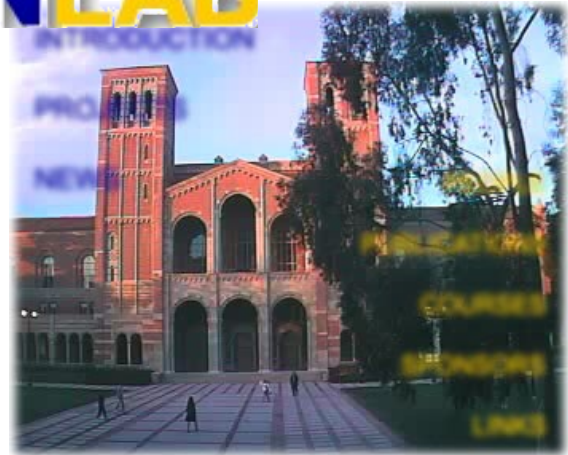


Fast Marching Method for generic Shape From Shading

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UCLAVISIONLAB

**RFIA 2006
January 2006, Tours, France**



Explicit generic SFS equation

Hypotheses:

- Lambertian Reflectance,
- Lighting:
 - Single punctual light source,
 - Located at the infinity or optical center,
 - Possibility to take into account the attenuation of the light due to the distance,
- Orthographic or perspective camera .

Explicit generic SFS equation

→ Particular case of the generic PDE:

$$\lambda \cdot F(\mathbf{u}) + \kappa_x \sqrt{|A_x \nabla \mathbf{u} + \mathbf{v}_x|^2 + K_x^2} + \mathbf{w}_x \cdot \nabla \mathbf{u} + c_x = 0$$

where $\kappa_x, K_x \geq 0$, $A_x \in \mathcal{M}_{2 \times 2}$, $\mathbf{v}_x, \mathbf{w}_x \in \mathbb{R}^2$, $c_x \in \mathbb{R}$, $\lambda \geq 0$
and F is a strictly increasing function.

[Prados:PhD'04]

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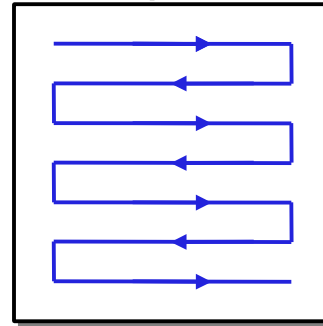
Particular characteristics:

- Dependency in \mathbf{u} (no only in $\nabla \mathbf{u}$).
- Solution not necessarily increasing along the characteristic curves...

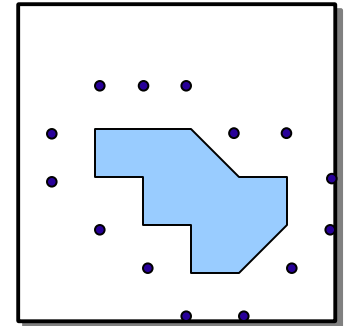
→ Current Fast Marching Methods do not apply.

Fast Marching Method (FMM)

- **One-pass method** for solving numerically **PDEs**
≠ iterative methods:
 - No threshold as stopping criterion
 - Optimal number of updates → Low computation time
 - Based on front propagation.



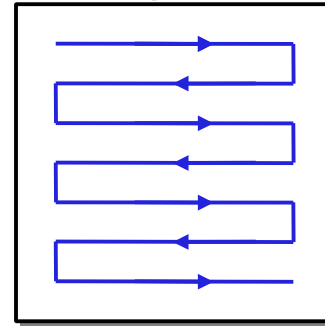
Iterative method



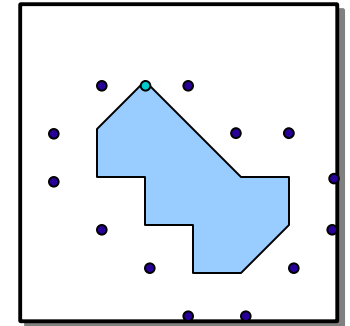
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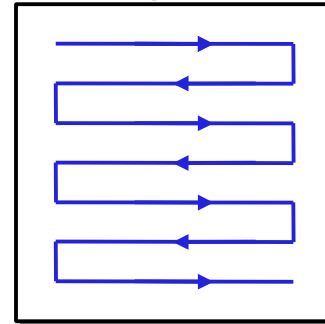
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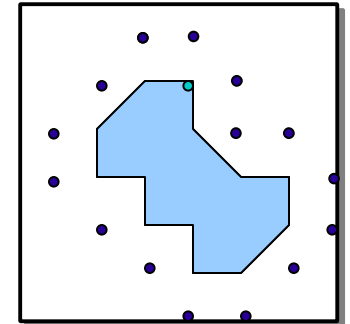
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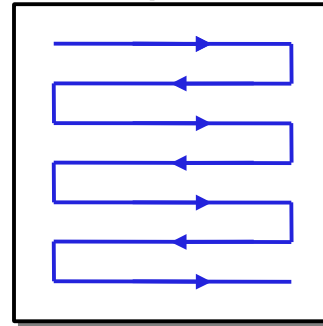
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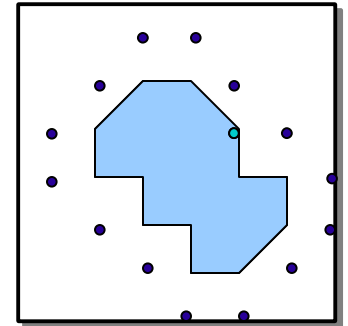
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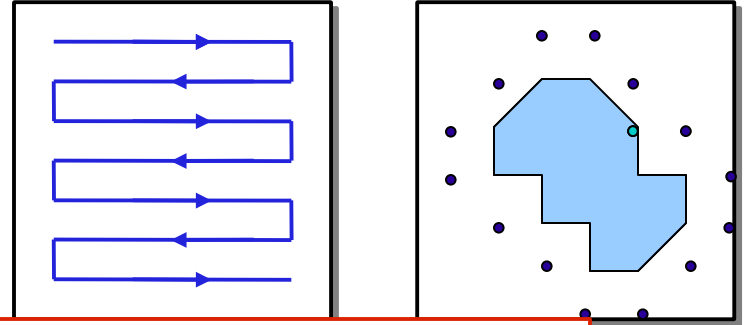


FMM

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IM

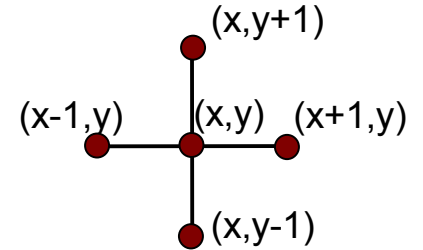
- **Many applications:**
 - Path planning [Kimmel-Sethian:01]
 - Geometric optics [Wenwang:03]
 - Image processing and computer vision [L.Cohen:05]
 - Extensive list, see [Sethian:99].

Fast Marching Method (FMM)

- **Basic method [Sethian:99, Dijkstra:59]**

$$|\nabla u| = g(x), \quad (\text{Eikonal equation})$$

4-neighborhood scheme.



- **Most recent extension: OUM [Sethian-Vladimirsky:03]**

$$\sup_a \{-f(x,a) - a \cdot \nabla u - 1\} = 0, \quad (2)$$

Scheme requiring very large neighborhood

(size depending on the anisotropy)

- **Our extension (new algorithm):**

$$\lambda F(u) + H(x, \nabla u) = 0, \quad (3)$$

where F is strictly increasing and H is convex

4-neighborhood scheme.

Our New Extension of the FMM

1. A New Numerical Scheme...
2. A New Causality...

A New Numerical Scheme

- **Preliminary step:**

Legendre Transform \longrightarrow equation as a supremum

$$\lambda F(u) + H(x, \nabla u) = 0$$



$$\lambda F(u) + \sup \{ -f(x, a) \cdot \nabla u(x) - \underbrace{l(x, a)}_{\text{Cost function}} \} = 0$$

Cost function

A New Numerical Scheme

We approximate $\nabla u(x) \approx \left(\frac{t - u(x + s_i e_i)}{-s_i} \right)$

$$t \lll u(x)$$
$$s_i \in \{+1, -1\}$$

And we choose the simplex, i.e. the s_i such that the scheme is increasing with respect to t (represent u)

$$\lambda F(t) + \sup_{a \in A} \left\{ -f(x, a) \cdot \left(\frac{t - u(x + s_i(x, a) e_i)}{-s_i(x, a)} \right) - l(x, a) \right\} = 0$$

We then choose $s_i = s_i(x, a) := \text{sign } f_i(x, a)$,

In other words, we choose the simplex which contains the optimal trajectory (see next section...).

A New Numerical Scheme

- Our scheme is provably **consistent** and **monotonic**
- Only **direct neighborhoods**
→ in SFS with regular mesh: 4-neighborhood.
- Equation depends on **u**

A New Causality and Reinterpretation

Key point: Distinction of the causality and of the simultaneous integration.

3. Causality: Theoretic propagation of the information;

→ ensures that we compute approximation of the viscosity solution !

The information propagation follows specific curves:

the optimal trajectories:

y_x^*

The solution can be computed by **direct integration along these curves:**

$$u(x) = \int_0^{\tau_x(\alpha_x^*)} l(y_x^*(\tau), \alpha_x^*(\tau)) d\tau + \varphi(y_x^*(\tau_x(\alpha_x^*))).$$

Trying to compute the solution curve after curve is **numerically unstable**

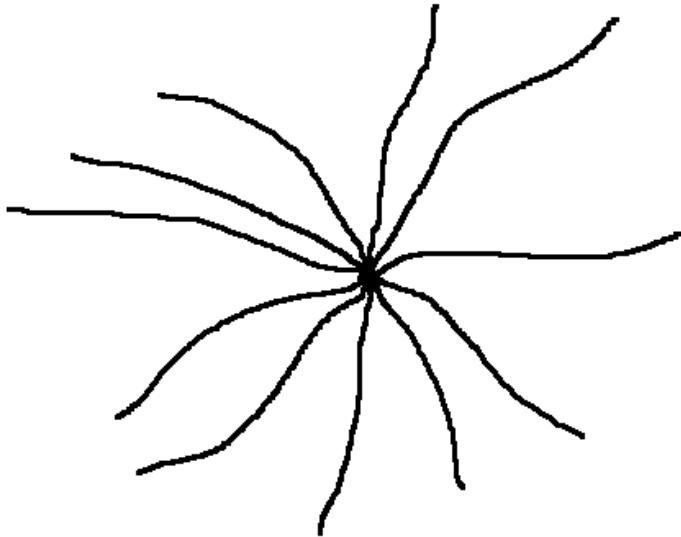
→ **Simultaneous integration along all the curves...**

Simultaneous integration: → ensures the numerical stability !

→ **Propagation front ;**

How to choose the propagation front?

Many propagation fronts following the optimal trajectories can be designed!

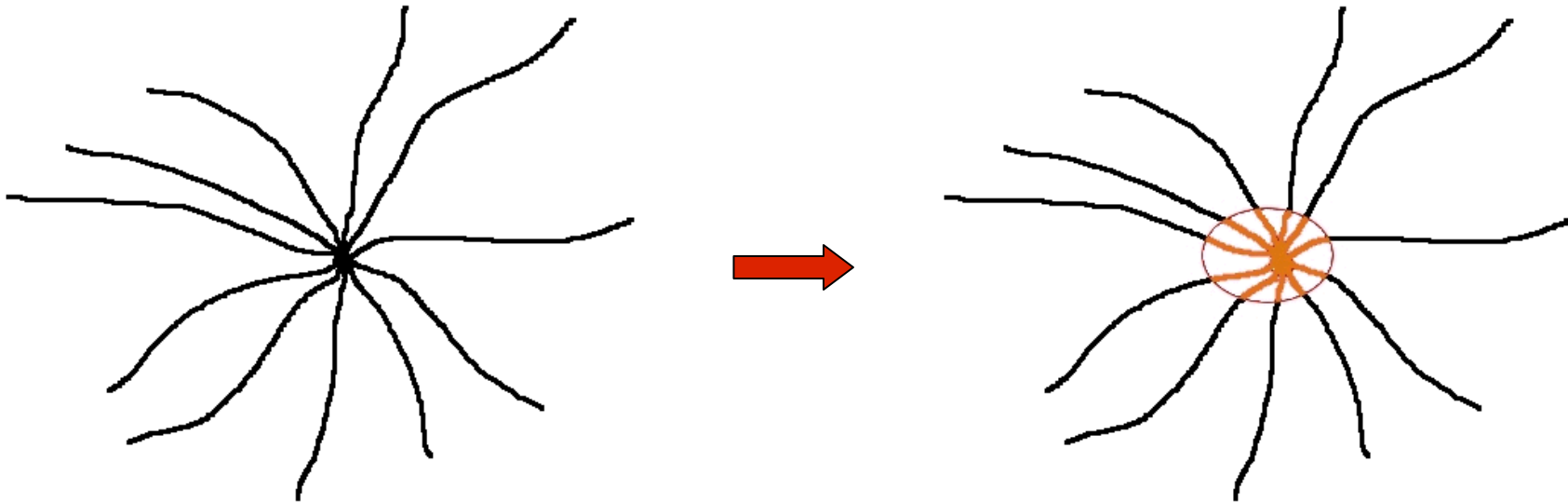


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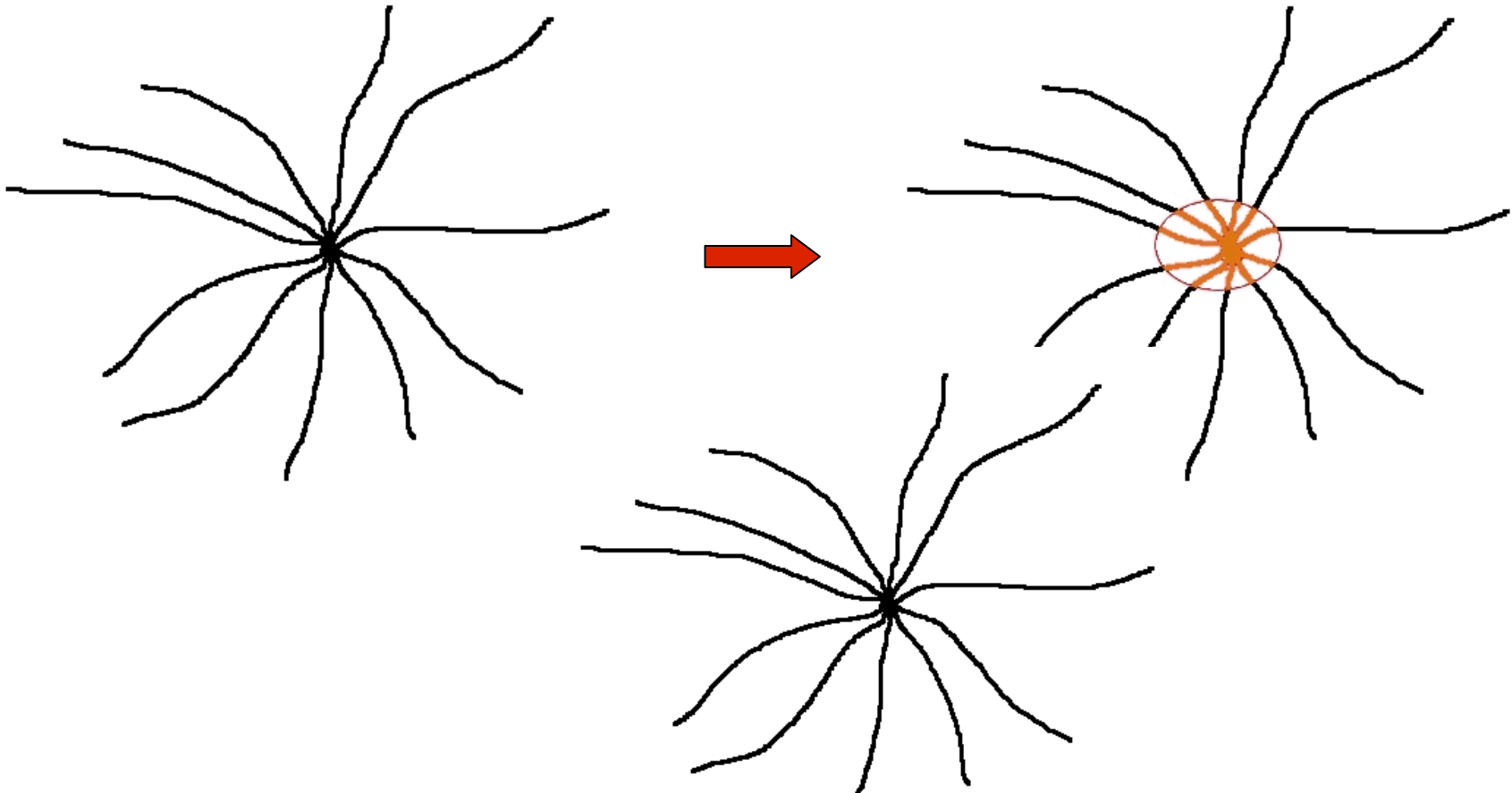


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Simultaneous integration:  **ensures the numerical stability !**

→ **Propagation front ;**

How to choose the propagation front?

Many propagation fronts following the optimal trajectories can be designed!

How to define the propagation front? idea: introduction of a “cost” C such that level sets of C correspond with propagated front.

Simultaneous integration:

→ All previous SFS methods choose

$$C = u \quad (u \text{ is the solution})$$

→ inconsistent with the causality

when the solution does not increase
along the optimal trajectories !

Optimal trajectory



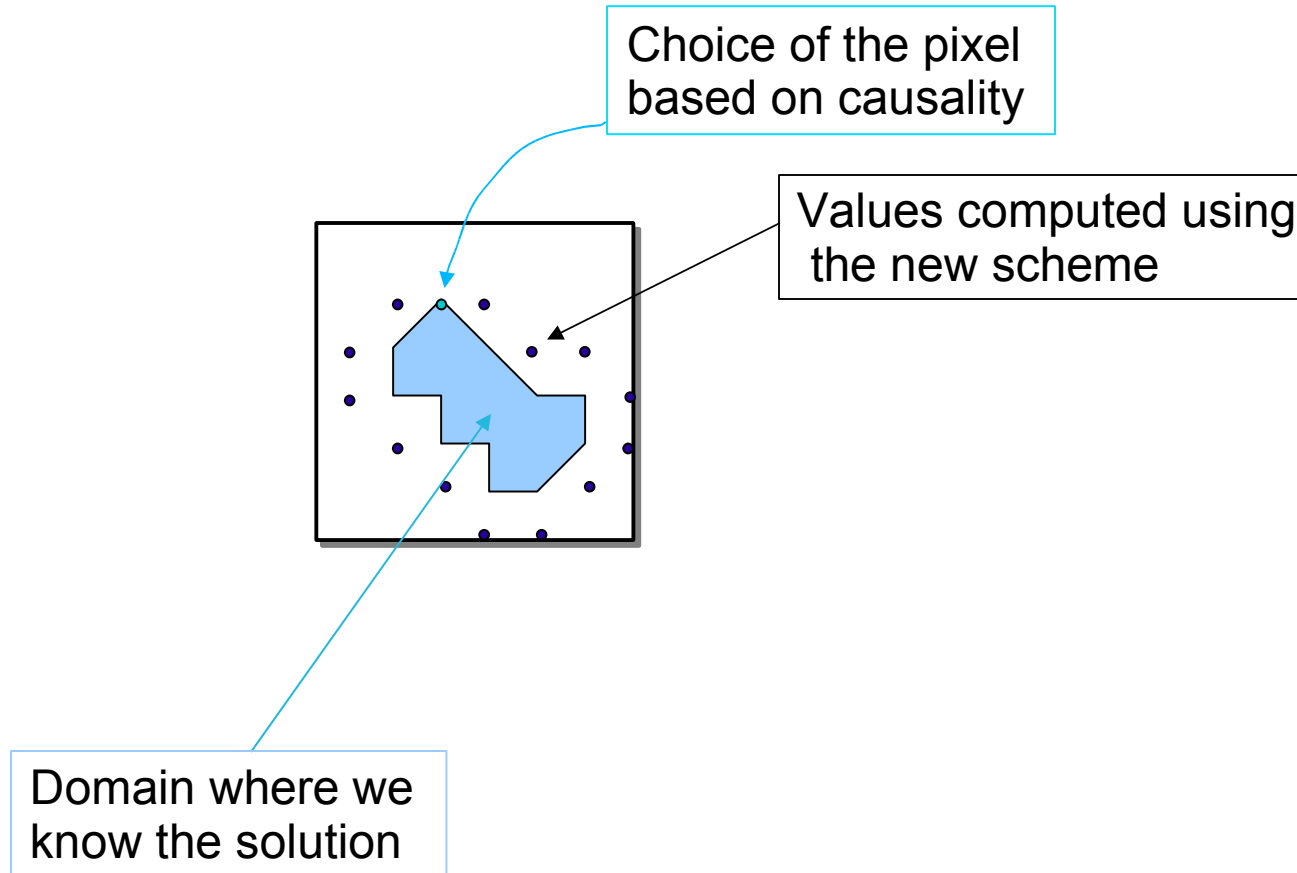
Level sets of u

→ We show how to define an appropriate cost C

- which is always consistent with optimal trajectories,
- which allows to define and compute simply and practically the update order...

→ This cost C is based on the notion of subsolution: $\psi \dots$

Fast Marching Method (FMM)



Algorithm

A = points acceptés
F = points éloignés
C = points considérés

1. Commencer avec tous les points de la grille dans **F** et poser $U(x) = V_\infty(x)$ (V_{infty} est définie dans la section 5).
2. Déplacer les points de la grille situés sur le bord $\partial\Omega$ et sur la cible \mathcal{T} dans **A**. Pour tous ces points, poser $U(x) = \varphi(x)$ ($U(x) = V_\infty(x)$, dans le cas de contraintes d'état).
3. Déplacer tous les points de la grille $x \in \mathbf{F}$ adjacents aux points Acceptés dans **C** et pour chacun de ces points, réévaluer $U(x)$ en utilisant le schéma de mise à jour (14).
4. Trouver le point de la grille $x_0 \in \mathbf{C} \cup \mathbf{F}$ qui a la plus petite valeur $U(x) - \psi(x)$. Déplacer x_0 dans **A**.
5. Déplacer tous les points Eloignés adjacents à x_0 dans **C**.
6. Réévaluer U en utilisant le schéma (14), pour tous les points Considérés adjacents à x_0 .
7. Si l'ensemble **C** n'est pas vide, retourner à l'étape 4.

A Practical and Detailed Example:

The classical Rouy/Tourin equation

Associated Modeling:

- Lambertian and homogeneous reflectance,
- Single *oblique* and far light source, $\mathbf{L} = (1, \gamma)$;
- orthographic projection.

Resulting PDE:
$$I(x) \sqrt{1 + |\nabla u(x)|^2} + \mathbf{1} \cdot \nabla u - \gamma = 0$$

where $I(x)$ is the image.

Associated cost function:
$$l(x, a) = I(x) \sqrt{1 + |a|^2} - \gamma ;$$

$l(x, a)$ and $H(x, 0) = I(x) - \gamma$ have arbitrary signs...

➡ Solution does not increase along the characteristic curves,

➡ All previous Fast Marching Methods do not directly apply !

Associated subsolution:
$$\psi(x) = -\frac{1}{\gamma} \mathbf{1} \cdot x.$$

Focus on the improvement due to the new Causality



original image



groundtruth

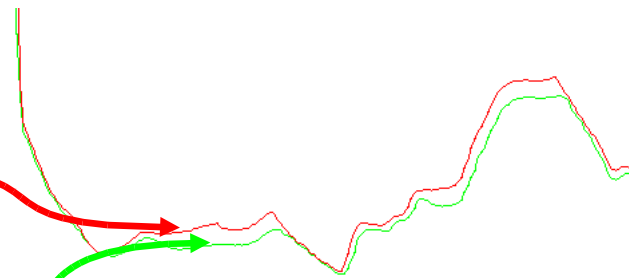
Computed solution

groundtruth

Reconstruction with the classical causality



oblique view

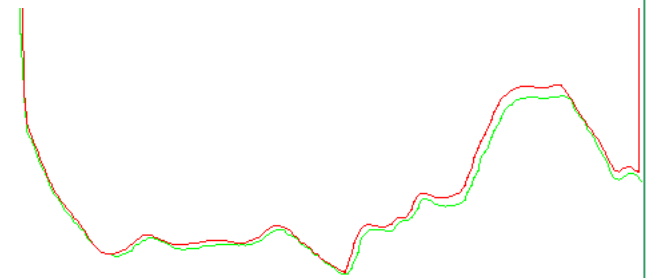


profile view

Reconstruction with the new and correct causality



oblique view



profile view

SFS with attenuation of light due to distance

Associated Modeling:

- Lambertian and homogeneous reflectance,
- Single *proximal* light source (optical center),
- perspective projection.

Resulting PDE:

$I(x)$ = image and f = focal length.

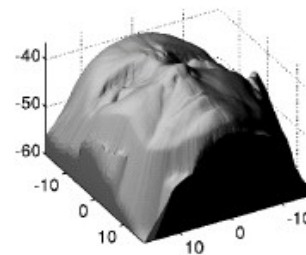
$$-e^{-2\mathbf{u}(x)} + J(x) \sqrt{f^2 |\nabla \mathbf{u}(x)|^2 + (\nabla \mathbf{u}(x) \cdot x)^2 + Q(x)^2} = 0,$$

$$J(x) = \frac{I(x)f^2}{Q(x)} \quad \text{and} \quad Q(x) = \sqrt{f^2/(|x|^2 + f^2)}.$$

➔ All previous Fast Marching Methods do not directly apply



image



reconstruction

References

- [Kimmel-Sethian:01] R. Kimmel and J.A. Sethian. Optimal algorithm for shape from shading and path planning. *JMIV*, 14(2):237–244, May 2001.
- [L.Cohen:05] L. Cohen. Minimal paths and fast marching methods for image analysis. In *Mathematical Models in Computer Vision: The Handbook*, Springer, 2005.
- [Sethian:99] J.A. Sethian. *Level Set Methods and Fast Marching Methods*. Cambridge University Press, 1999.
- [Sethian-Vladimirsky:03] J.A. Sethian and A. Vladimirsky. Ordered upwind methods for Hamilton–Jacobi equations: Theory and algorithms. *SIAM J. on Num. Ana.* 41(1), 2003
- [Prados:04] E. Prados. *Application of the theory of the viscosity solutions to the Shape From Shading problem*. PhD thesis, 2004.