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*A Γ -convergence approach for the detection of points
in 2-D images.*

Daniele Graziani, Laure BLANC-FÉRAUD, Gilles Aubert

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2

Thème COG



*Rapport
de recherche*

3 **A Γ -convergence approach for the detection of points in**
4 **2-D images.**

5 Daniele Graziani*, Laure BLANC-FÉRAUD*, Gilles Aubert†

6 Thème COG — Systèmes cognitifs
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9 **Abstract:** In this report we propose a new variational method to isolate points in a 2-
10 dimensional image. To this purpose we introduce a suitable functional whose minimizers
11 are given by the points we want to detect. Then in order to provide numerical experiments
12 we approximate this energy by means of a sequence of more treatable functionals by using
13 a Γ -convergence approach.

14 **Key-words:** points detection, curvature-depending functionals, divergence-measure fields,
15 Γ -convergence, biological 2-D images.

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17 2-D images.

18 **Résumé :**

19 **Mots-clés :**

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37 1 Introduction

38 Detecting fine structures, like points or curves in two or three dimensional images respec-
 39 tively, is an important issue in image analysis. In biological images a point may represent
 40 a viral particle whose visibility is compromised by the presence of other structures like cell
 41 membranes or some noise.

42 From a variational point of view the problem of isolating points is a difficult task, since it
 43 is not clear how these singularities must be classified in term of a some differential operator.
 44 Indeed, whereas these are usually defined as discontinuity without jump, we cannot use the
 45 gradient operator as in the classical problem of detecting contours. As a consequence the
 46 functional framework we have to deal with may be not clear.

47 One possible strategy to overcome this obstacle is considering these kind of pathology
 48 as a k -codimension object, meaning that they should be regarded as a singularity of a map
 49 $U : \mathbb{R}^{k+m} \rightarrow \mathbb{R}^k$, with $k \geq 2$ and $m \geq 0$ (see [7] for a complete survey on this subject). So
 50 that the detecting point case corresponds to the case $k = 2$ and $m = 0$.

51 In this direction, in [5], the authors have suggested a variational approach based on the
 52 theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are
 53 regarded as the topological singularities of a map $U : \mathbb{R}^2 \rightarrow \mathbb{S}^1$, where \mathbb{S}^1 is a unit sphere
 54 of \mathbb{R}^2 . So that it is crucial to construct, starting from the initial image $I : \mathbb{R}^2 \rightarrow \mathbb{R}$, an
 55 initial vector field $U_0 : \mathbb{R}^2 \rightarrow \mathbb{S}^1$ with a topological singularity of degree 1, where there is a
 56 point in the initial image I . How to do this in a rigorous way, it is a subject of a current
 57 investigation.

58 Here our purpose is to provide a lighter variational formulation, in which the singularity
 59 in the image is directly given in term of of a proper differential operator defined on vector
 60 fields. Another important difference is that in [5] points and curves are detected both as
 61 singularities, while in the present paper our aim is to isolate points and at same time remove
 62 any other singularities, like curves precisely, from the initial image.

63 In our model $\Omega \subset \mathbb{R}^2$ is an open set (the image domain), $\mathbb{B}(\Omega)$ is the σ -algebra of the all
 64 Borel sets of Ω , and the initial image $I : \mathbb{B}(\Omega) \rightarrow \mathbb{R}$ is a Radon measure.

65 In order to detect the singularities of the image, we have to find a functional space whose
 66 elements generate, in a suitable sense, a measure concentrated on points. Such a space is
 67 $\mathcal{DM}^p(\Omega)$, the space of the vector fields $U : \Omega \rightarrow \mathbb{R}^2$ whose distributional divergence is a
 68 Radon measure introduced in [4]. (see Subsection 2.2 for definitions and examples).

69 Unfortunately even if we are capable of fabricating an initial vector field U_0 (see below
 70 for such a fabrication) belonging to the space $\mathcal{DM}^p(\Omega)$, its singular set could contains
 71 several structures, we want to remove from the original image like for instance curve or
 72 points generated by some noise. Hence, after the initialization we have to clear away all the
 73 structures we are not interested in by building up, starting from the initial data U_0 , a new
 74 vector field U whose singularities are given by the points of the image I we want to isolate.

75 Thus, from one hand we have to force the concentration set of the divergence measure of
 76 U_0 to contain only the points we want to catch, and on the other hand we have to regularize
 77 the initial data U_0 outside the points of singularities. To this end we propose to minimize
 78 an energy involving a competition between a divergence term and the counting Hausdorff

79 measure \mathcal{H}^0 . More precisely the energy is the following

$$\int_{\Omega \setminus P} |\operatorname{div} U|^2 + \lambda \int_{\Omega} |U - U_0|^p + \mathcal{H}^0(P). \quad (1)$$

80 where $U \in L^{p,2}(\operatorname{div}; \Omega \setminus P)$ the space of the L^p vector field whose distributional divergence
 81 belongs to $L^2(\Omega \setminus P)$, P is the atomic set we want to target and λ is a positive weight. The
 82 first integral forces U to be regular outside P , while the term $\mathcal{H}^0(P)$ penalizes the presence
 83 of the singular curves and the noise in the image.

84 From a practical point of view, this choice allows us to handle with a first order differential
 85 operator and permits us to formulate the minimization problem in a common functional
 86 framework.

87 Clearly to initialize the minimization process we need of a vector field U_0 linked to the
 88 initial image I , belonging to the space $\mathcal{DM}^p(\Omega)$. Such a vector field can be provided by the
 89 gradient of weak solution of the classical Dirichlet problem with measure data.

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

90 In section 3 we recall the basic results of the theory of the elliptic equations with data
 91 measure.

92 In order to provide computer examples, we must approximate the functional (1) by means
 93 of a sequence of more convenient functionals from a numerical point of view.

94 The approximation, we suggest in this paper, is based on the so called Γ -convergence,
 95 the notion of variational convergence introduced by De Giorgi (see [13, 14]). This theory
 96 is designed to approximate a variational problem by a sequence of different variational
 97 problems. The most important feature of the Γ -convergence relies on the fact that it implies
 98 the convergence of minimizers of the approximating functionals to those of the limiting
 99 functional. In this work we provide a possible Γ -convergence approach for the detection of
 100 points.

101 Classically, in image analysis, the Γ -convergence has been used to deal with 1-dimensional
 102 objects like smooth boundaries. For instance in ([2, 3]) Ambrosio and Tortorelli have proven
 103 that the Mumford-Shah functional can be approximated by a sequence of elliptic functionals
 104 that are numerically more treatable.

The main difficulty here is related to the presence of a codimension 2 object, which is
 not a contour: the set P . In order to obtain a variational approximation close to the one
 provided in ([2, 3]), the crucial step is then to replace the term $\mathcal{H}^0(P)$ of a functional (1) by
 a more handy, from a variational point of view, functional involving a smooth boundary and
 his perimeter measure given by the 1-dimensional Hausdorff measure \mathcal{H}^1 . Following some
 suggestion from the paper of Braides and Malchiodi and Braides and March (see [9, 10])
 such a functional is given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

105 where D is a proper regular set containing the atomic set P , κ is the curvature of its
106 boundary, the constant $\frac{1}{4\pi}$ is a normalization factor, and β_ε infinitesimal as $\varepsilon \rightarrow 0$. Roughly
107 speaking the minimum of this functionals are achieved on the union of balls of small radius,
108 so that when $\beta_\varepsilon \rightarrow 0$ the functionals shrinks to the atomic measure $\mathcal{H}^0(P)$. On the other
109 hand this approach requires a non trivial and convenient, for a numerical point of view,
110 approximation a curvature-dependent functional. Such an approximation is based on a
111 celebrated conjecture due to De Giorgi (see [12]). By means of this argument it is possible
112 to substitute the curvature-depending functional with an integral functional involving the
113 Laplacian operator of smooth functions. Then it remains to approximate the \mathcal{H}^1 measure
114 and this can be done by retrieving a classical gradient approach used in [15, 16]. This strategy
115 allows to deal with a functional whose Euler-Lagrange equations can be discretized. A simple
116 and intuitive explanation of the construction of the complete approximating functionals will
117 be given in section 3.

118 The paper is organized as follows: section 1 is intended to remind the reader of mathe-
119 matical tools useful in the following. In section 2 we recall the notion of Γ -convergence and
120 we illustrate the two Γ -convergence results we need in the sequel. In section 3 we address the
121 construction of the approximating functionals. In section 4 we present the discrete model.
122 In section 5 we explain the detection strategy. Finally in the last section we give some
123 computer examples.

124 2 Preliminaries

125 2.1 Measure theory

126 We start by classical definitions of measure theory. For a general survey on measure theory
127 we refer, among others, to [1].

Definition 2.1. Let (X, Σ) be a measure space and $\mu : \Sigma \rightarrow [0, \infty)$. We say that μ is a positive measure if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise disjoint elements of Σ

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n).$$

128 We say that μ is finite if $\mu(X) < +\infty$ and that μ is σ -finite if X is the union of increasing
129 sequence of sets with finite measures.

130 **Definition 2.2.** Let (X, Σ) be a measure space and let $N \in \mathbb{N}$, $N \geq 1$.

(i) We say that $\mu : \Sigma \rightarrow \mathbb{R}^N$ is a measure, if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}$ of pairwise disjoint elements of the σ -algebra Σ

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n).$$

131 If $N = 1$ we say that μ is a real measure, if $N > 1$ we say that μ is a vector measure.

(ii) If μ is a measure, we define its total variation $|\mu|$ as follows

$$\forall E \in \Sigma \quad |\mu|(E) := \sup\left\{\sum_{n=0}^{\infty} |\mu(E_n)| : E_n \in \Sigma \text{ pairwise disjoint, } E = \bigcup_{n=0}^{\infty} E_n\right\}.$$

132 (iii) We say that μ is finite is $|\mu|(X) < +\infty$.

133 It can be proven that the total variation $|\mu|$ is a positive measure, more precisely, it is the
134 smallest positive measure such that $|\mu|(E) \geq |\mu(E)|$ for every $E \in \Sigma$. Clearly, if μ is a
135 positive measure, then $|\mu| = \mu$.

136 An important class of measures is the Radon measures. In order to introduce such a
137 concept, the notion of Borel set is necessary.

138 **Definition 2.3.** Let X be a locally compact and separable metric space. Let $\mathcal{A}(X)$ be the
139 collection of all open sets of X . We call the Borel σ -algebra on X the smallest σ -algebra
140 containing $\mathcal{A}(X)$ and we denoted it by $\mathbb{B}(X)$. We call its elements the Borel sets of X .

141 **Definition 2.4.** Let X be a locally compact and separable metric space and $\mathbb{B}(X)$ be the
142 Borel σ -algebra of X . Consider the measure space $(X, \mathbb{B}(X))$.

- 143 (i) A positive measure on $(X, \mathbb{B}(X))$ is a Borel measure.
- 144 (ii) A positive measure on each compact subset K of X will be called a positive Radon
 145 measure. If μ is an \mathbb{R}^N -valued measure defined on all the Borel subset of X s.t.
 146 $|\mu|$ is a Radon measure and $|\mu|(X) < +\infty$, we say that μ is an \mathbb{R}^N -valued finite
 147 Radon measure. We denote by $\mathcal{M}(X; \mathbb{R}^N)$ the space of the all \mathbb{R}^N -valued finite Radon
 148 measures.

149 2.2 Distributional divergence

150 In this subsection we recall the definition of the space $L^{p,q}(\operatorname{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$, introduced
 151 in [4].

152 Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field.

153 **Definition 2.5.** We say that $U \in L^{p,q}(\operatorname{div}; \Omega)$ if $U \in L^p(\Omega; \mathbb{R}^2)$ and if its distributional
 154 divergence $\operatorname{div}U \in L^q(\Omega)$. If $p = q$ the space $L^{p,q}(\operatorname{div}; \Omega)$ will be denoted by $L^p(\operatorname{div}; \Omega)$.

If $p = q = 2$ the space $L^2(\operatorname{div}; \Omega)$ equipped with the scalar product

$$(U, V) = \int_{\Omega} U \cdot V \, dx dy + \int_{\Omega} \operatorname{div}U \operatorname{div}V \, dx dy;$$

155 is an Hilbert space.

156 We say that $U \in L_{loc}^{p,q}(\operatorname{div}; \Omega)$ if $U \in L^{p,q}(\operatorname{div}; A)$ for every open set $A \subset\subset \Omega$.

Definition 2.6. For $U \in L^p(\Omega; \mathbb{R}^2)$, $1 \leq p \leq +\infty$, set

$$|\operatorname{div}U|(\Omega) := \sup\{\langle U, \nabla\varphi \rangle : \varphi \in C_0^1(\Omega), |\varphi| \leq 1\}.$$

We say that U is an L^p -divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$ if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega; \mathbb{R}^2)} + |\operatorname{div}U|(\Omega) < +\infty.$$

157 We say that $U \in \mathcal{DM}_{loc}^p(\Omega)$ if $U \in \mathcal{DM}^p(A)$ for every open set $A \subset\subset \Omega$.

158 **Remark 2.1.** If $U \in \mathcal{DM}^p(\Omega)$ then via Riesz Theorem it is possible to represent the distri-
 159 butional divergence of U by a Radon measure. More precisely there exists a Radon measure
 160 μ such that for every $\varphi \in C_0^1(\Omega)$ the following equality holds:

$$\int_{\Omega} U \cdot \nabla\varphi \, dx dy = - \int_{\Omega} \varphi \, d\mu.$$

161 For instance the field $U(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ belongs to $\mathcal{DM}_{loc}^1(\mathbb{R}^2)$ and its divergence
 162 measure is given by $-2\pi\delta_0$, where δ_0 is the Dirac mass.

163 Such a result can be proven by approximation. Let us define the following map:

$$U_{\varepsilon}(x, y) := \begin{cases} U(x, y) & \text{if } |x| \geq \varepsilon \\ (\frac{x}{\varepsilon^2}, \frac{y}{\varepsilon^2}) & \text{if } |x| < \varepsilon. \end{cases}$$

164 It is not difficult to check that u_ε is Lipschitz-map with divergence given by
 165 $\frac{2}{\varepsilon^2}\chi_{B(0,\varepsilon)}$.
 166 Then for every test function $\varphi \in C_0^1(\mathbb{R}^2)$ we have

$$\int U_\varepsilon \cdot \nabla \varphi dx dy = - \int \frac{2}{\varepsilon^2} \chi_{B(0,\varepsilon)} \varphi dx dy.$$

By applying the changing variable $x = \frac{x_1}{\varepsilon}$, $y = \frac{y_1}{\varepsilon}$ we obtain

$$\int U_\varepsilon \cdot \nabla \varphi dx dy = -2 \int \chi_{B(0,1)} \varphi\left(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon}\right) dx_1 dy_1,$$

so that, letting $\varepsilon \rightarrow 0$, by dominated convergence theorem we obtain

$$\int_\Omega U \cdot \nabla \varphi dx dy = -2\pi \varphi(0,0) = -2\pi \int_\Omega \varphi d\delta_0$$

167 3 The Dirichlet problem with measure data

168 For the initialization of our algorithm we must built a vector field U_0 which should be such
 169 that its divergence is singular on points of the image I . Therefore we will use the gradient
 170 of the solution of the following Dirichlet problem (applied with $\mu = I$)

$$171 \quad \begin{cases} \Delta f = \mu & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

172 where μ is a Radon measure

173 The study of a PDE with data measure is a very classical research topic in Analysis
 174 and there is a vast literature about this subject. In what follows we recall just some basic
 175 definitions and crucial results for our purpose.

Definition 3.1. *We say that f is a weak solution of the (3) if $f \in W_0^{1,1}(\Omega)$ and the following equality holds for every test $\varphi \in C_0^1(\Omega)$:*

$$\int_{\Omega} \nabla f \cdot \nabla \varphi = - \int_{\Omega} \varphi d\mu.$$

176 Note that if f is a weak solution, then its weak gradient is a divergence measure field,
 177 i.e. $U = \nabla f \in \mathcal{DM}^p(\Omega)$, and its divergence is equal to the measure data μ .

178 A classical example of a weak solution is provided by a fundamental solution of the
 179 Laplace equation. Let us consider the problem:

$$\begin{cases} \Delta f = \delta_0 & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

180 Then, by the same argument used in Remark 2.1, one can easily check that the distributional
 181 divergence of ∇f is given by δ_0 .

182 More generally classical results (see [17]) guarantee the existence of a unique solution of
 183 the problem (3). Concerning the regularity it is known then $f \in W^{1,p}$ with $p < 2$.

184 4 Γ -convergence

185 The key point of our strategy is the approximation of the functional (1) by means of more
186 regular functionals via Γ -convergence.

187 Therefore this section is devoted to a very simple presentation of the notion of Γ conver-
188 gence and the two crucial results for our goal: the Modica-Mortola's theorem (see [15, 16])
189 concerning the approximation of the perimeter measure and the De Giorgi conjecture (see
190 [12]) about the approximation of curvature depending functionals.

191 We start by recalling the definition of the Γ -convergence just in the special case we need
192 of. For a complete survey on this subject we refer to [8, 11] and references therein. In what
193 follows X stands for a metric space (X, d)

194 **Definition 4.1.** We say that a sequence $F_n : X \rightarrow [-\infty, +\infty]$ Γ -converges to a functional
195 $F : X \rightarrow [-\infty, +\infty]$ as $(n \rightarrow +\infty)$, with respect to the X -distance, if the following two
196 properties hold:

197 for every $u \in X$ and every $\{u_n\} \in X$, such that $u_n \rightarrow u$ with respect to X -distance,

$$F(u) \leq \liminf_{n \rightarrow +\infty} F_n(u_n), \quad (5)$$

198 for every $u \in X$ there exists $\{\bar{u}_n\} \subset X$, such that $\bar{u}_n \rightarrow u$ with respect to X -distance

$$F(u) = \lim_{n \rightarrow +\infty} F_n(\bar{u}_n). \quad (6)$$

199 The functional F is called the Γ -limit of F (with respect to the X -distance) and we write
200 $F = \Gamma - \lim_n F_n$.

201 The relevance of this notion is explained by the following fundamental theorem

Theorem 4.1. Let us suppose that $F = \Gamma - \lim_n F_n$ and let a compact set $K \subset X$ exist
such that $\inf_X F_n = \inf_K F_n$ for all n . Then

$$\min_X F = \lim_n \min_X F_n$$

202 .

203 Moreover, if u_n is a converging sequence such that $\lim_{n \rightarrow +\infty} F_\varepsilon(u_n) = \lim_{n \rightarrow +\infty} \min_X F_n$, then its
204 limit is a minimum point for F .

205 It is possible to extend the definition of Γ -convergence to a family depending on a real a
206 parameter. So that we can deal with Γ -limits of families $\{F_\varepsilon\}$ as $\varepsilon \rightarrow 0$.

207 **Definition 4.2.** We say that a sequence $F_\varepsilon : X \rightarrow [-\infty, +\infty]$ Γ -converges to a functional
208 $F : X \rightarrow [-\infty, +\infty]$ as $(\varepsilon \rightarrow 0)$, if for every sequence $\{\varepsilon_n\}$ converging to 0, as $n \rightarrow +\infty$,
209 the sequence F_{ε_n} Γ -converges to F .

210 4.1 Modica Mortola's approach

211 The Modica-Mortola theorem states that it is possible to approximate, in the Γ -convergence
212 sense, a perimeter measure by means of the following sequence of functionals

$$F_\varepsilon^1(u) := \begin{cases} \int_\Omega (\varepsilon |\nabla u|^2 + \frac{V(u)}{\varepsilon}) dx & \text{if } u \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

213 where $V(u) = u^2(1-u)^2$ is a double well potential. Besides in order to avoid that the
214 minimizers of the functional F_ε^1 are trivial, some constraint on the functions u_ε must be
215 added. In the original version of the Modica-Mortola's paper a volume constrained of the
216 type $\int_\Omega u dx dy = m$, was assumed.

217 Let us give an intuitive explanation of such a result. Since V has two absolute minimizers
218 at $u = 0, 1$, when ε is small, a local minimizer u_ε is closed to 1 on a part of Ω and close to
219 0 on the other part, making a rapid transition of order ε between 0 and 1. When $\varepsilon \rightarrow 0$ the
220 transition set shrinks to a set of dimension 1, so that u_ε goes to a function taking values u
221 into $\{0, 1\}$ and the functionals converge to the measure of the perimeter of the discontinuities
222 set of u .

223 By the way the Modica-Mortola's theorem still holds even without assuming the volume
224 constrain. It means that in general the Γ -limit could be given by a functional identically
225 equal to 0. The Modica-Mortola's Theorem is the following.

226 **Theorem 4.2.** *The functionals $F_\varepsilon^1 : L^1(\Omega) \rightarrow [0, +\infty]$ Γ -converge with respect to the
227 L^1 -distance to the following functional*

$$F^1(u) = \begin{cases} C_V \mathcal{H}^1(S_u) & \text{if } u \in \{0, 1\} \\ +\infty & \text{otherwise} \end{cases}$$

228 where, as usual, S_u denotes the set of the discontinuities of u and C_V is a suitable constant
229 depending on the potential V .

230 4.2 De Giorgi's conjecture

The aim of De Giorgi was finding a variational approximation of a curvature depending
functional of the type:

$$F^2(D) = \int_{\partial D} (1 + \kappa^2) d\mathcal{H}^1;$$

231 where D is a regular set and k is a curvature of its boundary ∂D .

232 Since ∂D can be represented as the discontinuity set of the function $u_0 = 1 - \chi_D$,
233 by Modica-Mortola's Theorem it follows that there is a sequence of non constant local
234 minimizers such that $u_\varepsilon \rightarrow u_0$ with respect to the L^1 -convergence such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(u_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

235 Furthermore looking at the Euler-Lagrange equation associated to a contour length term,
236 yields a contour curvature term κ , while the Euler-Lagrange equations for the functional
237 $F_\varepsilon^1(u)$ contains a term $2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon}$.

238 Then De Giorgi suggested to approximate the functional F^2 by adding to the Modica-
239 Mortola approximating functionals the following term

$$F_\varepsilon^2(u) = \int_{\Omega} (2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon})^2 dx.$$

240 In ([6]) Bellettini and Paolini proved the limsup inequality (6), while the validity of
241 liminf inequality (5) remains an open problem.

242 5 The approximating functionals

243 In this section we present the energy we deal with and the construction of the approximating
244 sequence.

The energy we are interested in is given by

$$\int_{\Omega \setminus P} |\operatorname{div} U|^2 + \lambda \int_{\Omega} |U - U_0|^p + \mathcal{H}^0(P).$$

245 where $U \in L^{p,2}(\operatorname{div}; \Omega \setminus P)$, $U_0 \in \mathcal{DM}_{loc}^p(\mathbb{R}^2)$ and finally P is an atomic set consisting of a
246 finite number N of points, i.e. $P = \{x_1, \dots, x_N\}$.

As pointed out in the introduction, the first step is to substitute the counting measure $\mathcal{H}^0(P)$ with a more treatable term given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x) \right) d\mathcal{H}^1(x);$$

247 where D is an union of regular simply connected sets $\{D_i\}$ with $i = 1, \dots, N$, such that
248 $x_i \in D_i$, $D_i \cap D_j = \emptyset$ for $i \neq j$. κ is the curvature of the boundary of the sets D_i , the
249 constant $\frac{1}{4\pi}$ is a normalization factor and β_ε is infinitesimal as $\varepsilon \rightarrow 0$.

250 To understand why we can approximate $\mathcal{H}^0(P)$ with $G_{\beta_\varepsilon}(D)$ one should note that the
251 solution of the following minimum problem

$$\min_{D \supset P} G_{\beta_\varepsilon}(D) \tag{7}$$

252 is given by $D = \bigcup_i^N B(x_i, \beta_\varepsilon)$, where x_i are the points of P . We sketch the argument in the
253 single point case.

254 By the Young's inequality we have

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi} \int_{\partial D} 2\kappa d\mathcal{H}^1(x)$$

255 and by applying the Gauss-Bonnet Theorem

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi} (2)(2\pi) = 1 = \mathcal{H}^0(P).$$

256 Finally a simple calculation shows that, if we evaluate the functional G_{β_ε} on $B(x_1, \beta_\varepsilon)$,
257 we obtain the value 1, i.e. the number of points in P , i.e. $\mathcal{H}^0(P)$. The N points case can
258 be recovered with minor changes by the same argument.

For what follows it is convenient to split the functional G_{β_ε} in two terms:

$$G_{\beta_\varepsilon}(D) = G_{\beta_\varepsilon}^1(D) + G_{\beta_\varepsilon}^2(D)$$

259 where

$$G_{\beta_\varepsilon}^1(D) = \frac{1}{4\pi} \int_{\partial D} \frac{1}{\beta_\varepsilon} d\mathcal{H}^1(x);$$

260 and

$$G_{\beta_\varepsilon}^2(D) := \frac{1}{4\pi} \int_{\partial D} \beta_\varepsilon \kappa^2(x) d\mathcal{H}^1(x).$$

261 We can write an intermediate approximation of the energy (1):

$$E_\varepsilon(U, D) = G_{\beta_\varepsilon}^1(D) + G_{\beta_\varepsilon}^2(D) + \int_{\Omega} (1 - \chi_D) |\operatorname{div}(U)|^2 + \lambda \int_{\Omega} |U - U_0|^p. \quad (8)$$

262 The advantage of such a formulation is that we know how to provide a variational ap-
 263 proximation of the perimeter measure $H^1|_{\partial D}$. Following the Modica-Mortola's approach
 264 such an approximation can be obtained by using the following measure:

$$\mu_\varepsilon(w, \nabla w) dx = \left(\varepsilon |\nabla w|^2 + \frac{V(w)}{\varepsilon} \right) dx,$$

265 where $V(w) = w^2(1-w)^2$ is a double well functional.

266 Next step is expressing the curvature term by means of the function w . Thanks to the
 267 De Giorgi's conjecture we can replace the term κ by the term $2\varepsilon \Delta w - \frac{V'(w)}{\varepsilon}$.

So that we can formally write the complete approximating functional:

$$\begin{aligned} \Phi_\varepsilon(U, w) : &= \int_{\Omega} w^2 |\operatorname{div}(U)|^2 dx + \frac{1}{4\pi} \int_{\Omega} \beta_\varepsilon \left(2\varepsilon \Delta w - \frac{V'(w)}{\varepsilon} \right)^2 dx + \frac{1}{\beta_\varepsilon} \int_{\Omega} \mu_\varepsilon(w, \nabla w) dx \\ &+ \lambda \int_{\Omega} |U - U_0|^p dx, \end{aligned} \quad (9)$$

268 where $U \in L^{p,2}(\operatorname{div}; \Omega)$ and w is smooth function equal to 1 on the boundary, i.e.
 269 $1 - w \in C_0^\infty(\Omega)$. We add the volume constraint $\frac{1}{\mathcal{L}^2(\Omega)} \int_{\Omega} w dx = 1$. This last condition
 270 prevents w from converging to the function constantly equal to 0 as $\varepsilon \rightarrow 0$.

271 Then if $(U_\varepsilon, w_\varepsilon)$ is a minimizing sequence of Φ_ε , then w_ε must be very close to the values
 272 1 when ε goes to 0, since the double well potential is positive except for $w_\varepsilon = 0, 1$ and w
 273 must be equal to 1 on the $\partial\Omega$. On the other hand, near the points where the divergence is
 274 very big w_ε must be close to 0. Besides when $\varepsilon \rightarrow 0$, $\beta_\varepsilon \rightarrow 0$ goes to 0 as well, so that the
 275 singular set D is given by an union of balls of a small radius β_ε .

276 Therefore, while the functions U_ε approximate a minimizer U of the original functional,
 277 the level set $\{w_\varepsilon = 0\}$ approximate the original singular set P .

The first variation of this functional leads to the following gradient flow system

$$\begin{aligned} \frac{\partial U}{\partial t} &= \nabla(w^2 \operatorname{div} U) - \lambda p |U - U_0|^{p-2} (U - U_0) \\ \frac{\partial w}{\partial t} &= -4 \frac{\Delta h}{\beta_\varepsilon} + \beta_\varepsilon h + \frac{2}{\varepsilon^2} \frac{1}{\beta_\varepsilon} V''(w) h - 2w |\operatorname{div} U|^2 \end{aligned} \quad (10)$$

278 where h is given by the equation

$$h = 2\varepsilon\Delta w - \frac{1}{\varepsilon}V'(w).$$

279 6 Detection

280 In our model the image contains an atomic Radon measure. Thus, in order to find an initial
 281 vector field which copies the singularities of the initial image, we use the gradient of the
 282 solution of the following Dirichlet problem:

$$\begin{cases} \Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

283 In this way we obtain a vector field whose divergence is singular on a proper set which
 284 contains the points we want to detect. In general this set could contain other structures.
 285 For instance if the initial image is a Radon measure concentrated both on points and on
 286 curves, the divergence of ∇f is singular on points and on curves at the same time. Besides
 287 if there is some noise in the image, it could be not clear how to differentiate the singular
 288 points due to the noise, from those we want to catch. As a consequence, by solving the
 289 problem (11), we get just a predetection, which has to be refined.

290 To this purpose we search for a minimizer of the energy $\Phi_\varepsilon(U, w)$ via solving equations
 291 (10) with initial data U_0 given by ∇f . So that we obtain a vector field U whose divergence
 292 is relevant only on the set P and a function w whose zeros are given by the set P .

293 6.1 Discretization

294 The image is an $N \times N$ vector. We endowed the space \mathbb{R}^{2N} with the standard scalar product
 295 and standard norm.

296 The discrete version of gradient operator is the following.

Let $I \in \mathbb{R}^{2N}$. Then the gradient ∇I is an element of the space $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ given by:

$$(\nabla I)_{i,j} = ((\nabla I)_{i,j}^1, (\nabla I)_{i,j}^2)$$

where

$$(\nabla I)_{i,j}^1 = \begin{cases} I_{i+1,j} - I_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N, \end{cases}$$

$$(\nabla I)_{i,j}^2 = \begin{cases} I_{i,j+1} - I_{i,j} & \text{if } j < N \\ 0 & \text{if } j = 0. \end{cases}$$

297 We also introduce the discrete version of the divergence operator simply defined as the
 298 adjoint operator of the gradient: $\text{div} = -\nabla^*$. More in details if $v \in \mathbb{R}^{2N}$ we have

$$(\operatorname{div} v)_{i,j} = \begin{cases} v_{i,j}^1 + v_{i,j}^2 & \text{if } i, j = 1 \\ v_{i,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } i = 1, 1 < j < N \\ v_{i,j}^1 - v_{i-1,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } 1 < i < N, 1 < j < N \\ -v_{i-1,j}^1 + v_{i,j}^2 - v_{i-1,j}^2 & \text{if } i = N, 1 < j < N \\ v_{i,j}^1 - v_{i-1,j}^1 + v_{i,j}^2 & \text{if } 1 < i < N, j = 1 \\ v_{i,j}^1 - v_{i-1,j}^1 - v_{i-1,j}^2 & \text{if } 1 < i < N, j = N \\ -(v_{i-1,j}^1 + v_{i-1,j}^2) & \text{if } i, j = N. \end{cases}$$

299 Then we can define the discrete version of the Laplacian operator as $\Delta I = \operatorname{div}(\nabla I)$.

300 6.2 Discretization in time

We simply replace $\frac{\partial U}{\partial t}$ and $\frac{\partial w}{\partial t}$ by $\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\delta t}$ and $\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\delta t}$ respectively. Then we write the system (10) in the form (for simplicity we omit the dependance on ε)

$$\begin{cases} U_1^{n+1} = \delta t \Phi_{U_1}(U_n, w_n) \\ U_2^{n+1} = \delta t \Phi_{U_2}(U_n, w_n) \\ w^{n+1} = \delta t \Phi_w(U_n, w_n). \end{cases}$$

301 We initialize our algorithm with $U(0) = \nabla f$, where f is the solution of the problem (11).
 302 To do this, we need to solve the Dirichlet problem (11) with measure data I , therefore we
 303 regularize the image by convolution with a Gaussian kernel G_σ with very small σ and then
 304 we solve, by classical finite differences method, the problem:

$$\begin{cases} \Delta f = I_\sigma & \text{on } \Omega \\ f = 0 & \partial\Omega, \end{cases} \quad (12)$$

305 where $I_\sigma = I * G_\sigma$.

306 To initialize our algorithm, we need of an initial guess on w . So we choose $w(0) = 1$.

307 7 Computer examples

308 7.1 Parameter settings

309 Before running our algorithm all the parameters have to be fixed. The most important are ε
 310 and β_ε , which govern the set D approximating points we want to detect. Those parameters
 311 are related by the condition $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\beta_\varepsilon} = 0$. Furthermore, since the mesh grid size is 1 and β_ε
 312 gives the radius of a balls centered in the singular point we want to detect, from a discrete
 313 point of view the smallest value we can take is $\frac{\sqrt{2}}{2}$. Then we use values ranging from 0.1
 314 and 0.5 for ε , and from 1.2 to 0.7 for β_ε .

315 Concerning the parameter λ we mainly used the small value $\lambda = 0.1$, in order to force
 316 the algorithm to regularize as much as possible the initial data U_0 .

317 Since we deal with small values of ε , in order to have some stability, we must take a
 318 small discretization time step, Practically we mainly used the value $\delta t = 1 \times 10^{-8}$.

319 Concerning the stop criterion we iterate the algorithm until $\max \left\{ \frac{|U_1^{n+1} - U_2^n|}{|U_1^n|}, \frac{|U_2^{n+1} - U_2^n|}{|U_2^n|}, \frac{|w^{n+1} - w^n|}{|w^n|} \right\}$
 320 ≤ 0.001 . Finally we set $p = 1.5$.

321 In all the computer examples the points are detected by means of the function w_ε . We
 322 display the level-set $\{w_\varepsilon \simeq 0\}$.

323 7.2 Commentaries

324 The figure 1 shows how resistant to the noise our model is. When the noise is larger the
 325 parameter ε must be as closer as possible to the ideal value 0. Besides it is possible to see
 326 that for small values of ε our detection is finer according to the continuous setting. More in
 327 details in the first row we display the initial image obtained by adding a gaussian noise to a
 328 binary image of five points. The increasing level of the noise corresponds to the increasing
 329 *PSNR* between the binary image and the noisy image. The second row shows the behavior
 330 of w for intermediate value of the parameters ε and β_ε . In the last row one can note that
 331 the smaller ε and β_ε are the finer the detection is.

332 In figure 2 in the first two rows we display the histograms of the gray level of I and w
 333 corresponding to the different values of *PSNR*. One can see that it is easier fixing a threshold
 334 value starting from the function w_ε than from the initial image I . In the last row we display
 335 the set $\{w_\varepsilon \simeq 0\}$ obtained by plotting the set $\{w_\varepsilon \leq \alpha\}$ with threshold value $\alpha = 0.5$

336 In figure 3 we test our algorithm on curves and points at the same time. In the first
 337 column we have a sequence of points and a curve with no boundary on Ω . In the second
 338 column we consider the same curve with boundary in Ω . The second column we display
 339 the function w_ε and in the last column the level set $\{w_\varepsilon \simeq 0\}$ once again obtained by
 340 fixing a threshold value $\alpha = 0.5$. The result is that, as desired, our algorithm is capable of
 341 eliminating the curve from the initial image. According to the continuous setting when ε
 342 takes values close to 0 the approximating energy (9) behaves similarly to the limit energy
 343 (1), so that the presence of the curve is penalized in the minimization process. Then the set
 344 $\{w_\varepsilon \simeq 0\}$ contains nothing else but points.

345 Finally in figure 4 we deal with a biological image. Our task is catching the finest
 346 structure present in the image. In both cases of figure 4 the isolated point is well detected,
 while the branch and the dashes of the cellule is not.

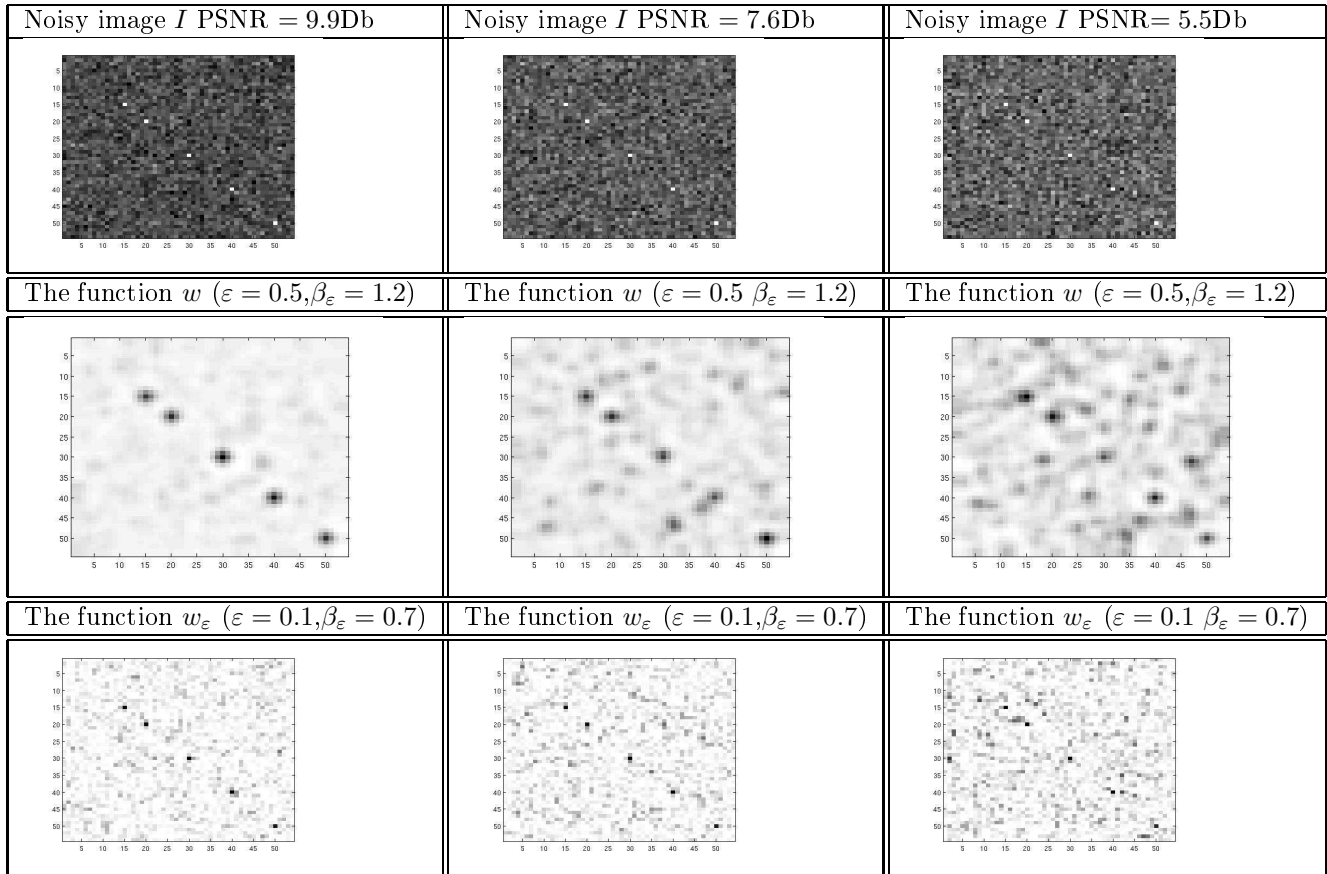


Figure 1: Synthetic image: we test our algorithm on noisy images. When the parameters ε and β_ε are small as much as possible the detection is finer.

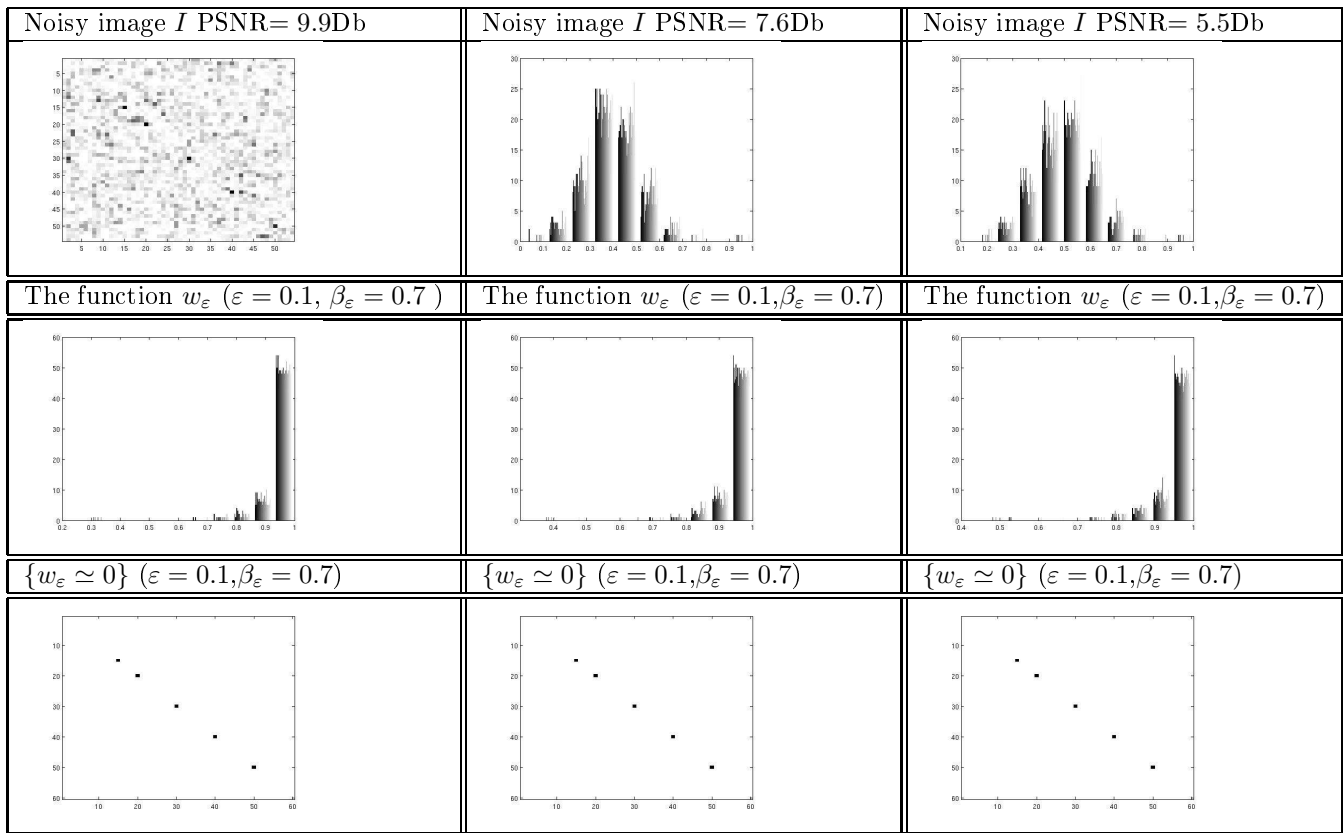


Figure 2: Synthetic images: The detection is refined by fixing a threshold value $\alpha = 0.5$ for the function w_ε .

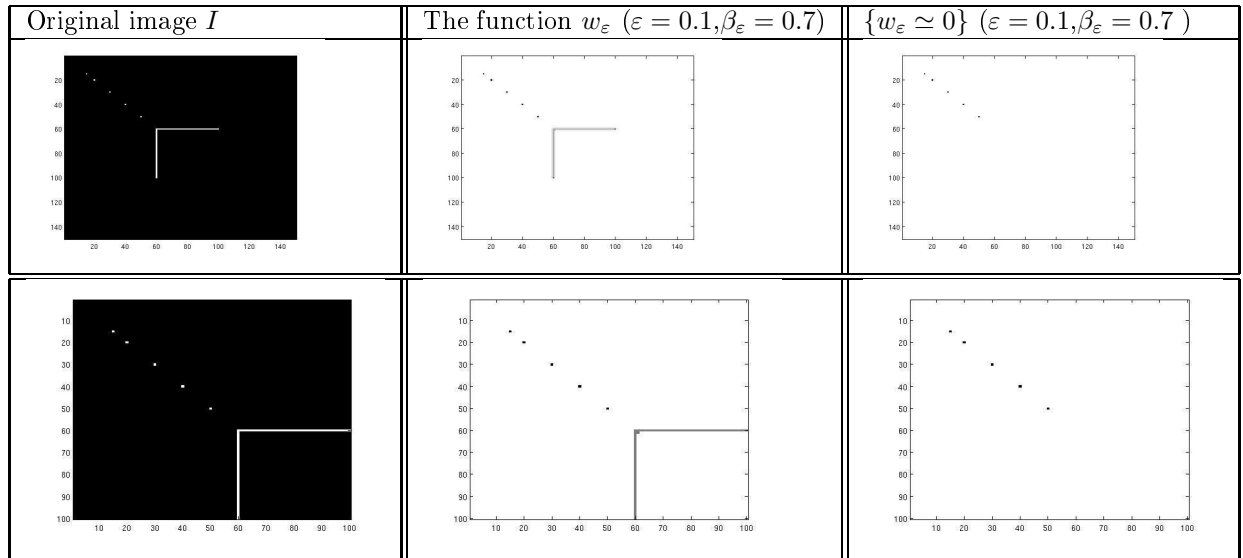


Figure 3: *Synthetic image: curve and points are present in the initial image. As expected our method is capable of removing the curve from the image.*

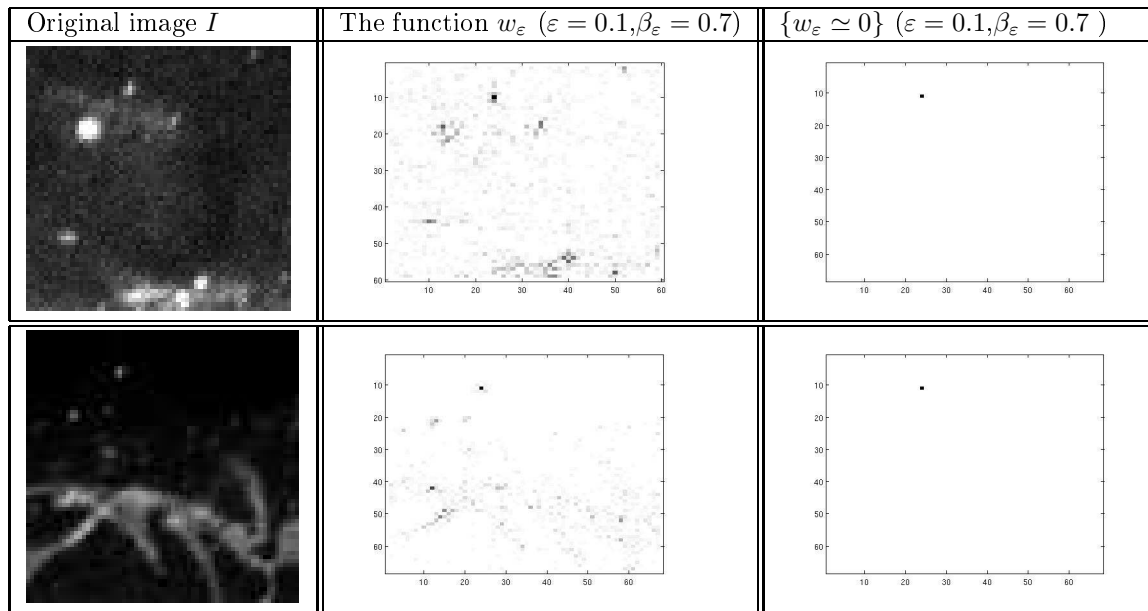


Figure 4: *Biological image: one isolated point, dashes and filament are present in the initial image. After the algorithm process we keep nothing else but the isolated point .*

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