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HAL Id: inria-00382154
https://inria.hal.science/inria-00382154

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N° 0001

October 2008

Rapport de recherche

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Abstract: We study network coding for multi-hop wireless networks. We focus the case of broadcasting, where one source transmits information to all the nodes in the network. Our goal is energy-efficient broadcasting, in other words, to minimize the number of transmissions for broadcasting to the entire network. In this report, we focus on lossy wireless networks, where the probability of successful transmission between two nodes, depends on the distance between the node. Our main result is that a proof of an asymptotic bound of the maximum broadcast rate between a source and the destinations. This result implies the asymptotic optimality of network coding with our hypothesis, with respect to energy-efficiency.

Key-words: wireless networks, network coding, broadcasting, multi-hop, min-cut, hypergraph

Performance du codage réseau dans les réseaux sans fil avec pertes

Résumé :   Nous étudions l'utilisation du codage réseau pour les réseaux ad-hoc multi-saut. Nous nous concentrons sur le cas de la diffusion, où une source transmet de l'information à tous les noeuds du rseau. Notre objectif est d'avoir une diffusion efficace en termes d'énergie, en d'autres termes, de minimiser le nombre nécessaire de transmissions pour diffuser l'ensemble du réseau. Dans ce rapport, nous étudions le cas des réseaux sans fil avec pertes de paquets, où la probabilité de transmission avec succès entre deux noeuds est fonction de la distance entre ces noeuds. Notre résultat principal est une preuve d'une borne asymptotique du débit maximal de transmission entre la source et les destinations. Ce résultat implique l'optimalité asymptotique du codage réseau, avec nos hypothses, du point de vue de l'efficacité en termes d'énergie.

Mots-clés :   réseaux sans fil, codage de réseau, diffusion, multi-sauts, coupe minimale, hypergraphe

Contents

1   Introduction

The concept of *network coding*, where intermediate nodes mix information from different flows, was introduced by seminal work from Ahlswede, Cai, Li and Yeung [3]. Since then, a rich literature has flourished for both theoretical and practical aspects, and several results have established network coding as an efficient method to broadcast data to the whole wireless networks (see Lun et al. [6] or Fragouli et al. [4] for instance), when efficiency consists in: minimizing the total number of packet transmissions for broadcasting from the source to all nodes of the network.

Among the results relative the energy-efficiency of wireless network coding, several results about wireless network coding are based on simplistic wireless models such as the unit disk graph model, where a transmission is received with 100 % success within the radio range and never outside this range, This is the case, for instance, of the results of Fragouli and Al. [4], or in our previous work [1, 2].

In this paper, we propose to extend some of the previously known results in the case of the lossy wireless networks. Specifically, we prove that asymptotically, is the maximum broadcast rate of the source with network coding is the average received rate by one node ; this implies asymptotic optimality with respect to energy efficiency.

2   Realistic Lossy Model

2.1   Topology and Notations

We consider that the network consists of a set V of n nodes randomly uniformly distributed into a square map of size $L \times L^1$. The node density is therefore $\nu = \frac{n}{L^2}$.

We assume that one source s is present in the network.

We assume a slotted model, and every node transmits data at same rate λ per slot, except for the source which transmits at a higher rate λ_s (defined later).

2.2   Transmission/Reception Model

We assume that a transmission made by a node can be received by a random node at distance r with probability $p(r)$. Function $p(r)$ is assumed to decrease with r and $\int_0^\infty p(r) r \, dr < \infty$. We don't assume any specific correlation between the reception of close nodes.

Notice that the special case of the unit disk graph, which is an unrealistic wireless model, corresponds to the case where when $p(r) = 1$ for $r \leq 1$ and $p(r) = 0$.

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[1]although our results can be extented directly to spaces of higher dimensions

3 Energy Efficiency of Network Coding

3.1 Framework

We consider the multicast of information from the source to several nodes, destination node, also denoted "terminal" nodes.

We will use recent results from Lun et al. [5] relative to network coding in wireless networks represented as hypergraphs: the key is to focus on the capacity of the min-cut of the hypergraph, between the source and one destination t.

More precisely, consider a cut, that is, a S, T partition of V, $V = \mathcal{T} \cup \mathcal{S}$ where the source is in $\mathcal{S}$ and the destination t is in $\mathcal{T}$. Considering the rates on the hypergraph, we apply the model found for in Lun et al. [5]:

- For a node y, we consider that there exists an hyperarc to any set of nodes X, with $X \subset V \setminus \{y\}$.

- The average transmission rate on this subset is denoted $p(X, y)$, and correspond to the rate of the event $\{ y$ transmits a packet and the set of receivers is exactly $X \}$

Then the capacity of the cut is the quantity:

$$C(\mathcal{T}, \mathcal{S}) \triangleq \sum_{y \in \mathcal{S}} A_{y,\mathcal{S}} \text{ with } A_{y,\mathcal{S}} \triangleq \sum_{X | X \subset V \setminus \{y\} \text{ and } X \cap \mathcal{T} \neq \emptyset} p(X, y) \qquad (1)$$

The min-cut is the cut of the network with minimum capacity, between the source and the destination: $C_{\min}(t, s)$ Following a frequent convention, we will also refer to the capacity of the min-cut, as simply "min-cut".

Under broad conditions given in [5], the maximum broadcast rate from the source to the destination t, is the capacity of the min-cut ; and this maximum broadcast rate may be achieved asymptotically.

3.2 Bound of the Cut in General Networks

Theorem 1. *Consider a source s, a destination t and a cut $\mathcal{T}, \mathcal{S}$. We have the lower bound:*

$$C(\mathcal{T}, \mathcal{S}) \geq \tilde{C}(\mathcal{T}, \mathcal{S}) \qquad (2)$$

where $\tilde{C}$ is defined as follows:

$$\tilde{C}(\mathcal{T}, \mathcal{S}) \triangleq \sum_{y \in \mathcal{S}} \max_{x \in \mathcal{T}} \{p(x, y)\} \qquad (3)$$

and where $p(x, y)$ is the average rate of successful transmissions from y to x.

Proof. By definition, we have

$$p(x, y) \triangleq \sum_{X | X \subset V \setminus \{y\} \text{ and } x \in X} p(X, y) \qquad (4)$$

Let $x \in \mathcal{T}$ and $y \in \mathcal{S}$. For any subset $X \subset V \setminus \{y\}$, $x \in X$ immediately implies that $X \cap \mathcal{T} \neq \emptyset$, hence the summation in eq. 4 includes more terms $p(X, y)$ than the summation in eq. 1 in $A_{y,\mathcal{S}}$, and as a result, we have: $A_{y,\mathcal{S}} \geq p(x, y)$.

Since the inequality is valid for all $x \in \mathcal{T}$, it follows that:

$$A_{y,\mathcal{S}} \geq \max_{x \in \mathcal{T}}\{p(x,y)\}$$

and the theorem is proven.   □

3.3   Bound of the Cut with the Loss Model

With our model of wireless transmission where the probability of reception is only dependant on the distance, decreasing with it, and from the definition of $p(x,y)$ we have the following corollary of the theorem 1:

Corollary 1.

$$C(\mathcal{T},\mathcal{S}) \geq \tilde{C}(\mathcal{T},\mathcal{S}) = \sum_{y \in \mathcal{S}} p(r(y,\mathcal{T})) \tag{5}$$

where $r(y,\mathcal{T}) \triangleq \min_{x \in \mathcal{T}}\{|x-y|\}$, $|x-y|$ is the euclidian distance between x and y.

This result may now be used for general networks: our aim is to compute the bound of the quantity $\tilde{C}_{\min}(t,s) \triangleq \min \tilde{C}(\mathcal{T},\mathcal{S})$ between t and s. To this aim, we split the network into an infinity of independent networks, called slices. We define δ as a small scalar value, and $p_i = p(i\delta) - p((i+1)\delta)$. Slice i is the graph obtained by considering the original network under the radius $(i+1)\delta$, meaning that nodes within $(i+1)\delta$ are neighbor and receive each other without loss (unit disk graph model). In slice i nodes transmit at rate p_i.

Lemma 1.   *We have the inequality*

$$\tilde{C}(\mathcal{T},\mathcal{S}) \geq \sum_{i=0}^{\infty} p_i C_i(\mathcal{T},\mathcal{S}) \tag{6}$$

where $C_i(\mathcal{T},\mathcal{S})$ is the capacity of the cut in the slice i graph. That is:

$$C_i(\mathcal{T},\mathcal{S}) = \sum_{y \in \mathcal{S}} 1_{B_y((i+1)\delta) \cap \mathcal{T} \neq \emptyset}$$

where $B_y(r)$ is the disk of radius r centered on y and 1_P is the indicator function of property P.

Proof. Consider any point $y \in \mathcal{S}$.

Let j be the unique integer that verifies $j\delta < r(y,\mathcal{T}) \leq (j+1)\delta$. This alos implies that $p(r(y,\mathcal{T})) \geq p((i+1)\delta)$

We have: for any $i \leq j$, y is not in range with any node of $\mathcal{T}$, whereas for any $i > j$, y is. Hence y contributes to the quantity C_i iff $i > j$.

As a result, the contribution of y in right hand side of 6 is $\sum_{i=j+1}^{\infty} p_i$.

Since:

$$\sum_{i=j+1}^{\infty} p_i = \sum_{i=j+1}^{\infty} (p(i\delta) - p((i+1)\delta)) = p((j+1)\delta) \leq p(r(y,\mathcal{T}))$$

By using this result for every $y \in \mathcal{S}$ with the corollary in Eq.5, we get Eq.6, and the lemma is proven.   □

3.4 Main Result

We will prove under asymptotic conditions where n, L, ν and m tend to infinity, and λ tends to zero, that:

Theorem 2. *The rate $\rho(t)$ at which node t receives innovative data which converges to $\lambda\nu\theta$ with $\theta = 2\pi \int_0^\infty p(r)rdr$.*

The asymptotic condition is that $\nu = \Omega(\log L)$ and $\nu = o(L)$. For all $\varepsilon > 0$ the rate $\rho(t)$ belongs to $[(1-\varepsilon)\lambda\nu\theta, (1+\varepsilon)\lambda\nu\theta]$. The second condition seems to be instrumental to our proof and we should expect no upper-bound limitation on the density factor ν.

Remark The quantity $\lambda\nu\theta$ is already an upper bound of the rate received by a node since it is the rate at which a node receive data from its closest node, assuming all packets are innovative.

The theorem 2 is equivalent to proving that the capacity of the min-cut converges to $\lambda\nu\theta$ (using the results of [5]).

In appendix we show that for all $\varepsilon > 0$ when $L \to \infty$ with $\nu = \Omega(\log L)$ and $\nu = o(L)$, with t and S remaining in a fixed position the minimum cut of the unit disk graph $C_L(t, S)$ is greater than $(1-\varepsilon)\pi\nu$ with probability tending to one.

Therefore for all $\varepsilon > 0$ when $L \to \infty$ with $O(\log L) < \nu < o(L)$ we have with probability tending to one, for all $\mathcal{T}$, $\mathcal{S}$:

$$C(\mathcal{T}, \mathcal{S}) \geq (1-\varepsilon)\nu\pi \sum_{i=0}^{i=k}(p(i\delta) - p((1+i)\delta))((1+i)\delta)^2$$

Since the superior limit of quantities $\sum_{i=0}^{i=k}(p(i\delta) - p((1+i)\delta))((1+i)\delta)^2$ for all δ and k is exactly $\theta = 2\pi \int_0^\infty p(r)rdr$. Therefore we prove the lower bound in our main result, that is $\forall \varepsilon > 0$ the probability that $\rho(t)$ exceeds $(1-\varepsilon)\lambda\nu\theta$ tends to one.

To terminate the proof of our main result, it remains top notice that since the average value $\rho(t)$ is smaller than $\lambda\nu\theta$, we necessarily have $\forall \varepsilon > 0 \lim P(\rho < (1+\varepsilon)\lambda\nu\theta) = 1$ and the theorem is proven.

Appendix

Theorem 3. *Let a random unit disk graph with a set V of n nodes in a square $L \times L$ with same data rate. Among them a terminal t and a set S of m sources in a central square $\ell \times \ell$ fixed. The average degree of a central node is $d = \pi\frac{n}{L^2}$ is in and $A \log L < d < BL^\beta$ for some $A, B > 0$ and $0 < \beta < 1$. We assume that $d \leq m$. When for all $\epsilon > 0$ when $L \to \infty$ the probability that the min-cut between t and S is greater than $(1-\varepsilon)d$, tends to one.*

Let $V = \mathcal{T} + \mathcal{S}$ with $t \in \mathcal{T}$ and $S \subset \mathcal{S}$. That provide a min cut. This minimum cut is the number of nodes in $\mathcal{S}$ that have a link to nodes of $\mathcal{T}$, *i.e.* are at distance smaller than one of $\mathcal{T}$. The cut $H(\mathcal{T}, \mathcal{S})$ is the set of such nodes.

We remark that the minimum cut is already smaller or equal to the degree of t that would be obtained by taking $\mathcal{T} = \{t\}$ and $\mathcal{S} = V - \{t\}$. Therefore with

probability tending to one the minimum cut is smaller than $(1+\varepsilon)d$. Therefore the conclusion of this theorem is that the minimum cut is between $(1-\varepsilon)d$ and $(1+\varepsilon)d$ with probability tending to one.

Lemma 2. *There exist a minimal cut $(\mathcal{T}, \mathcal{S})$ such that (i) $\mathcal{T}$ is connected, (ii) each connected components of $\mathcal{S}$ contains a source..*

Proof. We proceed in two steps.

Assume there exist a connected component $\mathcal{T}_0$ of $\mathcal{T}$ that does not contain node t. It is clear that the cut $H(\mathcal{T} - \mathcal{T}_0, \mathcal{S} + \mathcal{T}_0)$ will be smaller or equal than $H(\mathcal{T}, \mathcal{S})$ (no node connected to $\mathcal{T} - \mathcal{T}_0$ belongs to $\mathcal{T}_0$). Therefore we can merge all connected components of $\mathcal{T}$ which do not contain t with $\mathcal{S}$ without increasing the cut.

Now we proceed with the second step. We assume now that $\mathcal{T}$ is connected. If the set $\mathcal{S}$ contains a connected component $\mathcal{S}_1$ that does not contain any source, then the cut $H(\mathcal{T} + \mathcal{S}_1, \mathcal{S} - \mathcal{S}_1)$ is smaller than or equal to $H(\mathcal{T}, \mathcal{S})$. Merging $\mathcal{S}_1$ with $\mathcal{T}$ provide a connected subset since $\mathcal{S}_1$ and $\mathcal{T}$ are connected (otherwise $\mathcal{S}_1$ will be isolated, which contradicts the fact that the network is connected). We can therefore merge all connected component of $\mathcal{S}$ which do not contain a source with $\mathcal{T}$ and get the expected result. $\square$

We slice the square in vertical strip and in horizontal strip each strip is of width $1/2$ and of length L. When $L \to \infty$ thanks to Chernov bound, all the strips are connected graphs.

Strips that contain only nodes belonging to $\mathcal{T}$ are called T type, strips that contain only nodes belonging to $\mathcal{S}$ are C type. Strips that contains nodes from both sets are called CT type.

Lemma 3. *The number of vertical (horizontal) CT strips is smaller than $(1+\varepsilon)d$ with probability tending to one.*

Proof. Each CT strip contains at least one cut node. $\square$

Lemma 4. *When $L \to \infty$ with probability tending to one, there are only T and C strips don't coexist.*

Proof. Assume a two vertical strips one is C and one is T. Therefore every horizontal strip which has non empty intersection with the C and T vertical strip will be CT. Therefore there will be an $O(L)$ number of CT strips with probability tending to one, which contradicts the previous lemma. $\square$

From now we call NT situation the situation where there are no T strips and NC situation, when there are no C strips.

Lemma 5. *In NT situation, the set $\mathcal{T}$ is at distance smaller than $4(1+\varepsilon)d$ from t.*

Proof. In this case, all $\mathcal{T}$ is in CT strips. Let consider vertical strips. If one CT strip is further than $2(1+\varepsilon)d$ from the strip that contains node t, then the set $\mathcal{T}$ is disconnected. Indeed the average distance between the CT strips would be greater than one since there are less than $(1+\varepsilon)d$ such strips. This means that at least two consecutive CT strips would be at distance greater than one, which means that the $\mathcal{T}$ set is disconnected. Therefore the horizontal span of $\mathcal{S}$ around t does not exceed $2(1+\varepsilon)d$.

We end the proof by considering the vertical span of $\mathcal{T}$ via horizontal CT strips. $\qquad\square$

Lemma 6. *In NC situation the set $\mathcal{S}$ is at distance smaller than $4(1+\varepsilon)d$ from source set S.*

Proof. The proof is very similar of previous lemma proof. The set $\mathcal{S}$ is all in strips CT. Let a source $s \in S$ and let $\mathcal{S}_s$ be the connected component of $\mathcal{S}$ that contains node s. With the same arguments as in the previous proof we proof that both vertical and horizontal span around s of $\mathcal{S}_s$ cannot exceed $2(1+\varepsilon)d$. $\qquad\square$

From now for each node p in the network we define the partition of $N(p)$ into four sub-neighborhood: $N_u(p)$, $N_d(p)$, $N_\ell(p)$ and $N_r(p)$ by:

- Every node $z \in N_u(p)$ is neighbor of p (*i.e.* $|z - p| \leq 1$) and the angle of the vector $z - p$ is in the interval $[\frac{\pi}{4}, \frac{3\pi}{4}[$.

- Every node $z \in N_d(p)$ is neighbor of p and the angle of the vector $z - p$ is in the interval $]-3\frac{\pi}{4}, -\frac{\pi}{4}]$.

- Every node $z \in N_r(p)$ is neighbor of p and the angle of the vector $z - p$ is in the interval $]-\frac{\pi}{4}, \frac{\pi}{4}]$.

- Every node $z \in N_\ell(p)$ is neighbor of p and the angle of the vector $z - p$ is in the interval $[3\frac{\pi}{4}, 5\frac{\pi}{4}]$.

Lemma 7. *For all $\varepsilon > 0$ the probability that all nodes in network that are not neighbor of the border satisfy $\{|N_u(p)|, |N_d(p)|, |N_r(p)|, |N_\ell(p)|\} \subset [(1-\varepsilon)\frac{d}{4}, (1+\varepsilon)\frac{d}{4}]$ tends to one.*

Proof. This is a simple application of Chernov bound. $\qquad\square$

End of proof The min cut is greater than $(1-\varepsilon)d$ with probability tending to one. We first prove in NT condition. We define u_t, the node of $\mathcal{T}$ which has the largest vertical coordinate, and d_t the node of $\mathcal{T}$ which has the smallest vertical coordinate. Similarly we define ℓ_t and r_t the nodes which have respectively the largest and smallest horizontal coordinate. Notice that some of these nodes maybe identical. Nevertheless the sets $N_u(u_t)$, $N_d(d_t)$, $N_r(r_t)$ and $N_\ell(\ell_t)$ are disjoint and subset of $\mathcal{S}$, and therefore subset of $H(\mathcal{T}, \mathcal{S})$ since they are neighbor of nodes in $\mathcal{T}$. The nodes u_t, d_t, ℓ_t and r_t are at distance greater than one from network border thanks to previous lemmas. Therefore the sum of the size of the four sub-neighborhood is greater than $(1-\varepsilon)d$ with probability tending to one.

In NC situation the proof is a little more complex, since the definition of cut $H(\mathcal{T}, \mathcal{S})$ is not symmetric in $\mathcal{T}$ and $\mathcal{S}$. We first make the partition $\mathcal{S} = \mathcal{S}_0 + H(\mathcal{T}, \mathcal{S})$ where $\mathcal{S}_0$ is the subset of the nodes in $\mathcal{S}$ that are not neighbor of nodes in $\mathcal{T}$.

There are two cases: either $\mathcal{S}_0 = \emptyset$ or $\mathcal{S}_0 \neq \emptyset$. In the first case all sources are in $H(\mathcal{T}, \mathcal{S})$ and therefore $|H(\mathcal{T}, \mathcal{S})| \geq m \geq d$.

In the case $\mathcal{S}_0 \neq \emptyset$, we define the nodes tuple of nodes (u_s, d_s, ℓ_s, r_s) like (u_t, d_t, ℓ_t, r_t) but with respect to $\mathcal{S}_0$ instead of $\mathcal{T}$. Notice that the sets $N_u(u_s)$, $N_d(d_s)$, $N_r(r_s)$ and $N_\ell(\ell_s)$ are disjoints subset of $H(\mathcal{T}, \mathcal{S})$, because (i) they are subset of the complementary set of $\mathcal{S}_0$ which is $H(\mathcal{T}, \mathcal{S}) + \mathcal{T}$ and do not contain nodes in $\mathcal{T}$ since the nodes u_s, d_s, ℓ_s and r_t are not neighbor of nodes in $\mathcal{T}$. The

sum of the size of the subneighborhood is greater than $(1 - \varepsilon)d$ with probability tending to one, which terminates the proof.

Remark: the proof can be adapted for any unit-disk network map in any dimension (linear, cubic, *etc*).

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