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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## *Combinable Extensions of Abelian Groups*

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## Combinable Extensions of Abelian Groups

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Thème SYM — Systèmes symboliques  
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**Abstract:** The design of decision procedures for combinations of theories sharing some arithmetic fragment is a challenging problem in verification. One possible solution is to apply a combination method à la Nelson-Oppen, like the one developed by Ghilardi for unions of non-disjoint theories. We show how to apply this non-disjoint combination method with the theory of abelian groups as shared theory. We consider the completeness and the effectiveness of this non-disjoint combination method. For the completeness, we show that the theory of abelian groups can be embedded into a theory admitting quantifier elimination. For achieving effectiveness, we rely on a superposition calculus modulo abelian groups that is shown complete for theories of practical interest in verification.

**Key-words:** Satisfiability Procedure, Combination, Equational Reasoning, Union of Non-Disjoint Theories, Arithmetic, Abelian Groups

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## Extensions combinables des groupes abéliens

**Résumé :** La conception de procédures de décision pour la combinaison de théories partageant un fragment d'arithmétique est un défi dans le domaine de la vérification. Une solution possible consiste à appliquer une méthode de combinaison à la Nelson-Oppen, comme celle développée par Ghilardi pour l'union de théories non-disjointes. On montre comment appliquer cette méthode de combinaison non-disjointe avec la théorie des groupes abéliens comme théorie partagée. On considère la complétude et l'effectivité de cette méthode. Pour la complétude, on montre que la théorie des groupes abéliens peut se plonger dans une théorie admettant l'élimination des quantificateurs. Pour être effectif, on utilise un calcul de superposition modulo la théorie des groupes abéliens qui est montré complet pour des théories intéressantes en pratique dans le domaine de la vérification.

**Mots-clés :** procédure de satisfiabilité, combinaison, raisonnement équationnel, mélange de théories non-disjointes, groupes abéliens

## 1 Introduction

Decision procedures are the basic engines of the verification tools used to check the satisfiability of formulae modulo background theories, which may include axiomatizations of standard data-types such lists, arrays, bit-vectors, etc. Nowadays, there is a growing interest in applying theorem provers to construct decision procedures for theories of interest in verification [2, 1, 8, 4]. The problem of incorporating some reasoning modulo arithmetic properties inside theorem provers is particularly challenging. Many works are concerned with the problem of building-in certain equational axioms, starting from the seminal contributions by Plotkin [21] and by Peterson and Stickel [20]. The case of Associativity-Commutativity has been extensively investigated since it appears in many equational theories, and among them, the theory of abelian groups is a very good candidate as fragment of arithmetic. Recently, the standard superposition calculus [19] has been extended to a superposition calculus modulo the built-in theory of abelian groups [12]. This work paves the way for the application of a superposition calculus modulo a fragment of arithmetic to build decision procedures of practical interest in verification. However, practical problems are often expressed in a combination of theories where the fragment of arithmetic is shared by all the other theories involved. In this case the classical Nelson-Oppen combination method cannot be applied since the theories share some arithmetic operators. An extension of the Nelson-Oppen combination method to the non-disjoint case has been proposed in [11]. This non-disjoint combination framework has been recently applied to the theory of Integer Offsets [18]. In this paper, our aim is to consider a more expressive fragment by studying the case of abelian groups.

The contributions of the paper are twofold. First, we show that abelian groups satisfy all the properties required to prove the completeness, the termination and the effectiveness of the non-disjoint extension of the Nelson-Oppen combination method. To prove the completeness, we show the existence of an extension of the theory of abelian groups having quantifier elimination and that behaves the same w.r.t. the satisfiability of constraints. Second, we identify a class of theories that extend the theory of abelian groups and for which a simplified constraint-free (but many-sorted) version of the superposition calculus introduced in [12] is proved to be complete. This superposition calculus allows us to obtain effective decision procedures that can be plugged into the non-disjoint extension of the Nelson-Oppen combination method.

This paper is organized as follows. Section 2 briefly introduces the main concepts and the non-disjoint combination framework. In Section 3, we show some very useful properties in order to use the theory of abelian groups, namely  $AG$ , in the non-disjoint combination framework, especially we prove the quantifier elimination of a theory that is an extension of  $AG$ . In Section 4, we present a calculus modulo  $AG$ . In Section 5, we show its refutational completeness and we study how this calculus may lead to combinable decision procedures. Examples are given in Section 6. We conclude with some final remarks in Section 7. Most of the proofs are omitted and can be found in the appendix.

## 2 Preliminaries

We consider a many-sorted language. A *signature*  $\Sigma$  is a set of sorts, functions and predicate symbols (each endowed with the corresponding arity and sort). We assume that, for each sort  $s$ , the equality “ $=_s$ ” is a logical constant that does not occur in  $\Sigma$  and that is always interpreted as the identity relation over (the interpretation of)  $s$ ; moreover, as a notational convention, we will often omit the subscript and we will shorten  $=$  and  $\neq$  with  $\bowtie$ . Again, as a matter of convention, we denote with  $\Sigma^a$  the signature obtained from  $\Sigma$  by adding a set  $a$  of new constants (each of them again equipped with its sort), and with  $t\theta$  the application of a substitution  $\theta$  to a term  $t$ .  $\Sigma$ -atoms,  $\Sigma$ -literals,  $\Sigma$ -clauses, and  $\Sigma$ -formulae are defined in the usual way, i.e. they must respect the arities of function and predicate symbols and the variables occurring in them must also be equipped with sorts (well-sortedness). The empty clause is denoted by  $\square$ . A set of  $\Sigma$ -literals is called a  $\Sigma$ -constraint. Terms, literals, clauses and formulae are called *ground* whenever no variable appears in them; *sentences* are formulae in which free variables do not occur.

From the semantic side, we have the standard notion of a  $\Sigma$ -structure  $\mathcal{M}$ : it consists of non-empty pairwise disjoint domains  $M_s$  for every sort  $s$  and a sort- and arity-matching interpretation  $\mathcal{I}$  of the function and predicate symbols from  $\Sigma$ . The truth of a  $\Sigma$ -formula in  $\mathcal{M}$  is defined in any one of the standard ways. If  $\Sigma_0 \subseteq \Sigma$  is a subsignature of  $\Sigma$  and if  $\mathcal{M}$  is a  $\Sigma$ -structure, the  $\Sigma_0$ -reduct of  $\mathcal{M}$  is the  $\Sigma_0$ -structure  $\mathcal{M}|_{\Sigma_0}$  obtained from  $\mathcal{M}$  by forgetting the interpretation of the symbols from  $\Sigma \setminus \Sigma_0$ .

A collection of  $\Sigma$ -sentences is a  $\Sigma$ -theory, and a  $\Sigma$ -theory  $T$  admits (or has) *quantifier elimination* iff for every formula  $\varphi(\underline{x})$  there is a quantifier-free formula (over the same free variables  $\underline{x}$ )  $\varphi'(\underline{x})$  such that  $T \models \varphi(\underline{x}) \leftrightarrow \varphi'(\underline{x})$ .

In this paper, we are concerned with the (*constraint*) *satisfiability problem* for a theory  $T$ , which is the problem of deciding whether a  $\Sigma$ -constraint is satisfiable in a model of  $T$ . Notice that a constraint may contain variables: since these variables may be equivalently replaced by free constants, we can reformulate the constraint satisfiability problem as the problem of deciding whether a finite conjunction of ground literals in a simply expanded signature  $\Sigma^a$  is true in a  $\Sigma^a$ -structure whose  $\Sigma$ -reduct is a model of  $T$ .

### 2.1 A Brief Overview on Non-Disjoint Combination

Let us consider now the constraint satisfiability problem w.r.t. a theory  $T$  that is the union of the two theories  $T_1 \cup T_2$ , and let us suppose that we have at our disposal two decision procedures for the constraint satisfiability problem w.r.t.  $T_1$  and  $T_2$  respectively. It is known (cf. [5]) that such a problem without any other assumption on  $T_1$  and  $T_2$  is undecidable; nevertheless, the following theorem holds:

**Theorem 1 ([11])** *Consider two theories  $T_1, T_2$  in signatures  $\Sigma_1, \Sigma_2$  such that:*

1. *both  $T_1, T_2$  have a decidable constraint satisfiability problem;*
2. *there is some universal theory  $T_0$  in the signature  $\Sigma_0 := \Sigma_1 \cap \Sigma_2$  such that:*
  - (a)  *$T_1, T_2$  are both  $T_0$ -compatible;*

(b)  $T_1, T_2$  are both effectively Noetherian extensions of  $T_0$ .

Then the  $(\Sigma_1 \cup \Sigma_2)$ -theory  $T_1 \cup T_2$  also has a decidable constraint satisfiability problem.

The procedure underlying Theorem 1 basically extends the Nelson-Oppen combination method [17] to theories over non disjoint signatures, thus lifting the decidability of the constraint satisfiability problem from the component theories to their union.

The requirement (2a) of  $T_0$ -compatibility over the theories  $T_1$  and  $T_2$  means that there is a  $\Sigma_0$ -theory  $T_0^*$  such that (i)  $T_0 \subseteq T_0^*$ ; (ii)  $T_0^*$  has quantifier elimination; (iii) every  $\Sigma_0$ -constraint which is satisfiable in a model of  $T_0$  is satisfiable also in a model of  $T_0^*$ ; and (iv) every  $\Sigma_i$ -constraint which is satisfiable in a model of  $T_i$  is satisfiable also in a model of  $T_0^* \cup T_i$ , for  $i = 1, 2$ . This requirement guarantees the completeness of the combination procedure underlying Theorem 1 and generalizes the stable infiniteness requirement used for the completeness of the original Nelson-Oppen combination procedure.

The requirement (2b) on  $T_1, T_2$  of being effectively Noetherian extensions of  $T_0$  means the following: first of all (i)  $T_0$  is *Noetherian*, i.e., for every *finite* set of free constants  $\underline{a}$ , every infinite ascending chain  $\Theta_1 \subseteq \Theta_2 \subseteq \dots \subseteq \Theta_n \subseteq \dots$  of sets of ground  $\Sigma_0^{\underline{a}}$ -atoms is eventually constant modulo  $T_0$ , i.e. there is a  $\Theta_n$  in the chain such that  $T_0 \cup \Theta_n \models \Theta_m$ , for every natural number  $m$ . Moreover, we require to be capable to (ii) compute  $T_0$ -bases for both  $T_1$  and  $T_2$ , meaning that, given a finite set  $\Gamma_i$  of ground clauses (built out of symbols from  $\Sigma_i$  and possibly further free constants) and a finite set of free constants  $\underline{a}$ , we can always compute a finite set  $\Delta_i$  of *positive* ground  $\Sigma_0^{\underline{a}}$ -clauses such that (a)  $T_i \cup \Gamma_i \models C$ , for all  $C \in \Delta_i$  and (b) if  $T_i \cup \Gamma_i \models D$  then  $T_0 \cup \Delta_i \models D$ , for every positive ground  $\Sigma_0^{\underline{a}}$ -clause  $D$  ( $i = 1, 2$ ). Note that if  $\Gamma_i$  is  $T_i$ -unsatisfiable then w.l.o.g.  $\Delta_i = \{\square\}$ . Intuitively, the Noetherianity of  $T_0$  means that, fixed a finite set of constants, there exists only a finite number of atoms that are not redundant when reasoning modulo  $T_0$ ; on the other hand, the capability of computing  $T_0$ -bases means that, for every set  $\Gamma_i$  of ground  $\Sigma_i^{\underline{a}}$ -literals, it is possible to compute a finite “complete set” of logical consequences of  $\Gamma_i$  over  $\Sigma_0$ ; these consequences over the shared signature are exchanged between the satisfiability procedures of  $T_1$  and  $T_2$  in the loop of the combination procedure à la Nelson-Oppen, whose termination is ensured by the Noetherianity of  $T_0$ .

We depict in the algorithm below the combination procedure, where  $\Gamma_i$  denotes a set of ground literals built out of symbols of  $\Sigma_i$  (for  $i = 1, 2$ ), a set of shared free constants  $\underline{a}$  and possibly further free constants.

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**Algorithm 1** Extending Nelson-Oppen

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1. If  $T_0$ -basis $_{T_i}(\Gamma_i) = \Delta_i$  and  $\square \notin \Delta_i$  for each  $i \in \{1, 2\}$ , then
    - 1.1. For each  $D \in \Delta_i$  such that  $T_j \cup \Gamma_j \not\models D$ , ( $i \neq j$ ), add  $D$  to  $\Gamma_j$
    - 1.2. If  $\Gamma_1$  or  $\Gamma_2$  has been changed in 1.1, then rerun 1.
 Else **return** “unsatisfiable”
  2. If this step is reached, **return** “satisfiable”.
- 

In what follows we see how to apply this combination algorithm in order to show the decidability of the constraint satisfiability problem for the union of theories that share the theory of abelian groups, denoted from now on by



$AG$ . To this aim, we first show that  $AG$  is Noetherian (Section 3.2). Second, we exhibit a theory  $AG^* \supseteq AG$  that admits quantifier-elimination and whose behaviour w.r.t. the satisfiability of constraints is the same of  $AG$  (Section 3.3). Third, we see how to construct effectively Noetherian extensions of  $AG$  by using a superposition calculus (Section 5.1).

### 3 The Theory of Abelian Groups

In this section we focus on some properties that are particularly useful when trying to apply Theorem 1 to a combination of theories sharing  $AG$ .

$\boxed{AG}$  rules the behaviour of the binary function symbol  $+$ , of the unary function symbol  $-$  and of the constant  $0$ . More precisely,  $\Sigma_{AG} := \{0 : AG, - : AG \rightarrow AG, + : AG \times AG \rightarrow AG\}$ , and  $AG$  is axiomatized as follows:

$$\begin{array}{ll} \forall x, y, z \quad (x + y) + z = x + (y + z) & \forall x, y \quad x + y = y + x \\ \forall x \quad x + 0 = x & \forall x \quad x + (-x) = 0 \end{array}$$

From now on, given an expansion of  $\Sigma_{AG}$ , a generic term of sort  $AG$  will be written as  $n_1 t_1 + \dots + n_k t_k$ , where  $t_i$  is a term whose root symbol is different both from  $+$  and  $-$ ,  $t_1 - t_2$  is a shortening for  $t_1 + (-t_2)$ , and  $n_i t_i$  is a shortening for  $t_i + \dots + t_i$  ( $n_i$ )-times if  $n_i$  is a positive integer, or  $-t_i - \dots - t_i$  ( $-n_i$ )-times if  $n_i$  is negative.

#### 3.1 Unification in Abelian Groups

We will consider a superposition calculus using unification in  $AG$  with free symbols, which is known to be finitary [6]. In the following, we restrict ourselves to particular  $AG$ -unification problems with free symbols in which no variables of sort  $AG$  occur. By using a straightforward many-sorted extension of the Baader-Schulz combination procedure [3], one can show that an  $AG$ -equality checker is sufficient to construct a complete set of unifiers for these particular  $AG$ -unification problems with free symbols. Moreover, the following holds:

**Lemma 1** *Let  $\Gamma$  be a  $AG$ -unification problem with free symbols in which no variable of sort  $AG$  occurs, and let  $CSU_{AG}(\Gamma)$  be a complete set of  $AG$ -unifiers of  $\Gamma$ . For any  $\mu \in CSU_{AG}(\Gamma)$ , we have that 1.)  $VRan(\mu) \subseteq Var(\Gamma)$ , and that, 2.) for any  $AG$ -unifier  $\sigma$  of  $\Gamma$  such that  $Dom(\sigma) = Var(\Gamma)$ , there exists  $\mu \in CSU_{AG}(\Gamma)$  such that  $\sigma =_{AG} \mu(\sigma|_{VRan(\mu)})$ .*

#### 3.2 Noetherianity of Abelian Groups

Let us start by proving the Noetherianity of  $AG$ ; the problem of discovering effective Noetherian extensions of  $AG$  will be addressed in Section 5.1, after the introduction of an appropriate superposition calculus (Section 4).

**Proposition 1**  *$AG$  is Noetherian.*

*Proof.* Note that any equation is  $AG$ -equivalent to  $(\#) \sum_{i=1}^k n_i a_i = \sum_{j=1}^h m_j b_j$ , where  $a_i, b_j$  are free constants in  $\underline{a} \cup \underline{b}$  and  $n_i, m_j$  are positive integers, so we can

restrict ourselves to chains of sets of equations of the kind (#). Theorem 3.11 in [7] shows that  $AC$  is Noetherian, where  $AC$  is the theory of an associative and commutative  $+$  (thus  $\Sigma_{AC} = \{+\}$ ). From the definition of Noetherianity it follows that, if  $T$  is a Noetherian  $\Sigma$ -theory, any other  $\Sigma$ -theory  $T'$  such that  $T \subseteq T'$  is Noetherian, too. Clearly, the set of sentences over  $\Sigma_{AC}$  implied by  $AG$  extends  $AC$ ; hence any ascending chain of sets of equations of the kind (#) is eventually constant modulo  $AG$ , too.

In order to apply Theorem 1 to a combination of theories that share  $AG$ , we need to find an extension of  $AG$  that admits quantifier elimination and such that any constraint is satisfiable w.r.t. such an extension iff it is already satisfiable w.r.t.  $AG$ . A first, natural candidate would be  $AG$  itself. Unfortunately it is not the case: more precisely, it is known that  $AG$  cannot admit quantifier elimination (Theorem A.1.4 in [14]). On the other hand, it is possible to find an extension  $AG^*$  with the required properties:  $AG^*$  is the theory of divisible abelian groups with infinitely many elements of each finite order.

### 3.3 An Extension of Abelian Groups having Quantifier Elimination

Let  $D_n := \forall x \exists y ny = x$ , let  $O_n(x) := nx = 0$  and let  $L_{m,n} := \exists y_1, y_2, \dots, y_m \bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_{i=1}^m O_n(y_i)$ , for  $n, m \in \mathbb{N}$ .  $D_n$  expresses the fact that each element is divisible by  $n$ ,  $O_n(x)$  expresses that the element  $x$  is of order  $n$ , and  $L_{m,n}$  expresses the fact that there exist at least  $m$  elements of order  $n$ . The theory  $AG^*$  of divisible abelian groups with infinitely many elements of each finite order can be thus axiomatized by  $AG \cup \{D_n\}_{n>1} \cup \{L_{m,n}\}_{m>0, n>1}$ .

Now, instead of showing directly that  $AG^*$  admits quantifier elimination and satisfies exactly the same constraints that are satisfiable w.r.t.  $AG$ , we rely on a different approach. Let us start by introducing some more notions about structures and their properties. Given a  $\Sigma$ -structure  $\mathcal{M} = (M, \mathcal{I})$ , let  $\Sigma^M$  be the signature where we add to  $\Sigma$  constant symbols  $m$  for each element of  $M$ . The *diagram*  $\Delta(\mathcal{M})$  of  $\mathcal{M}$  is the set of all the ground  $\Sigma^M$ -literals that are true in  $\mathcal{M}$ . Given two  $\Sigma$ -structures  $\mathcal{M} = (M, \mathcal{I})$  and  $\mathcal{N} = (N, \mathcal{J})$ , a  $\Sigma$ -*embedding* (or, simply, an embedding) between  $\mathcal{M}$  and  $\mathcal{N}$  is a mapping  $\mu : M \rightarrow N$  among the corresponding support sets satisfying, for all the  $\Sigma^M$ -atoms  $\psi$ , the condition  $\mathcal{M} \models \psi$  iff  $\mathcal{N} \models \psi$  (here  $\mathcal{M}$  is regarded as a  $\Sigma^M$ -structure, by interpreting each additional constant  $a \in M$  into itself, and  $\mathcal{N}$  is regarded as a  $\Sigma^M$ -structure by interpreting each additional constant  $a \in M$  into  $\mu(a)$ ). If  $M \subseteq N$  and if the embedding  $\mu : M \rightarrow N$  is just the identity inclusion  $M \subseteq N$ , we say that  $\mathcal{M}$  is a *substructure* of  $\mathcal{N}$ . If it happens that, given three models of  $T$ :  $\mathcal{A}, \mathcal{M}, \mathcal{N}$  and two embeddings  $f : \mathcal{A} \rightarrow \mathcal{M}$  and  $g : \mathcal{A} \rightarrow \mathcal{N}$ , there always exists another model of  $T$ ,  $\mathcal{H}$ , and two embeddings  $h : \mathcal{M} \rightarrow \mathcal{H}$  and  $k : \mathcal{N} \rightarrow \mathcal{H}$  such that the composition  $f \circ h = g \circ k$ , we say that  $T$  has the *amalgamation property*. Finally if, given a  $\Sigma$ -theory  $T$  and a model  $\mathcal{M}$  for  $T$ , it happens that, for each  $\Sigma$ -sentence  $\psi$ ,  $\mathcal{M} \models \psi$  if and only if  $T \models \psi$ , then we say that  $T$  is a *complete theory*.

Now, in [14], Exercise 8 page 380, it is stated that  $AG^*$  is the so-called *model companion* of the theory  $AG$ , meaning that (i) for each model  $\mathcal{M}$  of  $AG^*$  the theory  $AG^* \cup \Delta(\mathcal{M})$  is a complete theory, (ii) every constraint that is satisfiable in a model of  $AG$  is satisfiable in a model of  $AG^*$  and (iii) every constraint that is satisfiable in a model of  $AG^*$  is satisfiable in a model of  $AG$ .

(of course, since  $AG \subset AG^*$ , condition (iii) gets trivial, but we report here for sake of completeness). At this point, since the behaviour of  $AG$  and  $AG^*$  is the same w.r.t. the satisfiability of constraints, the only condition that remains to be verified is that  $AG^*$  admits quantifier elimination. But:

**Theorem 2** *AG has the amalgamation property.*

**Corollary 1** *AG\* admits quantifier elimination.*

*Proof.*

In [10] it is shown that, if  $T$  is a universal theory and  $T^*$  is a model-companion of  $T$ , then the following are equivalent:

- (i)  $T^*$  has quantifier elimination;
- (ii)  $T$  has the amalgamation property.

Since  $AG$  has the amalgamation property, and  $AG^*$  is the model-companion of  $AG$ , we have that  $AG^*$  has quantifier elimination.

## 4 A Calculus for Abelian Groups

In [12] the authors give a superposition calculus in which the reasoning about elements of an abelian group is completely built-in. Our aim is to elaborate that calculus so that it provides a decision procedure for the satisfiability problem modulo theories modelling interesting data structures and extending  $AG$ . More precisely, we want to produce a calculus able to check the satisfiability in the models of  $AG$  of sets of literals in the form  $Ax(T) \cup G$ , where  $Ax(T)$  is a set of unit clauses, not necessarily ground, formalizing the behaviour of some data structure, and  $G$  is a set of ground literals. To that purpose, we eliminate the constraints from the calculus and we use a many-sorted language that extends the signature of the theory of abelian groups  $\Sigma_{AG}$  by additional function symbols. Moreover, we will adopt from now on the following assumption: we will consider only

unit clauses with no occurrence of variables of sort AG.    (\*)

Let us start to see more in detail the notations and the concepts used in the rules of the calculus.

First of all, we will reason over terms modulo an  $AG$ -rewriting system: quoting [12], the system  $R_{AG}$  consists of the rules (i)  $x+0 \rightarrow 0$ , (ii)  $-x+x \rightarrow 0$ , (iii)  $-(-x) \rightarrow 0$ , (iv)  $-0 \rightarrow 0$ , (v)  $-(x+y) \rightarrow (-x)+(-y)$ . Moreover, rewriting w.r.t.  $R_{AG}$  is considered *modulo AC*, namely the associativity and the commutativity of the  $+$ , thus, when rewriting  $\rightarrow_{R_{AG}}$ , we mean the relation  $=_{AC} \rightarrow_{R_{AG}} =_{AC}$ . The normal form of a term  $t$  w.r.t.  $R_{AG}$  will be often written as  $AG\text{-nf}(t)$ , and two terms  $t_1$  and  $t_2$  are equal modulo  $AG$  iff  $AG\text{-nf}(t_1) =_{AC} AG\text{-nf}(t_2)$ . Accordingly, we say that a substitution  $\sigma$  is in  $AG$ -normal form whenever all the terms occurring in the codomain of  $\sigma$  are in  $AG$ -normal form.

Moreover, we will consider an order  $\succ$  over terms that is total, well-founded, strict on ground terms and such that 1.  $\succ$  is  $AC$ -compatible, meaning that  $s' =_{AC} s \succ t =_{AC} t'$  implies  $s' \succ t'$ , 2.  $\succ$  orients all the rules of  $R_{AG}$ , meaning

that  $l\sigma \succ r\sigma$  for every rule  $l \rightarrow r$  of  $R_{AG}$  and all the grounding substitutions  $\sigma$ ;  
 3.  $\succ$  is monotonic on ground terms, meaning that for all ground terms  $s, t, u$ ,  $u[s]_p \succ u[t]_p$  whenever  $s \succ t$ . An ordering satisfying all the requirements above can be easily obtained considering an RPO ordering with a total precedence  $\succ_\Sigma$  on the symbols of the signature  $\Sigma$  such that  $f \succ_\Sigma - \succ_\Sigma + \succ_\Sigma 0$  for all symbols  $f$  in  $\Sigma$  and such that all the symbols have a lexicographic status, except  $+$ , whose status is multiset (see [9], where, in order to compare two terms, the arity of  $+$  is considered variable, but always greater than 1).

As a last convention, with a little abuse of notation, we will call *summand* any term whose root symbol is different from both  $+$  and  $-$ , notwithstanding its sort. In this way a *generic* term can be written in the shape  $n_1t_1 + \dots + n_k t_k$  (if it is of sort different from AG, it simply boils down to  $t_1$ ).

Now, we are ready to describe the calculus. We will rely basically on three rules, *Direct AG-superposition*, *Inverse AG-superposition* and *Reflection*, and, as in [12], we will apply the rules only in case the premises satisfy certain conditions as explained in the following. Moreover, from now on we assume that all the literals will be eagerly maintained in *AG-normal form*, meaning that they will be maintained as (dis)equations between terms in *AG-normal form*.

**Orientation for the left premises of direct AG-superposition** Let  $l = r$  be an equation; if it is on the sort *AG*, then it can be equivalently rewritten into  $e = 0$ . Thus the term  $e$  is a term of the form  $n_1t_1 + n_2t_2 + \dots + n_p t_p$ , where the  $t_i$  are non variable distinct summands, and the  $n_i$ 's are non zero integers. By splitting the summands into two disjoint sets, the equation  $e = 0$  can be rewritten as  $n_1t_1 + \dots + n_k t_k = -n_{k+1}t_{k+1} - \dots - n_p t_p$ . In the following, we will call any equation over AG in that form an *orientation* for  $e = 0$ . If  $l = r$  is an equation over a sort different from AG, then an *orientation* of  $l = r$  will be either  $l = r$  or  $r = l$ .

**Orientation for the left premises of inverse AG-superposition** Let  $e = 0$  be an equation over the sort AG. If  $e$  or  $-e$  is a term of the form  $s + e'$ , where  $s$  is a summand that occurs positively and  $e'$  is a generic term, then  $-s = e'$  is an *inverse orientation* for  $e = 0$ .

**Splitting of the right premises for direct AG-superposition** Let  $t$  be a non-variable subterm of either  $r$  or  $s$  in the literal  $r \bowtie s$ ; moreover, if  $s$  is of sort AG, we can freely assume that  $s$  is 0. If  $t$  is of sort AG, we ask that  $t$  is not immediately under  $+$  nor under  $-$ , and that the root of  $t$  is different from  $-$ . Thus, we can imagine that  $t$  is of the kind  $n_1s_1 + \dots + n_p s_p + t'$ , where all  $s_i$  are distinct summands, all  $n_i$  are positive integers and  $t'$  contains only negative summands. In this case,  $t_1 + t_2$  is a *splitting* for  $t$  if  $t_1$  is a term of the form  $k_1s_1 + \dots + k_p s_p$ , where  $0 \leq k_i \leq n_i$ , and  $t_2$  is  $(n_1 - k_1)s_1 + \dots + (n_p - k_p)s_p + t'$ . If  $t$  is not over the sort AG, then the only splitting admissible for  $t$  is  $t$  itself.

**Splitting of the right premises for inverse AG-superposition** Let  $t$  be a non variable subterm of either  $r$  or  $s$  in the literal  $r \bowtie s$ ; moreover, if  $s$  is of sort AG, we can freely assume that  $s$  is 0. Let  $t$  be of sort AG, and let  $t$  be not

immediately below  $+$  nor  $-$ . If  $t$  is of the form  $-s + t'$ , where  $s$  is a summand, then  $t_1 + t_2$  is an *inverse splitting* for  $t$  if  $t_1$  is  $-s$  and  $t_2$  is  $t'$ .

**AG-superposition rules** In the left premise  $l = r$  of the direct *AG*-superposition rule, it is assumed that  $l = r$  is an orientation of the literal. Similarly, in the right premise,  $D[t_1 + t_2]_p$  denotes that  $D|_p$  is a non-variable term that is not immediately below  $+$  or  $-$  with a splitting  $t_1 + t_2$ . Similarly, in the inverse *AG*-superposition rule,  $l = r$  and  $D|_p$  denote inverse orientation and splitting, respectively. The inference system, denoted by  $\mathcal{SP}_{AG}$ , is made of the following rules:

<i>Direct AG-superposition</i>	$\frac{l = r \quad D[t_1 + t_2]_p}{(D[r + t_2]_p)\mu_i} \quad (i)$
<i>Inverse AG-superposition</i>	$\frac{l = r \quad D[t_1 + t_2]_p}{(D[r + t_2]_p)\mu_i} \quad (ii)$
<i>Reflection</i>	$\frac{u' \neq u}{\square} \quad (iii)$

The condition (i) is that  $\mu_i$  is a most general solution of the *AG*-unification problem  $l =_{AG} t_1$ ; moreover the inference has to be performed whenever there is a ground instantiation of  $\mu_i, \theta$ , s.t., if  $nu = s$  is the *AG*-normal form of  $(l = r)\mu_i\theta$  and  $D'[nu]_q$  is the *AG*-normal form of  $(D[t_1 + t_2]_p)\mu_i\theta$  in which, in position  $q$ ,  $nu$  appears as subterm, then (a)  $u \succ s$ , (b)  $nu$  appears as subterm of the maximal term in  $D'$ .

The condition (ii) is that  $\mu_i$  is a most general solution of the *AG*-unification problem  $l =_{AG} t_1$ ; moreover the inference is needed to be performed whenever there is a ground instantiation of  $\mu_i, \theta$ , s.t., if  $-u = s$  is the *AG*-normal form of  $(l = r)\mu_i\theta$  and  $D'[-u]_q$  is the *AG*-normal form of  $(D[t_1 + t_2]_p)\mu_i\theta$  in which, in position  $q$ ,  $-u$  appears as subterm, then (a) either  $u$  is the maximal summand in  $s$  or  $u \succ s$ , (b)  $-u$  appears as subterm of the maximal term in  $D'$ .

The condition (iii) is that the *AG*-unification problem  $u =_{AG} u'$  has a solution (and  $\square$  is the syntactic convention for the empty clause).

Moreover, we assume that, after each inference step, the newly-derived literal is normalized modulo *AG*.

We point out that, thanks to Lemma 1(1.) and to our assumption (\*), at any step of a saturation no variable of sort *AG* is introduced, thus the resulting saturated set will consist of literals in which no variable of sort *AG* occurs. Moreover, we can note that the conditions on the inferences are, in general, far from being obvious to check. However, for our purposes, we will often perform inferences involving at least one ground literal. In that case, verifying all the conditions becomes easier.

## 5 Refutational Completeness of $\mathcal{SP}_{AG}$

In order to prove the refutational completeness of the calculus presented above, we will adapt the model generation technique presented in [12]. The idea behind this technique consists in associating to any saturated set of literals that does not contain the empty clause a model of terms identified modulo a rewriting system,

the latter being built according to some of the equations in the saturated set. Even if in our calculus no constrained literal will appear, in order to build the model of terms we will rely only on ground instances of the literals in the saturation that are *irreducible*. Moving from [12] and extending to the many-sorted case, we say that:

**Definition 1** *An equation  $s = t$  is in one-sided form whenever, (a) if  $s$  and  $t$  are of sort AG, the equation is in the form  $e = 0$ , and  $e$  is in AG-normal form; (b) if  $s$  and  $t$  are not of sort AG, both  $s$  and  $t$  are in AG-normal form.*

Whereas an equation over a sort different from AG has a unique one-sided form, an equation over the sort AG has two AG-equivalent one-sided forms, but in what follows it does not matter which of the two will be considered. Thus, from now on, when we will refer to equations, we will always assume that the equations are in one-sided form.

**Definition 2** *Let  $s$  be a term,  $\sigma$  be a grounding substitution such that both  $s$  and  $\sigma$  are in AG-normal form. Moreover, let  $R$  be a ground term rewriting system. We will say that the  $\text{maxred}_R(s\sigma)$  is*

- 0, if  $\text{AG-nf}(s\sigma)$  is  $R$ -irreducible;
- $\max PS$ , where  $PS$  is the following set of terms (ordered w.r.t.  $\succ$ ):  
 $PS := \{u \text{ is a summand} \mid \text{for some term } v \text{ and some } n \text{ in } \mathbb{Z}, \text{AG-nf}(s\sigma) \text{ is of the form } nu + v \text{ and } nu \text{ is } R\text{-reducible}\}.$

**Definition 3**<sup>1</sup> *Let  $s$  be a term in which no variable of sort AG occurs, let  $\sigma$  be a grounding substitution such that both  $s$  and  $\sigma$  are in AG-normal form, and let  $R$  be a ground TRS. The pair  $(s, \sigma)$  is irreducible w.r.t.  $R$  whenever:*

- $\text{AG-nf}(s\sigma)$  is  $R$ -irreducible, or
- if  $\text{AG-nf}(s\sigma)$  is  $R$ -reducible, let  $u$  be the  $\text{maxred}_R(s\sigma)$ . Then,  $(s, \sigma)$  is irreducible if  $s$  is not a variable and, for each term of the form  $t = f(t_1, \dots, t_n)$  such that  $s$  is of the form  $t+v$  or  $-t+v$  or  $t$  and such that  $u \succeq \text{AG-nf}(t\sigma)$ , each  $(t_i, \sigma)$  is irreducible.

If  $L$  is a literal, the pair  $(L, \sigma)$  is irreducible w.r.t.  $R$ :

- if  $L$  is an (dis)equation whose one-sided form is of the form  $e \bowtie 0$ , then  $(e, \sigma)$  is irreducible w.r.t.  $R$ ;
- if  $L$  is an (dis)equation whose one-sided form is of the form  $s \bowtie t$ , both  $(s, \sigma)$  and  $(t, \sigma)$  are irreducible w.r.t.  $R$ .

Before going on with the description of all the ingredients that are needed in order to show the completeness of the calculus, we want to point out a property that will be useful in the following.

<sup>1</sup>Here we are adapting, in case of absence of variables of sort AG, the definition of recursive irreducibility of [12], but in our context the two notions of recursive irreducibility and irreducibility are collapsing.

**Proposition 2** *Let  $s$  be a term in which no variable of sort AG occurs, let  $\sigma$  be a grounding substitution such that both  $s$  and  $\sigma$  are in AG-normal form, and let  $R$  be a ground TRS such that  $(s, \sigma)$  is irreducible w.r.t.  $R$ . Moreover, let  $\sigma =_{AG} \mu\pi$ , where  $\pi$  is another grounding substitution in AG-normal form and  $\mu$  is a substitution that does not have variables of sort AG in its range. Then  $(s\mu, \pi)$  is still irreducible w.r.t.  $R$ .*

To extract, from a given set of ground literals, a term rewriting system, we first of all transform all the equations in reductive normal form (see [12]):

**Definition 4** *A ground literal  $s \bowtie t$  in AG-normal form is in reductive form whenever  $s$  is of the form  $nu$ ,  $t$  is the form  $n_1v_1 + \dots + n_kv_k$  and  $n > 0$ ,  $n_i$  are non-zero integers,  $u$  and  $v_i$  are summands with  $u \succ v_i$ .*

Of course, if  $s$  and  $t$  are of sort different from AG, the definition above simply says that  $s \succ t$ ; moreover, it is always possible, given an equation, to obtain an equivalent one in reductive normal form. Now, a term rewriting system is obtained as follows:

**Definition 5** *Let  $S$  be a set of literals, let  $L$  be an equation with a ground instance  $L\sigma$ , let  $G$  be the reductive form of  $L\sigma$ :  $G \equiv nu = r$ . Then  $G$  generates the rule  $nu \rightarrow r$  if the following conditions are satisfied:*

- (i)  $(R_G \cup AG) \not\models G$ ;
- (ii)  $u \succ r$ ;
- (iii)  $nu$  is  $R_G$ -irreducible;
- (iv)  $(L, \sigma)$  is irreducible w.r.t.  $R_G$ .

where  $R_G$  is the set of rules generated by the reductive forms of the ground instances of  $S$  that are smaller than  $G$  w.r.t.  $\succ$ . Moreover, if  $n > 1$ , then also the rule  $-u \rightarrow (n-1)u - r$  is generated.

Now, exactly as in [12], we associate to a generic set of literals saturated under the rules of our calculus and that does not contain the empty clause,  $S$ , a structure  $I$  that is an AG-model for  $S$ .  $I$  is the equality Herbrand interpretation defined as the congruence on ground terms generated by  $R_S \cup AG$ , where  $R_S$  is the set of rules generated by  $S$  according to Definition 5. Since we are in a many-sorted context, the domain of  $I$  consists of different sets, one for each sort; since the rewriting rules in  $R_S \cup AG$  are sort-preserving, the congruence on the ground terms is well-defined. Applying the same kind of arguments used to prove Lemma 10 in [12], we have that  $R_S \cup AG$  is terminating and confluent, and it still holds that  $I \models s = t$  iff  $s \rightarrow_{R_S \cup AG}^* \tau \leftarrow_{R_S \cup AG}^* t$  for some term  $\tau$ . To show that  $I$  is really an AG-model for  $S$ , we can prove the following lemma:

**Lemma 2** *Let  $S$  be the closure under the calculus of a set of literals  $S_0$ , and let us assume that the empty clause does not belong to  $S$ . Let  $I$  be the model of terms derived from  $S$  as described above, and let  $Ir_{R_S}(S)$  be the set of ground instances  $L\sigma$  of  $L$  in  $S$  such that  $(L, \sigma)$  is irreducible w.r.t.  $R_S$ . Then (1)  $I \models Ir_{R_S}(S)$  implies that  $I \models S$ , and (2)  $I \models Ir_{R_S}(S)$ .*

From the lemma above, it follows immediately:

**Theorem 3** *The calculus  $SP_{AG}$  is refutational complete for any set of literals that do not contain variables of sort AG.*

## 5.1 Computing AG-bases

Let us go back, for the moment, to Theorem 1, and especially to condition (2b) that states that, in order to apply a combination procedure à la Nelson-Oppen to a pair of theories  $T_1$  and  $T_2$  sharing  $AG$ , we have to ensure that  $T_1$  and  $T_2$  are effectively Noetherian extensions of  $AG$ , i.e. we have to ensure the capability of computing  $AG$ -bases for  $T_1$  and  $T_2$ . Let us suppose that  $T_1$  and  $T_2$  are  $\Sigma_i$ -theory whose set of axioms is described by a finite number of unit clauses.

Now, for  $i = 1, 2$ , let  $\Gamma_i$  be a set of ground literals over an expansion of  $\Sigma_i \supseteq \Sigma_{AG}$  with the finite sets of fresh constants  $\underline{a}, \underline{b}_i$ , and suppose to perform a saturation w.r.t.  $\mathcal{SP}_{AG}$  adopting an RPO ordering in which the precedence is  $f \succ a \succ - \succ + \succ 0$  for every function symbol  $f$  in  $\Sigma_i^{\underline{b}_i}$  different from  $+, -, 0$ , every constant  $a$  in  $\underline{a}$  and that all the symbols have a lexicographic status, except  $+$ , whose status is multiset. Relying on the refutational completeness of  $\mathcal{SP}_{AG}$ , Proposition 3 shows how  $\mathcal{SP}_{AG}$  can be used in order to ensure that  $T_1$  and  $T_2$  are effectively Noetherian extensions of  $AG$ :

**Proposition 3** *Let  $S$  be a finite saturation of  $T_i \cup \Gamma_i$  w.r.t  $\mathcal{SP}_{AG}$  not containing the empty clause and suppose that, in every equation  $e = 0$  containing at least one of the constants  $a$  in  $\underline{a}$  as summand, the maximal summand is not unifiable with any other summand in  $e$ . Then the set  $\Delta_i$  of all the ground equations over  $\Sigma_{AG}^{\underline{a}}$  in  $S$  is an  $AG$ -basis for  $T_i$  w.r.t.  $\underline{a}$  ( $i = 1, 2$ ).*

## 6 Some Examples

Theorem 3 guarantees that  $\mathcal{SP}_{AG}$  is refutational complete, thus, if we want to turn it into a decision procedure for the constraint satisfiability problem w.r.t. a theory of the kind  $T \cup AG$ , it is sufficient to prove that any saturation under the rules of  $\mathcal{SP}_{AG}$  of a set of ground literals and the axioms of  $T$  is finite. Let us show some examples in which this is actually the case.

**Lists with Length** The theory of lists with length can be seen as the union of the theories  $T_L \cup T_\ell \cup AG$ , with  $T_L$  being the theory of lists and  $T_\ell$  being the theory that axiomatizes the behaviour of the function for the length; more formally:

$\boxed{T_L}$  has the many-sorted signature of the theory of lists:  $\Sigma_L$  is the set of function symbols  $\{\text{nil} : \text{LISTS}, \text{car} : \text{LISTS} \rightarrow \text{ELEM}, \text{cdr} : \text{LISTS} \rightarrow \text{LISTS}, \text{cons} : \text{ELEM} \times \text{LISTS} \rightarrow \text{LISTS}\}$  plus the predicate symbol  $\text{atom} : \text{LISTS}$ , and it is axiomatized as follows:

$$\begin{array}{ll} \forall x, y \text{ car}(\text{cons}(x, y)) = x & \forall x \neg \text{atom}(x) \Rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x \\ \forall x, y \text{ cdr}(\text{cons}(x, y)) = y & \forall x, y \neg \text{atom}(\text{cons}(x, y)) \\ & \text{atom}(\text{nil}) \end{array}$$

$\boxed{T_\ell}$  is the theory that gives the axioms for the function  $\text{length} \ell : \text{LISTS} \rightarrow \text{AG}$  and the constant  $(1 : \text{AG})$ :  $\ell(\text{nil}) = 0$ ;  $\forall x, y \ell(\text{cons}(x, y)) = \ell(y) + 1$ ;  $1 \neq 0$

Applying some standard reasoning (see, e.g. [18]), we can substitute  $T_L$  with the set of the purely equational axioms of  $T_L$ , say  $T_{L'}$ , and enrich a bit



the set of literals  $G$  to a set of literals  $G'$  in such a way  $T_L \cup T_\ell \cup AG \cup G$  is equisatisfiable to  $T_{L'} \cup T_{\ell'} \cup AG \cup G'$ . Let us choose as ordering an RPO with a total precedence  $\succ$  such that all the symbols have a lexicographic status, except  $+$ , whose status is multiset, and such that it respects the following requirements: (a)  $\text{cons} \succ \text{cdr} \succ \text{car} \succ c \succ e \succ \ell$  for every constant  $c$  of sort `LISTS` and every constant  $e$  of sort `ELEM`; (b)  $\ell \succ g \succ - \succ + \succ 0$  for every constant  $g$  of sort `AG`.

**Proposition 4** *For any set  $G$  of ground literals, any saturation of  $T_{L'} \cup T_{\ell'} \cup G'$  w.r.t.  $\mathcal{SP}_{AG}$  is finite.*

**Trees with Size** Let us reason about trees and their size. We can propose a formalization in which we need to reason about a theory of the kind  $T_T \cup T_{size} \cup AG$ , where  $T_T$  rules the behaviour of the trees and  $T_{size}$  constraints the behaviour of a function that returns the number of nodes of a tree. Thus we have:

$T_T$  has the mono-sorted signature  $\Sigma_T := \{\mathcal{E} : \text{TREES}, \text{binL} : \text{TREES} \rightarrow \text{TREES}, \text{binR} : \text{TREES} \rightarrow \text{TREES}, \text{bin} : \text{TREES} \times \text{TREES} \rightarrow \text{TREES}\}$ , and it is axiomatized as follows:

$$\begin{aligned} \forall x, y \text{ binL}(\text{bin}(x, y)) = x & \quad \forall x, y \text{ binR}(\text{bin}(x, y)) = y \\ \forall x \text{ bin}(\text{binL}(x), \text{binR}(x)) = x & \end{aligned}$$

$T_{size}$  is the theory that gives the axioms for the function  $\text{size} : \text{TREES} \rightarrow \text{AG}$ :  
 $\text{size}(\mathcal{E}) = 0; \quad \forall x, y \text{ size}(\text{bin}(x, y)) = \text{size}(x) + \text{size}(y)$

Let us now put as ordering an RPO with a total precedence  $\succ$  on the symbols of the signature such that all the symbols have a lexicographic status, except  $+$ , whose status is multiset, and such that it respects the following requirements: (a)  $\text{bin} \succ \text{binR} \succ \text{binL} \succ c \succ \text{size}$  for every constant  $c$  of sort `TREES`; (b)  $\text{size} \succ g \succ - \succ + \succ 0$  for every constant  $g$  of sort `AG`.

**Proposition 5** *For any set  $G$  of ground literals, any saturation of  $T_T \cup T_{size} \cup G$  w.r.t.  $\mathcal{SP}_{AG}$  is finite.*

**Application** (Algorithm 2.8 in [25]: *Left-Rotation of trees*) Using the procedure induced by the calculus  $\mathcal{SP}_{AG}$ , it is possible to verify, e.g. that the input tree  $x$  and the output tree  $y$  have the same size:

1.  $t := x$ ; 2.  $y := \text{binR}(t)$ ; 3.  $\text{binR}(t) := \text{binL}(y)$ ; 4.  $\text{binL}(y) := t$ ; 5. Return  $y$

In order to check that the size of  $x$  is exactly the one of  $y$ , we check for unsatisfiability modulo  $T_T \cup T_{size} \cup AG$  the following constraint (see, again [25]):

$$\begin{aligned} \text{binR}(t') = \text{binL}(\text{binR}(x')) \wedge \text{binL}(t') = \text{binL}(x') \wedge \text{binL}(y') = t' \\ \wedge \text{binR}(y') = \text{binR}(\text{binR}(x')) \wedge \text{size}(x') \neq \text{size}(y') \end{aligned}$$

where  $x', y'$  and  $t'$  are fresh constants that identify the trees on which the algorithm applies.

## 6.1 Applying the Combination Framework

In the section above we have shown some examples of theories that extend the theory of abelian groups and whose constraint satisfiability problem is decidable. We have proved that  $AG$  can be enlarged to  $AG^*$  and  $AG$  and  $AG^*$  behave the same w.r.t. the satisfiability of constraints; moreover we have checked that  $AG$  is a Noetherian theory. To guarantee now that the theories that have been studied can be combined together it is sufficient to show that they fully satisfy the requirement of being  $AG$ -compatible and effectively Noetherian extension of  $AG$  (requirements (2a) and (2b) of Theorem 1). The  $AG$ -compatibility both of lists with length and trees with size is easily ensured observing that a constraint is satisfied w.r.t.  $T_L \cup T_\ell \cup AG$  iff it is satisfied w.r.t.  $T_L \cup T_\ell \cup AG^*$  and, analogously, any constraint is satisfiable w.r.t.  $T_T \cup T_{size} \cup AG$  iff it is w.r.t.  $T_T \cup T_{size} \cup AG^*$ .

Moreover, checking the shape of the saturations produced, it is immediate to see that all the hypotheses required by Proposition 3 are satisfied when considering both the cases of lists with length and trees with size, turning  $\mathcal{SP}_{AG}$  not only into a decision procedure for the constraint satisfiability problem, but also into an effective method for deriving complete sets of logical consequences over the signature of abelian groups (namely, the  $AG$ -bases). This implies that also the requirement (2b) of being effectively Noetherian extensions of abelian groups is fulfilled for both lists with length and trees with size. To sum up, we have proved that the theories presented so far can be combined preserving the decidability of the constraint satisfiability problem.

## 7 Conclusion

The problem of integrating a reasoning modulo arithmetic properties into the superposition calculus has been variously studied, and different solutions have been proposed, both giving the possibility of reasoning modulo the linear rational arithmetic ([15]) and relying on an over-approximation of arithmetic via abelian groups ([12, 22]) or divisible abelian groups ([23, 24]).

We have focused on the second kind of approach, giving an original solution to the satisfiability problem in combinations of theories sharing the theory of abelian groups. We have shown that in this case all the requirements to apply the non-disjoint combination method are satisfied, and we have considered an appropriate superposition calculus modulo abelian groups in order to derive satisfiability procedures. This calculus relies on a non trivial adaptation the one proposed in [12]: We consider a many-sorted and constraint-free version of the calculus, in which we use a restricted form of unification in abelian groups with free symbols, and in which only literals are involved. Under these assumptions we have proved that the calculus is refutationally complete, but, as a side remark, we notice that the same kind of proof works also in case the rules are extended to deal with Horn clauses and also, exactly as it happens in [12], after the introduction of an appropriate rule for the Factoring, to deal with general clauses. Our focus on the unit clause case is justified by our interest in the application to particular theories whose formalization is actually through axioms of that form.

It is worth noticing that two combination methods are involved in our approach: the method for unification problems [3] and the non-disjoint extension of Nelson-Oppen for satisfiability problems [11].

The framework for the non-disjoint combination used here cannot be applied, as it is, to the case where we consider a combination of theories sharing the Presburger arithmetic, because the latter is not Noetherian. Another framework, able to guarantee the termination of the resulting procedure on all the inputs, should be designed for that case.

We envision several directions for future work. As a first direction, we would like to relax current restrictions on theories and saturation types to apply effectively the calculus in the non-disjoint combination method. At the moment, since the presence of variables of sort AG into the clauses is not allowed, the results in [18] are not subsumed by the present paper. That restriction is justified by technical reasons: an important issue would be to discard it, enlarging in this way the applicability of our results. As a second direction we foresee, it would be interesting to find general methods to ensure the termination of the calculus by developing, for instance, an automatic meta-saturation method [16], or by considering a variable-inactivity condition [1]. Finally, it would be interesting to study how our calculus can be integrated into Satisfiability Modulo Theories solvers, by exploiting for instance the general framework developed in [4].

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## A Unification in Abelian Groups

In this section, we rely on the combination algorithm for unification problems developed by Baader-Schulz, which makes use of unification of linear constant restriction [3]. Strictly speaking, we use a (straightforward) many-sorted extension of the Baader-Schulz combination algorithm, to deal with unification problems where function symbols in  $AG$  and free function symbols share the sort  $AG$ . Given a unification problem  $\Gamma$ , a set of free constants  $C$ , and  $<$  a linear constant restriction over  $C$ ,  $(\Gamma, C)$  denotes the  $E$ -unification problem  $\Gamma$  with free constants  $C$  and  $(\Gamma, C, <)$  denotes the  $E$ -unification problem  $(\Gamma, C)$  with the linear constant restriction  $<$  over  $C$ . A complete set of  $E$ -unifiers of  $(\Gamma, C)$  (resp.  $(\Gamma, C, <)$ ) is denoted by  $CSU_E(\Gamma, C)$  (resp.  $CSU_E^<(\Gamma, C)$ ).

The results we are using for  $AG$ -unification with free symbols over the shared sort  $AG$  are consequences of more general results stated in the following for  $E$ -unification with free symbols over the shared sort  $s_0$ .

**Definition 6** *Let  $\Sigma$  be a mono-sorted signature over the sort  $s_0$  and let  $E$  be an equational  $\Sigma$ -theory. Let  $\Sigma'$  be a many-sorted signature such that  $s_0$  is the unique symbol shared by  $\Sigma$  and  $\Sigma'$ , and let  $F$  be the empty equational  $\Sigma'$ -theory. A general  $E$ -unification problem is an  $E \cup F$ -unification problem. A general  $E$ -ground unification problem is a general  $E$ -unification problem in which no variable of sort  $s_0$  occurs.*

**Lemma 3** *Let  $(\Gamma, C, <)$  be an  $E$ -unification problem with linear constant restriction  $<$ , such that  $\Gamma$  is of the form  $\bigwedge_{i=1}^n x_i = t_i$ , where  $t_i$  is ground for  $i = 1, \dots, n$ . If  $E$  admits a convergent TRS, then it is possible to construct a  $CSU_E^<(\Gamma, C)$  which is either empty or a singleton whose unique element is a ground substitution.*

*Proof.* The repeated application of the rules given below allows us to obtain a solved form of the  $E$ -unification problem with constants  $(\Gamma, C)$ .

1.  $\Gamma \wedge x = s \wedge x = t \vdash \Gamma \wedge x = s \downarrow_R$  if  $x \in \text{Var}(\Gamma) \setminus C$  and  $s \downarrow_R = t \downarrow_R$
2.  $\Gamma \wedge x = s \wedge x = t \vdash \perp$  if  $x \in \text{Var}(\Gamma) \setminus C$  and  $s \downarrow_R \neq t \downarrow_R$
3.  $\Gamma \wedge x = s \vdash \Gamma$  if  $x \in C$  and  $x = s \downarrow_R$
4.  $\Gamma \wedge x = s \vdash \perp$  if  $x \in C$  and  $x \neq s \downarrow_R$

If the solved form is  $\perp$ , then  $CSU_E^<(\Gamma, C) = \emptyset$ . Otherwise, the solved form corresponds to a grounding substitution in  $R$ -normal form, say  $\sigma$ . To obtain solutions satisfying the linear constant restriction  $<$ , we still have to eliminate a constant  $c$  from the term  $x\sigma$  if  $x < c$ , which means that we have to find a substitution  $\mu$  such that there exists a term  $u$  satisfying  $x\sigma\mu =_E u$  and  $c \notin u$ . Since  $x\sigma$  is ground,  $\mu$  is the identity, and a term  $u$  with  $c \notin u$  exists iff  $c \notin x\sigma$ . Indeed, rewriting w.r.t  $R$  does not introduce new free constants, and so  $c \notin u$  implies  $c \notin u \downarrow_R = x\sigma \downarrow_R = x\sigma$ . Consequently,  $CSU_E^<(\Gamma, C) = \{\sigma\}$  if

$$\forall x \in \text{Var}(\Gamma) \setminus C \forall c \in C, x < c \Rightarrow c \notin x\sigma.$$

Otherwise,  $CSU_E^<(\Gamma, C) = \emptyset$ .

**Lemma 4** *If  $E$  is an equational theory admitting a convergent TRS, then general  $E$ -ground unification is finitary, and for any general  $E$ -ground unification problem  $\Gamma$ , we have:  $\forall \sigma \in CSU_{E \cup F}(\Gamma), VRan(\sigma) \subseteq Var(\Gamma)$ .*

*Proof.* The Baader-Schulz combination algorithm for unification [3] can be applied to solve an  $E \cup F$ -unification problem  $\Gamma$  in which no variable of sort  $s_0$  occurs. First, we purify the equations of  $\Gamma$  by applying repeatedly the following rules:

**VA:**  $\Gamma \wedge s[u]_\omega = t \vdash \Gamma \wedge s[x]_\omega = t \wedge x = u$  if  $u$  is a direct alien subterm of  $s$  occurring at position  $\omega$ , and  $x$  is new variable (of sort  $s_0$ )

**PurifEq:**  $\Gamma \wedge s = t \vdash \Gamma \wedge x = s \wedge x = t$  if  $s, t$  are pure (non-variable) terms, and  $x$  is a new variable (of sort  $s_0$ )

This purification process applied on  $\Gamma$  terminates and leads to an  $E \cup F$ -unification problem  $\Gamma_E \wedge \Gamma_F$ , such that  $\Gamma_E$  and  $\Gamma_F$  are respectively  $E$ -pure and  $F$ -pure. Then, the combination algorithm considers all possible cases of

- partitions of the set of shared variables  $V$  into  $V_E \oplus V_F$ ,
- identifications  $\xi$  of  $V_E \oplus V_F$  (i.e. idempotent substitutions whose domains and ranges are both included in  $V_E \oplus V_F$ ) such that  $x\xi = y\xi$  implies  $x, y \in V_E$  or  $x, y \in V_F$ ,
- linear orderings  $<$  over  $V_E\xi \oplus V_F\xi$

and calls unification algorithms with linear constant restriction known for  $E$  and  $F$  to solve the respective inputs  $(\Gamma_E\xi, V_F\xi, <)$  and  $(\Gamma_F\xi, V_E\xi, <)$ .

Since  $F$  is the empty theory, it is sufficient to consider the linear constant restrictions such that:

- (i) If  $x = t$  occurs in  $\Gamma_F$ , then  $x \in V_F$
- (ii) If  $x = t$  occurs in  $\Gamma_F$  and  $c \in t$  for some  $c \in V_E$ , then  $c < x$

since for the other linear constant restrictions, one get  $F$ -unification problems with no solutions. Thanks to (i), all abstraction variables occurring in  $\Gamma_E$  (introduced during the purification process) are necessarily in  $V_F$ , and so are considered as free constants in  $(\Gamma_E\xi, V_F\xi, <)$ . Therefore, Lemma 3 can be applied, and for each solution  $\sigma_E \in CSU_E^<(\Gamma_E\xi, V_F\xi)$  there are no new variables introduced. It is well-known that the same property holds also for the empty theory  $F$  and any solution  $\sigma_F \in CSU_F^<(\Gamma_F\xi, V_E\xi)$ . The set of all conjunctions of (the solved forms of)  $\xi$ ,  $\sigma_E$  and  $\sigma_F$  represents a disjunction of “dag” solved forms and after variable replacement we obtain a disjunction of solved forms equivalent to  $\Gamma_E \wedge \Gamma_F$ . The restrictions of these solved forms to  $Var(\Gamma)$  define a complete set of  $E \cup F$ -unifiers of  $\Gamma$  satisfying the required property.

**Lemma 5** *Let  $\Gamma$  be a  $E$ -unification problem, and let  $CSU_E(\Gamma)$  be a complete set of  $E$ -unifiers of  $\Gamma$  such that  $\forall \mu \in CSU_E(\Gamma), VRan(\mu) \subseteq Var(\Gamma)$ . For any  $E$ -unifier  $\sigma$  of  $\Gamma$  such that  $Dom(\sigma) = Var(\Gamma)$ , there exists  $\mu \in CSU_E(\Gamma)$  such that  $\sigma =_E \mu(\sigma|_{VRan(\mu)})$ .*

*Proof.* By definition of a complete set of  $E$ -unifiers, there exist  $\mu \in CSU_E(\Gamma)$  and a substitution  $\sigma'$  such that  $\forall x \in Var(\Gamma), x\sigma =_E x\mu\sigma'$ . By definition,  $VRan(\mu) \cap Dom(\mu) = \emptyset$ , and so  $\forall x \in VRan(\mu), x\mu = x$ . Since  $VRan(\mu) \subseteq Var(\Gamma)$ , we have:  $\forall x \in VRan(\mu), x\sigma =_E x\mu\sigma' = x\sigma'$  which means that  $\sigma|_{VRan(\mu)} =_E \sigma'|_{VRan(\mu)}$ . Hence,  $\sigma|_{VRan(\mu)}$  can replace  $\sigma'|_{VRan(\mu)}$  to instantiate terms in the range of  $\mu$  and so we get (1)  $\forall x \in Var(\Gamma), x\sigma =_E x\mu(\sigma|_{VRan(\mu)})$ . By assumption,  $Var(\Gamma) = Dom(\sigma)$ , and both  $Dom(\mu)$  and  $VRan(\mu)$  are included in  $Var(\Gamma)$ . Consequently, we have that (2) for any variable  $x$  not in  $Var(\Gamma)$ ,  $x\sigma = x = x\mu(\sigma|_{VRan(\mu)})$ . Finally, (1) and (2) imply that  $\sigma =_E \mu(\sigma|_{VRan(\mu)})$ .

Lemma 1 directly follows from Lemma 4 and Lemma 5.

## B Amalgamation Property of $AG$

In this section we prove that the theory of abelian groups  $AG$  has the amalgamation property. Though the result should be known, we are not able to provide a precise reference where the property is completely illustrated. For that reason, we prefer here to propose an argument by ours, moving from a basic construction in [13].

**Theorem 2.**  *$AG$  has the amalgamation property.*

*Proof.* The amalgamation property for abelian groups can be restated as follows: let  $H, K$  and  $A'$  be s.t. there exist two embeddings  $i : A' \rightarrow H$  and  $\iota : A' \rightarrow K$ . We want to show that there exists a group  $L$  such that:

1.  $L$  is abelian;
2. there exist two embeddings of abelian groups  $f : H \rightarrow L$  and  $g : K \rightarrow L$  such that the composition are equal, i.e.  $i \circ f = \iota \circ g$ .

Since  $i$  and  $\iota$  are embeddings from  $A'$  to  $H$  and  $K$  respectively, it means that there exist a subgroup of  $H$ , let us say  $A$ , and a subgroup of  $K$ , let us say  $B$ , such that  $A = i(A')$ ,  $B = \iota(A')$ , and such that  $A$  and  $B$  are isomorphic (the isomorphism is given by  $i^{-1} \circ \iota$ , or  $\iota^{-1} \circ i$ ). In order to find the subgroup  $L$  needed, we can start moving from  $H$  and  $K$ , adapting a case that can be found in [13].

**Building L** Let  $S$  be the set of all the finite words over the alphabet  $H \cup K$  and let  $\sim$  be the smallest equivalence relation on words such that, for all the words  $\alpha, \beta$ :

- E1** if  $1$  is  $1_H$  or  $1_K$ ,  $\alpha 1 \beta$  is equivalent to  $\alpha \beta$ ;
- E2** if  $h_1$  and  $h_2$  belong both to  $H$  and in  $H$   $h_1 h_2 = \hat{h}$ ,<sup>2</sup> then  $\alpha h_1 h_2 \beta$  is equivalent to  $\alpha \hat{h} \beta$ ;
- E3** if  $k_1$  and  $k_2$  belong both to  $K$  and in  $K$   $k_1 k_2 = \hat{k}$ , then  $\alpha k_1 k_2 \beta$  is equivalent to  $\alpha \hat{k} \beta$ ;
- E4** if  $a$  is an element of  $A$  and  $b$  is an element of  $B$  such that  $b = \iota(i^{-1}(a))$ , or, equivalently,  $a = i(\iota^{-1}(b))$ , then  $\alpha a \beta$  is equivalent to  $\alpha b \beta$ ;

<sup>2</sup>For sake of readability, we will not be very precise in using different symbols for the operation of concatenation of words over a given alphabet and the product of elements in a group: the context should be enough to avoid ambiguities.

**E5** if  $h$  is an element of  $H$  and  $k$  is an element of  $K$ , then  $\alpha hk\beta$  is equivalent to  $\alpha kh\beta$ .

We can consider the quotient of  $S$  with respect to  $\sim$ :  $S/\sim$ , and we can introduce a binary operation  $*$  over the equivalence classes in  $S/\sim$ , relying on the concatenation: if  $[f_1]$  and  $[f_2]$  are equivalence classes, then  $[f_1]*[f_2] := [f_1f_2]$ . It is easy to verify that

- the product  $*$  is well-defined: if  $f_1 \sim f'_1$  and  $f_2 \sim f'_2$ , then  $[f_1f_2] = [f'_1f'_2]$ ;
- the product  $*$  is associative;
- the (class of the) void word is the identity of the product;
- if  $f = g_1 \dots g_n$ , then  $f^{-1} := g_n^{-1} \dots g_1^{-1}$  is the representative of the inverse of  $[f]$  (being the  $g_i$ 's simply letters of the alphabet  $H \cup K$ ).

So, let us name  $L$  the group  $L := \langle S/\sim, * \rangle$ .

$L$  is commutative: by induction, it is easy to see that, if  $f_1 = g_1 \dots g_n$ ,  $f_2 = g'_1 \dots g'_m$ ,  $[f_1] * [f_2] = [f_2] * [f_1]$ .

**Building f and g** Our aim is now to build the appropriate embeddings of  $H$  and  $K$  into  $L$ . To this aim, we will rely on the fact that in  $L$  can be defined a *normal form* of the equivalence classes.

We will call the algorithm for the reduction of a word in its normal form a word process. Before starting to describe it, we introduce the definition of a word in canonical form. Relying on the decomposition of a group in right cosets of a subgroup<sup>3</sup>, we decompose  $H$  into right cosets of  $A$ , choosing the identity of  $H$ :  $1_H$  as the representative of  $A$ , and we decompose  $K$  into right cosets of  $B$ , choosing the identity of  $K$ :  $1_K$  as the representative of  $B$ . Thus  $H = A + Ah_1 + \dots + Ah_p + \dots$ , varying  $p$  into a set of indexes  $P$ , and  $K = B + Bk_1 + \dots + Bk_j + \dots$ , varying  $j$  into a set of indexes  $J$ . Since  $B$  is the isomorphic image of  $A'$  through  $\iota$  and  $A$  is the isomorphic image of  $A'$  through  $i$ , we shall write  $H = A' + A'h_1 + \dots + A'h_p + \dots$  and  $K = A' + A'k_1 + \dots + A'k_j + \dots$ . Thus, every element in  $H$  as a unique representation as  $h = ah_p$ , where  $a \in A'$  and  $h_p$  is one of the  $h$ 's that are representatives of the cosets, and, analogously, every element in  $K$  has a unique representation as  $k = ak_j$ , where  $a \in A'$  and  $k_j$  is one of the representatives (of course, if  $h$  or  $k$  is identified with an element in  $A'$ , then it will be represented -uniquely- with the corresponding  $a \in A'$ ).

We say that an element of  $L$  is in *canonical form* if it is of the form  $l = ah_pk_j$ , where  $a \in A'$ ,  $h_p \in H$ ,  $k_j \in K$  and  $h_p$  and  $k_j$  are the representatives of the cosets.

We are now ready to define a word process (that is, a set of operation that allows us to reduce a given word in  $L$  to its canonical form).

Let  $l = g_1g_2 \dots g_t$ , being the  $g_i$ 's elements of  $H$  or  $K$ ; we define a word process as follows:

$W_0 := \varepsilon$  (the empty word)

$W_1 :=$

- 1 if  $g_1$  is the identity of  $H$  or  $K$ ;

<sup>3</sup>If  $g$  is an element of the group  $G$  and  $N$  is a subgroup of  $G$ , the right coset  $Ng$  is  $\{ng \mid n \text{ is an element of } N\}$ .



- $ah$  if  $g_1 \in H$ ,  $g_1 = ah$  and  $ah$  is the presentation of  $g_1$  in the decomposition of  $H$  through the right cosets of  $A'$ ;
- $ak$  if  $g_1 \in K$ ,  $g_1 = ak$  and  $ak$  is the presentation of  $g_1$  in the decomposition of  $K$  through the right cosets of  $A'$ ;
- $a$  if  $g_1 \in A'$ ;

Notice that  $W_1$  is in canonical form. Suppose now that  $W_i$  is in canonical form:  $W_i = ah_ik_j$ .

$W_{i+1} :=$

- $W_i$  if  $g_{i+1}$  is the identity of  $H$  or  $K$ ;
- $\bar{a}h_ik_j$  where  $\bar{a} = aa'$ , if  $g_{i+1} = a'$ ,  $a' \in A'$ ;
- $\hat{a}h^*k_j$  if  $g_{i+1} = a'h'$ ,  $a'h'$  is the presentation of  $g_{i+1}$  in the decomposition of  $H$  through the right cosets of  $A'$ ,  $a^*h^*$  is the presentation of  $h_ik_j$  in the decomposition of  $H$  through the right cosets of  $A'$  and  $\hat{a} = aa'a^*$ ;
- $\tilde{a}h_ik^{**}$  if  $g_{i+1} = a''k''$ ,  $a''k''$  is the presentation of  $g_{i+1}$  in the decomposition of  $K$  through the right cosets of  $A'$ ,  $a^{**}k^{**}$  is the presentation of  $h_ik_j$  in the decomposition of  $K$  through the right cosets of  $A'$  and  $\tilde{a} = aa''a^{**}$ .

By construction, it is clear that, for every element  $l = g_1g_2 \dots g_t$  in  $L$ , the word process halts at step  $t$ , returning a word in canonical form. Moreover, since at every step  $i$  it holds that  $W_i g_{i+1} \sim W_{i+1}$ ,  $l$  and  $W_t$  belong to the same equivalence class. Finally, by induction it is easy to prove that, if  $l_1$  and  $l_2$  are words belonging to the same equivalence class, they have the same canonical form, and if  $l$  is in canonical form, then the word process leaves  $l$  unchanged.

Collecting everything together, we obtain that, once the cosets representatives have been chosen, in each equivalence class of elements in  $L$  there is one, and only one, element in canonical form:  $l = ahk$ , where  $a \in A'$ ,  $h$ ,  $k$  are coset representatives in  $H$  and  $K$ , respectively, different from the unity  $1_{A'}$  of  $A'$  and taken from some arbitrary but fixed selection of coset representatives.

Now, let us define  $f : H \rightarrow L$  as the map that associates, to each element  $h \in H$ , the class  $a'h_i$  in  $L$ , where  $h = a'h_i$  is the decomposition by coset representatives of  $h$  in  $H = A' + A'h_1 + \dots + A'h_i + \dots$ .

$f$  is clearly an homomorphism between groups; it is also injective because different elements have a different decomposition through coset representatives. These two properties exactly mean that  $f$  is an embedding of  $H$  into  $L$ .

Analogously, we can define  $g : K \rightarrow L$  as the map that associates, to each element  $k \in K$ , the class  $a'k_j$  in  $L$ , where  $h = a'k_j$  is the decomposition by coset representatives of  $k$  in  $K = A' + A'k_1 + \dots + A'k_j + \dots$ , and again  $f$  is an embedding of  $K$  into  $L$ .

Let us now consider an element  $a \in A'$ .  $a$  will be mapped into  $i(a)$  by the map  $i$  from  $A$  to  $H$ , and will be mapped by  $f$  into itself. On the other hand,  $a$  will be mapped into the element  $\iota(a)$ , that will be mapped by  $g$  again into  $a$ . In other words, both  $i \circ f$  and  $\iota \circ g$  act as the identity map on  $A'$ .

Thus we have proved there exist two embeddings  $f : H \rightarrow L$  and  $g : K \rightarrow L$  such that  $i \circ f = \iota \circ g$ .

## C Refutational Completeness of $\mathcal{SP}_{AG}$

Let us start proving the following proposition:

**Proposition 2.** *Let  $s$  be a term in which no variable of sort  $AG$  occurs, let  $\sigma$  be a grounding substitution such that both  $s$  and  $\sigma$  are in  $AG$ -normal form, and let  $R$  be a ground TRS such that  $(s, \sigma)$  is irreducible w.r.t.  $R$ . Moreover, let  $\sigma =_{AG} \mu\pi$ , where  $\pi$  is another grounding substitution in  $AG$ -normal form and  $\mu$  is a substitution that does not have variables of sort  $AG$  in its range. Then  $(s\mu, \pi)$  is still irreducible w.r.t.  $R$ . Proof.* Of course,  $AG\text{-nf}(s\sigma)$  is equal to  $AG\text{-nf}(s\mu\pi)$ , and  $\text{maxred}_R(s\sigma)$  is equal to  $\text{maxred}_R(s\mu\pi)$ .

If  $s$  is a variable, since  $(s, \sigma)$  is irreducible, it follows that  $s\sigma$  is  $R$ -irreducible, and so also  $(s\mu, \pi)$  is irreducible.

Suppose now that  $s$  is not a variable. Since the presence of variables of sort  $AG$  is forbidden, variables can occur in (proper) subterms of  $s$  of the kind  $f(x, t_1, \dots, t_n)$ , for  $f$  different from  $+$  and  $-$ . Let us focus our attention over such variables. We have only two cases to consider:

- $x\sigma$  is  $R$ -irreducible, and so the pair  $(x\mu, \pi)$  is irreducible. This implies that, when unfolding the definition of irreducibility, all the subterm of the kind  $x\mu$  in  $s\mu$  will satisfy, whenever needed, the requirement for the irreducibility of  $(s\mu, \pi)$ ;
- $x\sigma$  is  $R$ -reducible. Since  $(s, \sigma)$  is irreducible, it means that the occurrences of  $x\sigma$  are only in subterms that are deleted during the reduction in  $AG$ -normal form of  $s\sigma$ , thus implying that all the terms of the kind  $x\mu$  have no influence in the check of the irreducibility of  $(s\mu, \pi)$ .

Sometimes, it will be useful also the notion of irreducibility related to a certain term  $u$ :

**Definition 7** *Let  $s$  be a term in which no variable of sort  $AG$  occurs, let  $\sigma$  be a grounding substitution, let both  $s$  and  $\sigma$  be in  $AG$ -normal form, let  $u$  be a term and let  $R$  be a ground TRS. The pair  $(s, \sigma)$  is  $(u, \succeq)$ -irreducible:*

- if  $s$  is a variable, either  $u \prec s\sigma$ , or  $u \succeq s\sigma$  and  $s\sigma$  is  $R$ -irreducible;
- if  $s$  is not a variable, for for each term  $t = f(t_1, \dots, t_n)$  such that  $s$  is in the form  $t + v$  or  $-t + v$  and such that  $u \succeq AG\text{-nf}(t\sigma)$ , each  $(t_i, \sigma)$  is irreducible.

Before stating completely the proof of the refutational completeness of  $\mathcal{SP}_{AG}$ , we recall the following Lemma (its proof can be found in [12], but we restate it here for sake of completeness).

**Lemma 6** *Let  $M_{red}$  be the reductive form of some literal  $M\sigma$  in  $Ir_{R_S}(S)$  such that  $I \not\equiv M\sigma$ ; let  $M_{red}$  be not in the form  $t \neq t$ , and let  $s$  be the maximal summand in  $M_{red}$ .  $M_{red}$  is either (a) in the form  $ms = t$ , with  $s \succ t$ , or (b)  $ms \neq t$ , with  $s \succeq t$ . In both cases,  $ms$  is reducible by  $R_S$ .*

*Proof.* Indeed, suppose (a) holds. Since  $I \not\equiv ms = t$ ,  $M_{red}$  has generated no rule of  $R_S$ , thus, according to Definition 5, the only possibility is that  $ms$  is already reducible by  $R_{M_{red}}$ . Suppose, on the other hand, that (b) holds. The

fact that  $I \not\models ms \neq t$  means exactly that  $I \models ms = t$ , and so  $ms$  and  $t$  are joinable by  $R_S \cup R_{AG}$ ,  $ms \succ t$  and, since  $ms$  is in  $AG$ -normal form and is the maximal side,  $ms$  is reducible by  $R_S$ .

**Lemma 2.** *Let  $S$  be the closure under  $\mathcal{SP}_{AG}$  of a set of literals  $S_0$ , and let us assume that the empty clause does not belong to  $S$ . Let  $I$  be the model of terms derived from  $S$  as described in Section 5, and let  $Ir_{R_S}(S)$  be the set of ground instances  $L\sigma$  in  $S$  such that  $(L, \sigma)$  is irreducible w.r.t.  $R_S$ . Then:*

1.  $I \models Ir_{R_S}(S)$  implies that  $I \models S$ .
2.  $I \models Ir_{R_S}(S)$ .

*Proof.*

1. ([12]). For each ground instance  $L\sigma$  of a literal  $L$  in  $S$ , let us consider an other instance  $L\sigma'$  of  $L$ , where  $x\sigma'$  is the normal form of  $x\sigma$  w.r.t.  $R_S$  for every variable  $x$  of  $L$ . Naturally,  $L\sigma'$  is an other instance of  $S$  that is in  $Ir_{R_S}(S)$ . Since by hypothesis  $I \models Ir_{R_S}(S)$ , it follows that  $I \models L\sigma'$ , which implies, due to the definition of the congruence on  $I$ , that  $I \models L\sigma$ . In other words, we have proved that, for each ground instance  $L\sigma$  of a literal  $L$  in  $S$ ,  $I \models L\sigma$ , which immediately leads to the conclusion that  $I \models S$ .
2. The strategy here is to derive a contradiction from the existence of an irreducible literal that is not verified in  $I$ . In order to produce this contradiction, we rely on the technique presented in [12] and we adapt it to our context.

Let  $M_{red}$  be the minimal, w.r.t.  $\succ$ , literal that is the reductive form of some  $M\sigma$  in  $Ir_{R_S}(S)$  such that  $I \not\models M_{red}$ , and suppose that  $M$  is in the form  $e \bowtie t'$ . If the literal  $M_{red}$  is in the form  $t \neq t'$ , then an application of the rule Reflection to  $M$  could have been possible, thus producing the empty clause in  $S$ .

Otherwise, we will show that it is possible to perform some inferences involving  $M$  and producing a new literal to which it will be possible to apply the following schema of reasoning:

Proof Pattern Assume that (i) there exists a literal  $P$  such that  $(P, \sigma)$  is irreducible and such that it admits an orientation of the kind  $l = r$ , and (ii) there exists a position  $q$  in  $e$  such that  $e|_q$  is a non variable term that is not immediately below a  $+$  or a  $-$  and that admits a splitting  $e'_1 + e'_2$ , and that (iii) the following inferences are admissible:

$$\frac{l = r \quad e[e'_1 + e'_2]_q \bowtie t'}{(e[r + e'_2]_q \bowtie t')\mu_i}$$

where  $\mu_i$  ranges in the complete set of unifiers for the unification problem  $l =_{AG} e'_1$ , and where  $l\sigma =_{AG} e'_1\sigma$ . Moreover, assume that (iv)  $(AG\text{-nf}(e[r + e'_2]_q \bowtie t'), \sigma)$  is irreducible and that (v) the  $AG$ -normal form of  $(e[r + e'_2]_q \bowtie t')\sigma$  is smaller than  $M_{red}$  and not satisfied in  $I$ .

At this point, if  $\sigma =_{AG} \mu_j \sigma|_{VRan(\mu_j)}$  is the decomposition of the substitution  $\sigma$  according to Lemma 1, from Proposition 2 it follows that

also  $((AG\text{-nf}(e[r+e'_2]_q) \bowtie t')\mu_j, \sigma|_{VRan(\mu_j)})$  is irreducible, and relying on the confluence of the process of reduction in  $AG$ -normal form, it follows again that the reductive form of  $(e[r+e'_2]_q \bowtie t')\mu_j\sigma|_{VRan(\mu_j)}$  is smaller than  $M_{red}$  and it is still not verified in  $I$ .

Thus, every time we will be able to apply the proof pattern above, we will be able to prove the existence of an irreducible literal that is not verified in  $I$  and that is smaller than the minimal literal satisfying the same property, the wanted contradiction.

Let us enter now a little bit more into the details. From now on, we will strongly rely on some – very technical – lemmas that are stated in [12]. The extension to the many-sorted case is straightforward, so we will omit their proofs here, but we will quote them when needed. First of all, we focus more on the shape of  $M_{red}$ . Since it is not in the shape  $t \neq t$ , it can be in the form  $ms = t$ , with  $s \succ t$ , or  $ms \neq t$ , with  $s \succeq t$ . In both the cases, by Lemma 6,  $ms$  has to be reducible, and the rule reducing  $ms$  has to come from the reductive form  $P_{red}$  of some ground instance  $P\sigma$  of an equation  $P$  in  $S$ . Since  $P_{red}$  has produced a rule,  $P\sigma$  belongs to  $Ir_{R_S}(S)$ .

Moreover, adapting Lemma 44 of [12], it is possible to prove that there exists a subterm  $s'$  of  $ms$  such that: (i) if  $s'$  is in the form  $nu + v$ , then the rule  $nu \rightarrow r'$  is in  $R_S$  for some term  $r'$ , (ii) if  $s'$  is in the form  $-u + v$ , then the rule  $-u \rightarrow (n-1)u + r'$  is in  $R_S$  for some term  $r'$ , (iii) either  $s'$  is  $ms$  or  $s'$  occurs in  $ms$  in a position  $p \cdot i$  such that the topmost symbol of  $(ms)|_p$  is neither  $+$  nor  $-$ . From the considerations above, it follows that no rule with a left-hand side containing a term bigger than  $u$  can reduce  $s'$ . It is possible now to distinguish two cases:

- A) *The rule reducing  $ms$  is  $nu \rightarrow r'$ .*

In this case it is possible to prove (see Lemma 45 of [12]) that there exists an orientation  $l = r$  of the equation  $P$  such that  $AG\text{-nf}(l\sigma)$  is  $nu$  and  $AG\text{-nf}(r\sigma)$  is  $r'$ . Moreover  $(r, \sigma)$  is  $(u, \succeq)$ -irreducible. At this point it is convenient to focus on two different possibilities:

- \* A.1)  *$s'$  is  $ms$*

Then  $s$  is  $u$ , so  $M_{red}$  could be rewritten into  $mu \bowtie t$  for some  $m \geq n$ . We treat differently the case in which  $ms$ ,  $t$ ,  $nu$  and  $r'$  are of sort different from  $AG$  or of sort  $AG$ .

- A.1.1)  *$ms$ ,  $t$ ,  $nu$  and  $r'$  are of sort different from  $AG$ .*

This implies that  $n = m = 1$  and that the literal  $s \bowtie t$  is the instantiation of the literal  $M \equiv e_1 \bowtie e_2$  through  $\sigma$  (and, if needed, some steps of  $AG$ -normalization). Since  $s$  is reducible by  $R_S$  and since  $(M, \sigma)$  is irreducible w.r.t.  $R_S$ , it implies that  $e_1$  is not a variable, so the following inferences are possible:

$$\frac{l = r \quad e_1 \bowtie e_2}{(r \bowtie e_2)\mu_i}$$

where  $\mu_i$  ranges in the complete set of unifiers for the unification problem  $l =_{AG} e_1$ . It is easy to check that  $(r \bowtie e_2, \sigma)$  is irreducible, that  $(r \bowtie e_2)\sigma$  is not verified in  $I$  and that its

reduced form is smaller than  $M_{red}$ : at this point we can apply the proof pattern above, deriving the wanted contradiction.

· A.1.2)  $ms, t, nu$  and  $r'$  are of sort AG.

In this case  $M$  is in the form  $e \bowtie 0$  and  $AG\text{-nf}(e\sigma)$  is  $mu - t$ , for  $m \geq n$ . Moreover,  $u$  is the maximal summand of  $ms - t$  and  $(e, \sigma)$  is, by hypothesis, irreducible w.r.t.  $R_S$ . It is possible now to re-adapt Lemma 47 of [12] and to find a splitting  $e_1 + e_2$  of  $e$  such that  $(e_1 + e_2)\sigma =_{AG} e\sigma$ ,  $e_1\sigma$  is  $nu$ ,  $(e_2, \sigma)$  is  $(u, \succeq)$ -irreducible and the maximal summand of  $e_2\sigma$  is smaller or equal to  $u$ . So, the following inferences are allowed:

$$\frac{l = r \quad e_1 + e_2 \bowtie 0}{(r + e_2 \bowtie 0)\mu_i}$$

where, as before,  $\mu_i$  ranges in the complete set of unifiers for the unification problem  $l =_{AG} e_1$ . Again, it is easy to check that also  $(r + e_2 \bowtie 0, \sigma)$  is irreducible, that  $(r + e_2 \bowtie 0)\sigma$  is not verified in  $I$  and that its reduced form is smaller than  $M_{red}$ . Thus, as in the previous case, the proof pattern applies, deriving a contradiction.

\* A.2)  $s'$  is a proper subterm of  $ms$

More precisely,  $ms|_p$  is  $s'$  for some position  $p$  below some  $s$ . From now on, we will denote with  $e$  the left-hand side of the literal  $M$ , that is in one-sided form, notwithstanding its sort. Then, reproducing the argument of Lemmas 50 and 51 in [12] it is possible to find a position  $q$  in  $e$  such that  $e|_q =_{AG} s'$ ,  $(e|_q, \sigma)$  is irreducible w.r.t.  $R_S$  and, for all the terms  $r''$  in which no variable of sort AG occurs,  $e[r'']_q\sigma =_{AG} ms[r'']_p$  in case  $ms$  is of sort different from AG, or  $e[r'']_q\sigma =_{AG} ms[r'']_p - t$  in case  $ms$  is of sort AG. Moreover, if  $(r'', \sigma)$  is irreducible w.r.t.  $R_S$  and  $s' \succ AG\text{-nf}(r''\sigma)$ , then  $(e[r'']_q, \sigma)$  is irreducible w.r.t.  $R_S$ . We notice that also  $(e|_q, \sigma)$  is irreducible w.r.t.  $R_S$  and, recalling that  $s'$  is of the form  $nu + s''$  and that  $u$  is the maximal among the reducible summands of  $s'$ , it is possible to apply, properly adapted, Lemma 47 of [12], thus deriving the existence of a splitting  $e'_1 + e'_2$  of  $e|_q$  such that  $(e'_1 + e'_2)\sigma =_{AG} e|_q\sigma$ ,  $e'_1\sigma$  is  $nu$ , and  $(e'_2, \sigma)$  is  $(u, \succeq)$ -irreducible. Thus the following inferences are allowed:

$$\frac{l = r \quad e[e'_1 + e'_2]_q \bowtie t'}{(e[r + e'_2]_q \bowtie t')\mu_i}$$

where, as before,  $\mu_i$  ranges in the complete set of unifiers for the unification problem  $l =_{AG} e_1$ . The appropriate adaptation of Lemma 53 in [12] guarantees the irreducibility of  $(e[r + e'_2]_q \bowtie t', \sigma)$ ; moreover, since  $(e[r + e'_2]_q \bowtie t')\sigma$  is still not true in  $I$  and since its reduced form is smaller than  $M_{red}$ , we can apply again the proof pattern, obtaining the wished contradiction on the minimality of  $M_{red}$ .

– B) The rule reducing  $ms$  is  $-u \rightarrow (n - 1) + r'$ .

Since  $M_{red}$  is the reductive form of  $M\sigma$ ,  $s'$  cannot coincide with  $ms$  (it must be a proper subterm of  $s$ ). At this point, the contradiction on the minimality of  $M_{red}$  follows from arguments very similar to the

ones applied in the previous case A.2), with a proper adaptation of Lemmas 46, 48, 51 and 54 in [12].

### C.1 Computing $AG$ -bases

Let us fix the following data: let  $\Gamma(\underline{a}, \underline{b})$  be a set of ground literals over an expansion of  $\Sigma \supseteq \Sigma_{AG}$  with the finite sets of fresh constants  $\underline{a}, \underline{b}$ , let  $T \supseteq AG$  be a  $\Sigma$ -theory whose axioms are unit clauses. Suppose to perform a saturation w.r.t.  $\mathcal{SP}_{AG}$  adopting an RPO ordering in which the precedence is  $f \succ a \succ - \succ + \succ 0$  for every function symbol  $f$  in  $\Sigma^{\underline{b}}$  different from  $+, -, 0$ , every constant  $a$  in  $\underline{a}$  and that every symbol has a lexicographic status, except  $+$ , whose status is multiset. Proposition 3 shows how  $\mathcal{SP}_{AG}$  can be used in order to derive  $AG$ -bases:

**Proposition 3.** *Let  $S$  be a finite saturation of  $T \cup \Gamma$  w.r.t  $\mathcal{SP}_{AG}$  not containing the empty clause and suppose that, in every equation  $e = 0$  containing at least one of the constants  $a$  in  $\underline{a}$  as summand, the maximal summand is not unifiable with any other summand in  $e$ . Then the set  $\Delta$  of all the ground equations over  $\Sigma_{AG}^{\underline{a}}$  in  $S$  is a  $AG$ -basis for  $T$  w.r.t.  $\underline{a}$ .*

*Proof.*

Since  $T$  is axiomatized by unit clauses, it is in particular a Horn theory, and so it is convex. Thus, when looking for  $AG$ -bases for  $T$ , it will be sufficient to focus simply on the ground equations over the sort  $AG$  that are implied by  $T \cup \Gamma$ . At this point, we have to prove that, if  $e = 0$  is a ground equation over  $\Sigma_{AG}^{\underline{a}}$  implied by  $T \cup AG \cup \Gamma(\underline{a}, \underline{b})$ , then  $e = 0$  is already implied by  $AG \cup \Delta$ <sup>4</sup>. From the fact that  $\mathcal{SP}_{AG}$  is refutational complete, we have that  $T \cup AG \cup \Gamma(\underline{a}, \underline{b}) \models e = 0$  iff the saturation of  $Ax(T) \cup \Gamma(\underline{a}, \underline{b}) \cup \{e \neq 0\}$  contains the empty clause, and thus iff the saturation of  $S \cup \{e \neq 0\}$  contains the empty clause. Since  $S$  does not contain the empty clause, the only way to derive it is by reducing  $e \neq 0$  by means of equations contained in  $S$ .

Let us start analyzing into details when we perform a reduction of  $e \neq 0$  by an application of a direct superposition (the case of an application of an inverse superposition is very similar, and hence skipped). Thus, the maximal summand of  $e$  is of the kind  $ma$  for some integer  $m > 0$  and some constant  $a$  in  $\underline{a}$ , and the only inferences needed are inferences with equations in  $S$  admitting an orientation of the kind  $na = t$  for some positive integer  $n$ . The fact that the left-hand side of the orientation is already in the shape  $na$  is due to the fact that (i) the left-hand side of the orientation has to be unified with some subterm of  $ma$ , that is just a sum of constants, and that (ii) the presence of variables of sort  $AG$  is forbidden. Let us show, now that the term  $t$  can contain only constants. Suppose not; by the choice of the ordering, the maximal summand in  $na = t$  is necessarily a term  $s$  different from a constant.  $s$  cannot be ground, otherwise no orientation with left-hand side  $na$  will ever be performed, being in this case always  $t$  bigger than  $na$ . So  $s$  could only be a non ground term, different from a variable. But now the hypothesis that  $s$  cannot unify with any of the other summands in the equation  $na = t$  implies that, for every grounding  $\theta$ ,  $s\theta$  is always the maximal summand, even after the usual step of

<sup>4</sup>As well as when performing the calculus, when dealing with literal we will always consider them in one-sided form.

AG-normalization. So, also in this case the inference with the orientation  $na = t$  would be unnecessary in order to derive the empty clause.

Summing up, we have that  $e \neq 0$  can be reduced using only equations in  $\Delta$ , and any inference will produce new inequations  $e' \neq 0$  that, applying the same arguments above, can be reduced by only equations in  $\Delta$ .

Hence the saturation of  $T \cup \Gamma(\underline{a}, \underline{b}) \cup \{e \neq 0\}$  does not contain the empty clause iff the saturation of  $\Delta \cup \{e \neq 0\}$  does not contain it. Relying on the completeness of the calculus, we have obtained that  $T \cup AG \cup \Gamma(\underline{a}, \underline{b}) \models e = 0$  iff  $AG \cup \Delta \models e = 0$ .

## D Some examples

**Lists with Length** In order to prove its satisfiability of a set of ground literals  $G$  modulo  $T_L \cup T_\ell \cup AG$  we first of all replace all the literals in  $G \cup \{\text{atom}(\text{nil})\}$  in the form  $\neg \text{atom}(t)$  and  $\text{atom}(t')$  with respectively  $t = \text{cons}(sk_1, sk_2)$  and  $\forall x_0, x_1 t' \neq \text{cons}(x_0, x_1)$ , where  $t$  and  $t'$  are ground terms of sort LISTS and  $sk_1, sk_2$  are fresh constants of the appropriate sort. Let now  $T_{L'}$  be the subtheory of  $T_L$  whose axioms are just the first two (equational) axioms of  $T_L$ . We have (see e.g. [18]) that  $G$  is satisfiable w.r.t.  $T_L \cup T_\ell \cup AG$  if and only if  $G'$  is satisfiable w.r.t.  $T_{L'} \cup T_\ell \cup AG$ . So, applying at most some standard steps of flattening, we can focus our attention to sets of literals of the following kind ( $x$  is a variable of sort ELEM,  $y$  is a variable of sort LISTS,  $h, l, a, f, c, l_1, l_2, e, d, e_1, e_2, g$  are constants of the appropriate sorts), and the left-hand side of all the literals is the maximal one.

- |   |  |
|---|--|
| i.) equational axioms for lists             | iv.) ground literals over the sort LISTS             |
| a) $\text{car}(\text{cons}(x, y)) = x;$     | a) $\text{cons}(e, l) = c;$                          |
| b) $\text{cdr}(\text{cons}(x, y)) = y;$     | b) $\text{cdr}(f) = c;$                              |
| ii.) reduction for atom                     | c) $l_1 \bowtie l_2;$                                |
| a) $\text{cons}(x, y) \neq h;$              | v.) ground literals over the sort ELEM               |
| b) $\text{cons}(x, y) \neq \text{nil};$     | a) $\text{car}(h) = d;$                              |
| iii.) axioms for the length                 | b) $e_1 \bowtie e_2;$                                |
| a) $\ell(\text{nil}) = 0;$                  | vi.) ground literals over the sort AG                |
| b) $\ell(\text{cons}(x, y)) = \ell(y) + 1;$ | a) $ns \bowtie m_1 t_1 + n_2 t_2 + \dots + m_n t_n,$ |

where the literals in the group vi are in reductive normal form, and  $s$  and  $t_i$  of the kind  $\ell(a)$  or  $g$  (being the  $g$ 's simply constants of sort AG).

Let us choose as ordering a RPO with a total precedence  $\succ$  on the symbols of the signature such that all the symbols have a lexicographic status, except  $+$ , whose status is multiset, and such that respects the following requirements: (a)  $\text{cons} \succ \text{cdr} \succ \text{car} \succ c \succ e \succ \ell$  for every constant  $c$  of sort LISTS and every constant  $e$  of sort ELEM; (b)  $\ell \succ g \succ - \succ + \succ 0$  for every constant  $g$  of sort AG.

These requirements over the precedence guarantee that every compound term of sort LISTS is bigger than any constant of the same sort, any compound

term over the sort ELEM is bigger than any constant of sort ELEM, and that any term of the kind  $\ell(a)$  is bigger than any constant of sort AG.

**Proposition 4.** *For any set  $G$  of ground literals, any saturation of  $T_{L'} \cup T_\ell \cup G'$  w.r.t.  $\mathcal{SP}_{AG}$  is finite.*

The key observations, in order to prove termination, are that the non-ground set of literals is already saturated, every (dis)equation obtained by the application of a rule to ground factors is smaller in the ordering w.r.t. the biggest factor in the antecedent of the rule, and every application of a rule of the calculus to a ground and a non-ground literal produces a ground literal that is smaller than the ground factor. In other terms, every literal produced during the saturation phase is ground and it is strictly smaller than the biggest ground literal in the input set. Since the ordering on the literals is the multiset extension of a terminating ordering, it is terminating too.

**Trees with Size** In order to check for satisfiability modulo  $T_T \cup T_{size} \cup AG$  a set of ground literals  $G$ , we have to saturate a set of literals of the following kind (as usual, any set of ground literals can be seen, at most after the application of some standard steps of flattening, as a set of literals as below):

- |   |   |
|---|---|
| <p>i.) equational axioms for trees</p> <p>a) <math>\text{binL}(\text{bin}(x, y)) = x;</math><br/> b) <math>\text{binR}(\text{bin}(x, y)) = y;</math><br/> c) <math>\text{bin}(\text{binL}(x), \text{binR}(x)) = x;</math></p> | <p>iii.) ground literals over the sort TREES</p> <p>a) <math>\text{bin}(a, b) = c;</math><br/> b) <math>\text{binL}(d) = e;</math><br/> c) <math>\text{binR}(f) = g;</math><br/> d) <math>h_1 \bowtie h_2;</math></p> |
| <p>ii.) axioms for the size</p> <p>a) <math>\text{size}(\mathcal{E}) = 0;</math><br/> b) <math>\text{size}(\text{bin}(x, y)) = \text{size}(x) + \text{size}(y);</math></p>  | <p>iv.) ground literals over the sort AG</p> <p>a) <math>\text{size}(\mathcal{E}) = 0</math><br/> b) <math>ns \bowtie m_1 t_1 + n_2 t_2 + \dots + m_n t_n.</math></p>   |

where  $a, b, c, d, e, f, g, h_1$  and  $h_2$  are constants of sort TREES, the literals in the set (ivb) are in reductive form, and the summands  $s, t_1, \dots, t_n$  are in one of the following shapes: (i) constants of sort AG, (ii)  $\text{size}(a)$ , (iii)  $\text{size}(\text{binL}(b))$ , (iv)  $\text{size}(\text{binR}(c))$ .

Let us now put as ordering a RPO with a total precedence  $\succ$  on the symbols of the signature such that all the symbols have a lexicographic status, except  $+$ , whose status is multiset, and satisfying: (a)  $\text{bin} \succ \text{binR} \succ \text{binL} \succ c \succ \text{size}$  for every constant  $c$  of sort TREES; (b)  $\text{size} \succ g \succ - \succ + \succ 0$  for every constant  $g$  of sort AG.

Analogously to what happens in the previous example, these requirements over the precedence guarantee that every compound term of sort TREES is bigger than any constant of the same sort, and that any term of the kind  $\text{size}(a)$  is bigger than any constant of sort AG.

**Proposition 5.** *For any set  $G$  of ground literals, any saturation of  $T_T \cup T_{size} \cup G$  w.r.t.  $\mathcal{SP}_{AG}$  is finite.*



*Proof.* Let us start with a brief remark. When the calculus  $\mathcal{SP}_{AG}$  is applied to terms of sort TREES, it boils down to the standard superposition calculus. So, since any term of sort different from AG cannot contain subterms of sort AG, it is possible to use all the reduction rules of the standard superposition in order to simplify and shorten the saturation. In particular, we will apply in what follows the rule of *simplification*, which allows to substitute to a set of clauses of the kind  $S \cup \{C[l']_p, l = r\}$  a set of the kind  $S \cup \{C[r\theta]_p, l = r\}$  whenever, for some substitution  $\theta$ ,  $l' \equiv l\theta$ ,  $r\theta \prec l\theta$  and, for each literal  $L$  in  $C[l']_p$ ,  $(l\theta = r\theta) \prec L$ . Moreover, we can easily see that the standard rule of *deletion* and *strict subsumption* can be safely applied, without affecting the refutational completeness of  $\mathcal{SP}_{AG}$ .

Now, if we try to saturate a set of literals of kind as above, we discover that, apart from some tautologies that can be immediately deleted, there are produced five new kind of literals:

- $\text{bin}(z, \text{binR}(\text{bin}(z, t))) = \text{bin}(z, t)$ , from (ia) applied to (ic);
- $\text{bin}(\text{binL}(\text{bin}(z, t)), t) = \text{bin}(z, t)$ , from (ib) applied to (ic);
- $\text{size}(\text{binR}(t)) = -\text{size}(\text{binL}(t)) + \text{size}(t)$ , from (ic) applied to (iib) (\*);
- $\text{bin}(e, \text{binR}(d)) = d$ , from (iiib) applied to (ic);
- $\text{bin}(\text{binL}(f), g) = f$ , from(iiic) applied to (ic).

The literals of the first two kinds are deleted from the saturation because, applying one step of simplification, a tautology is produced (and immediately deleted). When performing any step of saturation involving the literals of the third kind, it is easy to restrict the number of possible inferences observing that, for every grounding and consequent reduction in AG-normal form, the summand that will always be the maximal one is the one obtained from the instantiation of  $\text{size}(\text{binR}(t))$ . So, the only possible orientations of that kind of literals are exactly the ones in the shape  $\text{size}(\text{binR}(t)) = -\text{size}(\text{binL}(t)) + \text{size}(t)$  ( $-\text{size}(\text{binR}(t)) = \text{size}(\text{binL}(t)) - \text{size}(t)$  in the inverse rule), and the only allowed splittings in  $t_1 + t_2$  are the ones of the kind  $t_1 \equiv \text{size}(\text{binR}(t))$  and  $t_2 \equiv \text{size}(\text{binL}(t)) - \text{size}(t)$ .

At this point it is easy to prove that the saturation is finite, because, when the literal of the kind (\*) has been produced, the set of non-ground literals is saturated; moreover, every other literal generated from now on during the saturation phase is ground and it is strictly smaller than the ground literal in the antecedent of the rule used to derive it. Since the ordering on the literals is the multiset extension of a terminating ordering, it is terminating too.



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