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# Stability Properties of Networks with Interacting TCP Flows

Carl Graham<sup>1</sup>, Philippe Robert<sup>2</sup>, and Maaïke Verloop<sup>3\*</sup>

<sup>1</sup> UMR 7641 CNRS — École Polytechnique, Route de Saclay, 91128 Palaiseau, France  
[carl@cmapx.polytechnique.fr](mailto:carl@cmapx.polytechnique.fr)

<sup>2</sup> INRIA Paris — Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay, France  
[Philippe.Robert@inria.fr](mailto:Philippe.Robert@inria.fr)  
<http://www-rocq.inria.fr/~robert>

<sup>3</sup> CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands  
[I.M.Verloop@cwi.nl](mailto:I.M.Verloop@cwi.nl)  
<http://www.cwi.nl/~maaike>

**Abstract.** The equilibrium distributions of a Markovian model describing the interaction of several classes of permanent connections in a network are analyzed. It has been introduced by Graham and Robert [5]. For this model each of the connections has a self-adaptive behavior in that its transmission rate along its route depends on the level of congestion of the nodes on its route. It has been shown in [5] that the invariant distributions are determined by the solutions of a fixed point equation in a finite dimensional space. In this paper, several examples of these fixed point equations are studied. The topologies investigated are rings, trees and a linear network, with various sets of routes through the nodes.

## 1 Introduction

Data transmission in the Internet network can be described as a self-adaptive system to the different congestion events that regularly occur at its numerous nodes. A connection, a TCP flow, in this network adapts its throughput according to the congestion it encounters on its path: Packets are sent as long as no loss is detected and throughput grows linearly during that time. On the contrary when a loss occurs, the throughput is sharply reduced by a multiplicative factor. This scheme is known as an Additive Increase and Multiplicative Decrease algorithm (AIMD).

Globally, the TCP protocol can be seen as a bandwidth allocation algorithm on the Internet. From a mathematical modelling perspective, the description is somewhat more difficult. While the representation of the evolution of the throughput of a single TCP flow has been the object of various rigorous works, there are few rigorous studies for modelling the evolution of a large set of TCP connections in a quite large network.

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A possible mathematical formulation which has been used is via an optimization problem: given  $K$  classes of connections, when there are  $x_k$  connections of class  $k \in \{1, \dots, K\}$ , their total throughput achieved is given by  $\lambda_k$  so that the vector  $(\lambda_k)$  is a solution of the following optimization problem

$$\max_{\lambda \in \Lambda} \sum_{k=1}^K x_k U_k(\lambda_k / x_k),$$

where  $\Lambda$  is the set of admissible throughputs which takes into account the capacity constraints of the network. The functions  $(U_k)$  are defined as *utility* functions, and various expressions have been proposed for them. See Kelly *et al.* [7], Massoulié [9] and Massoulié and Roberts [10]. With this representation, the TCP protocol is seen as an adaptive algorithm maximizing some criterion at the level of the network.

A different point of view has been proposed in Graham and Robert [5]. It starts on the local dynamics of the AIMD algorithm used by TCP and, through a scaling procedure, the global behavior of the network can then be described rigorously. It is assumed that there are  $K$  classes of *permanent* connections going through different nodes and with different characteristics. The loss rate of a connection using a given node  $j$ ,  $1 \leq j \leq J$ , is described as a function of the congestion  $u_j$  at this node. The quantity  $u_j$  is defined as the (possibly weighted) sum of the throughputs of *all* the connections that use node  $j$ . The interaction of the connections in the network is therefore expressed via the loss rate at each node.

It has been shown in Graham and Robert [5] that under a mean-field scaling, the evolution of a class  $k$  connection,  $1 \leq k \leq K$ , can be asymptotically described as the unique solution of an unusual stochastic differential equation. Furthermore, it has also been proved that the equilibrium distribution of the throughputs of the different classes of connections is in a one to one correspondence with the solution of a fixed point equation  $(\mathcal{E})$  of dimension  $J$  (the number of nodes).

Under “reasonable” conditions, there should be only one solution of  $(\mathcal{E})$  and consequently a unique stable equilibrium of the network. Otherwise this would imply that the state of the network could oscillate between several stable states. Although this is mentioned here and there in the literature, this has not been firmly established in the context of an IP network. It has been shown that multi-stability may occur in loss networks, see Gibbens *et al.* [4] and Marbukh [8] or in the context of a wireless network with admission control, see Antunes *et al.* [1]. Raghunathan and Kumar [12] presents experiments that suggest that a phenomenon of bi-stability may occur in a context similar to the one considered in this paper but for wireless networks.

It turns out that it is not easy to check in practice whether the fixed point equation  $(\mathcal{E})$  has a unique solution or not. The purpose of this paper is to investigate in detail this question for several topologies. The paper is organized as follows. Section 2 reviews the main definitions and results used in the paper. In addition, a simple criterion for the existence of a fixed-point solution is given.

Section 3 presents a uniqueness result for a tree topology under the assumptions that all connections use the root. Section 4 considers a linear network. Section 5 studies several scenarios for ring topologies and a uniqueness result is proved for connections going through one, two, or all the nodes. Two main approaches are used to prove uniqueness: monotonicity properties of the network and contraction arguments.

A general conjecture that we make is that when the loss rates are increasing in the level of congestion, this should be sufficient to imply the uniqueness of the equilibrium in a general network (together with regularity properties perhaps).

## 2 A Stochastic Fluid Picture

In this section, a somewhat simplified version of the stochastic model of interacting TCP flows of Graham and Robert [5] is presented.

### The case of a single connection

Ott *et al.* [11] presents a fluid model of a single connection. Via scalings with respect to the loss rate, Dumas *et al.* [3] proves various limit theorems for the resulting processes. The limiting picture of Dumas *et al.* [3] for the evolution of the throughput of single long connection is as follows.

If the instantaneous throughput at time  $t$  of the connection is  $W(t)$ , this process has the Markov property and its infinitesimal generator is given by

$$\Omega(f)(x) = af'(x) + \beta x(f(rx) - f(x)) \quad (1)$$

for  $f$  a  $C^1$ -function from  $\mathbb{R}_+$  to  $\mathbb{R}$ . For  $t \geq 0$ , the quantity  $W(t)$  should be thought as the instantaneous throughput of the connection at time  $t$ .

The Markov process  $(W(t))$  increases linearly at rate  $a$ . The constant  $a$  is related to the distance between the source and the destination. It increases proportionally to the round trip time  $RTT$ , typically

$$a = \frac{C_0}{C_1 + RTT},$$

for some constants  $C_0$  and  $C_1$ .

Given  $W(t) = x$ , the process  $(W(t))$  jumps from  $x$  to  $rx$  ( $r$  is usually  $1/2$ ) at rate  $\beta x$ . The expression  $\beta x$  represents the loss rate of the connection. Of course, the quantities  $a$ ,  $\beta$  and  $r$  depend on the parameters of the connection.

The density of the invariant distribution of this Markov process is given in the following proposition. It has been analyzed in Ott *et al.* [11] at the fluid level and by Dumas *et al.* [3], see also Guillemin *et al.* [6]. The transient behavior has been investigated in Chafai *et al.* [2].

**Proposition 1.** *The function*

$$H_{r,\rho}(w) = \frac{\sqrt{2\rho/\pi}}{\prod_{n=0}^{+\infty}(1 - r^{2n+1})} \sum_{n=0}^{+\infty} \frac{r^{-2n}}{\prod_{k=1}^n(1 - r^{-2k})} e^{-\rho r^{-2n} w^2/2}, \quad w \geq 0, \quad (2)$$

with  $\rho = a/\beta$ , is the density of the invariant distribution of the Markov process  $(W(t))$  whose infinitesimal generator is given by Equation (1). Furthermore, its expected value is given by

$$\int_0^{+\infty} w H_{r,\rho}(w) dw = \sqrt{\frac{2\rho}{\pi}} \prod_{n=1}^{+\infty} \frac{1 - r^{2n}}{1 - r^{2n-1}}. \quad (3)$$

### A Representation of Interacting Connections in a Network

The network has  $J \geq 1$  nodes and accommodates  $K \geq 1$  classes of permanent connections. For  $1 \leq k \leq K$ , the number of class  $k$  connections is  $N_k \geq 1$ , and one sets

$$N = (N_1, \dots, N_K), \text{ and } |N| = N_1 + \dots + N_K.$$

An *allocation matrix*  $A = (A_{jk}, 1 \leq j \leq J, 1 \leq k \leq K)$  with positive coefficients describes the use of nodes by the connections. In particular the route of a class  $k$  connection goes through node  $j$  only if  $A_{jk} > 0$ . In practice, the class of a connection is determined by the sequence set of nodes it is using.

If  $w_{n,k} \geq 0$  is the throughput of the  $n$ th class  $k$  connection,  $1 \leq n \leq N_k$ , the quantity  $A_{jk}w_{n,k}$  is the weighted throughput at node  $j$  of this connection. A simple example would be to take  $A_{jk} = 1$  or  $0$  depending on whether a class  $k$  connection uses node  $j$  or not. The total weighted throughput  $u_j$  of node  $j$  by the various connections is given by

$$u_j = \sum_{k=1}^K \sum_{n=1}^{N_k} A_{jk} w_{n,k}.$$

The quantity  $u_j$  represents the level of utilization/congestion of node  $j$ . In particular, the loss rate of a connection going through node  $j$  will depend on this variable.

For  $1 \leq k \leq K$ , the corresponding parameters  $a$  and  $\beta$  of Equation (1) for a class  $k$  connection are given by a non-negative number  $a_k$  and a function  $\beta_k : \mathbb{R}_+^J \rightarrow \mathbb{R}_+$ , so that when the resource vector of the network is  $u = (u_j, 1 \leq j \leq J)$  and if the state of a class  $k$  connection is  $w_k$ :

- Its state increases linearly at rate  $a_k$ . For example  $a_k = 1/R_k$  where  $R_k$  is the round trip time between the source and the destination of a class  $k$  connection.
- A loss for this connection occurs at rate  $w_k \beta_k(u)$  and in this case its state jumps from  $w_k$  to  $r_k w_k$ . The function  $\beta_k$  depends only on the utilization of all nodes used by class  $k$  connections. In particular, if a class  $k$  connection goes through the nodes  $j_1, j_2, \dots, j_{l_k}$ , one has

$$\beta_k(u) = \beta_k(u_{j_1}, u_{j_2}, \dots, u_{j_{l_k}}).$$

A more specific (and natural) choice for  $\beta_k$  would be

$$\beta_k(u) = \delta_k + \varphi_{j_1}(u_{j_1}) + \varphi_{j_2}(u_{j_2}) + \dots + \varphi_{j_{l_k}}(u_{j_{l_k}}), \quad (4)$$

where  $\varphi_{j_\ell}(x)$  is the loss rate at node  $j_\ell$  when its congestion level is  $x \geq 0$ , and  $\delta_k$  is the loss rate in a non-congested network. Another example is when the loss rate  $\beta_k$  depends only on the sum of the utilizations of the nodes used by class  $k$ , i.e.,

$$\beta_k(u) = \beta_k \left( \sum_{l=1}^{l_k} u_{j_l} \right). \quad (5)$$

### Asymptotic behaviour of typical connections

If  $(W_{n,k}(t))$  denotes the throughput of the  $n$ th class  $k$  connection,  $1 \leq n \leq N_k$ , then the vector

$$(W(t)) = ((W_{n,k}(t)), 1 \leq k \leq K, 1 \leq n \leq N_k, t \geq 0)$$

has the Markov property. As it stands, this Markov process is quite difficult to analyze. For this reason, a mean field scaling is used to get a more quantitative representation of the interaction of the flows. More specifically, it is assumed that the total number of connections  $\|N\|$  goes to infinity and that the total number of class  $k$  connections is of the order  $p_k \|N\|$ , where  $p_1 + \dots + p_K = 1$ .

For each  $1 \leq k \leq K$ , one takes a class  $k$  connection at random, let  $n_k$  be its index,  $1 \leq n_k \leq N_k$ . The process  $(W_{n_k,k}(t))$  represents the throughput of a “typical” class  $k$  connection. It is shown in Graham and Robert [5] that, as  $\|N\|$  goes to infinity and under mild assumptions, the process  $[(W_{n_k,k}(t)), 1 \leq k \leq K]$  converges in distribution to  $(\overline{W}(t)) = [(\overline{W}_k(t)), 1 \leq k \leq K]$ , where the processes  $(\overline{W}_k(t))$ , for  $1 \leq k \leq K$ , are independent and, for  $1 \leq k \leq K$ , the process  $(\overline{W}_k(t))$  is the solution of the following stochastic differential equation,

$$d\overline{W}_k(t) = a_k dt - (1 - r_k) \overline{W}_k(t-) \int \mathbb{1}_{\{0 \leq z \leq \overline{W}_k(t-) \beta_k(u_{\overline{W}}(t))\}} \mathcal{N}_k(dz, dt), \quad (6)$$

with  $u_{\overline{W}}(t) = (u_{\overline{W},j}(t), 1 \leq j \leq J)$  and, for  $1 \leq j \leq J$ ,

$$u_{\overline{W},j}(t) = \sum_{k=1}^K A_{jk} p_k \mathbb{E}(\overline{W}_k(t)),$$

where  $(\mathcal{N}_k, 1 \leq k \leq K)$  are i.i.d. Poisson point processes on  $\mathbb{R}_+^2$  with Lebesgue characteristic measure.

Because of the role of the deterministic function  $(u_{\overline{W}(t)})$  in these equations, the Markov property holds for this process but it is not *time-homogeneous*. The analogue of the infinitesimal generator  $\overline{\mathcal{Q}}_{k,t}$  is given by

$$\overline{\mathcal{Q}}_{k,t}(f)(x) = a_k f'(x) + x \beta_k(u_{\overline{W}}(t)) (f(r_k x) - f(r_k)).$$

The homogeneity holds when the function  $(u_{\overline{W}}(t))$  is equal to a constant  $u^*$ , which will be the case at equilibrium. In this case a class  $k$  connection behaves like a single isolated connection with parameters  $a = a_k$  and  $\beta = \beta_k(u^*)$ .

### The Fixed Point Equations

The following theorem gives a characterization of the invariant distributions for the process  $(\overline{W}(t))$ .

**Theorem 1.** *The invariant distributions for solutions  $(\overline{W}(t))$  of Equation (6) are in one-to-one correspondence with the solutions  $u \in \mathbb{R}_+^J$  of the fixed point equation*

$$u_j = \sum_{k=1}^K A_{jk} \phi_k(u), \quad 1 \leq j \leq J, \quad (7)$$

where

$$\phi_k(u) = p_k \sqrt{\frac{2}{\pi}} \left( \prod_{n=1}^{+\infty} \frac{1 - r_k^{2n}}{1 - r_k^{2n-1}} \right) \sqrt{\frac{a_k}{\beta_k(u)}}. \quad (8)$$

If  $u^*$  is such a solution, the corresponding invariant distribution has the density  $w \rightarrow \prod_{k=1}^K H_{r_k, \rho_k}(w_k)$  on  $\mathbb{R}_+^K$ , where  $\rho_k = a_k / \beta_k(u^*)$  and  $H_{r, \rho}$  is defined in Proposition 1.

The above theorem shows that if the fixed point equation (7) has several solutions, then the limiting process  $(\overline{W}(t))$  has several invariant distributions. Similarly, if equation (7) has no solution then, in particular,  $(\overline{W}(t))$  cannot converge to an equilibrium. These possibilities have been suggested in the Internet literature through simulations, like the cyclic behavior of some nodes in the case of congestion.

Under mild and natural assumptions, such as the loss rate being non-decreasing with respect to the utilization in the nodes, we show that for some specific topologies there exists a unique fixed point. We believe that such a uniqueness result will hold, in fact, for any network in general (under suitable regularity properties on the functions  $\beta_k$ ,  $k = 1, \dots, K$ ). Before proceeding to the examples, we first present an existence result that holds for a general network.

#### 2.1 An Existence Result

In this section the existence of a solution to the fixed-point equation (7) is proved for a quite general framework.

If  $u$  is a solution of Equation (7) and  $z_k = \phi_k(u)$ ,  $1 \leq k \leq K$ , then the vector  $z = (z_k)$  satisfies the relation  $u = Az$ , i.e.,  $u_j = A_{j1}z_1 + A_{j2}z_2 + \dots + A_{jK}z_K$ ,  $1 \leq j \leq J$ , and as well

$$z = \Phi(z) \stackrel{\text{def.}}{=} (\phi_k(Az), 1 \leq k \leq K). \quad (9)$$

The proposition below gives a simple criterion for the existence of a fixed point.

**Proposition 2.** *If the functions  $u \rightarrow \beta_k(u)$ ,  $1 \leq k \leq K$ , are continuous and non-decreasing, and if there exists a vector  $z^{(0)} \in \mathbb{R}_+^K$  such that the relations*

$$z^{(0)} \leq \Phi(z^{(0)}), \quad z^{(0)} \leq \Phi(\Phi(z^{(0)})), \quad \text{and} \quad \Phi(z^{(0)}) < \infty,$$

hold coordinate by coordinate, then there exists at least one solution for the fixed point Equation (9) and therefore also for Equation (7).

*Proof.* Define the sequence  $z^{(n)} = \Phi(z^{(n-1)})$ ,  $n = 1, 2, \dots$ . From  $z^{(0)} \leq \Phi(z^{(0)})$  and  $z^{(0)} \leq \Phi(\Phi(z^{(0)}))$ , it follows that  $z^{(0)} \leq z^{(1)}$  and  $z^{(0)} \leq z^{(2)}$ . Since the function  $\Phi$  is non-increasing, one gets that the relation

$$z^{(0)} \leq z^{(2)} \leq \dots \leq z^{(2n)} \leq \dots \leq z^{(2n+1)} \leq \dots \leq z^{(3)} \leq z^{(1)}$$

holds. Hence, there are  $z_*, z^* \in \mathbb{R}_+^K$  such that

$$\lim_{n \rightarrow \infty} z^{(2n)} = z_* \text{ and } \lim_{n \rightarrow \infty} z^{(2n+1)} = z^*,$$

with  $z_* \leq z^*$ . Since  $z^{(2n)} = \Phi(z^{(2n-1)})$  and  $z^{(2n+1)} = \Phi(z^{(2n)})$ , by continuity we also have that  $z^* = \Phi(z_*)$  and  $z_* = \Phi(z^*)$ .

Define the set  $D = \{z : z_* \leq z \leq z^*\}$ . Note that  $z^* \leq z^{(1)} = \Phi(z^{(0)}) < \infty$ , hence  $D$  is bounded. In addition, for  $z \in D$ ,

$$z_* = \Phi(z^*) \leq \Phi(z) \leq \Phi(z_*) = z^*,$$

since the function  $\Phi$  is non-increasing. One can therefore apply Brouwer fixed point theorem to  $\Phi$  restricted to the compact convex set  $D$ , and conclude that  $D$  contains at least one fixed point of the function  $\Phi$ . The proposition is proved.

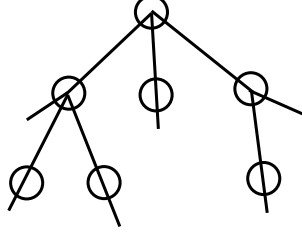
The conditions of Proposition 2 trivially hold when  $\beta_k$  is non-decreasing and  $\beta_k(0) > 0$  for all  $k = 1, \dots, K$ , since then  $\Phi(0) < \infty$ ,  $0 \leq \Phi(0)$  and  $0 \leq \Phi(\Phi(0))$ . In particular, when the function  $\beta_k$  is given by (4),  $\delta_k > 0$  is a sufficient condition for the existence of a fixed point.

### 3 Tree topologies

We consider a finite tree network. A connection starts in the root and then follows the tree structure until it leaves the network at some node. The set of routes is therefore indexed by the set of nodes, i.e., a connection following route  $G \in \mathcal{T}$  starts in the root and leaves the tree in node  $G$ .

The tree can be classically represented as a subset  $\mathcal{T}$  of  $\cup_{n \geq 0} \mathbb{N}^n$  with the constraint that if  $G = (g_1, \dots, g_p) \in \mathcal{T}$ , then, for  $1 \leq \ell \leq p$ , the element  $H = (g_1, \dots, g_\ell)$  is a node of the tree as well. In addition, node  $H$  is the  $g_\ell$ th child of generation (level)  $\ell$  and the ancestor of  $G$  for this generation. One writes  $H \subseteq G$  in this situation and  $H \vdash G$  when  $\ell = p - 1$ , i.e., when  $G$  is a daughter of  $H$ . The quantity  $u_{[H,G]}$  denotes the vector  $(u_P, P : H \subseteq P \subseteq G)$ . The root of the tree is denoted by  $\emptyset$ . Assume that  $A_{HG} = 1$  if route  $G$  uses node  $H$ , and 0 otherwise. Equation (7) writes in this case,

$$u_H = \sum_{G \in \mathcal{T}, H \subseteq G} \phi_G(u_{[\emptyset, G]}), \quad H \in \mathcal{T},$$



**Fig. 1.** Tree with connections starting at root node

which is equivalent to the recursive equations

$$u_H = \phi_H(u_{[\emptyset, H]}) + \sum_{G \in \mathcal{T}, H \vdash G} u_G, \quad H \in \mathcal{T}. \quad (10)$$

**Proposition 3.** *If the functions  $\beta_H, H \in \mathcal{T}$ , are continuous and non-decreasing, then there exists a unique solution for the fixed point equation (7).*

*Proof.* Let  $H$  be a maximal element on  $\mathcal{T}$  for the relation  $\subseteq$ , i.e.,  $H$  is a leaf, and denote by  $P(H)$  the parent of node  $H$ . Equation (10) then writes

$$u_H = \phi_H(u_{[\emptyset, P(H)]}, u_H). \quad (11)$$

The function  $\phi_H$  being non-increasing and continuous, for a fixed vector  $u_{[\emptyset, P(H)]}$ , there exists a unique solution  $u_H = F_H(u_{[\emptyset, P(H)]}) \geq 0$  to the above equation. Furthermore, the function  $u_{[\emptyset, P(H)]} \rightarrow F_H(u_{[\emptyset, P(H)]})$  is continuous and non-increasing. For such an  $H$ , for  $H' = P(H)$ , Relation (10) can then be written as

$$u_{H'} = \phi_{H'}(u_{[\emptyset, P(H')]}), u_{H'}) + \sum_{G \in \mathcal{T}, H' \vdash G} F_G(u_{[\emptyset, P(H')]}), u_{H'}).$$

Since  $\phi_{H'}$  and  $F_G$ , with  $G$  a leaf, are non-increasing and continuous, there exists a unique solution  $u_{H'} = F_{H'}(u_{[\emptyset, P(H')]})) \geq 0$  and the function

$$u_{[\emptyset, P(H')]} \rightarrow F_{H'}(u_{[\emptyset, P(H')]}),$$

is continuous and non-increasing. By induction (by decreasing level of nodes), one obtains that a family of continuous, non-increasing functions  $F_G, G \in \mathcal{T}, G \neq \emptyset$ , exists, such that, for a fixed vector  $u_{[\emptyset, P(G)]}$ ,  $u_G = F_G(u_{[\emptyset, P(G)]})$  is the unique solution of

$$u_G = \phi_G(u_{[\emptyset, P(G)]}, u_G) + \sum_{G' \in \mathcal{T}, G \vdash G'} F_{G'}(u_{[\emptyset, P(G)]}, u_G).$$

Equation (10) at the root then writes

$$u_\emptyset = \phi_\emptyset(u_\emptyset) + \sum_{G \in \mathcal{T}, \emptyset \vdash G} F_G(u_\emptyset),$$

and this equation has a unique solution  $\bar{u}_\emptyset$ . Now, one defines recursively (by increasing level of nodes)

$$\bar{u}_G = F_G(\bar{u}_{[\emptyset, P(G)]}), \quad G \in \mathcal{T}.$$

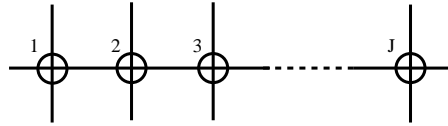
Then clearly  $(\bar{u}_G, G \in \mathcal{T})$  satisfies Relation (10) and is the unique solution.

## 4 Linear topologies

In this section we consider a linear network with  $J$  nodes and  $K = J + 1$  classes of connections. Class  $j$  connections,  $1 \leq j \leq J$ , use node  $j$  only, while class 0 connections use all  $J$  nodes. Assume  $A_{jk} = 1$  if class  $k$  uses node  $j$ , and 0 otherwise. Equation (7) is in this case

$$u_j = \phi_0(u) + \phi_j(u_j), \quad 1 \leq j \leq J, \quad (12)$$

with  $u = (u_1, \dots, u_J)$ .



**Fig. 2.** A linear network with  $J$  nodes and  $K = J + 1$  classes of connections

**Proposition 4.** *If the functions,  $\beta_k, 0 \leq k \leq J$ , are continuous and non-decreasing, then there exists a unique solution for the fixed point equation (7).*

*Proof.* Let  $\bar{\phi}_j(x) = x - \phi_j(x)$ ,  $x \in \mathbb{R}$ , which is continuous and non-decreasing. Hence, (12) can be rewritten as

$$u_j = \bar{\phi}_j^{-1}(\phi_0(u)) = \bar{\phi}_j^{-1}\left(\frac{\alpha_0}{\sqrt{\beta_0(u)}}\right), \quad 1 \leq j \leq J. \quad (13)$$

for some constant  $\alpha_0$ , see Equation (8). In addition, define the function  $\psi_j(x) = \bar{\phi}_j^{-1}(\alpha_0/\sqrt{x})$ ,  $x \in \mathbb{R}$ , which is continuous and non-increasing. From (13) we obtain the relation

$$\beta_0(u) = \beta_0(\psi_1(\beta_0(u)), \dots, \psi_J(\beta_0(u))).$$

Since  $\beta_0$  is non-decreasing and  $\psi_j$  is non-increasing, the fixed point equation  $\beta = \beta_0(\psi_1(\beta), \dots, \psi_J(\beta))$  has a unique solution  $\beta^* \geq 0$ . Hence, the Relation (13) has a unique fixed point, which is given by  $u_j^* = \bar{\phi}_j^{-1}(\alpha_0/\sqrt{\beta^*})$ .

## 5 Ring topologies

In this section, the topology of the network is based on a ring. Several situations are considered for the paths of the connections.

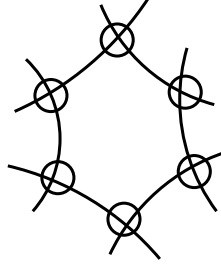
### Routes with two consecutive nodes

It is assumed that there are  $J$  nodes and  $K = J$  classes of connections and class  $j \in \{1, \dots, J\}$  uses two nodes: node  $j$  and  $j + 1$ . Assume  $A_{jk} = 1$  if class  $k$  uses node  $j$ , and 0 otherwise. Equation (7) is in this case

$$u_j = \phi_{j-1}(u_{j-1}, u_j) + \phi_j(u_j, u_{j+1}), \quad j = 1, \dots, J. \quad (14)$$

For  $y_j = \phi_j(u_j, u_{j+1})$ , the above equation can be rewritten as follows

$$y_j = \phi_j(y_{j-1} + y_j, y_j + y_{j+1}), \quad j = 1, \dots, J. \quad (15)$$



**Fig. 3.** Routes with two consecutive nodes

**Proposition 5.** *If the functions  $\beta_k$ ,  $1 \leq k \leq K$ , are continuous, non-decreasing and satisfy the assumptions of Proposition 2, then there exists a unique solution for the fixed point equation (7).*

*Proof.* From Proposition 2 we have that Equation (15) has at least one fixed point solution. Let  $x = (x_j : j = 1, 2, \dots, J)$  and  $y = (y_j : j = 1, 2, \dots, J)$  both be fixed points.

If the relation  $y_j < x_j$  holds for all  $j = 1, \dots, J$ , then the inequality

$$\phi_j(y_j + y_{j-1}, y_{j+1} + y_j) = y_j < x_j = \phi_j(x_j + x_{j-1}, x_{j+1} + x_j),$$

and the fact that the function  $\phi_j$  is non-increasing, give directly a contradiction.

Consequently, possibly up to an exchange of  $x$  and  $y$ , one can assume that there exists  $m \in \{1, \dots, J\}$  such that  $y_m \leq x_m$  and  $y_{m+1} \geq x_{m+1}$ . Define  $c_j = x_j - y_j$  and  $d_j = y_j - x_j$ . Hence,  $c_m \geq 0$  and  $d_{m+1} \geq 0$ . Without loss of

generality, it can be assumed that the classes are ordered such that  $d_{m-1} \leq d_{m+1}$ . Since the function  $\phi_m$  is non-increasing, and

$$\phi_m(y_m + y_{m-1}, y_m + y_{m+1}) = y_m \leq x_m = \phi_m(x_m + x_{m-1}, x_m + x_{m+1}),$$

we have that either

$$y_m + y_{m-1} \geq x_m + x_{m-1} \text{ and/or } y_m + y_{m+1} \geq x_m + x_{m+1},$$

i.e.,  $d_{m-1} \geq c_m$  and/or  $d_{m+1} \geq c_m$ . Because  $d_{m-1} \leq d_{m+1}$ , then, necessarily,  $d_{m+1} \geq c_m \geq 0$ . Hence

$$\begin{aligned} \phi_{m+1}(y_{m+1} + y_m, y_{m+1} + y_{m+2}) &= y_{m+1} \\ &\geq x_{m+1} = \phi_{m+1}(x_{m+1} + x_m, x_{m+1} + x_{m+2}). \end{aligned}$$

Since  $\phi_{m+1}$  is non-increasing, one has  $y_{m+1} + y_{m+2} \leq x_{m+1} + x_{m+2}$  and consequently  $d_{m+1} \leq c_{m+2}$ .

From  $0 \leq c_m \leq d_{m+1} \leq c_{m+2}$ , we obtain  $x_{m+2} \geq y_{m+2}$ , which, using the same steps as before, implies  $c_{m+2} \leq d_{m+3}$ . In particular, by induction it can be concluded that

$$c_j \leq d_{j+1} \leq c_{j+2} \leq d_{j+3}, \text{ for all } j = 1, \dots, J,$$

where the indices  $j+1$ ,  $j+2$ , and  $j+3$  are considered as modulo  $J$ . This implies that  $c_j = d_j = c$ , for all  $j = 1, \dots, L$ , and hence  $y_j + y_{j-1} = x_j + x_{j-1}$ , i.e.,

$$y_j = \phi_j(y_j + y_{j-1}, y_j + y_{j+1}) = \phi_j(x_j + x_{j-1}, x_j + x_{j+1}) = x_j,$$

for  $j = 1, \dots, J$ . We can conclude that the fixed point is unique.

The rest of this part will be devoted to a contraction argument that can be used to get a unique solution to the fixed point equation.

**Proposition 6.** *If the functions  $\beta_k$ ,  $1 \leq k \leq K$ , are Lipschitz, continuous differentiable, and non-decreasing, then there exists a unique solution for the fixed point equation (7).*

*Proof.* The proof consists in showing that (15) has a unique solution. By the Implicit function theorem, there exists a unique  $x_j(y_{j-1}, y_{j+1})$  such that,

$$x_j(y_{j-1}, y_{j+1}) = \phi_j(y_{j-1} + x_j(y_{j-1}, y_{j+1}), x_j(y_{j-1}, y_{j+1}) + y_{j+1}), \quad (16)$$

and this function  $(y_{j-1}, y_{j+1}) \rightarrow x_j(y_{j-1}, y_{j+1})$  is positive and continuous differentiable. Taking the partial derivative to  $y_{j-1}$  on both sides of this identity, one gets that

$$\begin{aligned} \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} &= \frac{\partial \phi_j(s_1, s_2)}{\partial s_1} \Big|_{s=s(y)} \times \left( 1 + \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} \right) \\ &\quad + \frac{\partial \phi_j(s_1, s_2)}{\partial s_2} \Big|_{s=s(y)} \times \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}}, \end{aligned}$$

with  $s(y) = (y_{j-1} + x_j(y_{j-1}, y_{j+1}), x_j(y_{j-1}, y_{j+1}) + y_{j+1})$ . Hence,

$$\frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} = \left[ \frac{\partial \phi_j(s_1, s_2)}{\partial s_1} \middle/ \left( 1 - \frac{\partial \phi_j(s_1, s_2)}{\partial s_1} - \frac{\partial \phi_j(s_1, s_2)}{\partial s_2} \right) \right] \bigg|_{s=s(y)} \leq 0.$$

A similar expression holds for  $\partial x_j(y_{j-1}, y_{j+1})/\partial y_{j+1} \leq 0$ , and one can conclude that

$$\begin{aligned} & \left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} \right| + \left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j+1}} \right| \\ &= - \left[ \left( \frac{\partial \phi_j(s_1, s_2)}{\partial s_1} + \frac{\partial \phi_j(s_1, s_2)}{\partial s_2} \right) \middle/ \left( 1 - \frac{\partial \phi_j(s_1, s_2)}{\partial s_1} - \frac{\partial \phi_j(s_1, s_2)}{\partial s_2} \right) \right] \bigg|_{s=s(y)}. \end{aligned} \quad (17)$$

If  $x_j(0, y_{j+1}) = 0$ , then by Relation (16), one gets that, for some constant  $\alpha_j$ , see Equation (8),

$$0 = \phi_j(0, y_{j+1}) = \alpha_j / \sqrt{\beta_j(0, y_{j+1})},$$

which holds only if  $y_{j+1} = \infty$ . Hence,  $x_j(0, y_{j+1}) > 0$ . Since  $x_j(y_{j-1}, y_{j+1})$  is continuous and positive, and  $x_j(0, y_{j+1}) > 0$ , one obtains that there exists an  $M_j^- > 0$  such that  $y_{j-1} + x_j(y_{j-1}, y_{j+1}) > M_j^-$  for all  $y_{j-1}, y_{j+1} \geq 0$ . Similarly, there exists an  $M_j^+ > 0$  such that  $x_j(y_{j-1}, y_{j+1}) + y_{j+1} > M_j^+$ . This gives the following upper bound,

$$\begin{aligned} - \frac{\partial \phi_j(s_1, s_2)}{\partial s_i} \bigg|_{s=s(y)} &= \frac{\alpha_j}{2} \frac{\partial \beta_j(s_1, s_2)}{\partial s_i} \bigg|_{s=s(y)} \beta_j(s(y))^{-3/2} \\ &\leq \frac{\alpha_j}{2} \frac{L}{(\beta_j(M_j^-, M_j^+))^{3/2}}, \end{aligned}$$

where we used that  $\beta_j$  is non-decreasing, Lipschitz continuous (with constant  $L$ ) and differentiable. From Equation (17)) one now obtains that there exists a constant  $0 < C < 1$  such that

$$\left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} \right| + \left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j+1}} \right| < C.$$

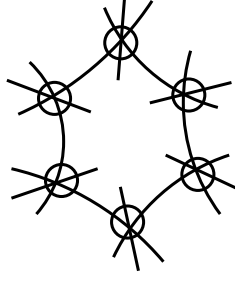
Hence, the mapping  $T : \mathbb{R}_+^J \rightarrow \mathbb{R}_+^J$  with  $T(y) = (x_j(y_{j-1}, y_{j+1}))$  for  $j = 1, \dots, J$  is a contraction, and has a unique fixed point  $(y_j^*)$ , i.e., Equation (15) has a unique solution.

### Routes with one node or two consecutive nodes

Consider now a ring with  $J$  nodes and  $K = 2J$  classes. Class  $j$  uses two nodes: nodes  $j$  and  $j + 1$ ,  $j = 1, \dots, J$ . Class  $0j$  uses one node: node  $j$ ,  $j = 1, \dots, J$ . We assume that  $A_{jk} = 1$  if and only if class  $k$  uses node  $j$ , and zero otherwise.

We focus on functions  $\beta_k$  that satisfy (5). Equation (7) is in this context

$$u_j = \phi_{0j}(u_j) + \phi_{j-1}(u_{j-1} + u_j) + \phi_j(u_j + u_{j+1}), \quad 1 \leq j \leq J.$$



**Fig. 4.** Routes with one node or two consecutive nodes

For  $y_j = \phi_j(u_j + u_{j+1})$  and  $y_{0j} = \phi_{0j}(u_j)$ ,  $j = 1 \dots, J$ , the above equation can be rewritten as follows, for  $j = 1, \dots, J$ ,

$$\begin{cases} y_j &= \phi_j(y_{j-1} + 2y_j + y_{j+1} + y_{0j} + y_{0j+1}), \\ y_{0j} &= \phi_{0j}(y_{0j} + y_{j-1} + y_j). \end{cases} \quad (18)$$

**Proposition 7.** *If the functions  $\beta_k$ ,  $1 \leq k \leq J$ ,  $01 \leq k \leq 0J$ , are Lipschitz, continuously differentiable, non-decreasing, and satisfy (5), then there exists a unique solution for the fixed point equation (7).*

*Proof.* By the Implicit function theorem, for each  $j$ , there exists a unique  $x_{0j}(t)$  satisfying the relation  $x_{0j}(t) = \phi_{0j}(x_{0j}(t) + t)$ , and this function is non-increasing and continuous differentiable. One now has to solve the equation

$$y_j = \phi_j(y_{j-1} + 2y_j + y_{j+1} + x_{0j}(y_{j-1} + y_j) + x_{0,j+1}(y_j + y_{j+1})). \quad (19)$$

From the fact that  $-1 \leq x'_{0j}(t) \leq 0$ , it can be easily checked that the right-hand side of Equation (19) is non-increasing in  $y_j$ . Hence, there exists a unique  $x_j(y_{j-1}, y_{j+1})$  such that  $y_j = x_j(y_{j-1}, y_{j+1})$  satisfies Equation (19), and this function  $(y_{j-1}, y_{j+1}) \rightarrow x_j(y_{j-1}, y_{j+1})$  is positive and continuous differentiable (by the Implicit function theorem). In particular,  $x_j(y_{j-1}, y_{j+1}) = \phi_j(f_j(y))$ , for all  $j = 1, \dots, L$ , with

$$\begin{aligned} f_j(y) &= y_{j-1} + 2x_j(y_{j-1}, y_{j+1}) + y_{j+1} \\ &\quad + x_{0j}(y_{j-1} + x_j(y_{j-1}, y_{j+1})) + x_{0,j+1}(x_j(y_{j-1}, y_{j+1}) + y_{j+1}). \end{aligned}$$

From this one can derive that

$$\begin{aligned} \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} &= \phi'_j(f_j(y)) [1 + x'_{0j}(y_{j-1} + x_j(y_{j-1}, y_{j+1}))] / \\ &\quad \left[ 1 - \phi'_j(f_j(y))(2 + x'_{0j}(y_{j-1} + x_j(y_{j-1}, y_{j+1})) \right. \\ &\quad \left. + x'_{0,j+1}(x_j(y_{j-1}, y_{j+1}) + y_{j+1})) \right] \leq 0, \end{aligned}$$

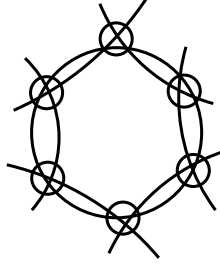
and a similar expression holds for  $\partial x_j(y_{j-1}, y_{j+1})/\partial y_{j+1} \leq 0$ . As in the proof of Proposition 6, an upper bound on  $-\phi'_j(f_j(y))$  can be obtained. This implies that there exists a constant  $0 < C < 1$  such that

$$\left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j-1}} \right| + \left| \frac{\partial x_j(y_{j-1}, y_{j+1})}{\partial y_{j+1}} \right| < C, \quad j = 1, \dots, J.$$

Hence, the mapping  $T : \mathbb{R}_+^J \rightarrow \mathbb{R}_+^J$  with  $T(y_1, \dots, y_J) = (x_j(y_{j-1}, y_{j+1}), j = 1, \dots, J)$  is a contraction, and has a unique fixed point  $(y_j^*)$ . One concludes that there exists a unique solution  $y_j^*$  and  $y_{0j}^* = x_{0j}(y_{j-1}^* + y_j^*)$ ,  $j = 1, \dots, J$ , of (18).

### Routes with two consecutive nodes and a complete route

Consider a ring with  $J$  nodes and  $K = J + 1$  classes. Class  $1 \leq j \leq J$  uses two nodes: node  $j$  and  $j + 1$  and class 0 uses all nodes  $1, \dots, J$ .



**Fig. 5.** Routes with two consecutive nodes and a complete route

We focus on functions  $\beta_k$  that satisfy (5). Equation (7) is in this context

$$u_j = \phi_0(u_1 + \dots + u_J) + \phi_{j-1}(u_{j-1} + u_j) + \phi_j(u_j + u_{j+1}).$$

For  $y_j = \phi_j(u_j + u_{j+1})$  and  $y_0 = \phi_0(u_1 + u_2 + \dots + u_J)$ , the above equation can be rewritten as follows

$$\begin{cases} y_j &= \phi_j(y_{j-1} + 2y_j + y_{j+1} + 2y_0), \quad j = 1, \dots, J, \\ y_0 &= \phi_0(Jy_0 + 2\sum_{j=1}^J y_j). \end{cases} \quad (20)$$

**Proposition 8.** *If the functions  $\beta_k$ ,  $0 \leq k \leq J$ , are continuous, non-decreasing, satisfy (5), and satisfy the assumptions of Proposition 2, then there exists a unique solution for the fixed point equation (7).*

*Proof.* From Proposition 2 we have that Equation (20) has at least one fixed point solution. Let  $x = (x_j : j = 0, 1, \dots, J)$  and  $y = (y_j : j = 0, 1, \dots, J)$  both be fixed points.

If the relation  $y_j < x_j$  holds for all  $j = 1, \dots, J$ , then

$$\phi_j(2y_j + y_{j-1} + y_{j+1} + 2y_0) = y_j < x_j = \phi_j(2x_j + x_{j-1} + x_{j+1} + 2x_0).$$

Since the function  $\phi_j$  is non-increasing, one gets that

$$2y_j + y_{j-1} + y_{j+1} + 2y_0 > 2x_j + x_{j-1} + x_{j+1} + 2x_0.$$

Summing over all  $j = 1, \dots, L$ , we obtain

$$4(y_1 + \dots + y_J) + 2Jy_0 > 4(x_1 + \dots + x_J) + 2Jx_0,$$

which implies that  $x_0 < y_0$ . However,

$$y_0 = \phi_0(2(y_1 + \dots + y_J) + Jy_0) \leq \phi_0(2(x_1 + \dots + x_J) + Jx_0) = x_0,$$

hence, we obtain a contradiction.

We can conclude that there is an  $m \in \{1, \dots, J\}$  such that  $y_m \leq x_m$  and  $y_{m+1} \geq x_{m+1}$ . To show that  $x = y$ , one proceeds along similar lines as in the proof of Proposition 6.

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