leq 1$ by the properties of ψ_h and η . The case $x \in D_h$ is similar. Let now
 393 $x \in \partial\Omega$ then $\delta_{D_h}(x) \geq 0$ and $1 - w_h(x) = \psi_h\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\left(1 - \eta\left(\frac{\delta_{D_h}(x)}{\varepsilon_h}\right)\right)$. From (45), it follows
 394 $\delta_{D_h}(x) \geq 2|\log \varepsilon_h|$ for h large enough, hence the claim follows. Then we take $\{(u_h, w_h)\}_h$
 395 as optimal sequence, where u_h is given as in Theorem 4.4

396 First of all we have to check that $\{(u_h, w_h)\}_h$ H-converges to u . For any $x \in \Omega \setminus P_{\nabla u}$
 397 we have that for h large enough $\delta_{D_h}(x) \geq 0$ and one can check that $w_h(x) \rightarrow 1$ for every
 398 $x \in \Omega \setminus P_{\nabla u}$. This implies that $\mathcal{L}^2(\{w_h < t\}) \rightarrow 0$ for every $t \in (0, 1)$. Now for every
 399 $t \in (0, 1)$ we write

$$\{w_h = t\} = (\{w_h = t\} \cap D_h) \cup (\{w_h = t\} \cap \Omega \setminus D_h). \quad (46)$$

400 Hence, since $w_h(x) \rightarrow 1$ for $x \in \Omega \setminus D_h$, for any $t \in (0, 1)$ there exists $h(t)$ such that
 401 $\{w_h = t\} \cap \Omega \setminus D_h = \emptyset$ for every $h \geq h(t)$. So that from (46) it follows that for every
 402 $t \in (0, 1)$, $\{w_h = t\} \rightarrow P_{\nabla u}$ when $h \rightarrow +\infty$. So we can conclude that (u_h, w_h) H-converges
 403 to u .

As in [4] we set

$$D_h^1 = \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h |\log \varepsilon_h|\}, \quad D_h^2 = \{x \in \Omega : \varepsilon_h |\log \varepsilon_h| < |\delta_{D_h}(x)| < 2|\varepsilon_h \log \varepsilon_h|\}.$$

Therefore we can write

$$\begin{aligned} & \int_{\Omega \setminus \{\nabla w_h = 0\}} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)\right)^2 (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) dx \\ &= \int_{D_h^1} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)\right)^2 (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) dx + \\ & \int_{D_h^2} \left(\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)\right)^2 (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) dx = \\ &= I_h + II_h. \end{aligned} \quad (47)$$

For $x \in D_h^1$, we have $\frac{|\delta_{D_h}|}{\varepsilon_h} < \log \varepsilon_h$ therefore $w_h(x) = \eta(\frac{|\delta_{D_h}|}{\varepsilon_h})$. By taking into account the definition of η we have $\frac{\varepsilon_h}{1+\varepsilon_h} \leq w_h \leq \frac{1}{1+\varepsilon_h}$. Moreover it easy to check that

$$\eta'_h(t) = \frac{1}{\varepsilon_h} \eta'(t) = \frac{1}{\varepsilon_h} \sqrt{W(t)}; \quad |\nabla w_h| = |\eta'_k(\delta_{D_h})|.$$

This, together with the coarea formula yields

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\{w_h=t\}} (\frac{1}{\beta_h} + \beta_h k^2) d\mathcal{H}^1 dt.$$

Now we have that $\partial D_h = \{w_h = \frac{1}{2}\}$ then

$$I_h = 2 \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} \int_{\partial D_h} (\frac{1}{\beta_h} + \beta_h k^2) d\mathcal{H}^1 dt + O(\varepsilon_h \log(\varepsilon_h)) \int_{\frac{\varepsilon_h}{1+\varepsilon_h}}^{\frac{1}{1+\varepsilon_h}} \sqrt{W(t)} dt.$$

404 Then

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt. \quad (48)$$

405 Moreover with the same argument and by using the definition of w_h one can check that

$$\lim_{h \rightarrow \infty} II_h = 0 \quad (49)$$

406 By (48) and (49) we have

407

$$\lim_{h \rightarrow \infty} \frac{1}{8\pi C} \int_{\Omega \setminus \{\nabla w_h=0\}} (\frac{1}{\beta_{\varepsilon_h}} + \beta_{\varepsilon_h} (\operatorname{div}(\frac{\nabla w_h}{|\nabla w_h|}))^2 (\varepsilon_h |\nabla w_{\varepsilon_h}|^2 + \frac{1}{\varepsilon_h} W(w_{\varepsilon_h}))) dx = \mathcal{H}^0(P_{\nabla u}). \quad (50)$$

408 Now let us examine the terms involving u_h . As in the proof of Theorem 4.3 we have

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h - U_0|^p dx \leq \int_{\Omega} |\nabla u - U_0|^p dx. \quad (51)$$

Furthermore, by taking into account that $w_h(x) = 1$ if $\delta_{D_h}(x) \geq 2\varepsilon_h |\log(\varepsilon_h)|$ and $w_h(x) = 0$ if $\delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|$, we have

$$\int_{\Omega} w_h^2 |\Delta u_h|^2 dx = \int_{\Omega \setminus D_h^0} |\Delta u_h|^2 dx,$$

409 where $D_h^0 = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log(\varepsilon_h)|\}$. By choosing ρ_h in such a way that
410 $\rho_h \leq 2\varepsilon_h |\log(\varepsilon_h)|$, we obtain

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} w_h^2 |\Delta u_h|^2 dx = \lim_{h \rightarrow +\infty} \int_{\Omega \setminus D_h^0} |\Delta u|^2 dx = \int_{\Omega \setminus P_{\nabla u}} |\Delta u|^2 dx = \int_{\Omega} |\Delta u|^2 dx. \quad (52)$$

411 Finally from the definition of w_h , it follows that $w_h = 1$ outside the disk $(D_h)_{2\log \varepsilon_h}$ and
 412 hence thanks to (32) and (33)

$$\limsup_{h \rightarrow +\infty} \frac{1}{\gamma_h} \int_{\Omega} (1 - w_h)^2 dx \leq \lim_{h \rightarrow +\infty} \mathcal{L}(D_{h2\log \varepsilon_h}) \frac{1}{\gamma_h} = 0. \quad (53)$$

413 The thesis follows by collecting (50), (51), (52) and (53) \square

414 5.4 Variational property

415 Also in this case we obtain, as a direct consequence of Theorems 5.1, 5.2 and 5.3, the
 416 convergence of the minimum problems. The proof is as in Theorem 4.4.

Theorem 5.4. *Let $h \in \mathbb{N}$ go to $+\infty$. Let \mathcal{G}_ε and \mathcal{F} be given respectively by (31) and (14). If $\{\varepsilon_h\}$ is a sequence of positive numbers converging to zero and $\{(u_h, w_h)\} \subset S(\Omega)$ such that*

$$\lim_{h \rightarrow +\infty} (\mathcal{G}_{\varepsilon_h}(u_{\varepsilon_h}, w_{\varepsilon_h}) - \inf_{S(\Omega)} \mathcal{F}_{\varepsilon_h}(u, w)) = 0,$$

417 *then there exists a subsequence $\{(u_{h_k}, w_{h_k})\} \subset S(\Omega)$ and a minimizer \bar{u} of $\mathcal{F}(u)$, with*
 418 *$\bar{u} \in \Delta \mathcal{AM}^{p,2}(\Omega)$, such that $\{(u_{h_k}, w_{h_k})\}_k$ H -converges to \bar{u} .*

6 De Giorgi's Conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

where D is a regular set and κ is a curvature of its boundary ∂D .

Since ∂D can be represented as the discontinuity set of the function $w_0 = 1 - \chi_D$, by Modica-Mortola's Theorem it follows that there is a sequence of non constant local minimizers such that $w_\varepsilon \rightarrow w_0$ with respect to the L^1 -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(w_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

Furthermore looking at the Euler-Lagrange equation associated to a contour length term, yields a contour curvature term κ , while the Euler-Lagrange equations for the functional $F_\varepsilon^1(w)$ contains a term $2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon}$.

Then De Giorgi suggested to approximate by Γ -convergence the functional F^2 by adding to the Modica-Mortola approximating functionals the following term

$$F_\varepsilon^2(w) = \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx.$$

In ([5]) Bellettini and Paolini have proven the limsup inequality, while the validity of liminf inequality for a modified version of the original conjecture has been proven by Röger and Shätzle (see [23]).

Inspired by the De Giorgi's conjecture (see [15] for the original statement) it appears natural to investigate, in the spirit of [9], the possibility of approximating the functional \mathcal{F} by means of a sequence \mathcal{F}_ε much more convenient from a numerical point view (see [19]):

$$\begin{aligned} \mathcal{F}_\varepsilon(u, w) &= \int_{\Omega} w^2 |\Delta u|^2 dx + \frac{1}{8\pi C} \left(\frac{\beta_\varepsilon}{2\varepsilon} \int_{\Omega} (2\varepsilon \Delta w - \frac{W'(w)}{\varepsilon})^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} (\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} W(w)) dx \right) \\ &+ \int_{\Omega} |\nabla u - U_0|^p dx + \int_{\Omega} \frac{1}{\gamma_\varepsilon} (1 - w^2) dx. \end{aligned}$$

The presence of the term $\frac{1}{2\varepsilon}$ will be clear in the proof. By the way we are able to prove only the Γ -limsup inequality.

Theorem 6.1. *Let $h \in \mathbb{N}$ go to $+\infty$. Let ε_h be a sequence of positive numbers with $\varepsilon_h \rightarrow 0$. For every $u \in \Delta \mathcal{AM}^{p,2}(\Omega)$, there exists a sequence $\{(u_h, w_h)\}_h \subset S(\Omega)$ H -converging to u such that*

$$\limsup_{h \rightarrow +\infty} \mathcal{F}_{\varepsilon_h}(u_h, w_h) \leq \mathcal{F}(u). \quad (54)$$

Proof. Let $\{(u_h, w_h)\}_h$ be the optimal sequence of Theorem 5.3. It is not difficult to see that for every $x \in D_h^1$ the function $\delta_h(x)$ is regular and using the definition of w_h and taking into account that $\eta' = \sqrt{W(\eta)}$ the following identity holds

$$2\varepsilon_h \Delta w_h - \frac{W'(w_h)}{\varepsilon_h} = 2\varepsilon_h \eta'_h \Delta \delta_{D_h}(x) + 2\varepsilon_h \eta''_h - \frac{W'(w_h)}{\varepsilon_h} = 2\varepsilon_h \eta'_h (\delta_{D_h}(x)).$$

For h large enough we also have $\Delta \delta_{D_h}(x) = \kappa(\{\delta_{D_h}(x) = t\})$. Besides on D_h^1 we have $w_h(x) = \eta(\frac{\delta_{D_h}(x)}{\varepsilon_h})$ and hence the level set $\{\delta_{D_h}(x) = t\}$ corresponds to the level set $\{w_h(x) = \eta(\frac{t}{\varepsilon_h})\}$ with $0 \leq \eta \leq 1$, so that we infer

$$\kappa(\{\delta_{D_h}(x) = t\}) = \operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right).$$

By proceeding as in the proof of Theorem 5.3 and taking into account the equality $2\varepsilon_h |\eta'_h(\delta_{D_h}(x))| = 2\varepsilon_h |\nabla w_h|$ we have

$$\begin{aligned} I_h &= \int_{D_h^1} \frac{\beta_{\varepsilon_h}}{2\varepsilon_h} \left(2\varepsilon_h \Delta w_h - \frac{W'(w_h)}{\varepsilon_h}\right)^2 + \frac{1}{\beta_{\varepsilon_h}} (\varepsilon_h |\nabla w_h|^2 + \frac{1}{\varepsilon_h} W(w_h)) dx \\ &= 2 \int_{D_h^1} \left(\beta_{\varepsilon_h} \left(\operatorname{div}\left(\frac{\nabla w_h}{|\nabla w_h|}\right)\right)^2 + \frac{1}{\beta_{\varepsilon_h}}\right) \sqrt{W(w_h)} |\nabla w_h| dx. \end{aligned}$$

Then as in the proof of Theorem 5.3 we conclude

$$\lim_{h \rightarrow +\infty} I_h = 8\pi \mathcal{H}^0(P_{\nabla u}) \int_0^1 \sqrt{W(t)} dt.$$

437 By the same calculation on D_h^2 one can check that the integral over D_h^2 vanishes as in the
438 proof of Theorem 5.3.

439 The other terms can be estimated exactly as in the proof of Theorem 5.3 and therefore
440 the thesis is achieved. \square

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