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A Hybrid Linear Logic for Constrained Transition Systems with Applications to Molecular Biology

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Abstract

Linear implication can represent state transitions, but real transition systems operate under temporal, stochastic or probabilistic constraints that are not directly representable in ordinary linear logic. We propose a general modal extension of intuitionistic linear logic where logical truth is indexed by constraints and hybrid connectives combine constraint reasoning with logical reasoning. The logic has a focused cut-free sequent calculus that can be used to internalize the rules of particular constrained transition systems; we illustrate this with an adequate encoding of the synchronous stochastic pi-calculus. We also present some preliminary experiments of direct encoding of biological systems in the logic.

1 Introduction

To reason about state transition systems, we need a logic of state. Linear logic [20] is such a logic and has been successfully used to model such diverse systems as: planning [39], Petri nets, CCS, the π-calculus [9, 27], concurrent ML [9], security protocols [5], multi-set rewriting, graph traversal algorithms [40], and games. Linear logic achieves this versatility by representing propositions as *resources* that are composed into elements of state using $\otimes$, which can then be transformed using the linear implication ($\multimap$). However, linear implication is timeless: there is no way to correlate two concurrent transitions. If resources have lifetimes and state changes have temporal, probabilistic or stochastic *constraints*, then the logic will allow inferences that may not be realizable in the system. The need for formal reasoning in constrained systems has led to the creation of specialized logistic systems such as Continuous Stochastic Logic (CSL) [2] or Probabilistic CTL [22], that pay a considerable encoding overhead for the state component of transitions in exchange for the constraint reasoning not provided by linear logic.

A prominent alternative to the logical approach is to use a suitably enriched process algebra; a short list of examples includes reversible CCS [14], bioambients [37], brane calculi [8], stochastic and probabilistic π-calculi, the PEPA algebra [23], and the κ-calculus [15]. Each process algebra comes equipped with an underlying algebraic semantics which is used to justify mechanistic abstractions of observed reality as processes. These abstractions are then animated by means of simulation and then compared with the observations. Process calculi do not however fill the need for formal logical reasoning for constrained transition systems. For example, there is no uniform language to encode different stochastic process algebras. Encoding the stochastic π calculus in CSL, for example, would be inordinately complex because CSL does not provide any direct means of encoding π-calculus dynamics such as the linear production and consumption of messages in a synchronous interaction.

Fortunately, there is a simple yet general method to add constraint reasoning to linear logic that reunites linguistic need with ability. It is an old idea—*labelled deduction* [41] with *hybrid* connectives [7]—applied to a new domain. Precisely, we parameterize ordinary logical truth on a *constraint domain*: $A@w$ stands for the truth of A under constraint w. We then use the hybrid connectives of *satisfaction* and *localisation* to perform generic symbolic reasoning on the constraints at the propositional level. We call the result *hybrid linear logic* (HyLL). No properties—except a basic monoidal structure—are assumed about the constraints from a proof-theoretic standpoint. Indeed, HyLL has a

generic cut-free (but cut admitting) sequent calculus that can be strengthened with a focusing restriction [1] to obtain a normal form for proofs. Any instance of HyLL that gives a semantic interpretation to the constraints continues to enjoy these proof-theoretic properties.

Focusing allows us to treat HyLL as a *logical framework* for constrained transition systems. Logical frameworks with hybrid connectives have been considered before; hybrid LF (HLF), for example, is a generic mechanism to add many different kinds of resource-awareness, including linearity, to ordinary LF [36]. However, HLF follows the usual LF methodology by keeping the logic of the framework minimal. Its proof objects are canonical (β-normal η-long) natural deduction terms, where canonicity is known to be brittle because of permutative equivalences [42]. With focusing we have more direct access to canonical proofs in the sequent calculus, so we can enrich the framework with any connectives that obey the focusing discipline [12]. This reduces the overhead of encodings; indeed, *representational adequacy* of an encoding in terms of (partial) focused derivations is routine. We illustrate this style of obtaining adequate encodings by encoding of the synchronous stochastic π-calculus ($S\pi$) in HyLL with the constraint domain of rates.

In addition to the novel stochastic component, our encoding of $S\pi$ is a conceptual improvement over other encodings of π calculi in linear logic [9, 27]: Our encoding performs a full propositional reflection of processes as in [27], but is first-order and adequate as in [9]. Being a logical framework, HyLL does not itself prescribe an operational semantics for the encoding of processes; thus, bisimilarity in continuous time Markov chains (CTMCs) is not the same as logical equivalence in stochastic HyLL, unlike in CSL [16]. This is not a deficiency; the *combination* of focused HyLL proofs and a proof search strategy tailored to a particular encoding is necessary to produce faithful symbolic executions. This is exactly analogous to $S\pi$ where it is the simulation rather than the transitions in the process calculus that is shown to be faithful to the CTMC semantics [32].

The sections of this paper are organized as follows: in sec. 2, we present the inference system (natural deduction and sequent calculus) for HyLL and describe the two main semantic instances: temporal and stochastic constraints. In sec. 3 we sketch the general focusing restriction on HyLL sequent proofs. In sec. 4 we give the encoding of $S\pi$ in stochastic HyLL, and show that the encoding is representationally adequate for focused proofs (theorems 23 and 25). In sec. 5 we present some preliminary experiments of direct encoding of biological systems in HyLL. We end with an overview of related (sec. 6) and future work (sec. 7).

2 Hybrid linear logic

In this section we define HyLL, a conservative extension of intuitionistic first-order linear logic (ILL) [20] where the truth judgements are labelled by worlds representing constraints. Like in ILL, propositions are interpreted as *resources* which may be composed into a *state* using the usual linear connectives, and the linear implication ($\multimap$) denotes a transition between states. The world label of a judgement represents a constraint on states and state transitions; particular choices for the worlds produce particular instances of HyLL. The common component in all the instances of HyLL is the proof theory, which we fix once and for all. We impose the following minimal requirement on the kinds of constraints that HyLL can deal with.

Definition 1. *A* constraint domain $\mathcal{W}$ *is a monoid structure* $\langle W, \cdot, \iota\rangle$*. The elements of* W *are called* worlds*, and the partial order* $\preceq : W \times W$*—defined as* $u \preceq w$ *if there exists* $v \in W$ *such that* $u \cdot v = w$*—is the* reachability relation *in* $\mathcal{W}$*.*

The identity world ι is $\preceq$-initial and is intended to represent the lack of any constraints. Thus, the ordinary ILL is embeddable into any instance of HyLL by setting all world labels to the identity. When needed to disambiguate, the instance of HyLL for the constraint domain $\mathcal{W}$ will be written HyLL($\mathcal{W}$).

Atomic propositions are written using minuscule letters ($a, b, \ldots$) applied to a sequence of *terms* ($s, t, \ldots$), which are drawn from an untyped term language containing term variables ($x, y, \ldots$) and function symbols ($f, g, \ldots$) applied to a list of terms.. Non-atomic propositions are constructed from the connectives of first-order intuitionistic linear logic and the two hybrid connectives *satisfaction* ($\mathtt{at}$), which states that a proposition is true at a given world ($w, u, v, \ldots$), and *localization* ($\downarrow$), which binds a name for the world the proposition is true at. The following grammar summarizes the syntax of HyLL propositions.

$$\begin{aligned} A, B, \ldots ::= {} & a\,\vec{t} \mid A \otimes B \mid \mathbf{1} \mid A \multimap B \mid A \,\&\, B \mid \top \mid A \oplus B \mid \mathbf{0} \mid !A \mid \forall x.\, A \mid \exists x.\, A \\ & \mid (A \;\mathtt{at}\; w) \mid \downarrow u.\, A \mid \forall u.\, A \mid \exists u.\, A \end{aligned}$$

Note that in the propositions $\downarrow u.\, A$, $\forall u.\, A$ and $\exists u.\, A$, the scope of the world variable u is all the worlds occurring in A. World variables cannot be used in terms, and neither can term variables occur in worlds; this restriction is important for the modular design of HyLL because it keeps purely logical truth separate from constraint truth. We let α range over variables of either kind.

The unrestricted connectives $\wedge$, $\vee$, $\supset$, *etc.* of intuitionistic (non-linear) logic can also be defined in terms of the linear connectives and the exponential ! using any of the available embeddings of intuitionistic logic into linear logic, such as Girard's embedding [20].

2.1 Natural deduction for HyLL

We start with the judgements from linear logic [21] and enrich them with a modal situated truth. We present the syntax of hybrid linear logic in a natural deduction style, using Martin-Löf's principle of separating judgements and logical connectives. Instead of the ordinary mathematical judgement "A is true", judgements of HyLL are of the form "A is true at world w", abbreviated as $A@w$. We use dyadic hypothetical derivations of the form $\Gamma\ ;\ \Delta$ |- $C@w$ where Γ and Δ are sets of judgements of the form $A@w$, with Δ being moreover a *multiset*. Γ is called the *unrestricted context*: its hypotheses can be consumed any number of times. Δ is a *linear context*: every hypothesis in it must be consumed singly in the proof.

The rules for the linear connectives are borrowed from [11] where they are discussed at length, so we omit a more thorough discussion here. The rules for the first-order quantifiers are completely standard. The unrestricted context Γ enjoys weakening and contraction; as usual, this is a theorem that is attested by the inference rules of the logic, and we omit its straightforward inductive proof. The notation $[w/u]A$ stands for the replacement of all free occurrences of the world variable u in A with the world w, avoiding capture.

Theorem 2 (structural properties)**.**

1. *If* $\Gamma\ ;\ \Delta \vdash C@w$*, then* $\Gamma, \Gamma'\ ;\ \Delta \vdash C@w$*. (weakening)*
2. *If* $\Gamma, A@u, A@u\ ;\ \Delta \vdash C@w$*, then* $\Gamma, A@u\ ;\ \Delta \vdash C@w$*. (contraction)*

The full collection of inference rules are in fig. 1. A brief discussion of the hybrid rules follows. To introduce the *satisfaction* proposition ($A\ \mathtt{at}\ w$) (at any world w'), the proposition A must be true in the world w. The proposition ($A\ \mathtt{at}\ w$) itself is then true at any world, not just in the world w. In other words, ($A\ \mathtt{at}\ w$) carries with it the world at which it is true. Therefore, suppose we know that ($A\ \mathtt{at}\ w$) is true (at any world w'); then, we also know that $A@w$. These two introduction and elimination rules match up precisely to (de)construct the information in the $A@w$ judgement. The other hybrid connective of *localisation*, $\downarrow$, is intended to be able to name the current world. That is, if $\downarrow u.\, A$ is true at world w, then the variable u stands for w in the body A. This interpretation is reflected in its introduction rule $\downarrow I$. For elimination, suppose we have a proof of $\downarrow u.\, A@w$ for some world w. Then, we also know $[w/u]A@w$.

For the linear and unrestricted hypotheses, substitution is no different from that of the usual linear logic.

Theorem 3 (substitution)**.**

1. *If* $\Gamma\ ;\ \Delta \vdash A@u$ *and* $\Gamma\ ;\ \Delta', A@u \vdash C@w$*, then* $\Gamma\ ;\ \Delta, \Delta' \vdash C@w$*.*
2. *If* $\Gamma\ ;\ \cdot \vdash A@u$ *and* $\Gamma, A@u\ ;\ \Delta \vdash C@w$*, then* $\Gamma\ ;\ \Delta \vdash C@w$*.*

Proof sketch. By structural induction on the second given derivation in each case. □

Note that the $\downarrow$ connective commutes with every propositional connective, including itself. That is, $\downarrow u.\, (A * B)$ is equivalent to $(\downarrow u.\, A) * (\downarrow u.\, B)$ for all binary connectives $*$, and $\downarrow u.\ * A$ is equivalent to $*(\downarrow u.\, A)$ for every unary connective $*$, assuming the commutation will not cause an unsound capture of u. It is purely a matter of taste where to place the $\downarrow$, and repetitions are harmless.

Theorem 4 (conservativity)**.** *Call a proposition or multiset of propositions* pure *if it contains no instance of the hybrid connectives, and let* Γ, Δ *and* A *be pure. Then,* $\Gamma\ ;\ \Delta \vdash A@w$ *in HyLL iff* $\Gamma\ ;\ \Delta \vdash A$ *in intuitionistic linear logic.*

Proof. By structural induction on the given HyLL derivation. □

Judgemental rules

$$\frac{}{\Gamma\,;\,A@w \vdash A@w}\ \text{hyp} \qquad \frac{}{\Gamma, A@w\,;\,\cdot \vdash A@w}\ \text{hyp!}$$

Multiplicative rules

$$\frac{\Gamma\,;\,\Delta \vdash A@w \quad \Gamma\,;\,\Delta' \vdash B@w}{\Gamma\,;\,\Delta,\Delta' \vdash A \otimes B@w}\ \otimes I \qquad \frac{\begin{array}{c}\Gamma\,;\,\Delta \vdash A \otimes B@w \\ \Gamma\,;\,\Delta', A@w, B@w \vdash C@w'\end{array}}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ \otimes E$$

$$\frac{}{\Gamma\,;\,\cdot \vdash \mathbf{1}@w}\ \mathbf{1}I \qquad \frac{\Gamma\,;\,\Delta \vdash \mathbf{1}@w \quad \Gamma\,;\,\Delta' \vdash C@w'}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ \mathbf{1}E$$

$$\frac{\Gamma\,;\,\Delta, A@w \vdash B@w}{\Gamma\,;\,\Delta \vdash A \multimap B@w}\ \multimap I \qquad \frac{\Gamma\,;\,\Delta \vdash A \multimap B@w \quad \Gamma\,;\,\Delta' \vdash A@w}{\Gamma\,;\,\Delta,\Delta' \vdash B@w}\ \multimap E$$

Additive rules

$$\frac{\Gamma\,;\,\Delta \vdash A@w \quad \Gamma\,;\,\Delta \vdash B@w}{\Gamma\,;\,\Delta \vdash A \,\&\, B@w}\ \&I \qquad \frac{\Gamma\,;\,\Delta \vdash A_1 \,\&\, A_2@w}{\Gamma\,;\,\Delta \vdash A_i@w}\ \&E_i$$

$$\frac{\Gamma\,;\,\Delta \vdash A_i@w}{\Gamma\,;\,\Delta \vdash A_1 \oplus A_2@w}\ \oplus I_i \qquad \frac{\Gamma\,;\,\Delta \vdash A \oplus B@w \quad \begin{array}{c}\Gamma\,;\,\Delta', A@w \vdash C@w' \\ \Gamma\,;\,\Delta', B@w \vdash C@w'\end{array}}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ \oplus E$$

$$\frac{}{\Gamma\,;\,\Delta \vdash \top@w}\ \top I \qquad \frac{\Gamma\,;\,\Delta \vdash \mathbf{0}@w}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ \mathbf{0}E$$

First-order rules

$$\frac{\Gamma\,;\,\Delta \vdash A@w}{\Gamma\,;\,\Delta \vdash \forall\alpha.\, A@w}\ \forall I^{\alpha} \qquad \frac{\Gamma\,;\,\Delta \vdash \forall\alpha.\, A@w}{\Gamma\,;\,\Delta \vdash [\tau/x]A@w}\ \forall E$$

$$\frac{\Gamma\,;\,\Delta \vdash [\tau/x]A@w}{\Gamma\,;\,\Delta \vdash \exists\alpha.\, A@w}\ \exists I \qquad \frac{\Gamma\,;\,\Delta \vdash \exists\alpha.\, A@w \quad \Gamma\,;\,\Delta', A@w \vdash C@w'}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ \exists E^{\alpha}$$

For $\forall I^{\alpha}$ and $\exists E^{\alpha}$, α is assumed to be fresh with respect to the conclusion.
For $\exists I$ and $\forall E$, τ stands for a term or world, as appropriate.

Exponential rules

$$\frac{\Gamma\,;\,\cdot \vdash A@w}{\Gamma\,;\,\cdot \vdash !A@w}\ !I \qquad \frac{\Gamma\,;\,\Delta \vdash !A@w \quad \Gamma, A@w\,;\,\Delta' \vdash C@w'}{\Gamma\,;\,\Delta,\Delta' \vdash C@w'}\ !E$$

Hybrid rules

$$\frac{\Gamma\,;\,\Delta \vdash A@w}{\Gamma\,;\,\Delta \vdash (A\ \mathtt{at}\ w)@w'}\ \mathtt{at}I \qquad \frac{\Gamma\,;\,\Delta \vdash (A\ \mathtt{at}\ w)@w'}{\Gamma\,;\,\Delta \vdash A@w}\ \mathtt{at}E$$

$$\frac{\Gamma\,;\,\Delta \vdash [w/u]A@w}{\Gamma\,;\,\Delta \vdash \downarrow u.\, A@w}\ \downarrow I \qquad \frac{\Gamma\,;\,\Delta \vdash \downarrow u.\, A@w}{\Gamma\,;\,\Delta \vdash [w/u]A@w}\ \downarrow E$$

Figure 1: Natural deduction for HyLL

.

2.2 Sequent calculus for HyLL

In this section, we give a sequent calculus presentation of HyLL and prove a cut-admissibility theorem. The sequent formulation in turn will lead to an analysis of the polarities of the connectives in order to get a focused sequent calculus that can be used to compile a logical theory into a system of derived inference rules with nice properties (sec. 3). For instance, if a given theory defines a transition system, then the derived rules of the focused calculus will exactly exhibit the same transitions. This is key to obtain the necessary representational adequacy theorems, as we shall see for the $S\pi$-calculus example chosen in this paper (sec. 4.1).

In the sequent calculus, we depart from the linear hypothetical judgement $\vdash$ which has only an "active" right-hand side to a sequent arrow $\Longrightarrow$ that has active zones on both sides. A rule that infers a proposition on the right of the sequent arrow is called a "right" rule, and corresponds exactly to the introduction rules of natural deduction. Dually, introductions on the left of the sequent arrow correspond to elimination rules of natural deduction; however, as all rules in the sequent calculus are introduction rules, the information flow in a sequent derivation is always in the same direction: from the conclusion to the premisees, incidentally making the sequent calculus ideally suited for proof-search.

The full collection of rules of the HyLL sequent calculus is in fig. 2. There are only two structural rules: the init rule infers an atomic initial sequent, and the copy rule introduces a contracted copy of an unrestricted assumption into the linear context (reading from conclusion to premise). Weakening and contraction are admissible rules:

Theorem 5 (structural properties)**.**

1. *If* $\Gamma\,;\Delta \Longrightarrow C@w$*, then* $\Gamma,\Gamma'\,;\Delta \Longrightarrow C@w$*. (weakening)*
2. *If* $\Gamma, A@u, A@u\,;\Delta \Longrightarrow C@w$*, then* $\Gamma, A@u\,;\Delta \Longrightarrow C@w$*. (contraction)*

Proof. By straightforward structural induction on the given derivations. □

The most important structural properties are the admissibility of the identity and the cut principles. The identity theorem is the general case of the init rule and serves as a global syntactic completeness theorem for the logic. Dually, the cut theorem below establishes the syntactic soundness of the calculus; moreover there is no cut-free derivation of $\cdot\,;\cdot \Longrightarrow \mathbf{0}@w$, so the logic is also globally consistent.

Theorem 6 (identity)**.** $\Gamma\,;A@w \Longrightarrow A@w$.

Proof. By induction on the structure of A (see sec. A.1). □

Theorem 7 (cut)**.**

1. *If* $\Gamma\,;\Delta \Longrightarrow A@u$ *and* $\Gamma\,;\Delta', A@u \Longrightarrow C@w$*, then* $\Gamma\,;\Delta,\Delta' \Longrightarrow C@w$.
2. *If* $\Gamma\,;\cdot \Longrightarrow A@u$ *and* $\Gamma, A@u\,;\Delta \Longrightarrow C@w$*, then* $\Gamma\,;\Delta \Longrightarrow C@w$.

Proof. By lexicographic structural induction on the given derivations, with cuts of kind 2 additionally allowed to justify cuts of kind 1. The style of proof sometimes goes by the name of *structural cut-elimination* [11]. See sec. A.2 for the details. □

We can use the admissible cut rules to show that the following rules are invertible: $\otimes L$, $\mathbf{1}L$, $\oplus L$, $\mathbf{0}L$, $\exists L$, $\multimap R$, $\&R$, $\top R$, and $\forall R$. In addition, the four hybrid rules, $\mathtt{at}R$, $\mathtt{at}L$, $\downarrow R$ and $\downarrow L$ are invertible. In fact, $\downarrow$ and $\mathtt{at}$ commute freely with all non-hybrid connectives:

Theorem 8 (Invertibility)**.** *The following rules are invertible:*

1. *On the right:* $\&R$, $\top R$, $\multimap R$, $\forall R$, $\downarrow R$ *and* $\mathtt{at}R$*;*
2. *On the left:* $\otimes L$, $\mathbf{1}L$, $\oplus L$, $\mathbf{0}L$, $\exists L$, $!L$, $\downarrow L$ *and* $\mathtt{at}L$.

Proof. See §A.3. □

Theorem 9 (Correctness of the sequent calculus)**.**

1. *If* $\Gamma\,;\Delta \Longrightarrow C@w$*, then* $\Gamma\,;\Delta \vdash C@w$*. (soundness)*
2. *If* $\Gamma\,;\Delta \vdash C@w$*, then* $\Gamma\,;\Delta \Longrightarrow C@w$*. (completeness)*

Judgemental rules

$$\frac{}{\Gamma\,;\, a\,\vec{t}\,@u \Longrightarrow a\,\vec{t}\,@u}\ \text{init} \qquad \frac{\Gamma, A@u\,;\,\Delta, A@u \Longrightarrow C@w}{\Gamma, A@u\,;\,\Delta \Longrightarrow C@w}\ \text{copy}$$

Multiplicatives

$$\frac{\Gamma\,;\,\Delta \Longrightarrow A@w \quad \Gamma\,;\,\Delta' \Longrightarrow B@w}{\Gamma\,;\,\Delta,\Delta' \Longrightarrow A \otimes B@w}\ \otimes R \qquad \frac{\Gamma\,;\,\Delta, A@u, B@u \Longrightarrow C@w}{\Gamma\,;\,\Delta, A \otimes B@u \Longrightarrow C@w}\ \otimes L$$

$$\frac{}{\Gamma\,;\,\cdot \Longrightarrow \mathbf{1}@w}\ \mathbf{1}R \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow C@w}{\Gamma\,;\,\Delta, \mathbf{1}@u \Longrightarrow C@w}\ \mathbf{1}L \qquad \frac{\Gamma\,;\,\Delta, A@w \Longrightarrow B@w}{\Gamma\,;\,\Delta \Longrightarrow A \multimap B@w}\ \multimap R$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow A@u \quad \Gamma\,;\,\Delta', B@u \Longrightarrow C@w}{\Gamma\,;\,\Delta,\Delta', A \multimap B@u \Longrightarrow C@w}\ \multimap L$$

Additives

$$\frac{}{\Gamma\,;\,\Delta \Longrightarrow \top@w}\ \top R \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow A@w \quad \Gamma\,;\,\Delta \Longrightarrow B@w}{\Gamma\,;\,\Delta \Longrightarrow A \,\&\, B@w}\ \&R$$

$$\frac{\Gamma\,;\,\Delta, A_i@u \Longrightarrow C@w}{\Gamma\,;\,\Delta,\Delta', A_1 \,\&\, A_2@u \Longrightarrow C@w}\ \&L_i$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow A_i@w}{\Gamma\,;\,\Delta \Longrightarrow A_1 \oplus A_2@w}\ \oplus R_i \qquad \frac{}{\Gamma\,;\,\Delta, \mathbf{0}@u \Longrightarrow C@w}\ \mathbf{0}L$$

$$\frac{\Gamma\,;\,\Delta, A@u \Longrightarrow C@w \quad \Gamma\,;\,\Delta, B@u \Longrightarrow C@w}{\Gamma\,;\,\Delta, A \oplus B@u \Longrightarrow C@w}\ \oplus L$$

Quantifiers

$$\frac{\Gamma\,;\,\Delta \Longrightarrow A@w}{\Gamma\,;\,\Delta \Longrightarrow \forall\alpha.\, A@w}\ \forall R^{\alpha} \qquad \frac{\Gamma\,;\,\Delta, [\tau/\alpha]A@u \Longrightarrow C@w}{\Gamma\,;\,\Delta, \forall\alpha.\, A@u \Longrightarrow C@w}\ \forall L$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow [\tau/\alpha]A@w}{\Gamma\,;\,\Delta \Longrightarrow \exists\alpha.\, A@w}\ \exists R \qquad \frac{\Gamma\,;\,\Delta, A@u \Longrightarrow C@w}{\Gamma\,;\,\Delta, \exists\alpha.\, A@u \Longrightarrow C@w}\ \exists L^{\alpha}$$

For $\forall R^{\alpha}$ and $\exists L^{\alpha}$, α is assumed to be fresh with respect to the conclusion. For $\exists R$ and $\forall L$, τ stands for a term or world, as appropriate.

Exponentials

$$\frac{\Gamma\,;\,\cdot \Longrightarrow A@w}{\Gamma\,;\,\cdot \Longrightarrow !A@w}\ !R \qquad \frac{\Gamma, A@u\,;\,\Delta \Longrightarrow C@w}{\Gamma\,;\,\Delta, !A@u \Longrightarrow C@w}\ !L$$

Hybrid connectives

$$\frac{\Gamma\,;\,\Delta \Longrightarrow A@u}{\Gamma\,;\,\Delta \Longrightarrow (A\ \mathtt{at}\ u)@v}\ \mathtt{at}R \qquad \frac{\Gamma\,;\,\Delta, A@u \Longrightarrow C@w}{\Gamma\,;\,\Delta, (A\ \mathtt{at}\ u)@v \Longrightarrow C@w}\ \mathtt{at}L$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow [w/u]A@w}{\Gamma\,;\,\Delta \Longrightarrow \downarrow u.\, A@w}\ \downarrow R \qquad \frac{\Gamma\,;\,\Delta, [v/u]A@v \Longrightarrow C@w}{\Gamma\,;\,\Delta, \downarrow u.\, A@v \Longrightarrow C@w}\ \downarrow L$$

Figure 2: The sequent calculus for HyLL

.

Proof. See §A.4. □

Corollary 10 (consistency)**.** *There is no proof of* $\cdot\,;\,\cdot \vdash \mathbf{0}@w$.

Proof. See §A.4. □

HyLL is conservative with respect to ordinary intuitionistic logic: as long as no hybrid connectives are used, the proofs in HyLL are identical to those in ILL [11]. The proof (omitted) is by simple structural induction.

Theorem 11 (conservativity)**.** *If* $\Gamma\,;\,\Delta \Longrightarrow_{HyLL} C@w$ *is derivable, contains no occurrence of the hybrid connectives* $\downarrow$, $\mathtt{at}$, $\forall u$ *or* $\exists u$, *and each element of* Γ *and* Δ *is of the form* $A@w$, *then* $\Gamma\,;\,\Delta \Longrightarrow_{ILL} C$.

In the rest of this paper we use the following derived connectives.

Definition 12 (modal connectives)**.**

$$\Box A \triangleq \downarrow u.\, \forall w.\, (A \;\mathtt{at}\; u \cdot w) \qquad \Diamond A \triangleq \downarrow u.\, \exists w.\, (A \;\mathtt{at}\; u \cdot w)$$
$$\rho_v A \triangleq \downarrow u.\, (A \;\mathtt{at}\; u \cdot v) \qquad \dagger A \triangleq \forall u.\, (A \;\mathtt{at}\; u)$$

The connective ρ represents a form of delay. Note its derived right rule:

$$\frac{\Gamma\,;\,\Delta \vdash A@w \cdot v}{\Gamma\,;\,\Delta \vdash \rho_v A@w}\ \rho R$$

The proposition $\rho_v A$ thus stands for an *intermediate state* in a transition to A. Informally it can be thought to be "v before A"; thus, $\forall v.\ \rho_v A$ represents *all* intermediate states in the path to A, and $\exists v.\ \rho_v A$ represents *some* such state. The modally unrestricted proposition $\dagger A$ represents a resource that is consumable in any world; it is mainly used to make transition rules applicable at all worlds.

It is worth remarking that HyLL proof theory can be seen as at least as powerful as (the linear restriction of) intuitionistic S5 [41]:

Theorem 13 (HyLL is S5)**.** *The following sequent is derivable:* $\cdot\,;\ \Diamond A@w \Longrightarrow \Box\Diamond A@w$.

Proof. See §A.5. □

Obviously HyLL is more expressive as it allows direct manipulation of the worlds using the hybrid connectives: for example, the ρ connective is not definable in S5.

2.3 Temporal constraints

As a pedagogical example, consider the constraint domain $\mathcal{T} = \langle \mathbf{R}^+, +, 0\rangle$ representing instants of time. This domain can be used to define the lifetime of resources, such as keys, sessions, or delegations of authority. Delay (defn. 12) in HyLL($\mathcal{T}$) represents intervals of time; $\rho_d A$ means "A will become available after delay d", similar to metric tense logic [35]. This domain is very permissive because addition is commutative, resulting in the equivalence of $\rho_u \rho_v A$ and $\rho_v \rho_u A$. The "forward-looking" connectives G and F of ordinary tense logic are precisely $\Box$ and $\Diamond$ of defn. 12. In addition to the future connectives, this domain also admits past connectives if we add saturating subtraction (*i.e.*, $a - b = 0$ if $b \geq a$) to the language of worlds. We can then define the duals H and P of G and F as:

$$H\,A \triangleq \downarrow u.\, \forall w.\, (A \;\mathtt{at}\; u - w)$$
$$P\,A \triangleq \downarrow u.\, \exists w.\, (A \;\mathtt{at}\; u - w)$$

While this domain does not have any branching structure like CTL, it is expressive enough for many common idioms because of the branching structure of derivations involving $\oplus$. CTL reachability ("in some path in some future"), for instance, is the same as our $\Diamond$; similarly, CTL stability ("in all paths in all futures") is the same as $\Box$. There is some loss of expressive power, however; for instance, in CTL steadiness ("in some path for all futures") is distinct from stability, whereas the best approximation in HyLL is $\exists w.\ \Box(A \;\mathtt{at}\; u \cdot w)$.

On the other hand, the availability of linear reasoning makes certain kinds of reasoning in HyLL much more natural than in ordinary temporal logics. One important example is of *oscillation* between states in systems with kinetic feedback. In a temporal specification language such as BIOCHAM [10], only finite oscillations are representable using a nested syntax, while in HyLL we use a simple bi-implication; for example, the oscillation between A and B with delay d is represented by the rule $\dagger(A \multimap \rho_d\, B) \,\&\, (B \multimap \rho_d\, A)$ (or $\dagger(A \multimap \Diamond\, B) \,\&\, (B \multimap \Diamond\, A)$ if the oscillation is aperiodic). If HyLL($\mathcal{T}$) were extended with constrained implication and conjunction in the style of CILL [39] or η [17], then we can define localized versions of $\Box$ and $\Diamond$, such as "A is true everywhere/somewhere in an interval". They would also allow us to define the "until" and "since" operators of linear temporal logic [24].

2.4 Stochastic constraints

Transitions in practice rarely have precise delays. Phenomenological and experimental evidence is used to construct a probabilistic model of the transition system where the delays are specified as probability distributions of continuous variables. To simplify matters, we shall only consider real-valued random variables with a fairly traditional presentation in this paper; the generalisation to arbitrary measure spaces is well known.

Fact 14 (see *e.g.* [6])**.** *If F and G are the probability distributions*[1] *of the real-valued random variables X and Y respectively, i.e.,* $\Pr[X < t] = F(t)$ *and* $\Pr[Y < t] = G(t)$*, then the distribution of $X + Y$ is the* convolution *of F and G, written $F * G$:*

$$\Pr[X + Y < t] = (F * G)(t) \triangleq \int_{\mathbf{R}} F(t - x)\, dG(x).$$

Convolution is associative and commutative, and has for its identity the Heavyside step function $\Theta(x) \triangleq (1+\mathrm{sign}(x))/2$.

Because we are modelling a state transition system that only moves forward in time, our random variables (representing the delays of a transition) are always positive. Positively supported probability distributions (*i.e.*, distributions that are non-zero only on positive values) of random variables, together with convolution as the operator and the Heavyside step function as identity, form a commutative monoid that we shall call the *stochastic constraint domain*. (We avoid simple generalizations to the full real line, or Lebesgue measures in general, though they would present no problems.)

Definition 15. *The* stochastic domain $\mathcal{S}$ *is the monoid* $\langle \mathbf{D}, *, \Theta \rangle$ *where* $\mathbf{D}$ *is the space of positively supported probability distributions of one dimensional real-valued random variables.*

The standard model of stochastic transition systems is continuous time Markov chains (CTMCs) where the delays of transitions between states are distributed according to the Markov assumption of memorylessness, *i.e.*, the distributions are exponential distributions. The convolution of two exponential distributions produces a hypoexponential distribution, a variety of phase type distribution [29], on which convolution is closed. Hypoexponentials describe the time to absorption of a CTMC assuming that it is entered only in a single initial state and every non-absorbing state has a unique next state. This is sufficient for the purposes of the encoding in sec. 4 because our traces are linear. We consider the Heavyside step function as describing the time to absorption of a degenerate CTMC that starts in the absorbing state.

Definition 16. *A* rate *is either a hypoexponential distribution or the Heavyside step function. Let $\mathcal{R}$ be the submonoid of $\mathcal{S}$ where the distributions are rates. The instance HyLL($\mathcal{R}$) will sometimes be called "stochastic hybrid linear logic".*

3 Focusing

As HyLL is intended to represent transition systems adequately, it is crucial that HyLL derivations in the image of an encoding have corresponding transitions. However, transition systems are generally specified as rewrite algebras

[1] We use the convention of calling the cumulative distribution function the (probability) distribution.

over an underlying congruence relation. These congruences have to be encoded propositionally in HyLL, so a HyLL derivation will generally require several inference rules to implement a single transition; moreover, several trivially different reorderings of these "micro" inferences would correspond to the same transition. It is therefore futile to attempt to define an operational semantics directly on HyLL inferences.

We restrict the syntax to focused derivations [1], which ignores many irrelevant rule permutations in a sequent proof and divides the proof into clear *phases* that define the grain of atomicity. The logical connectives are divided into two classes, *negative* and *positive*, and rule permutations for connectives of like polarity are confined to *phases*. A *focused derivation* is one in which the positive and negative rules are applied in alternate maximal phases in the following way: in the *active* phase, all negative rules are applied (in irrelevant order) until no further negative rule can apply; the phase then switches and one positive proposition is selected for *focus*; this focused proposition is decomposed under focus (*i.e.*, the focus persists unto its sub-propositions) until it becomes negative, and the phase switches again.

As noted before, the logical rules of the hybrid connectives $\mathtt{at}$ and $\downarrow$ are invertible, so they can be considered to have both polarities. It would be valid to decide a polarity for each occurrence of each hybrid connective independently; however, as they are mainly intended for book-keeping during logical reasoning, we define the polarity of these connectives in the following *parasitic* form: if its immediate subformula is positive (resp. negative) connective, then it is itself positive (resp. negative). These connectives therefore become invisible to focusing. This choice of polarity can be seen as a particular instance of a general scheme that divides the $\downarrow$ and $\mathtt{at}$ connectives into two polarized forms each. To complete the picture, we also assign a polarity for the atomic propositions; this restricts the shape of focusing phases further [13], and is crucial to our intended use. The full syntax of positive ($P, Q, \ldots$) and negative ($M, N, \ldots$) propositions is as follows:

$$
\begin{array}{rcl}
P, Q, \ldots & ::= & p\,\vec{t} \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid !\,N \mid \exists\alpha.\, P \mid \downarrow u.\, P \mid (P \;\mathtt{at}\; w) \mid \Downarrow N \\
N, M, \ldots & ::= & n\,\vec{t} \mid N \,\&\, N \mid \top \mid P \multimap N \mid \forall\alpha.\, N \mid \downarrow u.\, N \mid (N \;\mathtt{at}\; w) \mid \Uparrow P
\end{array}
$$

The two syntactic classes refer to each other via the new connectives $\Uparrow$ and $\Downarrow$. Sequents in the focusing calculus are of the following forms.

$$
\left.\begin{array}{l}
\Gamma\,;\,\Delta\,;\,\Omega \Longrightarrow \cdot\,;\,P@w \\
\Gamma\,;\,\Delta\,;\,\Omega \Longrightarrow N@w\,;\,\cdot
\end{array}\right\}\text{active}
\qquad
\left.\begin{array}{l}
\Gamma\,;\,\Delta\,;\,[N@u] \Longrightarrow \cdot\,;\,P@w \\
\Gamma\,;\,\Delta\,;\,[N@u] \Longrightarrow M@w\,;\,\cdot \\
\Gamma\,;\,\Delta \Longrightarrow [P@w]
\end{array}\right\}\text{focused}
$$

In each case, Γ and Δ contain only negative propositions (*i.e.*, of the form $N@u$) and Ω only positive propositions (*i.e.*, of the form $P@u$). The full collection of inference rules are in fig. 3. The sequent form $\Gamma\,;\,\Delta\,;\,\cdot \Longrightarrow \cdot\,;\,P@w$ is called a *neutral sequent*; from such a sequent, a left or right focused sequent is produced with the rules lf, cplf or rf. Focused logical rules are applied (non-deterministically) and focus persists unto the subformulas of the focused proposition as long as they are of the same polarity; when the polarity switches, the result is an active sequent, where the propositions in the innermost zones are decomposed in an irrelevant order until once again a neutral sequent results.

Soundness of the focusing calculus with respect to the ordinary sequent calculus is immediate by simple structural induction. In each case, if we forget the additional structure in the focused derivations, then we obtain simply an unfocused proof. We omit the obvious theorem. Completeness, on the other hand, is a hard result. We omit the proof because focusing is by now well known for linear logic, with a number of distinct proofs via focused cut-elimination (see *e.g.* the detailed proof in [13]). The hybrid connectives pose no problems because they allow all cut-permutations.

Theorem 17 (focusing completeness)**.** *Let Γ^- and $C^-@w$ be negative polarizations of Γ and $C@w$ (that is, adding $\Uparrow$ and $\Downarrow$ to make C and each proposition in Γ negative) and Δ^+ be a positive polarization of Δ. If $\Gamma\,;\,\Delta \Longrightarrow C@w$, then $\cdot\,;\,\cdot\,;\,!\,\Gamma^-, \Delta^+ \Longrightarrow C^-@w\,;\,\cdot$.*

4 Encoding the synchronous stochastic π-calculus

In this section, we shall illustrate the use of HyLL($\mathcal{R}$) as a logical framework for constrained transition systems by encoding the syntax and the operational semantics of the synchronous stochastic π-calculus ($S\pi$), which extends the ordinary π-calculus by assigning to every channel and internal action an *inherent* rate of synchronization. HyLL($\mathcal{R}$) can therefore be seen as a formal language for expressing $S\pi$ executions (traces).

Focused logical rules

$$\frac{}{\Gamma\,;\,[n\,\vec{t}@w] \Longrightarrow \Downarrow n\,\vec{t}@w}\ \mathrm{li} \qquad \frac{\Gamma\,;\,\Delta\,;\,P@u \Longrightarrow \cdot\,;\,Q@w}{\Gamma\,;\,\Delta\,;\,[\Uparrow P@u] \Longrightarrow Q@w}\ \Uparrow L \qquad \frac{\Gamma\,;\,\Delta\,;\,[N_i@u] \Longrightarrow Q@w}{\Gamma\,;\,\Delta\,;\,[N_1 \,\&\, N_2@u] \Longrightarrow Q@w}\ \&L_i$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow [P@u] \quad \Gamma\,;\,\Xi\,;\,[N@u] \Longrightarrow Q@w}{\Gamma\,;\,\Delta,\Xi\,;\,[P \multimap N@u] \Longrightarrow Q@w}\ \multimap L \qquad \frac{\Gamma\,;\,\Delta\,;\,[[\tau/\alpha]N@u] \Longrightarrow Q@w}{\Gamma\,;\,\Delta\,;\,[\forall\alpha.\,N@u] \Longrightarrow Q@w}\ \forall L$$

$$\frac{\Gamma\,;\,\Delta\,;\,[[v/u]N@v] \Longrightarrow Q@w}{\Gamma\,;\,\Delta\,;\,[\downarrow u.\,N@v] \Longrightarrow Q@w}\ \downarrow LF \qquad \frac{\Gamma\,;\,\Delta\,;\,[N@u] \Longrightarrow Q@w}{\Gamma\,;\,\Delta\,;\,[(N\ \mathtt{at}\ u)@v] \Longrightarrow Q@w}\ \mathtt{at}LF$$

$$\frac{}{\Gamma\,;\,\Uparrow p\,\vec{t}@w \Longrightarrow [p\,\vec{t}@w]}\ \mathrm{ri} \qquad \frac{\Gamma\,;\,\Delta\,;\,\cdot \Longrightarrow N@w\,;\,\cdot}{\Gamma\,;\,\Delta \Longrightarrow [\Downarrow N@w]}\ \Downarrow R \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow [P@w] \quad \Gamma\,;\,\Xi \Longrightarrow [Q@w]}{\Gamma\,;\,\Delta,\Xi \Longrightarrow [P \otimes Q@w]}\ \otimes R$$

$$\frac{}{\Gamma\,;\,\cdot \Longrightarrow [\mathbf{1}@w]}\ \mathbf{1}R \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow [P_i@w]}{\Gamma\,;\,\Delta \Longrightarrow [P_1 \oplus P_2@w]}\ \oplus R_i \qquad \frac{\Gamma\,;\,\cdot\,;\,\cdot \Longrightarrow N@w\,;\,\cdot}{\Gamma\,;\,\cdot \Longrightarrow [!N]@w}\ !R$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow [[\tau/\alpha]P@w]}{\Gamma\,;\,\Delta \Longrightarrow [\exists\alpha.\,P@w]}\ \exists R \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow [[w/u]P@w]}{\Gamma\,;\,\Delta \Longrightarrow [\downarrow u.\,P@w]}\ \downarrow RF \qquad \frac{\Gamma\,;\,\Delta \Longrightarrow [P@u]}{\Gamma\,;\,\Delta \Longrightarrow [(P\ \mathtt{at}\ u)@w]}\ \mathtt{at}RF$$

Active logical rules (R of the form $\cdot\,;\,Q@w$ or $N@w\,;\,\cdot$, and L of the form $\Gamma\,;\,\Delta\,;\,\Omega$)

$$\frac{\mathsf{L}, P@u, Q@u \Longrightarrow \mathsf{R}}{\mathsf{L}, P \otimes Q@u \Longrightarrow \mathsf{R}}\ \otimes L \qquad \frac{\mathsf{L} \Longrightarrow \mathsf{R}}{\mathsf{L}, \mathbf{1}@u \Longrightarrow \mathsf{R}}\ \mathbf{1}L \qquad \frac{\mathsf{L}, P@u \Longrightarrow \mathsf{R} \quad \mathsf{L}, Q@u \Longrightarrow \mathsf{R}}{\mathsf{L}, P \oplus Q@u \Longrightarrow \mathsf{R}}\ \oplus L \qquad \frac{}{\mathsf{L}, \mathbf{0}@u \Longrightarrow \mathsf{R}}\ \mathbf{0}L$$

$$\frac{\mathsf{L}, [v/u]P@v \Longrightarrow \mathsf{R}}{\mathsf{L}, \downarrow u.\,P@v \Longrightarrow \mathsf{R}}\ \downarrow LA \qquad \frac{\mathsf{L}, P@u \Longrightarrow \mathsf{R}}{\mathsf{L}, (P\ \mathtt{at}\ u)@v \Longrightarrow \mathsf{R}}\ \mathtt{at}LA \qquad \frac{\mathsf{L}, P@u \Longrightarrow \mathsf{R}}{\mathsf{L}, \exists\alpha.\,P@u \Longrightarrow \mathsf{R}}\ \exists L^{\alpha}$$

$$\frac{\Gamma, N@u\,;\,\Delta\,;\,\Omega \Longrightarrow \mathsf{R}}{\Gamma\,;\,\Delta\,;\,\Omega, !N@u \Longrightarrow \mathsf{R}}\ !L \qquad \frac{\Gamma\,;\,\Delta, N@w\,;\,\Omega \Longrightarrow \mathsf{R}}{\Gamma\,;\,\Delta\,;\,\Omega, \Downarrow N@w \Longrightarrow \mathsf{R}}\ \Downarrow L \qquad \frac{\Gamma\,;\,\Delta, \Uparrow p\,\vec{t}\,;\,\Omega \Longrightarrow \mathsf{R}}{\Gamma\,;\,\Delta\,;\,\Omega, p\,\vec{t}@w \Longrightarrow \mathsf{R}}\ \mathrm{lp}$$

$$\frac{\mathsf{L} \Longrightarrow M@w\,;\,\cdot \quad \mathsf{L} \Longrightarrow N@w\,;\,\cdot}{\mathsf{L} \Longrightarrow M \,\&\, N@w\,;\,\cdot}\ \&R \qquad \frac{}{\mathsf{L} \Longrightarrow \top@w\,;\,\cdot}\ \top R \qquad \frac{\mathsf{L}, P@w \Longrightarrow N@w\,;\,\cdot}{\mathsf{L} \Longrightarrow P \multimap N@w\,;\,\cdot}\ \multimap R$$

$$\frac{\mathsf{L} \Longrightarrow [w/u]N@w\,;\,\cdot}{\mathsf{L} \Longrightarrow \downarrow u.\,N@w\,;\,\cdot}\ \downarrow RA \qquad \frac{\mathsf{L} \Longrightarrow N@u}{\mathsf{L} \Longrightarrow (N\ \mathtt{at}\ u)@w}\ \mathtt{at}RA \qquad \frac{\mathsf{L} \Longrightarrow N@u\,;\,\cdot}{\mathsf{L} \Longrightarrow \forall\alpha.\,N@u\,;\,\cdot}\ \forall R^{\alpha}$$

$$\frac{\mathsf{L} \Longrightarrow \cdot\,;\,P@w}{\mathsf{L} \Longrightarrow \Uparrow P@w\,;\,\cdot}\ \Uparrow R \qquad \frac{\mathsf{L} \Longrightarrow \cdot\,;\,\Downarrow n\,\vec{t}@w}{\mathsf{L} \Longrightarrow n\,\vec{t}@w\,;\,\cdot}\ \mathrm{rp}$$

Focusing decisions (L of the form $\Gamma\,;\,\Delta$)

$$\frac{\Gamma\,;\,\Delta\,;\,[N@u] \Longrightarrow Q@w \quad N \text{ not } \Uparrow p\,\vec{t}}{\Gamma\,;\,\Delta, N@u\,;\,\cdot \Longrightarrow \cdot\,;\,Q@w}\ \mathrm{lf} \qquad \frac{\Gamma, N@u\,;\,\Delta\,;\,[N@u] \Longrightarrow Q@w}{\Gamma, N@u\,;\,\Delta\,;\,\cdot \Longrightarrow \cdot\,;\,Q@w}\ \mathrm{cplf}$$

$$\frac{\Gamma\,;\,\Delta \Longrightarrow [P@w] \quad P \text{ not } \Downarrow n\,\vec{t}}{\Gamma\,;\,\Delta\,;\,\cdot \Longrightarrow \cdot\,;\,P@w}\ \mathrm{rf}$$

Figure 3: Focusing rules for HyLL.

Interactions

$$\frac{}{!x(y).\ P + M \mid ?x.\ Q + M' \xrightarrow{\text{rate}(x)} P \mid Q\,y}\ \textsc{SYN} \qquad \frac{}{\tau_r.\ P \xrightarrow{r} P}\ \textsc{INT}$$

$$\frac{P \xrightarrow{r} P'}{P \mid Q \xrightarrow{r} P' \mid Q}\ \textsc{PAR} \qquad \frac{\forall x_s.\left(P\,x \xrightarrow{r} Q\,x\right)}{\nu_s\ P \xrightarrow{r} \nu_s\ Q}\ \textsc{RES} \qquad \frac{P \xrightarrow{r} Q \quad P \equiv P' \quad Q \equiv Q'}{P' \xrightarrow{r} Q'}\ \textsc{CONG}$$

Congruence

$$\frac{}{P \mid 0 \equiv P} \qquad \frac{}{P \mid Q \equiv Q \mid P} \qquad \frac{}{P \mid (Q \mid R) \equiv (P \mid Q) \mid R} \qquad \frac{}{\nu_r\ 0 \equiv 0} \qquad \frac{X_n \triangleq P \in E}{E \vdash X_n\,x_1 \cdots x_n \equiv P\,x_1 \cdots x_n}$$

$$\frac{}{\nu_r(\lambda x.\ \nu_s(\lambda y.\ P)) \equiv \nu_s(\lambda y.\ \nu_r(\lambda x.\ P))} \qquad \frac{\forall x_r.\ (P\,x \equiv Q\,x)}{\nu_r\ P \equiv \nu_r\ Q} \qquad \frac{}{\nu_r(\lambda x.\ P \mid Q(x)) \equiv P \mid \nu_r\ Q}$$

$$\frac{P \equiv P'}{P \mid Q \equiv P' \mid Q} \qquad \frac{P \equiv P'}{!x(m).\ P \equiv !x(m).\ P'} \qquad \frac{\forall n.\ (P\,n \equiv Q\,n)}{?x.\ P \equiv ?x.\ Q} \qquad \frac{P \equiv P'}{\tau_r.\ P \equiv \tau_r.\ P'}$$

$$\frac{}{M + N \equiv N + M} \qquad \frac{}{M + (N + K) \equiv (M + N) + K} \qquad \frac{M \equiv M'}{M + N \equiv M' + N} \qquad \frac{M \equiv N}{M + N \equiv M}$$

Figure 4: Interactions and congruence in $S\pi$. The environment E is elided in most rules.

For the rest of this section we shall use $r, s, t, \ldots$ instead of $u, v, w, \ldots$ to highlight the fact that the worlds represent rates, with the understanding that $\cdot$ is convolution (defn. 14) and ι is Θ. We don't directly use rates because the syntax and transitions of $S\pi$ are given generically for a π-calculus with labelled actions, and it is only the interpretation of the labels that involves probabilities.

We first summarize the syntax of $S\pi$, which is a minor variant of a number of similar presentations such as [34]. For hygienic reasons we divide entities into the syntactic categories of *processes* ($P, Q, \ldots$) and *sums* ($M, N, \ldots$), defined as follows. We also include environments of recursive definitions (E) for constants.

$$\begin{array}{lll}
\textit{(Processes)} & P, Q, \ldots & ::= \nu_r\ P \mid P \mid Q \mid 0 \mid X_n\,x_1 \cdots x_n \mid M \\
\textit{(Sums)} & M, N, \ldots & ::= !x(y).\ P \mid ?x.\ P \mid \tau_r.\ P \mid M + N \\
\textit{(Environments)} & E & ::= E, X_n \triangleq P \mid \cdot
\end{array}$$

$P \mid Q$ is the parallel composition of P and Q, with unit 0. The restriction $\nu_r\ P$ abstracts over a free channel x in the process $P\,x$. We write the process using higher-order abstract syntax [31], *i.e.*, P in $\nu_r\ P$ is (syntactically) a function from channels to processes. This style lets us avoid cumbersome binding rules in the interactions because we reuse the well-understood binding structure of the λ-calculus. A similar approach was taken in the earliest encoding of (ordinary) π-calculus in (unfocused) linear logic [27], and is also present in the encoding in CLF [9].

A sum is a non-empty choice (+) over terms with *action prefixes*: the output action $!x(y)$ sends y along channel x, the input action $?x$ reads a value from x (which is applied to its continuation process), and the internal action τ_r has no observable I/O behaviour. Replication of processes happens via guarded recursive definitions [28]; in [38] it is argued that they are more practical for programming than the replication operator !. In a definition $X_n \triangleq P$, X_n denotes a (higher-order) defined constant of arity n; given channels $x_1, \ldots, x_n$, the process $X_n\,x_1 \cdots x_n$ is synonymous with $P\,x_1 \cdots x_n$. The constant X_n may occur on the right hand side of any definition in E, including in its body P, as long as it is prefixed by an action; this prevents infinite recursion without progress.

Interactions are of the form $E \vdash P \xrightarrow{r} Q$ denoting a transition from the process P to the process Q, in a global environment E, by performing an action at rate r. Each channel x is associated with an inherent rate specific to the channel, and internal actions τ_r have rate r. The restriction $\nu_r\ P$ defines the rate of the abstracted channel as r.

The full set of interactions and congruences are in fig. 4. We generally omit the global environment E in the rules as it never changes. It is possible to use the congruences to compute a normal form for processes that are a parallel composition of sums and each reaction selects two suitable sums to synchronise on a channel until there are no further reactions possible; this refinement of the operational semantics is used in $S\pi$ simulators such as SPiM [33].

Definition 18 (syntax encoding)**.**

1. *The encoding of the process P as a positive proposition, written $[\![P]\!]_{\mathrm{p}}$, is as follows (*`sel` *is a positive atom and* `rt` *a negative atom).*

$$[\![P \mid Q]\!]_{\mathrm{p}} = [\![P]\!]_{\mathrm{p}} \otimes [\![Q]\!]_{\mathrm{p}} \qquad [\![\nu_r\, P]\!]_{\mathrm{p}} = \exists x.\ !(\mathtt{rt}\ x\ \mathtt{at}\ r) \otimes [\![P\ x]\!]_{\mathrm{p}}$$
$$[\![0]\!]_{\mathrm{p}} = \mathbf{1} \qquad [\![X_n\, x_1 \cdots x_n]\!]_{\mathrm{p}} = X_n\, x_1 \cdots x_n$$
$$[\![M]\!]_{\mathrm{p}} = \Downarrow(\mathtt{sel} \multimap [\![M]\!]_{\mathrm{s}})$$

2. *The encoding of the sum M as a negative proposition, written $[\![M]\!]_{\mathrm{s}}$, is as follows (*`out`*,* `in` *and* `tau` *are positive atoms).*

$$[\![M + N]\!]_{\mathrm{s}} = [\![M]\!]_{\mathrm{s}} \,\&\, [\![N]\!]_{\mathrm{s}} \qquad [\![!x(m).\ P]\!]_{\mathrm{s}} = \Uparrow(\mathtt{out}\ x\ m \otimes [\![P]\!]_{\mathrm{p}})$$
$$[\![?x.\ P]\!]_{\mathrm{s}} = \forall n.\ \Uparrow(\mathtt{in}\ x\ n \otimes [\![P\ n]\!]_{\mathrm{p}}) \qquad [\![\tau_r.\ P]\!]_{\mathrm{s}} = \Uparrow(\mathtt{tau}\ r \otimes [\![P]\!]_{\mathrm{p}})$$

3. *The encoding of the definitions E as a context, written $[\![E]\!]_{\mathrm{e}}$, is as follows.*

$$[\![E, X_n \triangleq P]\!]_{\mathrm{e}} \;=\; [\![E]\!]_{\mathrm{e}}\,, \dagger\, \forall x_1, \ldots, x_n.\ X_n\, x_1 \cdots x_n \multimapboth [\![P\ x_1 \cdots x_n]\!]_{\mathrm{p}}$$
$$[\![\cdot]\!]_{\mathrm{e}} \;=\; \cdot$$

where $P \multimapboth Q$ is defined as $(P \multimap \Uparrow Q) \,\&\, (Q \multimap \Uparrow P)$.

The encoding of processes is positive, so they will be decomposed in the active phase when they occur on the left of the sequent arrow, leaving a collection of sums. The encoding of restrictions will introduce a fresh unrestricted assumption about the rate of the restricted channel. Each sum encoded as a processes undergoes a polarity switch because $\multimap$ is negative; the antecedent of this implication is a *guard* `sel`. This pattern of guarded switching of polarities prevents unsound congruences such as $!x(m).\ !y(n).\ P \equiv !y(n).\ !x(m).\ P$ that do not hold for the synchronous π calculus.[2] This guard also *locks* the sums in the context: the $S\pi$ interaction rules INT and SYN discard the non-interacting terms of the sum, so the environment will contain the requisite number of `sel`s only when an interaction is in progress. The action prefixes themselves are also synchronous, which causes another polarity switch. Each action releases a token of its respective kind—`out`, `in` or `tau`—into the context. These tokens must be consumed by the interaction before the `act` token becomes available again. For each action, the (encoding of the) continuation process is also released into the context.

The proof of the following congruence lemma is omitted. Because the encoding is (essentially) a $\otimes/\&$ structure, there are no distributive laws in linear logic that would break the process/sum structure.

Theorem 19 (congruence)**.**
$E \vdash P \equiv Q$ iff both $[\![E]\!]_{\mathrm{e}}@\iota\ ;\ \cdot\ ;\ [\![P]\!]_{\mathrm{p}}@\iota \Longrightarrow \cdot\ ;\ [\![Q]\!]_{\mathrm{p}}@\iota$ and $[\![E]\!]_{\mathrm{e}}@\iota\ ;\ \cdot\ ;\ [\![Q]\!]_{\mathrm{p}}@\iota \Longrightarrow \cdot\ ;\ [\![P]\!]_{\mathrm{p}}@\iota$.

Now we encode the interactions. Because processes were lifted into propositions, we can be parsimonious with our encoding of interactions by limiting ourselves to the atomic interactions SYN and INT (below); the PAR, RES and CONG interactions will be ambiently implemented by the logic. Because there are no concurrent interactions—only one interaction can trigger at a time in a trace—the interaction rules must obey a locking discipline. We represent this lock as the proposition `act` that is consumed at the start of an interaction and produced again at the end. This lock also carries the net rate of the prefix of the trace so far: that is, an interaction $P \xrightarrow{r} Q$ will update the lock from $\mathtt{act}@s$ to $\mathtt{act}@s \cdot r$. The encoding of individual atomic interactions must also remove the `in`, `out` and `tau` tokens introduced in context by the interacting processes.

Definition 20 (interaction)**.**
Let $\mathtt{inter} \triangleq \dagger(\mathtt{act} \multimap \Uparrow\mathtt{int} \,\&\, \Uparrow\mathtt{syn})$ *where* `act` *is a positive atom and* `int` *and* `syn` *are as follows:*

$$\mathtt{int} \triangleq (\mathtt{sel}\ \mathtt{at}\ \iota) \otimes \Downarrow\forall r.\ \big((\mathtt{tau}\ r\ \mathtt{at}\ \iota) \multimap \rho_r \Uparrow\mathtt{act}\big)$$
$$\mathtt{syn} \triangleq (\mathtt{sel} \otimes \mathtt{sel}\ \mathtt{at}\ \iota) \otimes \Downarrow\forall x, r, m.\ \big((\mathtt{out}\ x\ m \otimes \mathtt{in}\ x\ m\ \mathtt{at}\ \iota) \multimap \Downarrow(\mathtt{rt}\ x\ \mathtt{at}\ r) \multimap \rho_r \Uparrow\mathtt{act}\big).$$

The number of interactions that are allowed depend on the number of instances of `inter` in the linear context: each focus on `inter` implements a single interaction. If we are interested in all finite traces, we will add `inter` to the unrestricted context so it may be reused as many times as needed.

[2] Note: $(x \multimap a \otimes (x \multimap b \otimes c)) \multimap (x \multimap b \otimes (x \multimap a \otimes c))$ is not provable in linear logic.

Suppose $\mathsf{L} = \mathtt{rt}\,x@r, \mathtt{inter}@\iota$ and $\mathsf{R} = (\llbracket S \rrbracket_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@t$. (All judgements $@\iota$ omitted.)

$$
\begin{array}{rl}
 & \mathsf{L}\,;\ \wr Q \wr, \wr R\,a \wr, \Uparrow \mathtt{act}@s \cdot r\,;\cdot \Longrightarrow \cdot\,;\mathsf{R} \\
\hline
5 & \mathsf{L}\,;\ \ \wr Q \wr, \Uparrow \mathtt{out}\,x\,a, \Uparrow \mathtt{in}\,x\,a, \wr R\,a \wr, \\
 & \qquad \forall x, r, m.\,((\mathtt{out}\,x\,m \otimes \mathtt{in}\,x\,m\ \mathtt{at}\ \iota) \multimap \Downarrow(\mathtt{rt}\,x\ \mathtt{at}\ r) \multimap \rho_r\ \mathtt{act})@s \quad ;\cdot \Longrightarrow \cdot\,;\mathsf{R} \\
\hline
4 & \mathsf{L}\,;\ \ \Uparrow \mathtt{out}\,x\,a, \wr Q \wr, \mathtt{sel} \multimap \forall y.\,\Uparrow(\mathtt{in}\,x\,y \otimes \llbracket R\,y \rrbracket_{\mathrm{p}}), \\
 & \qquad \Uparrow \mathtt{sel}, \forall x, r, m.\,((\mathtt{out}\,x\,m \otimes \mathtt{in}\,x\,m\ \mathtt{at}\ \iota) \multimap \Downarrow(\mathtt{rt}\,x\ \mathtt{at}\ r) \multimap \rho_r\ \mathtt{act})@s \quad ;\cdot \Longrightarrow \cdot\,;\mathsf{R} \\
\hline
3 & \mathsf{L}\,;\ \ \mathtt{sel} \multimap \Uparrow(\mathtt{out}\,x\,a \otimes \llbracket Q \rrbracket_{\mathrm{p}}), \mathtt{sel} \multimap \forall y.\,(\mathtt{in}\,x\,y \otimes \llbracket R\,y \rrbracket_{\mathrm{p}}), \\
 & \qquad \Uparrow \mathtt{sel}, \Uparrow \mathtt{sel}, \forall x, r, m.\,((\mathtt{out}\,x\,m \otimes \mathtt{in}\,x\,m\ \mathtt{at}\ \iota) \multimap \Downarrow(\mathtt{rt}\,x\ \mathtt{at}\ r) \multimap \rho_r\ \mathtt{act})@s \quad ;\cdot \Longrightarrow \cdot\,;\mathsf{R} \\
\hline
2 & \mathsf{L}\,;\Uparrow \mathtt{act}@s, \mathtt{sel} \multimap \Uparrow(\mathtt{out}\,x\,a \otimes \llbracket Q \rrbracket_{\mathrm{p}}), \mathtt{sel} \multimap \forall y.\,\Uparrow(\mathtt{in}\,x\,y \otimes \llbracket R\,y \rrbracket_{\mathrm{p}})\,;[\mathtt{inter}] \Longrightarrow \mathsf{R} \\
\hline
1 & \mathsf{L}\,;\Uparrow \mathtt{act}@s, \mathtt{sel} \multimap \Uparrow(\mathtt{out}\,x\,a \otimes \llbracket Q \rrbracket_{\mathrm{p}}), \mathtt{sel} \multimap \forall y.\,\Uparrow(\mathtt{in}\,x\,y \otimes \llbracket R\,y \rrbracket_{\mathrm{p}})\,;\cdot \Longrightarrow \cdot\,;\mathsf{R} \\
\hline\hline
 & \mathsf{L}\,;\Uparrow \mathtt{act}@s, \wr !x(a).\ Q \mid ?x.\ R \wr\ ;\cdot \Longrightarrow \cdot\,;\mathsf{R}
\end{array}
$$

Steps		
1: focus on $\mathtt{inter} \in \mathsf{L}$	3: $\mathtt{sel}$ for output + full phases	5: cleanup
2: select $\mathtt{syn}$ from $\mathtt{inter}$, active rules	4: $\mathtt{sel}$ for input + full phases	

Figure 5: Example interaction in the $S\pi$-encoding.

4.1 Representational adequacy.

Adequacy consists of two components: completeness and soundness. Completeness is the property that every $S\pi$ execution is obtainable as a HyLL derivation using this encoding, and is the comparatively simpler direction (see thm. 23). Soundness is the reverse property, and is false for unfocused HyLL as such. However, it *does* hold for focused proofs (see thm. 25). In both cases, we reason about the following canonical sequents of HyLL.

Definition 21. *The* canonical context of P, *written* $\wr P \wr$, *is given by:*

$$
\wr X_n\,x_1 \cdots x_n \wr = \Uparrow X_n\,x_1 \cdots x_n \qquad \wr P \mid Q \wr = \wr P \wr, \wr Q \wr \qquad \wr 0 \wr = \cdot \qquad \wr \nu_r\ P \wr = \wr P\,a \wr
$$
$$
\wr M \wr = \mathtt{sel} \multimap \llbracket M \rrbracket_{\mathrm{s}}
$$

For $\wr \nu_r\ P \wr$, *the right hand side uses* a fresh *channel* a *that is not free the rest of the sequent it occurs in.*

As an illustration, take $P \triangleq !x(a).\ Q \mid ?x.\ R$. We have:

$$
\wr P \wr = \ \mathtt{sel} \multimap \Uparrow(\mathtt{out}\,x\,a \otimes \llbracket Q \rrbracket_{\mathrm{p}}), \mathtt{sel} \multimap \forall y.\,\Uparrow(\mathtt{in}\,x\,y \otimes \llbracket R\,y \rrbracket_{\mathrm{p}})
$$

Obviously, the canonical context is what would be emitted to the linear zone at the end of the active phase if $\llbracket P \rrbracket_{\mathrm{p}}$ were to be present in the left active zone.

Definition 22. *A neutral sequent is* canonical *iff it has the shape*

$$
\llbracket E \rrbracket_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\Uparrow \mathtt{act}@s, \wr P_1 \mid \cdots \mid P_k \wr @\iota\ ;\cdot \Longrightarrow \cdot\,;(\llbracket Q \rrbracket_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@t
$$

where $\mathtt{rates}$ *contains elements of the form* $\mathtt{rt}\ x@r$ *defining the rate of the channel* x *as* r, *and all free channels in* $\llbracket E \rrbracket_{\mathrm{e}}, \wr P_1 \mid \cdots \mid P_k \mid Q \wr$ *have a single such entry in* $\mathtt{rates}$.

Figure 5 contains an example of a derivation for a canonical sequent involving P. Focusing on any (encoding of a) sum in $\wr P \wr @\iota$ will fail because there is no $\mathtt{sel}$ in the context, so only $\mathtt{inter}$ can be given focus; this will consume the $\mathtt{act}$ and release two copies of $(\mathtt{sel}\ \mathtt{at}\ \iota)$ and the continuation into the context. Focusing on the latter will fail now (because $\mathtt{out}\,x\,m$ and $\mathtt{in}\,x\,m$ (for some m) are not yet available), so the only applicable foci are the two sums that can now be "unlocked" using the $\mathtt{sel}$s. The output and input can be unlocked in an irrelevant order, producing two tokens $\mathtt{in}\,x\,a$ and $\mathtt{out}\,x\,a$. Note in particular that the witness a was chosen for the universal quantifier in the encoding

of $?x.\ Q$ because the subsequent consumption of these two tokens requires the messages to be identical. (Any other choice will not lead to a successful proof.) After both tokens are consumed, we get the final form $\mathtt{act}@s \cdot r$, where r is the inherent rate of x (found from the `rates` component of the unrestricted zone). This sequent is canonical and contains $\wr Q \mid R\, a \int$.

Our encoding therefore represents every $S\pi$ action in terms of "micro" actions in the following rigid order: one micro action to determine what kind of action (internal or synchronization), one micro action per sum to select the term(s) that will interact, and finally one micro action to establish the contract of the action. Thus we see that focusing is crucial to maintain the semantic interpretation of (neutral) sequents. In an unfocused calculus, several of these steps could have partial overlaps, making such a semantic interpretation inordinately complicated. We do not know of any encoding of the π calculus that can provide such interpretations in unfocused sequents without changing the underlying logic. In CLF [9] the logic is extended with explicit monadic staging, and this enables a form of adequacy [9]; however, the encoding is considerably more complex because processes and sums cannot be fully lifted and must instead be specified in terms of a lifting computation. Adequacy is then obtained via a permutative equivalence over the lifting operation. Other encodings of π calculi in linear logic, such as [19] and [3], concentrate on the easier asynchronous fragment and lack adequacy proofs anyhow.

Theorem 23 (completeness)**.** *If $E \vdash P \xrightarrow{r} Q$, then the following canonical sequent is derivable.*

$$[\![E]\!]_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\ \Uparrow\mathtt{act}@s, \wr P \int @\iota\ ;\ \cdot \Longrightarrow \cdot\ ;\ ([\![Q]\!]_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@s \cdot r.$$

Proof. By structural induction of the derivation of $E \vdash P \xrightarrow{r} Q$. Every interaction rule of $S\pi$ is implementable as an admissible inference rule for canonical sequents. For CONG, we appeal to thm. 19. □

Completeness is a testament to the expressivity of the logic – all executions of $S\pi$ are also expressible in HyLL. However, we also require the opposite (soundness) direction: that every canonical sequent encodes a possible $S\pi$ trace. The proof hinges on the following canonicity lemma.

Lemma 24 (canonical derivations)**.** *In a derivation for a canonical sequent, the derived inference rules for* `inter` *are of one of the two following forms (conclusions and premises canonical).*

$$\frac{}{[\![E]\!]_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\ \Uparrow\mathtt{act}@s, \wr P \int @\iota\ ;\ \cdot \Longrightarrow \cdot\ ;\ ([\![P]\!]_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@s}$$

$$\frac{[\![E]\!]_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\ \Uparrow\mathtt{act}@s \cdot r, \wr Q \int @\iota\ ;\ \cdot \Longrightarrow \cdot\ ;\ ([\![R]\!]_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@t}{[\![E]\!]_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\ \Uparrow\mathtt{act}@s, \wr P \int @\iota\ ;\ \cdot \Longrightarrow \cdot\ ;\ ([\![R]\!]_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@t}$$

where: either $E \vdash P \xrightarrow{r} Q$, or $E \vdash P \equiv Q$ with $r = \iota$.

Proof. This is a formal statement of the phenomenon observed earlier in the example (fig. 5): $[\![R]\!]_{\mathrm{p}} \otimes \mathtt{act}$ cannot be focused on the right unless $P \equiv R$, in which case the derivation ends with no more foci on `inter`. If not, the only elements available for focus are `inter` and one of the congruence rules $[\![E]\!]_{\mathrm{e}}$ in the unrestricted context. In the former case, the derived rule consumes the $\Uparrow\mathtt{act}@s$, and by the time `act` is produced again, its world has advanced to $s \cdot r$. In the latter case, the definition of a top level X_n in $\wr P \int$ is (un)folded (without advancing the world). The proof proceeds by induction on the structure of P. □

Lemma 24 is a strong statement about HyLL derivations using this encoding: every partial derivation using the derived inference rules represents a prefix of an $S\pi$ trace. This is sometimes referred to as *full adequacy*, to distinguish it from adequacy proofs that require complete derivations [30]. The structure of focused derivations is crucial because it allows us to close branches early (using init). It is impossible to perform a similar analysis on unfocused proofs for this encoding; both the encoding and the framework will need further features to implement a form of staging [9, Chapter 3].

Corollary 25 (soundness)**.**
If $[\![E]\!]_{\mathrm{e}}, \mathtt{rates}, \mathtt{inter}@\iota\ ;\ \Uparrow\mathtt{act}@\iota, \wr P \int @\iota\ ;\ \cdot \Longrightarrow \cdot\ ;\ ([\![Q]\!]_{\mathrm{p}}\ \mathtt{at}\ \iota) \otimes \mathtt{act}@r$ is derivable, then $E \vdash P \xrightarrow{r}{}^{*} Q$.

Proof. Directly from lem. 24. □

4.2 Stochastic correctness

So far the HyLL($\mathcal{R}$) encoding of $S\pi$ represents any $S\pi$ trace symbolically. However, not every symbolic trace of an $S\pi$ process can be produced according to the operational semantics of $S\pi$. This is the main difference between HyLL (and $S\pi$) and the approach of CSL [2], where the truth of a proposition is evaluated against a CTMC, which is why equivalence in CSL is identical to CTMC bisimulation [16]. In this section we sketch how the execution could be used directly on the canonical sequents to produce only correct traces (proofs). The proposal in this section should be seen by analogy to the execution model of $S\pi$ simulators such as SPiM [32], although we do not use the Gillespie algorithm.

The main problem of simulation is determining which of several competing enabled actions in a canonical sequent to select as the "next" action from the *race condition* of the actions enabled in the sequent. Because of the focusing restriction, these enabled actions are easy to compute. Each element of $\wr P \wr$ is of the form $\mathtt{sel} \multimap [\![M]\!]_{\mathrm{s}}$, so the enabled actions in that element are given precisely by the topmost occurrences of $\Uparrow$ in $[\![M]\!]_{\mathrm{s}}$. Because none of the sums can have any restricted channels (they have all been removed in the active decomposition of the process earlier), the rates of all the channels will be found in the `rates` component of the canonical sequent.

The effective rate of a channel x is related to its inherent rate by scaling by a factor proportional to the *activity* on the channel, as defined in [32]. Note that this definition is on the *rate constants* of exponential distributions, not the rates themselves. The distribution of the minimum of a list of random variables with exponential distribution is itself an exponential distribution whose rate constant is the sum of those of the individual variables. Each individual transition on a channel is then weighted by the contribution of its rate to this sum. The choice of the transition to select is just the ordinary logical non-determinism. Note that the rounds of the algorithm do not have an associated *delay* element as in [32]; instead, we compute (symbolically) a distribution over the delays of a sequence of actions.

Because stochastic correctness is not necessary for the main adequacy result in the previous subsection, we leave the details of simulation to future work.

5 Direct encoding of molecular biology

Models of molecular biology have a wealth of examples of transition systems with temporal and stochastic constraints. In a biochemical reaction, molecules can interact to form other molecules or undergo internal changes such as phosphorylation, and these changes usually occur as parts of networks of interacting processes with continuous kinetic feedback. $S\pi$ has been used in a number of such models; since we have an adequate encoding of $S\pi$, we can use these models via the encoding.

However, biological systems can also be encoded directly in HyLL. As an example, consider a simplified *repressilator* gene network consisting of two genes, each causing the production of a protein that represses the other gene by negative feedback. This is a simplification of the three-gene network constructed in [18]. We note that each gene can be in an "on" (activated) or an "off" (deactivated) state, represented by the unary predicates `on` and `off`. Molecules of the transcribed proteins are represented with the unary predicate `prot`. Transitions in the network are encoded as axioms.

Example: the repressilator, using temporal constraints The system consists of the following components:

- *Repression*: Each protein molecule deactivates the next gene in the cycle after (average) deactivation delay d

  $$\mathtt{repress}\, a\, b \stackrel{\text{def}}{=} \mathtt{prot}\, a \otimes \mathtt{on}\, b \multimap \rho_d(\mathtt{off}\, b \otimes \mathtt{prot}\, a).$$

- *Reactivation*: When a gene is in the "off" state, it eventually becomes "on" after an average delay of r:

  $$\mathtt{react} \stackrel{\text{def}}{=} \forall a.\, \mathtt{off}\, a \multimap \rho_r\, \mathtt{on}\, a.$$

  It is precisely this reactivation that causes the system to oscillate instead of being bistable.

- *Transcription*: When a gene is "on", it transcribes RNA for its protein taking average delay t, after which it continues to be "on" and a molecule of the protein is formed.

  $$\mathtt{trans} \stackrel{\text{def}}{=} \forall a.\, \mathtt{on}\, a \multimap \rho_t(\mathtt{on}\, a \otimes \mathtt{prot}\, a).$$

- *Dissipation*: If a protein does not react with a gene, then it dissipates after average delay s:

$$\mathtt{diss} \stackrel{\text{def}}{=} \forall a.\, \mathtt{prot}\, a \multimap \rho_s\, \mathbf{1}.$$

The system consists of a repression cycle for genes a and b, and the other processes:

$$\mathtt{system} \stackrel{\text{def}}{=} \mathtt{repress\,a\,b}, \mathtt{repress\,b\,a}, \mathtt{react}, \mathtt{trans}, \mathtt{diss}.$$

Examples of valid sequents are (0 is the initial instant of time):

$$\dagger\, \mathtt{system}@0\ ;\ \underbrace{\rho_{r+t}\, \mathtt{on\,a}@0, \mathtt{off\,b}@0}_{\text{initial state}} \Longrightarrow \underbrace{\rho_{r+t+d}\, \mathtt{off\,a} \otimes \top@0}_{\text{final state}}$$

From $\mathtt{off\,b}$ we get $\mathtt{on\,b} \otimes \mathtt{prot\,b}$ after interval $r+t$; then $\mathtt{prot\,b}$ together with $\mathtt{on\,a}$ forms $\mathtt{prot\,b} \otimes \mathtt{off\,a}$ after a further delay d.

Example: stochastic repressilator We now revisit our example but this time using rates. Note that the encodings can be very similar in the temporal and stochastic fragments of our logic; the only differences being the interpretation of the constraints: Here, d, t, r and s are interpreted as rates.

$$\begin{aligned}
\mathtt{repress}\, a\, b &\stackrel{\text{def}}{=} \mathtt{prot}\, a \otimes \mathtt{on}\, b \multimap \rho_d(\mathtt{off}\, b \otimes \mathtt{prot}\, a)\\
\mathtt{trans} &\stackrel{\text{def}}{=} \forall a.\, \mathtt{on}\, a \multimap \rho_t(\mathtt{on}\, a \otimes \mathtt{prot}\, a).\\
\mathtt{react} &\stackrel{\text{def}}{=} \forall a.\, \mathtt{off}\, a \multimap \rho_r\, \mathtt{on}\, a.\\
\mathtt{diss} &\stackrel{\text{def}}{=} \forall a.\, \mathtt{prot}\, a \multimap \rho_s\, \mathbf{1}.
\end{aligned}$$

Suppose we want to show that in the two-gene repressilator, the state $\mathtt{on}(a) \otimes \mathtt{off}(b)$ can oscillate to $\mathtt{off}(a) \otimes \mathtt{on}(b)$. The proof looks as below, with one sub-proofs named P, and most of the worlds and a second sub-proof elided:

$$\dfrac{\dfrac{}{\mathtt{off\,b} \Longrightarrow \mathtt{off\,b}} \quad \dfrac{\dfrac{}{\mathtt{on\,b} \Longrightarrow \mathtt{on\,b}} \quad \dfrac{\dfrac{}{\mathtt{on\,a}, \rho_r\, \rho_t\, \mathtt{prot\,b} \Longrightarrow \rho_r\, \rho_t\, \rho_d\, \mathtt{off\,a}}\, P \quad \cdots}{\mathtt{on\,a}, \rho_r\, \rho_t(\mathtt{on\,b} \otimes \mathtt{prot\,b}) \Longrightarrow \exists k.\ \rho_k(\mathtt{off\,a} \otimes \mathtt{on\,b})}\, \mathtt{trans}}{\mathtt{on\,a}, \rho_r\, \mathtt{on\,b} \Longrightarrow \exists k.\ \rho_k(\mathtt{off\,a} \otimes \mathtt{on\,b})}\, \mathtt{react}}{\mathtt{on\,a}, \mathtt{off\,b} \Longrightarrow \exists k.\ \rho_k(\mathtt{off\,a} \otimes \mathtt{on\,b})}$$

$$P = \dfrac{\dfrac{\mathtt{on\,a} \Longrightarrow \rho_r\, \rho_t\, \mathtt{on\,a} \quad \rho_r\, \rho_t\, \mathtt{prot\,b} \Longrightarrow \rho_r\, \rho_t\, \mathtt{prot\,b}}{\mathtt{on\,a}, \rho_r\, \rho_t\, \mathtt{prot\,b} \Longrightarrow \rho_r\, \rho_t(\mathtt{on\,a} \otimes \mathtt{prot\,b})}\, \otimes I \quad \dfrac{}{\rho_r\, \rho_t\, \rho_d\, \mathtt{off\,a} \Longrightarrow \rho_r\, \rho_t\, \rho_d\, \mathtt{off\,a}}}{\mathtt{on\,a}, \rho_r\, \rho_t\, \mathtt{prot\,b} \Longrightarrow \rho_r\, \rho_t\, \rho_d\, \mathtt{off\,a}}\, \mathtt{repress\,b\,a}$$

In this proof we are using the transition rules at many different worlds. This is allowed because the rules are prefixed with † and therefore available at all worlds. Importantly, in the first premise of P we need to show that $\mathtt{on\,a} \Longrightarrow \rho_r\, \rho_t\, \mathtt{on\,a}$. This is only possible if the rate of a self-transition on $\mathtt{on\,a}$ is $r \cdot t$. Of course, this is not derivable from the rest of the theory (and may not actually be true), so it must be added as a new rule; it is the contract that must be satisfied by the repressilator in order for it to oscillate in the desired fashion.

All existing methods for modelling biology have algebraic foundations and none treats logic as the primary inferential device. In this section, we have sketched a mode of use of HyLL that lets one represent the biological elements directly in the logic. Note, however, that unlike formalisms such as the brane or κ-calculi, we do not propose HyLL as a new idealisation of biology. Instead, as far as systems biology is concerned, our proposal should be seen as a uniform language to encode biological systems; providing genuine means to reason about them is left for future work.

6 Related work

Logically, the HyLL sequent calculus is a variant of labelled deduction, a very broad topic not elaborated on here. The combination of linear logic with labelled deduction isn't new to this work. In the η-logic [17] the constraint domain is intervals of time, and the rules of the logic generate constraint inequalities as a side-effect; however its sole aim is the representation of proof-carrying authentication, and it does not deal with genericity or focusing. The main feature of η not in HyLL is a separate constraint context that gives new constrained propositions. HyLL is also related to the Hybrid Logical Framework (HLF) [36] which captures linear logic itself as a labelled form of intuitionistic logic. Encoding constrained π calculi directly in HLF would be an interesting exercise: we would combine the encoding of linear logic with the constraints of the process calculus. Because HLF is a very weak logic with a proof theory based on natural deduction, it is not clear whether (and in what forms) an adequacy result in HyLL can be transferred to HLF.

Constrained temporal logics such as CSL and PCTL [22] are popular for logical reasoning in constrained domains. In such logics, truth is defined in terms of correctness with respect to a constrained forcing relation on the constraint algebra. While such logics have been very successful in practice with efficient tools, the proof theory of these logics is very complex. Indeed, such modal logics generally cannot be formulated in the sequent calculus, and therefore lack cut-elimination and focusing. In contrast, HyLL has a very traditional proof theoretic pedigree, but lacks such a close correspondence between logical and algebraic equivalence. Probably the most well known and relevant stochastic formalism not already discussed is that of stochastic Petri-nets [26], which have a number of sophisticated model checking tools, including the PRISM framework [25]. Recent advances in proof theory suggest that the benefits of model checking can be obtained without sacrificing proofs and proof search [4].

7 Conclusion and future work

We have presented HyLL, a hybrid extension of intuitionistic linear logic with a simple notion of situated truth, a traditional sequent calculus with cut-elimination and focusing, and a modular and instantiable constraint system that can be directly manipulated using hybrid connectives. We have shown how to obtain representationally adequate encodings of constrained transition systems, such as the synchronous stochastic π-calculus in a suitable instance of HyLL. We have also given some simple examples of direct encoding of biological systems, viewed as transition systems, in HyLL, using either temporal or stochastic constraints.

Several instantiations of HyLL besides the one in this paper seem interesting. For example, we can already use disjunction ($\oplus$) to explain disjunctive states, but it is also possible to obtain a more extensional branching by treating the worlds as points in an arbitrary partially-ordered set instead of a monoid. Another possibility is to consider lists of worlds instead of individual worlds – this would allow defining periodic availability of a resource, such as one being produced by an oscillating process. The most interesting domain is that of discrete probabilities: here the underlying semantics is given by discrete time Markov chains instead of CTMCs, which are often better suited for symbolic evaluation [43].

An important open question is whether a general logic such as HyLL can serve as a framework for specialized logics such as CSL and PCTL. A related question is what benefit linearity truly provides for such logics – linearity is obviously crucial for encoding process calculi that are inherently stateful, but CSL requires no such notion of single consumption of resources.

In the κ-calculus, reactions in a biological system are modeled as reductions on graphs with certain state annotations. It appears (though this has not been formalized) that the κ-calculus can be embedded in HyLL even more naturally than $S\pi$, because a solution—a multiset of chemical products—is simply a tensor of all the internal states of the binding sites together with the formed bonds. One important innovation of κ is the ability to extract semantically meaningful "stories" from simulations. We believe that HyLL provides a natural formal language for such stories.

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A Proofs

A.1 Identity principle

Theorem 26 (Identity principle)**.** *The following rule is derivable.*

$$\frac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\ \text{init*}$$

Proof. By induction on the structure of A. We have the following cases.
case A is an atom $p\,\vec{t}$. Then, $\Gamma\,;\,p\,\vec{t}@w \Longrightarrow p\,\vec{t}@w$ by init.
case A is $B \,\&\, C$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{i.h.}}{\Gamma\,;\,B \,\&\, C@w \Longrightarrow B@w}\ \&L_1 \qquad \dfrac{\dfrac{}{\Gamma\,;\,C@w \Longrightarrow C@w}\ \text{i.h.}}{\Gamma\,;\,B \,\&\, C@w \Longrightarrow C@w}\ \&L_2}{\Gamma\,;\,B \,\&\, C@w \Longrightarrow B \,\&\, C@w}\ \&R$$

case A is $\top$.

$$\frac{}{\Gamma\,;\,\top@w \Longrightarrow \top@w}\ \top R$$

case A is $B \oplus C$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{i.h.}}{\Gamma\,;\,B@w \Longrightarrow B \oplus C@w}\ \oplus R_1 \qquad \dfrac{\dfrac{}{\Gamma\,;\,C@w \Longrightarrow C@w}\ \text{i.h.}}{\Gamma\,;\,C@w \Longrightarrow B \oplus C@w}\ \oplus R_2}{\Gamma\,;\,B \oplus C@w \Longrightarrow B \oplus C@w}\ \oplus L$$

case A is $\mathbf{0}$.

$$\frac{}{\Gamma\,;\,\mathbf{0}@w \Longrightarrow \mathbf{0}@w}\ \mathbf{0}L$$

case A is $B \multimap C$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{i.h.} \quad \dfrac{}{\Gamma\,;\,C@w \Longrightarrow C@w}\ \text{i.h.}}{\Gamma\,;\,B \multimap C@w, B@w \Longrightarrow C@w}\ \multimap L}{\Gamma\,;\,B \multimap C@w \Longrightarrow B \multimap C@w}\ \multimap R$$

case A is $B \otimes C$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{i.h.} \quad \dfrac{}{\Gamma\,;\,C@w \Longrightarrow C@w}\ \text{i.h.}}{\Gamma\,;\,B@w, C@w \Longrightarrow B \otimes C@w}\ \otimes R}{\Gamma\,;\,B \otimes C@w \Longrightarrow B \otimes C@w}\ \otimes L$$

case A is $\mathbf{1}$.

$$\dfrac{\dfrac{}{\Gamma\,;\,\cdot \Longrightarrow \mathbf{1}@w}\ \mathbf{1}R}{\Gamma\,;\,\mathbf{1}@w \Longrightarrow \mathbf{1}@w}\ \mathbf{1}L$$

case A is $\forall x.\, B$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{i.h.}}{\Gamma\,;\,\forall\alpha.\, B@w \Longrightarrow B@w}\ \forall L}{\Gamma\,;\,\forall\alpha.\, B@w \Longrightarrow \forall\alpha.\, B@w}\ \forall R^{\alpha}$$

case A is $\exists x.\, B$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, B@w \Longrightarrow B@w}\ \text{i.h.}}{\Gamma\,;\, B@w \Longrightarrow \exists\alpha.\, B@w}\ \exists R}{\Gamma\,;\, \exists\alpha.\, B@w \Longrightarrow \exists\alpha.\, B@w}\ \exists L^{\alpha}$$

case A is $!\,B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, B@w\,;\, B@w \Longrightarrow B@w}\ \text{i.h.}}{\Gamma, B@w\,;\, \cdot \Longrightarrow B@w}\ \text{copy}}{\Gamma, B@w\,;\, \cdot \Longrightarrow\, !\,B@w}\ !R}{\Gamma\,;\, !\,B@w \Longrightarrow\, !\,B@w}\ !L$$

case A is $\downarrow u.\, B$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, [w/u]B@w \Longrightarrow [w/u]B@w}\ \text{i.h.}}{\Gamma\,;\, \downarrow u.\, B@w \Longrightarrow [w/u]B@w}\ \downarrow L}{\Gamma\,;\, \downarrow u.\, B@w \Longrightarrow \downarrow u.\, B@w}\ \downarrow R$$

case A is $(B\ \mathtt{at}\ v)$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, B@v \Longrightarrow B@v}\ \text{i.h.}}{\Gamma\,;\, (B\ \mathtt{at}\ v)@w \Longrightarrow B@v}\ \mathtt{at}L}{\Gamma\,;\, (B\ \mathtt{at}\ v)@w \Longrightarrow (B\ \mathtt{at}\ v)@w}\ \mathtt{at}R$$

□

A.2 Cut admissibility

Theorem 27 (Cut admissibility)**.** *The following two rules are admissible.*

$$\dfrac{\Gamma\,;\, \Delta \Longrightarrow A@w \quad \Gamma\,;\, \Delta', A@w \Longrightarrow C@w'}{\Gamma\,;\, \Delta, \Delta' \Longrightarrow C@w'}\ \text{cut}$$

$$\dfrac{\Gamma\,;\, \cdot \Longrightarrow A@w \quad \Gamma, A@w\,;\, \Delta \Longrightarrow C@w'}{\Gamma\,;\, \Delta \Longrightarrow C@w'}\ \text{cut!}$$

Proof. Name the two premise derivations $\mathcal{D}$ and $\mathcal{E}$ respectively. The proof proceeds by induction on the structure of the derivations $\mathcal{D}$ and $\mathcal{E}$, and more precisely on a lexicographic order that allows the induction hypothesis to be used whenever:

1. The cut formula becomes strictly smaller (in the subformula relation), or
2. The cut formula remains the same, but an instance of cut is used to justify an instance of cut!.
3. The cut formula remains the same, but the derivation $\mathcal{D}$ is strictly smaller, or
4. The cut formula remains the same, but the derivation $\mathcal{E}$ is strictly smaller, or

In each case, we consider derivations to be identical that differ in such a way that one can be derived from the other simply by weakening and contracting the unrestricted contexts of their respective sequents. The lexicographic order is well-founded because the given derivations $\mathcal{D}$ and $\mathcal{E}$ are finite, and cut! is used at most once per subformula of A (see "copy cuts" below). All the cuts break down into the following four major categories.

Atomic cuts where the formula A is an atom $p\,(\vec{t})$. We have the following two cases;

Case. $\mathcal{D}$ is:

$$\dfrac{}{\Gamma\,;\, p\,(\vec{t})@w \Longrightarrow p\,(\vec{t})@w}\ \text{init}$$

Then the result of the cut has the same conclusion as that of $\mathcal{E}$.

Case. $\mathcal{E}$ is

$$\frac{}{\Gamma\,;\,p\,(\vec{t})@w \Longrightarrow p\,(\vec{t})@w}\ \text{init}$$

Then the result of the cut has the same conclusion as that of $\mathcal{D}$.

Principal cuts where a non-atomic cut formula A is introduced by a final right rule in $\mathcal{D}$ and a final left-rule in $\mathcal{E}$. We have the following cases.

Case. A is $A_1 \,\&\, A_2$, and:

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Gamma\,;\,\Delta \Longrightarrow A_1@w \quad \mathcal{D}_2 :: \Gamma\,;\,\Delta \Longrightarrow A_2@w}{\Gamma\,;\,\Delta \Longrightarrow A_1 \,\&\, A_2@w}\ \&R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta', A_i@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', A_1 \,\&\, A_2@w \Longrightarrow C@w'}\ \&L_i$$

Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}_i$ and $\mathcal{E}'$.

Case. A is $A_1 \oplus A_2$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta \Longrightarrow A_i@w}{\Gamma\,;\,\Delta \Longrightarrow A_1 \oplus A_2@w}\ \oplus R_i \qquad \mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1 :: \Gamma\,;\,\Delta', A_1@w \Longrightarrow C@w' \\ \mathcal{E}_2 :: \Gamma\,;\,\Delta', A_2@w \Longrightarrow C@w'\end{array}}{\Gamma\,;\,\Delta', A_1 \oplus A_2@w \Longrightarrow C@w'}\ \oplus L$$

Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}_i$.

Case. A is $A_1 \multimap A_2$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta, A_1@w \Longrightarrow A_2@w}{\Gamma\,;\,\Delta \Longrightarrow A_1 \multimap A_2@w}\ \multimap R \qquad \mathcal{E} = \frac{\mathcal{E}_1 :: \Gamma\,;\,\Delta'_1 \Longrightarrow A_1@w \quad \mathcal{E}_2 :: \Gamma\,;\,\Delta'_2, A_2@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta'_1, \Delta'_2, A_1 \multimap A_2 \Longrightarrow C@w'}\ \multimap L$$

Then:

$\Gamma\,;\,\Delta, A_1@w, \Delta'_2 \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}_2$.
$\Gamma\,;\,\Delta, \Delta'_1, \Delta'_2 \Longrightarrow C@w'$ cut on $\mathcal{E}_1$ and above.

Case. A is $A_1 \otimes A_2$, and:

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Gamma\,;\,\Delta_1 \Longrightarrow A_1@w \quad \mathcal{D}_2 :: \Gamma\,;\,\Delta_2 \Longrightarrow A_2@w}{\Gamma\,;\,\Delta_1, \Delta_2 \Longrightarrow A_1 \otimes A_2@w}\ \otimes R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta', A_1@w, A_2@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', A_1 \otimes A_2@w \Longrightarrow C@w'}\ \otimes L$$

Then:

$\Gamma\,;\,\Delta', \Delta_2, A_1@w \Longrightarrow C@w'$ cut on $\mathcal{D}_2$ and $\mathcal{E}'$.
$\Gamma\,;\,\Delta', \Delta_1, \Delta_2 \Longrightarrow C@w'$ cut on $\mathcal{D}_1$ and above.

Case. A is $\mathbf{1}$, and:

$$\mathcal{D} = \frac{}{\Gamma\,;\,\cdot \Longrightarrow \mathbf{1}@w}\ \mathbf{1}R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta' \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \mathbf{1}@w \Longrightarrow C@w'}\ \mathbf{1}L$$

The result of the cut is the conclusion of $\mathcal{E}'$.

Case. A is $\forall x.\, B$, and:

$$\mathcal{D} = \frac{\mathcal{D}'(\alpha) :: \Gamma\,;\,\Delta \Longrightarrow B@w}{\Gamma\,;\,\Delta \Longrightarrow \forall\alpha.\, B@w}\ \forall R^{\alpha} \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta', [\tau/\alpha]B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \forall\alpha.\, B@w \Longrightarrow C@w'}\ \forall L$$

Let a be any parameter. Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'(\tau)$ and $\mathcal{E}'$.

Case. A is $\exists x.\, B$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta \Longrightarrow [\tau/\alpha]B@w}{\Gamma\,;\,\Delta \Longrightarrow \exists\alpha.\, B@w}\ \exists R \qquad \mathcal{E} = \frac{\mathcal{E}'(\alpha) :: \Gamma\,;\,\Delta', B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \exists\alpha.\, B@w \Longrightarrow C@w'}\ \exists L^{\alpha}$$

Let a be any parameter. Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}'(\alpha)$.

Case. A is $!\,B$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\cdot \Longrightarrow B@w}{\Gamma\,;\,\cdot \Longrightarrow !\,B@w}\ !R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma, B@w\,;\,\Delta' \Longrightarrow C@w'}{\Gamma\,;\,\Delta', !\,B@w \Longrightarrow C@w'}\ !L$$

Then:

$\Gamma\,;\,\Delta' \Longrightarrow C@w'$ cut! on $\mathcal{D}'$ and $\mathcal{E}'$.

Case. A is $\downarrow u.\, B$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta \Longrightarrow [w/u]B@w}{\Gamma\,;\,\Delta \Longrightarrow \downarrow u.\, B@w}\ \downarrow R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta', [w/u]B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \downarrow u.\, B@w \Longrightarrow C@w'}\ \downarrow L$$

Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}'$.

Case. A is $(B\ \mathtt{at}\ v)$, and:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta \Longrightarrow B@v}{\Gamma\,;\,\Delta \Longrightarrow (B\ \mathtt{at}\ v)@w}\ \mathtt{at}R \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\,\Delta', B@v \Longrightarrow C@w'}{\Gamma\,;\,\Delta', (B\ \mathtt{at}\ v)@w \Longrightarrow C@w'}\ \mathtt{at}L$$

Then:

$\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}'$.

Copy cuts where the cut formula in $\mathcal{E}$ was transferred using copy, i.e.:

$$\mathcal{D} :: \Gamma\,;\,\cdot \Longrightarrow A@w \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma, A@w\,;\,\Delta', A@w \Longrightarrow C@w'}{\Gamma, A@w\,;\,\Delta' \Longrightarrow C@w'}\ \text{copy}$$

Here,

$\Gamma, A@w\,;\,\cdot \Longrightarrow A@w$ weakening on $\mathcal{D}$.
$\Gamma, A@w\,;\,\Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}$ and $\mathcal{E}'$.
$\Gamma\,;\,\Delta' \Longrightarrow C@w'$ cut! on $\mathcal{D}$ and above.

The first cut is applied on a variant of $\mathcal{D}$ that differs from $\mathcal{D}$ only in terms of a weaker unrestricted context. In the last step, a cut was used to justify a cut!, which is allowed by the lexicographic order.

Left-commutative cuts where the cut formula A is a side formula in the derivation $\mathcal{D}$. The following is a representative case.

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\,\Delta, D@w'', E@w'' \Longrightarrow A@w}{\Gamma\,;\,\Delta, D \otimes E@w'' \Longrightarrow A@w}\ \otimes L \qquad \mathcal{E} :: \Gamma\,;\,\Delta', A@w \Longrightarrow C@w'.$$

Here,

$\Gamma\,;\,\Delta, D@w'', E@w'', \Delta' \Longrightarrow C@w'$ cut on $\mathcal{D}'$ and $\mathcal{E}$.
$\Gamma\,;\,\Delta, \Delta', D \otimes E@w'' \Longrightarrow C@w'$ $\otimes L$.

Right-commutative cuts where the cut formula A is a side formula in the derivation $\mathcal{E}$. The following is a representative case.

$$\mathcal{D} :: \Gamma\,;\,\Delta \Longrightarrow A@w \qquad \mathcal{E} = \frac{\mathcal{E}_1 :: \Gamma\,;\,\Delta', A@w \Longrightarrow D@w' \quad \mathcal{E}_2 :: \Gamma\,;\,\Delta', A@w \Longrightarrow E@w'}{\Gamma\,;\,\Delta', A@w \Longrightarrow D \,\&\, E@w'}\ \&R$$

Here,

$$\begin{array}{ll}
\Gamma\,;\,\Delta, \Delta' \Longrightarrow D@w' & \text{cut on } \mathcal{D} \text{ and } \mathcal{E}_1. \\
\Gamma\,;\,\Delta, \Delta' \Longrightarrow E@w' & \text{cut on } \mathcal{D} \text{ and } \mathcal{E}_2. \\
\Gamma\,;\,\Delta, \Delta' \Longrightarrow D \,\&\, E@w' & \&R.
\end{array}$$

This completes the inventory of all possible cuts. □

A.3 Invertibility

Theorem 28 (Invertibility)**.** *The following rules are invertible:*

1. *On the right:* $\&R$, $\top R$, $\multimap R$, $\forall R$, $\downarrow R$ *and* $@R$;
2. *On the left:* $\otimes L$, $\mathbf{1}L$, $\oplus L$, $\mathbf{0}L$, $\exists L$, $!L$, $\downarrow L$ *and* $\mathtt{at}L$.

Proof. Each inversion is shown to be admissible using a suitable cut.
Case of $\&R$:

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow A_1 \,\&\, A_2@w \quad \dfrac{\dfrac{}{\Gamma\,;\,A_i@w \Longrightarrow A_i@w}\ \text{init*}}{\Gamma\,;\,A_1 \,\&\, A_2@w \Longrightarrow A_i@w}\ \&L_i}{\Gamma\,;\,\Delta \Longrightarrow A_i@w}\ \text{cut}$$

Case of $\top R$: trivial.
Case of $\multimap R$:

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow A \multimap B@w \quad \dfrac{\dfrac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\ \text{init*} \quad \dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\ \text{init*}}{\Gamma\,;\,A \multimap B@w, A@w \Longrightarrow B@w}\ \multimap L}{\Gamma\,;\,\Delta, A@w \Longrightarrow B@w}\ \text{cut}$$

Case of $\forall R$:

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \forall\alpha.\, A@w \quad \dfrac{\dfrac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\ \text{init*}}{\Gamma\,;\,\forall\alpha.\, A@w \Longrightarrow A@w}\ \forall L}{\Gamma\,;\,\Delta \Longrightarrow A@w}\ \text{cut}$$

Case of $\downarrow R$:

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \downarrow u.\, A@w \quad \dfrac{\dfrac{}{\Gamma\,;\,[w/u]A@w \Longrightarrow [w/u]A@w}\ \text{init*}}{\Gamma\,;\,\downarrow u.\, A@w \Longrightarrow [w/u]A@w}\ \downarrow L}{\Gamma\,;\,\Delta \Longrightarrow [w/u]A@w}\ \text{cut}$$

Case of $\mathtt{at}R$:

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow (A\ \mathtt{at}\ v)@w \quad \dfrac{\dfrac{}{\Gamma\,;\,A@v \Longrightarrow A@v}\ \text{init*}}{\Gamma\,;\,(A\ \mathtt{at}\ v)@w \Longrightarrow A@v}\ \mathtt{at}L}{\Gamma\,;\,\Delta \Longrightarrow A@v}\ \text{cut}$$

Case of $\otimes L$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\,\text{init*} \quad \dfrac{}{\Gamma\,;\,B@w \Longrightarrow B@w}\,\text{init*}}{\Gamma\,;\,A@w, B@w \Longrightarrow A \otimes B@w}\,\otimes R \quad \Gamma\,;\,\Delta, A \otimes B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta, A@w, B@w \Longrightarrow C@w'}\,\text{cut}$$

Case of $\mathbf{1}L$:

$$\dfrac{\dfrac{}{\Gamma\,;\,\cdot \Longrightarrow \mathbf{1}@w}\,\mathbf{1}R \quad \Gamma\,;\,\Delta, \mathbf{1}@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta \Longrightarrow C@w'}\,\text{cut}$$

Case of $\oplus L$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,A_i@w \Longrightarrow A_i@w}\,\text{init*}}{\Gamma\,;\,A_i@w \Longrightarrow A_1 \oplus A_2@w}\,\oplus R_i \quad \Gamma\,;\,\Delta, A_1 \oplus A_2@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta, A_i@w \Longrightarrow C@w'}\,\text{cut}$$

Case of $\mathbf{0}L$: trivial.
Case of $\exists L$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\,\text{init*}}{\Gamma\,;\,A@w \Longrightarrow \exists\alpha.\,A@w}\,\exists R \quad \Gamma\,;\,\Delta, \exists x.\,A@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta, A@w \Longrightarrow C@w'}\,\text{cut}$$

Case of $!L$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A@w\,;\,A@w \Longrightarrow A@w}\,\text{init*}}{\Gamma, A@w\,;\,\cdot \Longrightarrow A@w}\,\text{copy}}{\Gamma, A@w\,;\,\cdot \Longrightarrow\, !A@w}\,!R \quad \dfrac{\Gamma\,;\,\Delta, !A@w \Longrightarrow C@w'}{\Gamma, A@w\,;\,\Delta, !A@w \Longrightarrow C@w'}\,\text{weaken}}{\Gamma, A@w\,;\,\Delta \Longrightarrow C@w'}\,\text{cut}$$

Case of $\downarrow L$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,[w/u]A@w \Longrightarrow [w/u]A@w}\,\text{init*}}{\Gamma\,;\,[w/u]A@w \Longrightarrow \downarrow u.\,A@w}\,\downarrow R \quad \Gamma\,;\,\Delta, \downarrow u.\,A@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta, [w/u]A@w \Longrightarrow C@w'}\,\text{cut}$$

Case of $\mathtt{at}L$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,A@v \Longrightarrow A@v}\,\text{init*}}{\Gamma\,;\,A@v \Longrightarrow (A\,\mathtt{at}\,v)@w}\,\mathtt{at}R \quad \Gamma\,;\,\Delta, (A\,\mathtt{at}\,v)@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta, A@v \Longrightarrow C@w'}\,\text{cut}$$

□

A.4 Correctness and consistency

Theorem 29 (Correctness of the sequent calculus)**.**

1. *If* $\Gamma\,;\,\Delta \Longrightarrow C@w$*, then* $\Gamma\,;\,\Delta \vdash C@w$*. (soundness)*
2. *If* $\Gamma\,;\,\Delta \vdash C@w$*, then* $\Gamma\,;\,\Delta \Longrightarrow C@w$*. (completeness)*

Proof. The right rules of the sequent calculus and the introduction rules of natural deduction coincide. Therefore, for (1), we need only to show that the judgemental and left rules of the sequent calculus are admissible in natural deduction, and for (2), only to show that the judgemental and elimination rules of natural deduction are admissible in the sequent calculus. The following are the main cases.

$\Longrightarrow/\vdash$ case. (init)

$$\dfrac{}{\Gamma\,;\, p\,(\vec{t})@w \vdash p\,(\vec{t})@w}\ \text{hyp}$$

$\Longrightarrow/\vdash$ case. (copy)

$$\dfrac{\dfrac{}{\Gamma, A@w\,;\, \cdot \vdash A@w}\ \text{hyp!} \qquad \Gamma, A@w\,;\, \Delta, A@w \vdash C@w'}{\Gamma, A@w\,;\, \Delta \vdash C@w'}\ \text{subst}$$

$\Longrightarrow/\vdash$ case. ($\&L_i$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, A_1 \,\&\, A_2@w \vdash A_1 \,\&\, A_2@w}\ \text{hyp}}{\Gamma\,;\, A_1 \,\&\, A_2@w \vdash A_i@w}\ \&E_i \qquad \Gamma\,;\, \Delta, A_i@w \vdash C@w'}{\Gamma\,;\, \Delta, A_1 \,\&\, A_2@w \vdash C@w'}\ \text{subst}$$

$\Longrightarrow/\vdash$ case. ($\oplus L$)

$$\dfrac{\dfrac{}{\Gamma\,;\, A_1 \oplus A_2@w \vdash A_1 \oplus A_2@w}\ \text{hyp} \qquad \Gamma\,;\, \Delta, A_1@w \vdash C@w' \qquad \Gamma\,;\, \Delta, A_2@w \vdash C@w'}{\Gamma\,;\, \Delta, A_1 \oplus A_2@w \vdash C@w'}\ \oplus E$$

$\Longrightarrow/\vdash$ case. ($\mathbf{0}L$)

$$\dfrac{\dfrac{}{\Gamma\,;\, \mathbf{0}@w \vdash \mathbf{0}@w}\ \text{hyp}}{\Gamma\,;\, \Delta, \mathbf{0}@w \vdash C@w'}\ \mathbf{0}E$$

$\Longrightarrow/\vdash$ case. ($\otimes L$)

$$\dfrac{\dfrac{}{\Gamma\,;\, A \otimes B@w \vdash A \otimes B@w}\ \text{hyp} \qquad \Gamma\,;\, \Delta, A@w, B@w \vdash C@w'}{\Gamma\,;\, \Delta, A \otimes B@w \vdash C@w'}\ \otimes E$$

$\Longrightarrow/\vdash$ case. ($\mathbf{1}L$)

$$\dfrac{\dfrac{}{\Gamma\,;\, \mathbf{1}@w \vdash \mathbf{1}@w}\ \text{hyp} \qquad \Gamma\,;\, \Delta \vdash C@w'}{\Gamma\,;\, \Delta, \mathbf{1}@w \vdash C@w'}\ \mathbf{1}E$$

$\Longrightarrow/\vdash$ case. ($\multimap L$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, A \multimap B@w \vdash A \multimap B@w}\ \text{hyp} \qquad \Gamma\,;\, \Delta \vdash A@w}{\Gamma\,;\, A \multimap B@w \vdash B@w}\ \multimap E \qquad \Gamma\,;\, \Delta', B@w \vdash C@w'}{\Gamma\,;\, \Delta, \Delta', A \multimap B@w \vdash C@w'}\ \text{subst}$$

$\Longrightarrow/\vdash$ case. ($\forall L$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\, \forall\alpha.\, A@w \vdash \forall\alpha.\, A@w}\ \text{hyp}}{\Gamma\,;\, \forall\alpha.\, A@w \vdash [\tau/\alpha]A@w}\ \forall E \qquad \Gamma\,;\, \Delta, [\tau/\alpha]A@w \vdash C@w'}{\Gamma\,;\, \Delta, \forall\alpha.\, A@w \vdash C@w'}\ \text{subst}$$

$\Longrightarrow/\vdash$ case. ($\exists L$)

$$\dfrac{\dfrac{}{\Gamma\,;\,\exists\alpha.\,A@w \vdash \exists\alpha.\,A@w}\,\text{hyp} \qquad \Gamma\,;\,\Delta, A@w \vdash C@w'}{\Gamma\,;\,\Delta, \exists\alpha.\,A@w \vdash C@w'}\,\exists E^{\alpha}$$

$\Longrightarrow/\vdash$ case. ($!L$)

$$\dfrac{\dfrac{}{\Gamma\,;\,!\,A@w \vdash !\,A@w}\,\text{hyp} \qquad \Gamma, !\,A@w\,;\,\Delta \vdash C@w'}{\Gamma\,;\,\Delta, !\,A@w \vdash C@w'}\,!E$$

$\Longrightarrow/\vdash$ case. ($\downarrow L$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,\downarrow u.\,A@w \vdash \downarrow u.\,A@w}\,\text{hyp}}{\Gamma\,;\,\downarrow u.\,A@w \vdash [w/u]A@w}\,\downarrow E \qquad \Gamma\,;\,\Delta, [w/u]A@w \vdash C@w'}{\Gamma\,;\,\Delta, \downarrow u.\,A@w \vdash C@w'}\,\text{subst}$$

$\Longrightarrow/\vdash$ case. ($\mathtt{at}\ L$)

$$\dfrac{\dfrac{\dfrac{}{\Gamma\,;\,(A\ \mathtt{at}\ v)@w \vdash (A\ \mathtt{at}\ v)@w}\,\text{hyp}}{\Gamma\,;\,(A\ \mathtt{at}\ v)@w \vdash A@v}\,\mathtt{at}\ E \qquad \Gamma\,;\,\Delta, A@v \vdash C@w'}{\Gamma\,;\,\Delta, (A\ \mathtt{at}\ v)@w \vdash C@w'}\,\text{subst}$$

$\vdash/\Longrightarrow$ case. (hyp)

$$\dfrac{}{\Gamma\,;\,A@w \Longrightarrow A@w}\,\text{init*}$$

$\vdash/\Longrightarrow$ case. (hyp!)

$$\dfrac{\dfrac{}{\Gamma, A@w\,;\,A@w \Longrightarrow A@w}\,\text{init*}}{\Gamma, A@w\,;\,\cdot \Longrightarrow A@w}\,\text{copy}$$

$\vdash/\Longrightarrow$ case. ($\&E_i$)

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow A_1 \,\&\, A_2@w \qquad \dfrac{\dfrac{}{\Gamma\,;\,A_i@w \Longrightarrow A_i@w}\,\text{init*}}{\Gamma\,;\,A_1 \,\&\, A_2@w \Longrightarrow A_i@w}\,\&L_i}{\Gamma\,;\,\Delta \Longrightarrow A_i@w}\,\text{cut}$$

$\vdash/\Longrightarrow$ case. ($\oplus E$)

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow A \oplus B@w \qquad \dfrac{\Gamma\,;\,\Delta', A@w \Longrightarrow C@w' \quad \Gamma\,;\,\Delta', B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', A \oplus B@w \Longrightarrow C@w'}\,\oplus L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\,\text{cut}$$

$\vdash/\Longrightarrow$ case. ($\mathbf{0}E$)

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \mathbf{0}@w \qquad \dfrac{}{\Gamma\,;\,\Delta', \mathbf{0}@w \Longrightarrow C@w'}\,\mathbf{0}L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\,\text{cut}$$

$\vdash/\Longrightarrow$ case. ($\otimes E$)

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow A \otimes B@w \qquad \dfrac{\Gamma\,;\,\Delta', A@w, B@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', A \otimes B@w \Longrightarrow C@w'}\,\otimes L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\,\text{cut}$$

$\vdash/\Longrightarrow$ case. $(\mathbf{1}E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \mathbf{1}@w \quad \dfrac{\Gamma\,;\,\Delta' \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \mathbf{1}@w \Longrightarrow C@w'}\;\mathbf{1}L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\;\text{cut}$$

$\vdash/\Longrightarrow$ case. $(\forall E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \forall\alpha.\,A@w \quad \dfrac{\dfrac{}{\Gamma\,;\,[\tau/\alpha]A@w \Longrightarrow [\tau/\alpha]A@w}\;\text{init*}}{\Gamma\,;\,\forall\alpha.\,A@w \Longrightarrow [\tau/\alpha]A@w}\;\forall L}{\Gamma\,;\,\Delta \Longrightarrow [\tau/\alpha]A@w}\;\text{cut}$$

$\vdash/\Longrightarrow$ case. $(\exists E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \exists\alpha.\,A@w \quad \dfrac{\Gamma\,;\,\Delta', A@w \Longrightarrow C@w'}{\Gamma\,;\,\Delta', \exists\alpha.\,A@w \Longrightarrow C@w'}\;\exists L^{\alpha}}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\;\text{cut}$$

$\vdash/\Longrightarrow$ case. $(!E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow\, !A@w \quad \dfrac{\Gamma, A@w\,;\,\Delta' \Longrightarrow C@w'}{\Gamma\,;\,\Delta', !A@w \Longrightarrow C@w'}\;!L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow C@w'}\;\text{cut}$$

$\vdash/\Longrightarrow$ case. $(\downarrow E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow \downarrow u.\,A@w \quad \dfrac{\dfrac{}{\Gamma\,;\,\Delta', [w/u]A@w \Longrightarrow [w/u]A@w}\;\text{hyp}}{\Gamma\,;\,\Delta', \downarrow u.\,A@w \Longrightarrow [w/u]A@w}\;\downarrow L}{\Gamma\,;\,\Delta, \Delta' \Longrightarrow [w/u]A@w}\;\text{cut}$$

$\vdash/\Longrightarrow$ case. $(\mathtt{at}\ E)$

$$\dfrac{\Gamma\,;\,\Delta \Longrightarrow (A\ \mathtt{at}\ v)@w \quad \dfrac{\dfrac{}{\Gamma\,;\,A@v \Longrightarrow A@v}\;\text{init*}}{\Gamma\,;\,(A\ \mathtt{at}\ v)@w \Longrightarrow A@v}\;\mathtt{at}\ L}{\Gamma\,;\,\Delta \Longrightarrow A@v}\;\text{cut}$$

□

Corollary 30 (Consistency of HyLL)**.** *There is no proof of* $\cdot\,;\,\cdot \vdash \mathbf{0}@w$.

Proof. Suppose $\cdot\,;\,\cdot \vdash \mathbf{0}@w$ is derivable. Then, by the completeness and cut-admissibility theorems on the sequent calculus, $\cdot\,;\,\cdot \Longrightarrow \mathbf{0}@w$ must have a cut-free proof. But, we can see by simple inspection that there can be no cut-free proof of $\cdot\,;\,\cdot \Longrightarrow \mathbf{0}@w$, as this sequent cannot be the conclusion of any rule of inference in the sequent calculus. Therefore, $\cdot\,;\,\cdot \vdash \mathbf{0}@w$ is not derivable. □

A.5 Connection to IS5

Theorem 31 (HyLL is intuitionistic S5)**.** *The following sequent is derivable:* $\cdot\,;\ \Diamond A@w \Longrightarrow \Box\Diamond A@w$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot\,;\,A@a \Longrightarrow A@a}\;\text{init*}}{\cdot\,;\,A@a \Longrightarrow (A\ \mathtt{at}\ a)\ \mathtt{at}\ b}\;\mathtt{at}R}{\cdot\,;\,A@a \Longrightarrow \exists v.\,(A\ \mathtt{at}\ v)@b}\;\exists R}{\cdot\,;\,(A\ \mathtt{at}\ a)@w \Longrightarrow (\exists v.\,(A\ \mathtt{at}\ v)\ \mathtt{at}\ b)@w}\;\mathtt{at}L,\ \mathtt{at}R}{\cdot\,;\,(A\ \mathtt{at}\ a)@w \Longrightarrow \forall u.\,(\exists v.\,(A\ \mathtt{at}\ v)\ \mathtt{at}\ u)@w}\;\forall R^{b}}{\cdot\,;\,\exists u.\,(A\ \mathtt{at}\ u)@w \Longrightarrow \forall u.\,(\exists v.\,(A\ \mathtt{at}\ v)\ \mathtt{at}\ u)@w}\;\exists L^{a}}{\cdot\,;\ \Diamond A@w \Longrightarrow \Box\Diamond A@w}\;\text{defn}$$

□