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Abstract: A *good edge-labelling* of a graph G is a labelling of its edges such that, for any ordered pair of vertices (x, y), there do not exist two paths from x to y with increasing labels. This notion was introduced in [**?**] to solve wavelength assignment problems for specific categories of graphs. In this paper, we aim at characterizing the class of graphs that admit a good edge-labelling. First, we exhibit infinite families of graphs for which no such edge-labelling can be found. We then show that deciding if a graph admits a good edge-labelling is NP-complete. Finally, we give large classes of graphs admitting a good edge-labelling: forests, C_3-free outerplanar graphs, planar graphs of girth at least 6, subcubic $\{C_3, K_{2,3}\}$-free graphs.

Key-words: graph theory, complexity, edge-labelling, planar graphs, matching-cut, channel assignment.

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Bon étiquetages des arêtes des graphes

Résumé : Un bon étiquetage des arêtes d'un graphe est un étiquetage de ses arêtes tel que, pour toute paire de sommets (x, y), il n'existe pas deux chemins de x vers y dont les étiquettes sont croissantes. Cette notion a été introduite dans [**?**] pour résoudre des problèmes d'allocation de fréquence pour des classes particulières de graphes. Dans cet article, nous tentons de caractériser la classe des graphes qui admettent un bon étiquetage. Tout d'abord, nous produisons des familles infinies de graphes pour lesquelles aucun bon étiquetage n'existe. Nous montrons par la suite que le problème de décider si un graphe admet un bon étiquetage est NP-Complet. Finalement, nous donnons de larges classes de graphes admettant un bon étiquetage : les forêts, les graphes sans triangles, les graphes planaires de maille au moins 6, les graphes subcubiques sans triangles ni $K_{2,3}$

Mots-clés : théorie des graphes, complexité, coloration d'arêtes, graphes planaires, coupe parallle, allocation de fréquence.

Good edge-labelling of graphs

J. Araujo, N. Cohen, F. Giroire and F. Havet

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A *good edge-labelling* of a graph G is a labelling of its edges such that, for any ordered pair of vertices (x, y), there do not exist two paths from x to y with increasing labels. This notion was introduced in [2] to solve wavelength assignment problems for specific categories of graphs. In this paper, we aim at characterizing the class of graphs that admit a good edge-labelling. First, we exhibit infinite families of graphs for which no such edge-labelling can be found. We then show that deciding if a graph admits a good edge-labelling is NP-complete. Finally, we give large classes of graphs admitting a good edge-labelling: forests, C_3-free outerplanar graphs, planar graphs of girth at least 6, subcubic graphs.

1 Introduction

A classical and widely studied problem in WDM (Wavelength Division Multiplexing) networks is the Routing and Wavelength Assignment (RWA) problem [8, 9, 1]. It consists in finding routes, and their associated wavelength as well, to satisfy a set of traffic requests while minimising the number of used wavelengths. This is a difficult problem which is, in general, NP-hard. Thus, it is often split into two distinct problems: First, routes are found for the requests. Then, in a second step, these routes are taken as an input. Wavelengths must be associated to them in such a way that two routes using the same fiber do not have the same wavelength. The last problem can be reformulated as follows: Given a digraph and a set of dipaths, corresponding to the routes for the requests, find the minimal number of wavelengths w needed to assign different wavelengths to dipaths sharing an edge. This problem can be seen as a colouring problem of the *conflict graph* which is defined as follows: It has one vertex per dipath and two vertices are linked by an edge if their corresponding dipaths share an edge. In [2], Bermond et al. studied the RWA problem for UPP-DAG which are acyclic digraphs (or DAG) in which there is at most one dipath from one vertex to another. In such digraph the routing is forced and thus the unique problem is the wavelength assignment one.

In their paper, they introduce the notion of good edge-labelling. An *edge-labelling* of a graph G is a function $\phi : E(G) \to \mathbb{R}$. A path is *increasing* if the sequence of its edge labels is non-decreasing. An edge-labelling of G is *good* if, for any two distinct vertices u, v, there is at most one increasing (u, v)-path. Bermond et al. [2] showed that the conflict graph of a

set of dipaths in a UPP-DAG has a *good edge-labelling*. Conversely, for any graph admitting a good edge-labelling one can exhibit a family of dipaths on a UPP-DAG whose conflict graph is precisely this graph. Bermond et al. [2] then use the existence of graphs with a good edge-labelling and large chromatic number to prove that there exist sets of requests on UPP-DAGs with load 2 (an edge is shared by at most two paths) requiring an arbitrarily large number of wavelengths.

To obtain other results on this problem, it may be useful to identify the *good* graphs which admit a good labelling and the *bad* ones which do not. Bermond et al.[2] noticed that C_3 and $K_{2,3}$ are bad. J.-S. Sereni [11] asked whether every $\{C_3, K_{2,3}\}$*-free graph* (i.e., with no C_3 nor $K_{2,3}$ as a subgraph) is good. In Section 3, we answer this question in the negative. We give an infinite family of bad graphs none of which is the subgraph of another.

Furthermore, in Section 4, we prove that determining if a graph has a good edge-labelling is NP-complete using a reduction of Not-All-Equal 3-SAT.

In Section 5, we show large classes of good graphs: forests, C_3-free outerplanar graphs, planar graphs of girth at least 6. To do so, we use the notion of *critical* graph which is a bad graph every proper subgraph of which is good. Clearly, a good edge-labelling of a graph induces a good edge-labelling of all its subgraphs. So every bad graph must contain a critical subgraph. We establish several properties of critical graphs. In particular, we show that they have no *matching-cut*. Hence, a result of Farley and Proskurowski [6] (Theorem 16) implies that a critical graph G has a least $\frac{3}{2}|V(G)| - \frac{3}{2}$ edges unless G is C_3 or $K_{2,3}$.

In Section 6, we use the characterization of graphs with no matching-cut and $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges given by Bonsma [3] to slightly improve this result. We show that a critical graph G has a least $\frac{3}{2}|V(G)| - \frac{1}{2}$ edges unless G is C_3 or $K_{2,3}$.

Finally, we present avenues for future research.

2 Preliminaries

In this section, we give some technically useful propositions. Their proof are straightforward and left to the reader.

A path is *decreasing* if the sequence of its edge labels is non-increasing. Then, a path $u_1u_2\dots u_k$ is decreasing if and only if its reversal $u_ku_{k-1}\dots u_1$ is increasing. Hence an edge-labelling is good if and only if for any two distinct vertices u, v, there is at most one decreasing (u,v)-path. Equivalently, an edge-labelling is good if and only if for any two distinct vertices u, v, there is at most one increasing (u,v)-path and at most one decreasing (u,v)-path.

Let x and y be two vertices of G. Two distinct (x,y)-paths P and Q are *independent* if $V(P) \cap V(Q) = \{x,y\}$. Observe that in an edge-labelled graph G, there are two vertices (u,v) with two increasing (u,v)-paths if and only if there are two vertices (u,v) with two increasing independent (u,v)-paths. Hence the definition of good edge-labelling may be expressed in terms of independent paths.

Proposition 1 *An edge-labelling is good if and only if for any two distinct vertices u and v, there are no two increasing independent (u,v)-paths.*

As above the definition may also be in terms of decreasing independent paths.

Proposition 2 *Let G be a graph. Then the following assertions are equivalent:*

(i) *G admits a good edge-labelling in $\mathbb{R}$;*

(ii) *G admits an injective good edge-labelling;*

(iii) *G admits a good edge-labelling in $\mathbb{R} \cup \{-\infty, +\infty\}$;*

(iv) *G admits a good edge-labelling in $\mathbb{Z}$.*

In the paper, we sometimes use these propositions without referring explicitly to them.

3 Bad graphs

A path of length one is both increasing and decreasing, and a path of length two is either increasing or decreasing. So C_3 has clearly no good edge-labelling. Also $K_{2,3}$ does not admit a good edge-labelling since there are three paths of length two between the two vertices of degree 2. Hence, in any edge-labelling, two of them are increasing or two of them are decreasing.

Extending this idea, we now construct an infinite family of bad graphs, none of which is the subgraph of another. The construction of this family is based on the graphs H_k defined below. These graphs play the same role as a path of length two because they have two vertices u and v such that any good edge-labelling of H_k has either a (u, v)-increasing path or a (v, u)-increasing path.

For any integer $k \geq 3$, let H_k be the graph defined by

$$\begin{aligned} V(H_k) &= \{u, v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_i \mid 1 \leq i \leq k\}, \\ E(H_k) &= \{uu_i \mid 1 \leq i \leq k\} \cup \{u_i v_i \mid 1 \leq i \leq k\} \cup \{v_i v \mid 1 \leq i \leq k\}, \\ &\quad \cup \{v_i u_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

with $u_{k+1} = u_1$. See Figure 1.

Observe that the graph H_k has no $K_{2,3}$ as a subgraph.

Proposition 3 *Let $k \geq 3$. For every good edge-labelling, the graph H_k has either an increasing (u, v)-path or an increasing (v, u)-path.*

Proof: Suppose, by way of contradiction, that H_k has a good edge-labelling ϕ having no increasing (u, v)-path and no increasing (v, u)-path. By Proposition 2, we may assume that ϕ is injective.

A key component in this proof is the following observation which follows easily from the fact that ϕ is good.

Observation 3.1 *If $x_1x_2x_3x_4x_1$ is a 4-cycle. Then, either $\phi(x_4x_1) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_1x_2)$, $\phi(x_2x_3) < \phi(x_3x_4)$, $\phi(x_1x_4) < \phi(x_3x_4)$, or all those inequalities are reversed.*

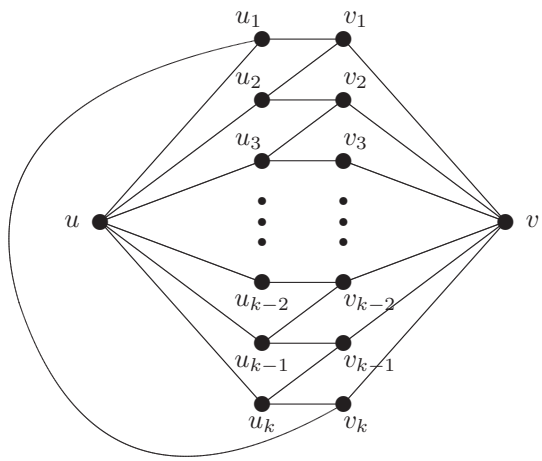

Figure 1: Graph H_k

By symmetry, we may assume that $\phi(uu_1) < \phi(u_1v_1)$. By Observation 3.1, $\phi(v_1u_2) < \phi(uu_2)$ and $\phi(uu_1) < \phi(uu_2)$. Then, since $vv_1u_2v_1u$ is not increasing, $\phi(u_2v_1) < \phi(v_1v)$. Again by Observation 3.1, $\phi(v_2v) < \phi(u_2v_2)$. Thus since uu_2v_2v is not increasing $\phi(uu_2) < \phi(u_2v_2)$.

Applying the same reasoning iteratively, we obtain that $\phi(uu_2) < \phi(uu_3)$ and $\phi(uu_3) < \phi(u_3v_3)$ And so on, iteratively, $\phi(uu_1) < \phi(uu_2) < \cdots < \phi(uu_k) < \phi(uu_1)$, a contradiction. $\square$

For convenience we denote by H_2 the path of length 2 with end vertices u and v. Let i, j, k be three integers greater than 1. The graph $J_{i,j,k}$ is the graph obtained from disjoint copies of H_i, H_j and H_k by identifying the vertices u of the three copies and the vertices v of the three copies.

Proposition 4 *Let i, j, k be three integers greater than 1. Then $J_{i,j,k}$ is bad.*

Proof: Suppose, by way of contradiction, that $J_{i,j,k}$ admits a good edge-labelling. By Proposition 3, in each of the subgraphs H_i, H_j and H_k, there is either an increasing (u, v)-path or an increasing (v, u)-path. Hence in $J_{i,j,k}$, there are either two increasing (u, v)-paths or two increasing (v, u)-paths, a contradiction. $\square$

4 NP-completeness

In this section, we prove that it is an NP-complete problem to decide if a graph admits a good edge-labelling. We give a reduction from the NOT-ALL-EQUAL (NAE) 3-SAT Problem [10] which is defined as follows:

Instance: A set V of variables and a collection $\mathcal{C}$ of clauses over V such that each clause has exactly 3 literals.

Question: Is there a truth assignment such that each clause has at least one true and at least one false literal?

Theorem 5 *The following problem is NP-complete.*
Instance: *A graph G.*
Question: *Does G have a good edge-labelling?*

Proof: We will actually reduce to the NAE 3-SAT Problem without repetition (i.e. a variable appears at most once in each clause) which is equivalent to NAE 3-SAT Problem (with repetition). (For each variable x, we introduce two other variables y and z. Then the second (third) occurrence of x in a clause is replaced by y (z). Then, x, y, z are forced to have the same truth assignment by adding $\bar{x} \vee y \vee z$, $x \vee \bar{y} \vee z$, and $x \vee y \vee \bar{z}$ to the instance.)

Let $V = \{x_1, \ldots, x_n\}$ and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be an instance I of the NAE 3-SAT Problem without repetition. We shall construct a graph G_I in such a way that I has an answer yes for the NAE 3-SAT Problem if and only if G has a good edge-labelling.

For each variable x_i, $1 \leq i \leq n$, we create a variable graph VG_i defined as follows (See Figure 2.):

$$\begin{aligned}
V(VG_i) &= \{v_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\
&\quad \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}. \\
E(VG_i) &= \{v_k^{i,j} v_{k+1}^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 3\} \cup \{v_4^{i,j} v_1^{i,j+1} \mid 1 \leq j \leq m-1\} \\
&\quad \cup \{v_k^{i,j} r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{v_k^{i,j} s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \\
&\quad \cup \{v_4^{i,j} r_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} r_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\} \\
&\quad \cup \{v_4^{i,j} s_1^{i,j} \mid 1 \leq j \leq m\} \cup \{v_k^{i,j+1} s_{k+1}^{i,j} \mid 1 \leq j \leq m-1, 1 \leq k \leq 3\}.
\end{aligned}$$

For each clause $C_j = l_1 \vee l_2 \vee l_3$, $1 \leq j \leq m$, we create a clause graph CG_j defined as follows (See Figure 3.):

$$\begin{aligned}
V(CG_j) &= \{c^j, b_1^j, b_2^j, b_3^j\} \\
E(CG_j) &= \{c^j b_1^j, c^j b_2^j, c^j b_3^j\}
\end{aligned}$$

Now, for each literal l_k, $1 \leq k \leq 3$, if l_k is the non-negated variable x_i, we identify b_k^j, c^j and b_{k+1}^j (index k is taken modulo 3) with $v_1^{i,j}$, $v_2^{i,j}$ and $v_3^{i,j}$, respectively. Otherwise, if l_k is the negated variable $\bar{x}_i$, we identify b_k^j, c^j and b_{k+1}^j with $v_3^{i,j}$, $v_2^{i,j}$ and $v_1^{i,j}$, respectively.

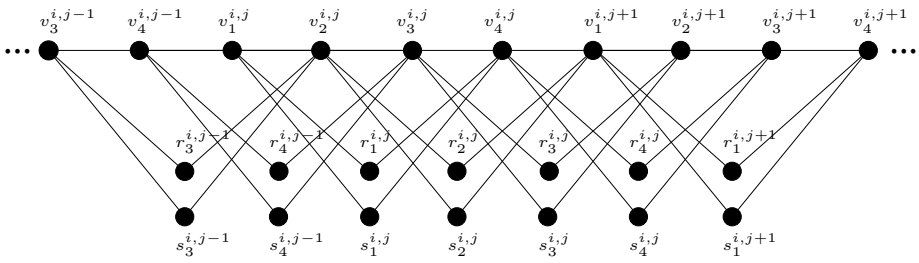

Figure 2: The variable graph VG_i

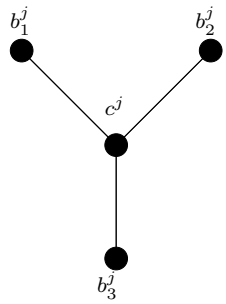

Figure 3: The clause graph CG_j.

Let us first show that, if G_I has a good edge-labelling ϕ, then there is a truth assignment such that each clause of I has at least one true literal and at least one false literal.

By Proposition 2, we may assume that ϕ is injective.

Claim 5.1 *Let $1 \leq i \leq n$. If $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ then $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$.*

Proof: By induction on j. A path of length two is necessarily increasing or decreasing. Now $v_1^{i,j}$ is joined to $v_4^{i,j}$ by two paths of length two via $r_1^{i,j}$ and $s_1^{i,j}$. Since ϕ is good, one of these two paths is increasing and the other one is decreasing. In addition, the path $v_1^{i,j}v_2^{i,j}v_3^{i,j}v_4^{i,j}$ is neither increasing nor decreasing so $\phi(v_2^{i,j}v_3^{i,j}) > \phi(v_3^{i,j}v_4^{i,j})$.

Applying three times this reasoning, we derive $\phi(v_3^{i,j}v_4^{i,j}) < \phi(v_4^{i,j}v_1^{i,j+1})$, $\phi(v_4^{i,j}v_1^{i,j+1}) > \phi(v_1^{i,j+1}v_2^{i,j+1})$ and finally $\phi(v_1^{i,j+1}v_2^{i,j+1}) < \phi(v_2^{i,j+1}v_3^{i,j+1})$. □

Hence we define the truth assignment Λ by $\Lambda(x_i) = true$ if $\phi(v_1^{i,1}v_2^{i,1}) < \phi(v_2^{i,1}v_3^{i,1})$ and $\Lambda(x_i) = false$ otherwise.

Let us show that each clause C_j has at least one true literal or one false literal. Set $C_j = l_1 \vee l_2 \vee l_3$. First observe that, by construction, for all $1 \leq k \leq 3$, l_k is true if $\phi(b_k^j c^j) < \phi(b_{k+1}^j c^j)$ and l_k is false otherwise. Now the three literals are not all true otherwise, $\phi(b_1^j c^j) < \phi(b_2^j c^j) < \phi(b_3^j c^j) < \phi(b_1^j c^j)$, a contradiction. And they are not all false, otherwise $\phi(b_1^j c^j) > \phi(b_2^j c^j) > \phi(b_3^j c^j) > \phi(b_1^j c^j)$, a contradiction. Hence C_j has at least one true literal and one false literal.

Reciprocally, let us now show that, if there is a truth assignment Λ such that each clause of I has at least one true literal and at least one false literal, then G_I has a good edge-labelling.

The idea is to find a good edge-labelling ϕ satisfying the following property $(\star)$: If $\Lambda(x_i) = true$, $\phi(v_1^{i,j}v_2^{i,j}) < \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$ and if $\Lambda(x_i) = false$, $\phi(v_1^{i,j}v_2^{i,j}) > \phi(v_2^{i,j}v_3^{i,j})$ for all $1 \leq j \leq m$.

Let $C_j = l_1 \vee l_2 \vee l_3$ be clause. To satisfy $(\star)$, we must label the edges of VG_j such that $\phi(b_k^j c^j) < \phi(b_{k+1}^j c^j)$ if l_k is true and $\phi(b_k^j c^j) > \phi(b_{k+1}^j c^j)$ if l_k is false. As C_j has at least one true and one false literal, there is a unique way to label the three edges $c^j b_k^j$, $1 \leq k \leq 3$, with $\{-1, 0, +1\}$ so that the condition $(\star)$ is fulfilled.

Let us now extend this edge-labelling to the remaining edges of each of the variable graphs VG_i. First, for all $1 \leq j \leq m$ and $1 \leq k \leq 4$, assign -3 and $+3$ alternatively on the edges of the cycle of length four containing both $r_k^{i,j}$ and $s_k^{i,j}$ so that $\phi(v_k^{i,j}r_k^{i,j}) = -3$. Then, if $\Lambda(x_i) = true$, set $\phi(v_3^{i,j}, v_4^{i,j}) = -2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = 2$ for all $1 \leq j \leq m$, and, if $\Lambda(x_i) = false$, set $\phi(v_3^{i,j}, v_4^{i,j}) = 2$ and $\phi(v_4^{i,j}, v_1^{i,j+1}) = -2$ for all $1 \leq j \leq m$.

We claim that ϕ is a good edge-labelling of G_I. Suppose, by way of contradiction, that there is a pair of vertices (x, y) such that two independent increasing (x, y)-paths P_1 and P_2 exist.

A set of two independent paths starting at a vertex of $R = \{r_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\} \cup \{s_k^{i,j} \mid 1 \leq j \leq m, 1 \leq k \leq 4\}$ contains one increasing path (the one starting with

the edge labelled -3) and one decreasing path (the one starting with the edge labelled 3). Hence x and y are not in R.

In addition, the union of P_1 and P_2 cannot be one of the four-cycles formed by the edges incident to $r_k^{i,j}$ and $s_k^{i,j}$ for some i,j and k.

Without loss of generality, we may assume that P_1 is at least as long has P_2. As cycles formed by two graphs GV_i and GV_j are of length at least 6, P_1 has length at least 3. Now one can see that P_1 may not contains any vertex of R because every path of length at least 3 with no end in R is not increasing (nor decreasing).

Hence P_1 must contain at least three consecutive edges on one of the paths $Q_i = VG_i - R$. So P_1 is not increasing, a contradiction. □

Remark 6 A standard dynamic programming approach yields a polynomial-time algorithm to decide if a graph of bounded tree-width has a good edge-labelling. It suffices to store if there is a good edge-labelling such that the ordering of edge labels corresponds to a given order (for all possible orderings of edges in the bag (there are at most $\left(\frac{k(k+1)}{2}\right)!$ of them).

However, we do not know what is the complexity of the problem when restricted to planar graphs.

Problem 7 *Does there exist a polynomial-time algorithm that decides if a given planar graph has a good edge-labelling?*

5 Classes of good graphs

Recall that a graph G is *critical* if it is bad but each of its proper subgraph is good. To prove that every graph in a class $\mathcal{C}$ closed under subgraphs has a good edge-labelling, it suffices to prove that $\mathcal{C}$ contains no critical graph.

Lemma 8 *Every critical graph is* 2*-connected.*

Proof: Suppose by way of contradiction that a critical graph has a cut-vertex x. Let $C_1, \ldots, C_k$ be the components of $G - x$ and $G_i = G[C_i \cup \{x\}]$, $1 \leq i \leq k$. As G is critical, each of the G_i admits a good edge-labelling. Then the union of these edge-labellings form a good edge-labelling of G because there are two independent (u, v)-paths in G only if there exists i, $1 \leq i \leq k$, so that u and v are in $V(G_i)$. This contradicts the fact that G has no good edge-labelling. □

Corollary 9 *Every forest F admits a good edge-labelling.*

Proof: No forest contains a non-trivial 2-connected subgraph, and so contains no critical subgraph. □

Let $G = (V, E)$ be a graph. An *edge-2-cut* of G is a set of two adjacent vertices u and v such that $G - \{u, v\}$ has more connected components than G.

Lemma 10 *Let G be a connected graph and $\{u, v\}$ an edge-2-cut in G so that $G - \{u, v\}$ has two connected components C_1 and C_2. If $G\langle C_1 \cup \{u, v\}\rangle$ and $G\langle C_2 \cup \{u, v\}\rangle$ have a good edge-labelling then G has a good edge-labelling.*

Proof: Let ϕ_1 and ϕ_2 be good edge-labellings of $G\langle C_1 \cup \{u, v\}\rangle$ and $G\langle C_2 \cup \{u, v\}\rangle$ respectively such that $\phi_1(uv) = \phi_2(uv)$.

Then the union of ϕ_1 and ϕ_2 is a good edge-labellign of G. Indeed, suppose by way of contradiction, that there exists x and y and two indepedent increasing (x, y)-paths P_1 and P_2. W. l. o. g. , we may assume that $x \in V(G_1)$. At least of the path, say P_1, contains at least one edge e_1 in $E(G_2) \setminus \{uv\}$ since ϕ_1 is good.

Assume first that $y \in V(G_1)$. Then P_1 must go through u and v. Let Q_2 be the shortest (u, v)-subpath of P containing e_1. Then Q_2 is either increasing or decreasing. Hence since uv is both increasing and decreasing, there are either two increasing paths or two decreasing paths in G_2. This contradicts the fact that ϕ_2 is good.

Assume now that $y \in C_2$. Then since P_1 and P_2 are independent without loss of generality, P_1 goes through u and P_2 goes through v. Let Q_1 be the (x, u)-subpath of P_1, R_1 be the (u, y)-subpath of P_1, let Q_2 be the (x, v)-subpath of P_2 and R_2 be the (v, y)-subpath of P_2.

If $\phi(uv)$ is larger than the label of the last edge of Q_1 then R_1uv and R_2 are two increasing paths in G_1, a contradiction. If not $\phi(uv)$ is smaller than the abel of the first edge of R_1 and vuR_1 and R_2 are two increasing paths in G_2, a contradiction.

□

Let $G = (V, E)$ be a graph. An *edge-cut* is a set of edge between a set of vertices S and its complement $\overline{S}$. Formally, for any $S \subset V$, the edge-cut $[S, \overline{S}]$ is the set $\{uv \in E \mid u \in S \text{ and } v \in \overline{S})$. An edge cut which is also a matching is called a *matching-cut*.

Lemma 11 *Let G be a graph and $[S, \overline{S}]$ a matching-cut in G. If $G\langle S\rangle$ and $G\langle \overline{S}\rangle$ have a good edge-labelling then G has a good edge-labelling.*

Proof: Let ϕ_1 be a good edge-labelling of $G\langle S\rangle$ and ϕ_2 be a good edge-labelling of $G\langle \overline{S}\rangle$ (in $\mathbb{R}$). Then the edge-labelling ϕ of G defined by $\phi(e) = \phi_1(e)$ if $e \in E(G\langle S\rangle)$, $\phi(e) = \phi_2(e)$ if $e \in E(G\langle \overline{S}\rangle)$ and $\phi(e) = +\infty$ if $e \in [S, \overline{S}]$ is good.

Indeed, suppose by way of contradiction, that it is not good. Then there exist two vertices u and v and two independent increasing (u, v)-paths P_1 and P_2. Since ϕ_1 and ϕ_2 are good, then without loss of generality, we may assume that $u \in S$ and $v \in \overline{S}$. For $i = 1, 2$ P_i contains an edge of v_iw_i in $[S, \overline{S}]$. Now as v_1w_1 and v_2w_2 are labelled $+\infty$ and incident to no edges labelled $+\infty$, v_1w_1 must be the last edge of P_1 and v_2w_2 the last edge of P_2. So $w_1 = v = w_2$ which is impossible as $[S, \overline{S}]$ is a matching. □

Corollary 12 *A critical graph has no matching-cut.*

Corollary 13 *Every C_3-free outerplanar graph admits a good edge-labelling.*

Proof: An easy result of Eaton and Hull [5] states that a C_3-free outerplanar graph has either a 1-vertex or two adjacent 2-vertices. This implies that it has a matching-cut. Hence by Corollary 12 no C_3-free outerplanar graph is critical, which yields the result. □

A graph is *subcubic* if every vertex has degree at most three.

Lemma 14 *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a matching cut.*

Proof: Let G be a subcubic graph $\{C_3, K_{2,3}\}$. If G has no cycle, then every edge forms a matching-cut. Suppose now that G has a cycle. Let C be a cycle of smallest length in G. If C is a connected component of G (in particular if $C = G$) then any pair of non-adjacent edges of C form a matching-cut.

If not, let us show that $[V(C), \overline{V(C)}]$ is a matching cut. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two distinct edges in this set with $x_1, x_2 \in V(C)$. Then $x_1 \neq x_2$ because these two vertices have degree (at most) 3 and two neighbours in $V(C)$. Suppose by way of contradiction that $y_1 = y_2$. Then x_1 and x_2 are not adjacent since G is C_3-free. Furthermore, there the two (x_1, x_2)-paths along C are of length at most 2 otherwise C would not be a smallest cycle. Hence C is a cycle of length 4 and the graph induced by $V(C) \cup \{y_1\}$ is a $K_{2,3}$, a contradiction. □

Corollary 12 and Lemma 14 immediately implies that the sole subcubic critical graphs are C_3 and $K_{2,3}$.

Corollary 15 *Every subcubic $\{C_3, K_{2,3}\}$-free graph has a good edge-labelling.*

Farley and Proskurowski [6] proved that every (multi)graph G on n vertices with less than $\frac{3}{2}(n-1)$ edges has a matching-cut.

Theorem 16 (Farley and Proskurowski [6]) *Let G be a multigraph. If $|E(G)| < \frac{3}{2}|V(G)| - \frac{3}{2}$ then G has a matching-cut.*

Corollary 12 and Theorem 16 yield immediately the following.

Corollary 17 *Every critical graph has at least $\lceil \frac{3}{2}|V(G)| - \frac{3}{2} \rceil$ edges.*

Corollary 18 *Every planar graph of girth at least 6 has a good edge-labelling.*

Proof: It is well known that a planar graph G with girth at least 6 has at most $\frac{3}{2}|V(G)| - 3$ edges. Indeed, recall the Euler's formula for a connected planar graph G: $|V(G)| - |E(G)| + |F(G)| = 2$, with $|F(G)|$ the number of faces of G. Now, as G has girth at least 6, $6|F(G)| \leq 2|E(G)|$. Thus $12 - 6|V(G)| + 6|E(G)| = 6|F(G)| \leq 2|E(G)|$. Hence $|E(G)| \leq \frac{3}{2}|V(G)| - 3$.

Thus no planar graph of girth 6 is critical by Corollary 17. □

6 Good edge-labelling of ABC-graphs

Corollary 17 states that every critical graph has at least $\lceil \frac{3}{2}(|V(G)|-1) \rceil$ edges. This is tight since if G is C_3 or $K_{2,3}$ then $|E(G)| = \lceil \frac{3}{2}(|V(G)|-1) \rceil$. We will now show that those two graphs are the unique critical ones satisfying this equality.

Farley and Proskurowski [6] constructed a class of multigraphs G called *ABC-graphs* with no matching-cut having $\lceil \frac{3}{2}(|V(G)|-1) \rceil$ edges. The definition of ABC-graphs is based on the following three operations:

- An *A-operation* on vertex u introduces vertices v and w and edges uv, uw and vw.
- A *B-operation* on edge uv introduces vertices w_1 and w_2 and edges uw_1, vw_1, uw_2 and vw_2, and removes edge uv.
- A *C-operation* on vertices u and v ($u = v$ is allowed) introduces vertex w and edges uw and vw.

Note that the C-operation is the only operation that can introduce parallel edges.

An *ABC-graph* is a graph that can be obtained from K_1 with a sequence of A- and B-operations and at most one C-operation.

It is easy to check that ABC-graphs have no matching-cut. In addition, solving a conjecture of Farley and Proskurowski [6], Bonsma [3, 4] showed that they are the unique extremal examples, i.e., satisfying $|E(G)| = \lceil \frac{3}{2}(|V(G)|-1) \rceil$.

Theorem 19 (Bonsma [3, 4]) *Let G be a graph with such that $|E(G)| = \lceil \frac{3}{2}(|V(G)|-1) \rceil$. Then G has no matching-cut if and only if G is an ABC-graph.*

Observe that the C-operation is the only one that changes the parity of the order. Hence an ABC-graph with an odd number of vertices is obtained from K_1 with a sequence of A- and B-operations and no C-operation.

Proposition 20 *Let G be a graph obtained from a graph H by a B-operation on some edge uv. If H has a good edge-labelling then G has a good edge-labelling.*

Proof: Let ϕ be a good edge-labelling of H. By Proposition 2, we may assume that ϕ is injective. Moreover, we may assume that ϕ is integer-valued and $\phi(uv) = 0$. Hence setting $\phi(uw_1) = \phi(w_2v) = \frac{1}{2}$ and $\phi(uw_2) = \phi(xw_1) = -\frac{1}{2}$, we obtain a good edge-labelling of G. $\square$

Lemma 21 *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G has a good edge-labelling.*

Proof: As every A-operation (with the exception of the transition $K_1 \to C_3$) creates a cut-vertex, G is obtained from C_3 with a sequence of B-operations. But a B-operation on

C_3 at any edge creates a $K_{2,3}$ and a B-operation on $K_{2,3}$ at any edge creates the graph G_1 depicted in Figure 4. Now this graph admits a good edge-labelling (See Figure 4). Hence an easy induction and Proposition 20 implies the result because if $G \notin \{C_3, K_{2,3}\}$ then it is obtained from G_1 with a sequence of B-operations. □

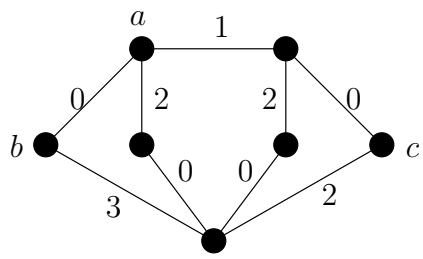

Figure 4: The graph G_1 and a good edge-labelling

Corollary 22 *Let G be an ABC-graph with an odd number of vertices. If G is $\{C_3, K_{2,3}\}$-free then G is good.*

We now would like to prove an analogous statement to the one of Corollary 22 but for ABC-graphs with an even number of vertices.

Let G be a graph and x, y be two distinct vertices of G. An (x, y)-*better edge-labelling* of G is an good edge-labelling of G such that there is no increasing (x, y)-path. Clearly, if x and y are adjacent or if x and y have two neighbours in common then G has no (x, y)-better edge labelling. A graph is *friendly* if it has a good-edge labelling and for any pair (x, y) of non-adjacent vertices with at most one neighbour in common there exists an (x, y)-better edge-labelling.

Proposition 23 *Let G be a graph obtained from a graph H by a B-operation on some edge uv. If H is friendly then G is friendly.*

Proof: Let w_1, w_2 be the vertices created by the B-operation. Let x and y be two non-adjacent vertices of G having at most one neighbour in common. Then $|\{x, y\} \cap \{w_1, w_2\}| \leq 1$.

- Suppose first that $\{x, y\} \cap \{w_1, w_2\} = \emptyset$. Then x and y are not adjacent in H.

  Assume first that they have at most one commom neighbour in H. Let ϕ be an integer-valued (x, y)-better edge-labelling of H such that $\phi(uv) = 0$. Setting $\phi(uw_1) = \phi(w_2v) = \frac{1}{2}$ and $\phi(uw_2) = \phi(vw_1) = -\frac{1}{2}$, we obtain an (x, y)-better edge-labelling of H.

  Assume now that x and y have two common neighbours in H. As they do not have two common neighbours in G, we can suppose w.l.o.g. that $x = u$ and $N(x) \cap N(y) = \{v, w\}$

for some vertex w. Let ϕ be a good edge-labelling of H. By Proposition 2, we may assume that ϕ is integer-valued and $\phi(uv) = 0$. Moreover, free to consider $-\phi$, we may assume that $\phi(vy) > 0$, so uvy is increasing. Hence in $H \setminus uv$ there is no increasing (u, y)-path. Moreover there is no increasing (u, v)-path (resp. increasing (v, u)-path) in $H \setminus uv$, otherwise there would be two such paths on H because uv (resp. vu) is an increasing path of H. Hence setting $\phi(uw_1) = \phi(w_2v) = +\infty$ and $\phi(uw_2) = \phi(xw_1) = -\infty$, we obtain an (x, y)-better edge-labelling of H.

- Suppose now that $|\{x, y\} \cap \{w_1, w_2\}| = 1$. Without loss of generality, we may assume that $x = w_1$ and y is not adjacent to v.

  Assume first that v and y have at most one common neighbour in H. Let ϕ be a (v, y)-better edge-labelling of H. Then in $H \setminus uv$, there is no increasing (u, v)-path nor increasing (v, u)-path. Hence setting $\phi(uw_1) = \phi(w_2v) = +\infty$ and $\phi(uw_2) = \phi(xw_1) = -\infty$, we obtain an (w_1, y)-better edge-labelling of H. Indeed there is no incresing (w_1, y)-path, through u since $\phi(uw_1) = +\infty$, nor through v since there is no increasing (v, y)-path in H.

  Assume now that v and y have two common neighbours in H.

  - Suppose that y is adjacent to u. Let ϕ be a good edge-labelling of H. By Proposition 2, we may assume that ϕ is injective and integer-valued and $\phi(uv) = 0$.

    Setting $\phi(uw_1) = \phi(w_2v) = \frac{1}{2}$ and $\phi(uw_2) = \phi(xw_1) = -\frac{1}{2}$, we obtain a a (w_1, y)-better edge-labelling of H. In addition, free to consider $-\phi$, we may assume that $\phi(uy) < 0$ and so $\phi(uy) \leq -1$. Setting $\phi(uw_1) = \phi(w_2v) = -\frac{1}{2}$ and $\phi(uw_2) = \phi(vw_1) = \frac{1}{2}$, we obtain a (w_1, y)-better edge-labelling of H. Indeed suppose for a contradiction that there is a (w_1, y)-increasing path P :

    * If u is the second vertex of P then $P - u$ is an increasing (u, y). Since $\phi(uy) \leq -1$, P is not uy. So there are two increasing (u, y)-paths in H a contradiction.
    * If v is the second vertex of P then the path in H obtained from P by replacing w_1 with u is an increasing (u, y)-path because the labels of the edges of P are positive. This path is distinct from uy, a contradiction.

  - Suppose that y is not adjacent to u. Let t_1 and t_2 be the two common neighbours of v and y. Let ϕ be a good edge-labelling of H. By Proposition 2, we may assume that ϕ is injective and integer-valued and $\phi(uv) = 0$. Moreover, without loss of generality, we may assume that vt_1y is increasing and vt_2y is decreasing. In addition, free to consider $-\phi$, we may assume that $\phi(vt_1) < 0$ and so $\phi(vt_1) \leq -1$. Setting $\phi(uw_1) = \phi(w_2v) = \frac{1}{2}$ and $\phi(uw_2) = \phi(xw_1) = -\frac{1}{2}$, we obtain a (w_1, y)-better edge-labelling of H. Indeed suppose for a contradiction that there is a (w_1, y)-increasing path P :

* If v is the second vertex of P then $P - w_1$ is an increasing (v, y). Since $\phi(vt_1) \leq -1$, P is not vt_1y. So there are two increasing (v, t)-path in H a contradiction.
* If u is the second vertex of P then the path in H obtained from P by replacing w_1 with v is an increasing (v, y)-path because the labels of the edges of P are positive. This path is distinct from vt_1y, a contradiction.

□

One can now generalise Lemma 21.

Lemma 24 *Let G be a 2-connected ABC-graph with an odd number of vertices. If $G \notin \{C_3, K_{2,3}\}$ then G is friendly.*

Proof: Similarly as in the proof of Lemma 21, it suffices to prove that G_1 is friendly. Then an easy induction and Proposition 23 yield the result.

Let x and y be two vertices of G_1 that are not adjacent and that have at most one neighbour in common, and let ϕ be the labelling of G_1 given in Figure 4 : ϕ or $-\phi$ is a (x, y)-better edge labelling. □

Corollary 25 *Every $\{C_3, K_{2,3}\}$-free ABC-graph with an odd number of vertices is friendly.*

Proof: Let x and y be two non-adjacent vertices of G having at most one common neighbour.

Assume first that x and y are in a same connected 2-component C. By Lemma 24, C has an (x, y)-better labelling and, by Corollary 22, $G \setminus E(C)$ has a good edge-labelling. The union of these two edge-labellings is clearly an (x, y)-better labelling of G.

Suppose now that the 2-connected components containing x do not contain y. Let G_1 be the graph induced by the union of the 2-connected components containing x and $G_2 = G \setminus E(G_1)$. By Corollary 22, the two graphs G_1 and G_2 admit good edge-labellings ϕ_1 and ϕ_2, respectively. Free to add a huge number to all the labels of ϕ_1, we may assume that $\min\{\phi_1(e) \mid e \in E(G_1)\} > \max\{\phi_2(e) \mid e \in E(G_2)\}$. Then the union of ϕ_1 and ϕ_2 is an (x, y)-better labelling of G. □

Lemma 26 *Let G be an ABC-graph with an even number of vertices. If G is $\{C_3, K_{2,3}\}$-free then G is good.*

Proof: We prove this lemma by induction on the number of vertices (or equivalently the number of A-, B- or C-operations). An even ABC-graph is obtained from K_1 with a sequence of A- and B-operations and exactly one C-operation. If G is not 2-connected then we get the result as in the proof of Lemma 8. So we may assume that G is 2-connected. In this case there can be no A-operation made after a C-operation. Consider a sequence of operations such that the C-operation is done as late as possible.

If the C-operation is the ultimate one, say it is done on u and v, and introduces w. Note that $u \neq v$ since G has no multiple edges. Since G is $\{C_3, K_{2,3}\}$-free then u and v are not adjacent and u and v have at most one neighbour in common. Hence by Corollary 25, $G-w$ admits a (u,v)-better edge-labelling ϕ (in $\mathbb{R}$). Setting $\phi(uw) = -\infty$ and $\phi(wv) = +\infty$ we obtain a good edge-labelling of G.

If the C-operation is the penultimate one, then it is followed by a B-operation on one of the introduced edges. These two operations together may be seen as a single one on u and v that introduces the vertices t_1, t_2 and w and the edges ut_1, ut_2, t_1w, t_2w and wv.

Note that u and v are not adjacent since G is $K_{2,3}$-free. Assume first that u and v have at most one neighbour in common. By Corollary 25, $G - \{t_1, t_2, w\}$ admits a (u,v)-better edge-labelling ϕ. Let M be the maximum value of ϕ. Then setting $\phi(ut_1) = \phi(t_2w) = -\infty$, $\phi(ut_2) = \phi(t_1w) = M+1$ and $\phi(vw) = M+2$, we obtain a good edge-labelling of G.

Assume now that u and v have at least two common neighbours. Since G is $K_{2,3}$-free, then u and v have exactly two common neighbours x_1 and x_2. By Corollary 22, $G - \{t_1, t_2, w\}$ admits a good edge-labelling ϕ. By Proposition 2, we may assume that ϕ is injective. Without loss of generality, we may suppose that $\phi(vx_1) > \phi(vx_2)$. Let us set $\phi(ut_1) = \phi(t_2w) = +\infty$, $\phi(ut_2) = \phi(t_1ww) = -\infty$ and $\phi(vw) = \frac{1}{2}(\phi(vx_1) + \phi(vx_2))$. We claim that ϕ is a good edge-labelling of G. Indeed suppose, by way of contradiction, that it is not the case. Then there exist two vertices a and b and two independent increasing (a,b)-paths P_1 and P_2. Since ϕ is a good edge-labelling of $G - \{t_1, t_2, w\}$ one of these two paths, say P_1 must go through w. Moreover since $\phi(t_1w) = -\infty$ and $\phi(t_2w) = +\infty$ w, and $d(w) = 3$, then either wt_1 is the first edge of P_1 or t_2w is the last edge of P_1. Free to consider $-\phi$ instead of ϕ, we may assume that we are in the first case. So $P_1 = (w, t_1, u)$, because $\phi(ut_1) = +\infty$. Now the first edge of P_2 is wv. Hence $Q_2 = P_2 - w$ is an increasing (v,u)-path and vx_2 is not the first edge of Q_2 since $\phi(wv) > \phi(vx_2)$. Note that by Observation 3.1, vx_2u is increasing because $\phi(vx_1) > \phi(vx_2)$. So, in $G - \{t_1, t_2, w\}$, there are two distinct increasing (v,u)-paths. This contradicts the fact that ϕ is a good edge-labelling of $G - \{t_1, t_2, w\}$.

If there are exactly two B-operations after the C-operation, and if u and v are not adjacent then by the induction hypothesis and Proposition 20, G has a good edge-labelling. If u and v are adjacent, then uv is an edge-2-cut. Let C_1 be the component of $G - \{u,v\}$ containing w (i.e., the set of vertices added with the C-operations and the following B-operations). Let $G_1 = G\langle C_1 \cup \{u,v\}\rangle$ and $G_2 = G\langle V(G) \setminus C_1\rangle$. Note that G_1 is obtained from a triangle by performing two B-operations and thus is the graph G_1 depicted Figure 4 which has a good edge-labelling. Similarly, G_2 is the graph G taken before peforming the C-operation, is so$\{C_3, K_{2,3}\}$-free and by Corollary 22 has a good edge-labelling. Hence by Lemma 10, G has a good edge-labelling.

If there are at least three B-operations after the C-operation, then by the induction hypothesis and Lemma 20, G has a good edge-labelling. □

Corollary 22 and Lemma 26 immediately yield the following.

Theorem 27 *Let G be a critical graph. If $G \notin \{C_3, K_{2,3}\}$ then $|E(G)| \geq \frac{3}{2}(|V(G)| - \frac{1}{2})$.*

7 Conclusions and further research

The *average degree* of a graph G is $Ad(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$

Theorem 27 implies that for any $c < 3$ there is a finite number of critical graphs with average degree at most c. Actually, we conjecture that the only ones are C_3 and $K_{2,3}$.

Conjecture 28 *Let G be a critical graph. Then $Ad(G) \geq 3$ unless $G \in \{C_3, K_{2,3}\}$.*

More generally for any $c < 4$, we conjecture the following.

Conjecture 29 *For any $c < 4$, there exists a finite list of graphs $\mathcal{L}$ such that if G is a critical graph with $Ad(G) \leq c$ then $G \in \mathcal{L}$.*

The constant 4 in the above conjecture would be tight. Indeed, for all k, the graph $J_{2,2,k}$ defined in Section 3 is critical: it is bad according to Proposition 4. Moreover one can easily show that for any edge e, $H_k \setminus e$ has a good edge-labelling with no (u, v)-increasing path and no (v, u)-increasing (just follow the constraint as in the proof of Proposition 3). Extending this labelling by labelling the two H_2 with $-\infty$ and $+\infty$ so that one of them is an increasing (u, v)-path and the other one an increasing (v, u)-path we obtain a good edge-labelling of $J_{2,2,k} \setminus e$. Furthermore $Ad(J_{2,2,k}) = \frac{8k+8}{2k+4} = 4 - \frac{4}{k+2}$. Last, one can easily see that if $k \neq k'$ then $J_{2,2,k}$ is not a subgraph of $J_{2,2,k'}$.

Theorem 27 says that if a graph has no dense subgraphs then it has a good edge-labelling. On the opposite direction one may wonder what is the minimum density ensuring a graph to be bad. Or equivalently,

Problem 30 What is the maximum number $g(n)$ of edges of a good graph on n vertices?

Clearly we have $g(n) = ex(n, \mathcal{C})$ where $\mathcal{C}$ is the set of critical graphs. As $K_{2,3}$ is critical then $g(n) \leq ex(n, K_{2,3}) = \frac{1}{\sqrt{2}} n^{3/2} + O(n^{4/3})$ by a result of Füredi [7].

The hypercubes show that g is super-linear. Indeed the hypercube H_k is obtained from two disjoints copies of H_{k-1} by adding a perfect matching between them. Hence an easy induction and Lemma 11 shows that H_k has a good edge-labelling. Since H_k has 2^k vertices and $2^{k-1}k$ edges, $g(2^k) \geq 2^{k-1}k$, so $g(n) \geq \frac{1}{2} n \log n$.

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