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Percolation and Connectivity in AB Random Geometric Graphs

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Abstract

We study a generalization to the continuum of the AB percolation model on discrete lattices. Let $\mathcal{P}_\lambda, \mathcal{P}_\mu$ be independent Poisson point processes in \mathbb{R}^d , $d \geq 2$, of intensities λ, μ respectively. The AB random geometric graph $G(\lambda, \mu, r)$ is a graph whose vertex set is \mathcal{P}_λ with edges between any two points $X_i, X_j \in \mathcal{P}_\lambda$ provided there exists a $Y \in \mathcal{P}_\mu$ such that $|X_k - Y| \leq r$, $k = i, j$. We investigate percolation and connectivity in AB random geometric graphs.

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1 Introduction

A variant of the usual independent percolation model that has been of interest is the AB percolation model ([4, 11]). Given a graph L , each vertex is given a mark A or B independent of other vertices. Edges between vertices with similar marks (A or B) are removed. The resulting random sub-graph is the AB graph model. Percolation in this model simply means an infinite path of vertices in the lattice with marks alternating between A and B i.e., $ABABABA\dots$. This model has been studied on lattices and some related graphs. The AB percolation model behaves quite differently compared to the usual percolation model. For example, it is known that AB percolation does not occur in \mathbb{Z}^2 . Our aim is to study a generalization of this model in the continuum.

A natural analog of the above model to the continuum is a graph whose vertex set is an independently marked Poisson point process \mathcal{P} of intensity η in \mathbb{R}^d . Unless otherwise stated, we will assume throughout the paper that $d \geq 2$. Each spatial point is given an independent mark A with probability p , or mark B with probability $1 - p$. Denote by $\mathcal{P}_1, \mathcal{P}_2$ the independent Poisson point processes consisting of the points of \mathcal{P} with marks A, B respectively. Two points $X_i \in \mathcal{P}_i, i = 1, 2$, share an edge if $|X_1 - X_2| \leq 2r$, where $r > 0$ is a constant and $|x - y|$ is the Euclidean distance between x and y . This suggests a more generic model where $\mathcal{P}_1, \mathcal{P}_2$ are independent Poisson point processes with intensities λ, μ respectively. Alternately one can consider the vertex set to be the points of \mathcal{P}_1 and draw edges between any two points of \mathcal{P}_1 that are connected via a point of \mathcal{P}_2 as above. Percolation concerns existence of infinite components or giant components in the graph. We show that the model exhibits phase transition with regard to percolation. Additionally we characterize the region (λ, μ) for which percolation occurs. In the percolation context we will refer to the graph as the AB Poisson Boolean model or more simply the AB Boolean model.

The set up for the study of connectivity in AB random geometric graphs is as follows. For each $n \geq 1$, let \mathcal{P}_{1n} and \mathcal{P}_{2n} be independent Poisson processes in the unit cube in \mathbb{R}^d of intensities n and cn respectively, where $c > 0$ is a fixed constant. The graph $G_n(r)$ has vertex set \mathcal{P}_{1n} with an edge between any two points $X_1, X_2 \in \mathcal{P}_{1n}$, if there is a point $Y \in \mathcal{P}_{2n}$ such that $|X_i - Y| \leq r$,

$i = 1, 2$. Of interest is the asymptotic behavior of the connectivity threshold $r_n := \inf\{r > 0 : G_n(r) \text{ is connected}\}$. We show that for $r_n = \frac{\log(n/\beta)}{cn\theta_d}$, $\beta > 0$, the number of isolated nodes converges in distribution to a Poisson random variable with mean β . We also derive sufficient conditions for G_n to be connected or disconnected eventually almost surely. An interesting corollary of our results is that the threshold for connectivity and the threshold for the graph not to have isolated nodes are different (see Remark 3.2). These two thresholds coincide for usual random geometric graphs ([10]).

Our motivation to study AB random geometric graphs comes from application to wireless communication. In models of ad-hoc wireless networks, the nodes are assumed to be communicating entities that are distributed randomly in space. Edges between any two nodes in the graph represents the ability of the two nodes to communicate effectively with each other. A pair of nodes share an edge if the distance between the nodes is less than a certain cutoff $r > 0$, that is determined by the transmission power. Percolation and connectivity thresholds for such a model have been used to derive, for example, the capacity of wireless networks ([3, 5]). Consider a transmission scheme called the frequency division half duplex, where each node transmits at a frequency f_1 and receives at frequency f_2 or vice-versa ([13]). Thus nodes with transmission-reception frequency pair (f_1, f_2) can communicate only with nodes that have transmission-reception frequency pair (f_2, f_1) that are located within the cutoff distance r . Another example where such a model would be applicable is in communication between communication units deployed at different levels, for example underwater, surface and in air. Units in a level can communicate only with those at the next level and vice-versa within a certain range. A third example is in secure communication in wireless sensor networks with two types of nodes, tagged and normal. Upon deployment, each tagged node broadcasts a key over a predetermined secure channel, which is received by all normal nodes that are within transmission range. Two tagged nodes can then communicate provided there is a normal node that has received a key from both these tagged nodes, that is, it is within transmission range of both the tagged nodes.

The following generalization of the discrete AB percolation model has been studied on various

graphs by Kesten *et. al.* (see [1, 7, 8]). Mark each vertex or site of a graph L independently as 0 or 1 with probability p and $1 - p$ respectively. Given any infinite sequence $w \in \{0, 1\}^\infty$, the question is whether w occurs in the graph L or not. The sentences $(1, 0, 1, 0\dots), (0, 1, 0, 1\dots)$ correspond to AB percolation and the sequence $(1, 1, 1\dots)$ corresponds to usual percolation. More generally Kesten *et al* answer whether all (or almost all) infinite sequences are seen in L or not. The graphs for which the answer is in affirmative are \mathbb{Z}^d for d large, triangular lattice and \mathbb{Z}_{cp}^2 , the close-packed graph of \mathbb{Z}^2 . Our results provide partial answers to these questions in the continuum.

The rest of the paper is organized as follows. Sections 2, 3 define and state our main theorems on percolation and connectivity of AB random geometric graphs respectively. Sections 4,5 contain the proofs of these results.

2 Percolation in the AB Poisson Boolean Model

Let $\mathcal{P}_\lambda = \{X_i\}_{i \geq 1}$ and $\mathcal{P}_\mu = \{Y_i\}_{i \geq 1}$ be independent Poisson point processes in \mathbb{R}^d , $d \geq 2$ with intensities λ and μ respectively. For any $r > 0$, let $B_x(r)$ denote the closed ball of radius r centered at x .

Definition 2.1. *The AB Poisson Boolean model $C(\lambda, \mu, r) := (\mathcal{P}_\lambda, E(\lambda, \mu, r))$ is the graph with vertex set \mathcal{P}_λ and edge set*

$$E(\lambda, \mu, r) := \{\langle X_i, X_j \rangle : X_i, X_j \in \mathcal{P}_\lambda, B_{X_i}(r) \cap B_Y(r) \neq \emptyset, B_{X_j}(r) \cap B_Y(r) \neq \emptyset, \text{ for some } Y \in \mathcal{P}_\mu\}.$$

The edges in all the graphs that we consider are undirected, that is, $\langle X_i, X_j \rangle \equiv \langle X_j, X_i \rangle$. Note that if we replace the condition $B_{X_i}(r) \cap B_Y(r) \neq \emptyset$ by $B_{X_i}(r_1) \cap B_Y(r_2) \neq \emptyset$ the resultant graph will be $C(\lambda, \mu, \frac{r_1+r_2}{2})$. Thus our definition also includes the case of differing radii. For any locally-finite point process \mathcal{X} , we define the coverage process $\mathcal{C}(\mathcal{X}, r) := \cup_{X_i \in \mathcal{X}} B_{X_i}(r)$. We abbreviate $C(\mathcal{P}_\lambda, r)$ by $C(\lambda, r)$. Equivalently, we shall use $C(\lambda, r)$ to represent the graph with vertex set \mathcal{P}_λ and edge set $E(\lambda, r) = \{\langle X_i, X_j \rangle : |X_i - X_j| \leq 2r\}$. In this usual continuum percolation model ([9]), the

interest is in the existence of an infinite component in $C(\lambda, r)$. It is known that there exists critical intensity $\lambda_c(r) \in (0, \infty)$ such that an infinite component exists iff $\lambda > \lambda_c(r)$. Let $\theta(\lambda, \mu, r)$ be the probability that there exists an infinite component in $C(\lambda, \mu, r)$, i.e, percolation occurs. It follows from the zero-one law that $\theta(\lambda, \mu, r) \in \{0, 1\}$. We are interested in characterizing the region formed by (λ, μ, r) in which percolation occurs i.e, $\theta(\lambda, \mu, r) = 1$. Define $\mu_c(\lambda, r) := \sup\{\mu : \theta(\lambda, \mu, r) = 0\}$.

We first make two simple observations. For any λ , coupling $C(\lambda, \mu, r)$ as a subgraph of $C(\lambda + \mu, r)$, it is easy to see that $\mu_c(\lambda, r) + \lambda \geq \lambda_c(r)$. Thus for $\lambda < \lambda_c(r)$, we have the lower bound $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$. On the other hand $\langle X_i, X_j \rangle \in E(\lambda, \mu, r)$ implies that $|X_i - X_j| \leq 4r$. Hence, $C(\lambda, \mu, r)$ has an infinite component only if $C(\lambda, 2r)$ has an infinite component. Thus $\mu_c(\lambda, r) = \infty$ if $\lambda \leq \lambda_c(2r)$. However it is not clear that $\mu_c(\lambda, r) < \infty$ for $\lambda > \lambda_c(2r)$. We answer this in affirmative for $d = 2$.

Theorem 2.1. *Fix $\lambda, r > 0$ and let $\mu_c(\lambda, r)$ be as defined above. We then have for $d = 2$, $\mu_c(\lambda, r) < \infty$ if $\lambda > \lambda_c(2r)$.*

Thus the AB Boolean model exhibits a *phase transition* in the plane. However, the above theorem does not tell us how to choose a μ for a given $\lambda > \lambda_c(2r)$ for $d = 2$ such that AB percolation happens, or if indeed there is a phase transition for $d \geq 3$. The following proposition gives an upper bound for $\mu_c(\lambda, r)$ for large λ . In order to state the proposition, we need some notation.

For a vector $\bar{r} = (r_1, \dots, r_k) \in \mathbb{R}_+^k$, define $r_0 = \inf_{1 \leq i, j \leq k} \{r_i + r_j\}$. For $d = 2$, we mimic the construction of [9, Fig 3.2, Pg 87]. Consider the triangular site percolation model(\mathbb{T}) with each edge being of length $r_0/2$. Around each vertex place a “flower” formed by circular arcs. Flowers centered at different vertices are disjoint. Let $a(2, r)$ be the area of a flower. Declare each vertex of the triangle (or equivalently its flower) “open” with probability p and closed otherwise, independently of other vertices. For $d \geq 3$, suppose we cover \mathbb{R}^d by disjoint cubes centered at points of $\mathbb{Z}^{*d} = \sqrt{\frac{r_0}{5}}\mathbb{Z}^d$ of equal side-length $\sqrt{\frac{r_0}{5}}$. Let \mathbb{Z}^{*d} be equipped with the usual graph structure and $a(d, r_0)$ be the volume of a cube. Declare each vertex of \mathbb{Z}^{*d} open with probability p and closed otherwise independently of others. This is the standard independent site percolation model on \mathbb{Z}^d . Let $p_c(d)$ denote the critical percolation probability in the above percolation models. It is known that

$p_c(2) = \frac{1}{2}$ and $p_c(d) < 1$, for $d \geq 3$ ([4]).

We obtain an upper bound for $\mu_c(\lambda, r)$ as a special case of a more general result. Fix $k \in \mathbb{N}$. For $i = 1, \dots, k$, let \mathcal{P}_{λ_i} be independent Poisson point processes of intensity λ_i . A sentence $\omega = \{w(i)\}_{i \geq 1} \in \{1, 2, \dots, k\}^\infty$ is said to exist if there exists a sequence $\{X_i\}_{i \geq 1} \subset \mathbb{R}^d$ such that $X_i \in \mathcal{P}_{\lambda_{w(i)}}$ and $|X_i - X_{i+1}| \leq r_{w(i)} + r_{w(i+1)}$ for $i \geq 1$.

Proposition 2.1. *If $\prod_{i=1}^k (1 - e^{-\lambda_i a(d, r)}) > p_c(d)$, then almost surely every sentence occurs. In particular, for any λ satisfying $a(d, 2r)\lambda > \log(\frac{1}{1-p_c(d)})$, we have that $\mu_c(\lambda, r) \leq \inf\{\mu > 0 : (1 - e^{-\lambda a(d, 2r)})(1 - e^{-\mu a(d, 2r)}) > p_c(d)\} < \infty$.*

Remark 2.1. *A simple calculation tells us that $a(2, 2) \simeq 0.8227$ and $\lambda \simeq 0.843$ is the least λ such that the condition in above proposition is satisfied. Using these, the upper bound in the above proposition is $\mu_c(0.85, 1) < 6.2001$.*

The above proposition can be used to show existence of AB percolation in the natural analog of the discrete AB percolation model discussed in the introduction. Let \mathcal{P}_λ be the Poisson point process of intensity $\lambda > 0$ defined above. Let $\{m_i\}_{i \geq 1}$ be a sequence of i.i.d. marks distributed as $m \in \{1, 2, \dots, k\}$ with probability mass function $\{p_i; 1 \leq i \leq k\}$. Let ω be an infinite sentence formed using k -characters as defined earlier. We say that ω occurs in \mathcal{P}_λ if there exists a sequence $\{X_i\}_{i \geq 1} \subset \mathcal{P}_\lambda$ such that $m_i = w(i)$ and $|X_i - X_{i+1}| \leq r_{w(i)} + r_{w(i+1)}$ for all $i \geq 1$. Let $a(d, r)$ be as defined above Proposition 2.1.

Corollary 2.1. *If $\prod_{i=1}^k (1 - e^{-\lambda p_i a(d, r)}) > p_c(d)$, then almost surely every sentence occurs. In particular, for any λ satisfying $a(d, 2r)\lambda > 2 \log(\frac{1}{1-\sqrt{p_c(d)}})$, there exists a $p(\lambda) < \frac{1}{2}$ such that $\theta(\lambda p, \lambda(1-p), r) = 1$ for all $p \in (p(\lambda), \frac{1}{2}]$.*

Remark 2.2. *It can be shown that the number of infinite components in the AB Boolean model is at most one. The proof of this fact follows along the same lines as the proof in Poisson Boolean model (see [9, Proposition 3.3, Proposition 3.6]) since it relies on the ergodic theorem, the topology of infinite components and not on the specific nature of the infinite components.*

3 Connectivity in AB Random Geometric Graphs

As explained in the introduction, we will work with an equivalent, but slightly different model so as to be consistent with the literature on random geometric graphs. We will redefine some of the notations used in Section 2. Let \mathcal{P}_λ and \mathcal{P}_μ denote independent Poisson point processes on $U = [0, 1]^d$, $d \geq 2$, of intensities λ, μ respectively. Let O denote the origin. We also nullify some of the technical complications arising out of boundary effects by choosing to work with the toroidal metric on the unit cube. The metric used will be the toroidal ℓ_2 metric on U defined as

$$d(x, y) := \inf\{|x - y + z| : z \in \mathbb{Z}^d\}, \quad x, y \in U. \quad (3.1)$$

$B(x, r)$ will now denote the ball of radius r around the point $x \in U$ with respect to the metric d .

Define the AB random geometric graph $G(\lambda, \mu, r)$ to be the graph with vertex set \mathcal{P}_λ and edge set

$$E(\lambda, \mu, r) := \{\langle X_i, X_j \rangle : X_i, X_j \in \mathcal{P}_\lambda, d(X_i, Y) \leq r, d(X_j, Y) \leq r \text{ for some } Y \in \mathcal{P}_\mu\}.$$

When the context is clear, we shall omit the reference to λ, μ, r in G and E . Alternatively, we shall also use $X_i \sim X_j$ to denote existence of an edge between X_i, X_j when the underlying graph is unambiguous. For any point process \mathcal{P} , and $A \subset \mathbb{R}^d$, denote by $\mathcal{P}(A)$ the number of points of \mathcal{P} in A . Also we denote the Palm version of the Poisson point process, $\mathcal{P}_\lambda \cup \{x\}$ by \mathcal{P}_λ^x . Recall that for any locally finite point process $\mathcal{X} \subset U$, the coverage process was defined as $\mathcal{C}(\mathcal{X}, r) := \cup_{X_i \in \mathcal{X}} B_{X_i}(r)$ and abbreviate $\mathcal{C}(\mathcal{P}_\lambda, r)$ by $\mathcal{C}(\lambda, r)$. The degree of $x \in \mathcal{P}_\lambda^x$ is given by $\text{deg}(x) := \sum_{X_i \in \mathcal{P}_\lambda} 1_{[\langle X_i, x \rangle \in E]} = \#\{X_i \in \mathcal{P}_\lambda : X_i \in \mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r)\}$.

Our goal in this section is to study the *connectivity threshold* in the sequence of graphs $G_n := G(n, cn, r_n)$ as $n \rightarrow \infty$. This constant c can be thought of as a measure of the relative denseness or sparseness of \mathcal{P}_λ with respect to \mathcal{P}_μ (see Remark 3.1 below). It is easier to first consider the critical radius required to eliminate isolated nodes. Let $W(\lambda, \mu, r) := \sum_{X_i \in \mathcal{P}_\lambda} 1_{[\text{deg}(X_i)=0]}$ be the number of isolated nodes in $G(\lambda, \mu, r)$ and define $M(\lambda, \mu) := \sup\{r \geq 0 : W(\lambda, \mu, r) > 0\}$. We will

call M to be the *largest nearest neighbor radius* in the AB random geometric graph. $W(n, cn, r_n)$ and $M(n, cn)$ shall be abbreviated to $W_n(r_n)$ and M_n respectively.

First we compute $\mathbb{E}(W_n(r_n))$ and look for regimes of r_n under which it stabilizes asymptotically. By Palm calculus for Poisson point processes we have,

$$\mathbb{E}(W(\lambda, \mu, r)) = \lambda \int_U \mathbb{E}(1_{[\text{deg}(x)=0]}) dx = \lambda \mathbb{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_O(r), r)) = 0).$$

Let $\theta_d = \|B_O(1)\|$ where $\|\cdot\|$ denotes the Lebesgue measure in \mathbb{R}^d . For $x \in U$,

$$\begin{aligned} \mathbb{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_O(r), r)) = 0) &= \mathbb{P}(\mathcal{P}_\mu \cap B_O(r) \cap \mathcal{C}(\lambda, r) = \emptyset) \\ &= \exp\{-\mu \mathbb{E}(\|B_O(r) \cap \mathcal{C}(\lambda, r)\|)\} \\ &= \exp\{-\mu \theta_d r^d (1 - \exp\{-\lambda \theta_d r^d\})\}. \end{aligned} \quad (3.2)$$

The above derivation is similar to that in [6, Chapter 3] and the details have been provided in the appendix. Thus

$$\mathbb{E}(W(\lambda, \mu, r)) = \lambda \exp\{-\mu \theta_d r^d (1 - \exp\{-\lambda \theta_d r^d\})\}. \quad (3.3)$$

Now we are ready to sketch the regimes of r_n under which $\mathbb{E}(W_n(r_n))$ stabilizes asymptotically. Let $\beta > 0$ and define

$$r_n^d = \frac{\log(n/\beta)}{cn\theta_d}, \quad n \geq 1. \quad (3.4)$$

It follows from (3.3) that with the choice of r_n as in (3.4),

$$\mathbb{E}(W_n(r_n)) \rightarrow \beta, \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

The following theorem gives convergence of the number of isolated nodes and the largest nearest neighbour radius in $G(n, cn, r_n)$ for the above choice of r_n . Let \xrightarrow{d} denote convergence in distribution and $Po(\beta)$ be a Poisson random variable with mean β .

Theorem 3.1. Let $\beta > 0$ and suppose that r_n satisfies (3.4). Then as $n \rightarrow \infty$,

$$W_n(r_n) \xrightarrow{d} \text{Po}(\beta), \text{ and} \quad (3.6)$$

$$\mathbb{P}(M_n \leq r_n) \rightarrow e^{-\beta}. \quad (3.7)$$

Remark 3.1. Define $W^0(\lambda, \mu, r) = \sum_{Y_i \in \mathcal{P}_\mu \cap U} 1_{[\mathcal{P}_\lambda(B_{Y_i}(r))=0]}$, i.e., the number of \mathcal{P}_μ nodes isolated from \mathcal{P}_λ nodes. From Palm calculus for Poisson point processes, we have that

$$\mathbb{E}(W^0(\lambda, \mu, r)) = \mu \int_U \mathbb{E}(1_{[\mathcal{P}_\lambda(B_x(r))=0]}) dx = \mu \exp\{-\lambda \theta_d r^d\}.$$

Suppose we denote $W^0(\lambda_n, \mu_n, r_n)$ by W_n^0 , then for the choice of r_n^d as in (3.4), we get that

$$\lim_{n \rightarrow \infty} \mathbb{E}(W_n^0) = \begin{cases} 0 & c < 1 \\ \beta & c = 1 \\ \infty & c > 1. \end{cases} \quad (3.8)$$

Thus there is a trade off between the relative density of the nodes and the radius required to stabilise the expected number of isolated nodes.

Let $\phi(a) = \arccos(\frac{\sqrt{c}}{2\sqrt{a}})$ and for any $a > 0$ define

$$\eta(a, c) = \begin{cases} \frac{1}{\pi}(2\phi(a) - \sin(2\phi(a))) & d = 2 \\ (1 - (\frac{c}{2^d a})^{\frac{1}{d}})^d & d \geq 3. \end{cases} \quad (3.9)$$

Define $a(c) := \inf\{a : a\eta(a, c) > 1\}$. The next theorem gives bounds on the connectivity threshold in the AB random geometric graphs.

Theorem 3.2. Let r_n^d be as defined in (3.4). Define $a_n^*(c) := \inf\{a : G(n, cn, a^{\frac{1}{d}} r_n) \text{ is connected}\}$.

Then almost surely,

$$\max\{1, \frac{c}{2^d}\} \leq \liminf_{n \rightarrow \infty} a_n^*(c) \leq \limsup_{n \rightarrow \infty} a_n^*(c) \leq a(c).$$

A sequence of events $\{A_n\}_{n \geq 1}$ is said to *occur with high probability (whp)* if $\mathbb{P}(A_n) \rightarrow 1$ as $n \rightarrow \infty$.

Remark 3.2. *It is easily seen that $a(c) \leq (\frac{2+c}{2})^d$ with equality holding for all $d \geq 3$. Now consider the case when $1 < a < \frac{c}{2^d}$. Then from Theorem 3.2, $G(n, cn, a^{\frac{1}{d}}r_n)$ is disconnected whp. However, since $ar_n^d = \frac{\log \frac{n}{n^\epsilon - a}}{cn\theta_d} > \frac{\log \frac{n}{\epsilon}}{cn\theta_d}$, for sufficiently large n and any $\epsilon > 0$, it follows from Theorem 3.1 that $G(n, cn, a^{\frac{1}{d}}r_n)$ contains no isolated nodes with whp. Thus unlike the usual random geometric graphs, in the case of the AB random geometric graphs the threshold for connectivity and the threshold to eliminate isolated nodes can be different.*

4 Proofs for Section 2

Proof of Proposition 2.1

Let $\lambda > \lambda_c(2r)$. The proof adapts the idea used in [2] of coupling the continuum percolation model to a discrete percolation model. For $l > 0$, let $l\mathbb{L}^2$ be the expanded two-dimensional integer lattice $l\mathbb{Z}^2$ with the usual graph structure i.e, $x, y \in l\mathbb{Z}^2$ share an edge if $|x - y| = l$. Denote the edge-set by $l\mathbb{E}^2$. For an edge e denote the mid-point of the edge by (x_e, y_e) . For every horizontal e , we shall define three rectangles $R_{ei}, i = 1, 2, 3$ as follows : R_{e1} is the rectangle $[x_e - 3l/4, x_e - l/4] \times [y_e - l/4, y_e + l/4]$; R_{e2} is the rectangle $[x_e - l/4, x_e + l/4] \times [y_e - l/4, y_e + l/4]$ and R_{e3} is the rectangle $[x_e + l/4, x_e + 3l/4] \times [y_e - l/4, y_e + l/4]$. Let $R_e = \cup_i R_{ei}$. The corresponding rectangles for vertical edges are defined similarly. The reader can refer to Figure 1.

Due to continuity of $\lambda_c(2r)$ (see [9, Theorem 3.7]), there exists $r_1 < r$ such that $\lambda > \lambda_c(2r_1)$. We shall now define some random variables associated with horizontal edges and the corresponding definitions for vertical edges are similar. Let A_e be the indicator random variable for the event that there exists a left-right crossing of R_e by a component of $\mathcal{C}(\lambda, 2r_1)$ and top-down crossings of R_{e1}

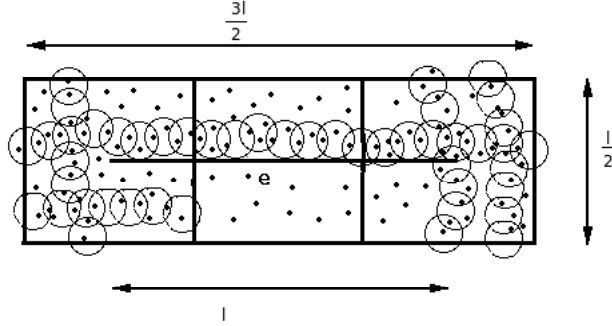


Figure 1: An horizontal edge e that satisfies the condition for $B_e = 1$. The balls are of radius $2r$, centered at points of \mathcal{P}_λ and the adjacent centers are of at most distance r_1 . The dots are the points of \mathcal{P}_μ .

and R_{e3} by a component of $\mathcal{C}(\lambda, 2r_1)$. Let B_e be the indicator random variable for the event that $\{A_e = 1\}$ and that the non-empty intersection of any two balls in the left-right crossing of R_e and in every left-right and top-down crossing of R_{e1} and R_{e3} when expanded to radius $2r$ contains a point of \mathcal{P}_μ .

Declare an edge in \mathbb{L}^2 to be open if $B_e = 1$. We first show that for $\lambda > \lambda_c(2r)$ there exists a μ, l such that \mathbb{L}^2 percolates (Step 1). The next step is to show that this implies percolation in the continuum model (Step 2).

STEP 1: The random variables $\{B_e\}_{e \in \mathbb{L}^2}$ are 1-dependent, i.e., B_e 's indexed by two non-adjacent edges are independent. Hence, given edges $e_1, \dots, e_n \in \mathbb{L}^2$, there exists $\{k_j\}_{j=1}^m \subset \{1, \dots, n\}$ with $m \geq n/4$ such that $\{B_{e_{k_j}}\}_{1 \leq j \leq m}$ are i.i.d. Bernoulli random variables. Hence,

$$\mathbb{P}(B_{e_i} = 0, 1 \leq i \leq n) \leq \mathbb{P}(B_{e_{k_j}} = 0, 1 \leq j \leq m) \leq \mathbb{P}(B_e = 0)^{n/4}. \quad (4.1)$$

We need to show that for a given $\epsilon > 0$ there exists l, μ , for which $\mathbb{P}(B_e = 0) < \epsilon$ for any $e \in \mathbb{L}^2$.

Fix an edge e . Observe that

$$\begin{aligned} \mathbb{P}(B_e = 0) &= \mathbb{P}(A_e = 0) + \mathbb{P}(B_e = 0|A_e = 1) \mathbb{P}(A_e = 1) \\ &\leq \mathbb{P}(A_e = 0) + \mathbb{P}(B_e = 0|A_e = 1). \end{aligned} \quad (4.2)$$

Since $\lambda > \lambda_c(2r_1)$, $C(\lambda, 2r_1)$ percolates. Hence by [9, Corollary 4.1], we can and do choose a l large enough so that

$$\mathbb{P}(A_e = 0) < \frac{\epsilon}{2}. \quad (4.3)$$

Now consider the second term on the right in (4.2). Given $A_e = 1$, there exist crossings as specified in the definition of A_e in $C(\lambda, 2r_1)$. Any two balls with a non-empty intersection are centered at a distance of at most $4r_1$. Now the width of the lens of intersection of two balls of radius $2r$ whose centers are at most $4r_1 (< 4r)$ apart is bounded below by a constant, say $b(r, r_1) > 0$. Hence if we cover R_e with disjoint squares of diagonal-length $b(r, r_1)/3$, then every lens of intersection will contain at least one such square. Let $S_j, j = 1, \dots, N(b)$, be the disjoint squares of diagonal-length $b(r, r_1)/3$ that cover R_e . Note that

$$\begin{aligned} \mathbb{P}(B_e = 1|A_e = 1) &\geq \mathbb{P}(\mathcal{P}_\mu \cap S_j \neq \emptyset, 1 \leq j \leq N(b)) \\ &= (1 - e^{-\mu b(r, r_1)^2/18})^{N(b)} \rightarrow 1, \text{ as } \mu \rightarrow \infty. \end{aligned}$$

Thus for the choice of l satisfying (4.3), we can choose a μ large enough so that

$$\mathbb{P}(B_e = 0|A_e = 1) < \frac{\epsilon}{2}. \quad (4.4)$$

From (4.2) - (4.4), we get $\mathbb{P}(B_e = 0) < \epsilon$. Hence given any $\epsilon > 0$, it follows from (4.1) that there exists l, μ large enough so that $\mathbb{P}(B_{e_i}, 1 \leq i \leq n) \leq \epsilon^{n/4}$. That $l\mathbb{L}^2$ percolates now follows from a standard Peierl's argument as in [4, pp. 17, 18].

STEP 2: Suppose that $B_e = B_f = 1$ for two adjacent edges e, f . Then, there exists crossings I_e and I_f of R_e and R_f respectively in $C(\lambda, 2r_1)$. Since e, f are adjacent, $R_{e_i} = R_{f_j}$ for some

$i, j \in \{1, 3\}$. Hence there exists a crossing J of R_{ei} in $C(\lambda, 2r_1)$ that intersects both I_e and I_f . Hence by definition, when the balls comprising the crossing J, I_e, I_f are expanded to radius $2r$, then every pairwise intersection of these balls will contain at least one point of \mathcal{P}_μ . This implies that I_e and I_f belong to the same AB component in $G(\lambda, \mu, r)$. Therefore $G(\lambda, \mu, r)$ percolates when \mathbb{L}^2 does. \square

Proof of Proposition 2.1. We declare a cube (for $d \geq 3$), or flower (when $d = 2$) open, if it contains at least one point each from \mathcal{P}_{λ_i} for all $i = 1, \dots, k$. This is clearly an independent percolation model on \mathbb{Z}^{*d} , $d \geq 3$ (or \mathbb{T} for $d = 2$) with probability $\prod_{i=1}^k (1 - e^{-\lambda_i a})$ of a cube (or flower) being open. The proof is complete by noting that the critical probability for percolation on \mathbb{Z}^{*d} (or \mathbb{T}) is $p_c(d)$. \square

Proof of Corollary 2.1. The statement for general k is a trivial consequence of Proposition 2.1. Consider the case $k = 2$. As $(1 - e^{-\lambda a/2}) > \sqrt{p_c(d)}$, by continuity, there exists an $\epsilon > 0$ such that for all $p \in (1/2 - \epsilon, 1/2 + \epsilon)$ we have $(1 - e^{-\lambda p a}) > \sqrt{p_c(d)}$. Thus for all $p \in (1/2 - \epsilon, 1/2]$, we get that $(1 - e^{-\lambda p a})(1 - e^{-\lambda(1-p)a}) > p_c(d)$. Hence by Proposition 2.1, $\theta(\lambda p, \lambda(1-p), r) = 1$. \square

5 Proofs for Section 3

The *total variation distance* between two integer valued random variables ψ, ζ is given as follows:

$$d_{TV}(\psi, \zeta) = \sup_{A \subset \mathbb{Z}} |\mathbb{P}(\psi \in A) - \mathbb{P}(\zeta \in A)|. \quad (5.5)$$

The following estimate in the spirit of Theorem 6.7([10]) shall be our main tool in proving Poisson convergence of $W_n(r_n)$. Recall that $\mathcal{C}(\mathcal{X}, r)$ is the coverage process formed by \mathcal{X} and balls of radius r and that the balls $B_x(r)$ are defined with respect to the toroidal metric.

Lemma 5.1. *Let $r > 0$ and $\lambda > 0$. Define integrals I_i , $i = 1, 2$ by*

$$\begin{aligned} I_1 &:= \lambda^2 \int_U dx \int_{B_x(5r)} dy \mathbb{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r)) = 0) \mathbb{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_y(r), r)) = 0), \\ I_2 &:= \lambda^2 \int_U dx \int_{B_x(5r)} dy \mathbb{P}(\mathcal{P}_\lambda^x(\mathcal{C}(\mathcal{P}_\mu \cap B_y(r), r)) = 0 = \mathcal{P}_\lambda^y(\mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r))). \end{aligned} \quad (5.6)$$

Then,

$$d_{TV}(W(\lambda, \mu, r), Po(\mathbb{E}(W(\lambda, \mu, r)))) \leq \min\left(3, \frac{1}{\mathbb{E}(W(\lambda, \mu, r))}\right) (I_1 + I_2).$$

A outline of the proof of the above Lemma which follows along the same lines as the proof of Theorem 6.7 ([10]) is given in the appendix.

Proof of Theorem 3.1. (3.7) follows easily from (3.6) by noting that

$$\mathbb{P}(M(\lambda, \mu) \leq r) = \mathbb{P}(W(\lambda, \mu, r) = 0).$$

Hence, the proof is complete if we show (3.6). Let I_{in} , $i = 1, 2$ be the integrals defined in (5.6) with (λ, μ, r) replaced by $(\lambda_n, \mu_n) = (n, cn)$ and r_n satisfying ((3.4). Since $\mathbb{E}(W_n(n, cn, r_n)) \rightarrow \beta$ as $n \rightarrow \infty$, the result follows from Lemma 5.1 if we show that $I_{in} \rightarrow 0$, $i = 1, 2$. Using (3.2), (3.3) and (3.5) we get for some finite positive constant C that

$$I_{1n} = \int_U dx \int_{B_x(5r_n)} dy (\mathbb{E}(W_n(r_n)))^2 \leq C(5r_n)^d \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We now compute the integrand in the inner integral in I_{2n} . Let $\Gamma(x, r) = \|B_0(r) \cap B_x(r)\|$. For

$x, y \in U$,

$$\begin{aligned}
& \mathbb{P}(\{\mathcal{P}_\lambda^x(\mathcal{C}(\mathcal{P}_\mu \cap B_y(r), r)) = 0\} \cap \{\mathcal{P}_\lambda^y(\mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r)) = 0\}) \\
&= \mathbb{P}(\mathcal{P}_\mu \cap B_y(r) \cap (\mathcal{C}(\lambda, r) \cup B_x(r)) = \emptyset, \mathcal{P}_\mu \cap B_x(r) \cap (\mathcal{C}(\lambda, r) \cup B_y(r)) = \emptyset) \\
&= \mathbb{P}(\mathcal{P}_\mu \cap B_y(r) \cap \mathcal{C}(\lambda, r) = \emptyset, \mathcal{P}_\mu \cap B_x(r) \cap B_y(r) = \emptyset, \mathcal{P}_\mu \cap B_x(r) \cap \mathcal{C}(\lambda, r) = \emptyset) \\
&= \mathbb{P}(\mathcal{P}_\mu \cap (B_x(r) \setminus B_y(r) \cup B_y(r) \setminus B_x(r)) \cap \mathcal{C}(\lambda, r) = \emptyset, \mathcal{P}_\mu \cap B_x(r) \cap B_y(r) = \emptyset) \\
&= \mathbb{P}(\mathcal{P}_\mu \cap (B_x(r) \setminus B_y(r) \cup B_y(r) \setminus B_x(r)) \cap \mathcal{C}(\lambda, r) = \emptyset) \mathbb{P}(\mathcal{P}_\mu \cap B_x(r) \cap B_y(r) = \emptyset) \\
&= \exp\{-2\mu\|B_x(r) \setminus B_y(r)\|(1 - \exp\{-\lambda\theta_d r^d\})\} \exp\{-\mu\Gamma(x - y, r)\} \\
&\leq \exp\{-2\mu\|B_x(r) \setminus B_y(r)\|(1 - \exp\{-\lambda\theta_d r^d\})\} \exp\{-\mu\Gamma(x - y, r)(1 - \exp\{-\lambda\theta_d r^d\})\} \\
&= \exp\{-\mu\theta_d r^d(1 - \exp\{-\lambda\theta_d r^d\})\} \exp\{-\mu\|B_x(r) \setminus B_y(r)\|(1 - \exp\{-\lambda\theta_d r^d\})\} \\
&\leq \exp\{-\mu\theta_d r^d(1 - \exp\{-\lambda\theta_d r^d\})\} \exp\{-\mu\eta r^{d-1}\|y - x\|(1 - \exp\{-\lambda\theta_d r^d\})\},
\end{aligned}$$

where the last inequality follows from the following lower bound (see [10, Eqn 8.21]): There exists $\eta > 0$ such that for $\|y - x\| \leq 5r$, $\|B_x(r) \setminus B_y(r)\| \geq \eta r^{d-1}\|y - x\|$. Thus for $\lambda_n = n$, $\mu_n = cn$ and r_n satisfying (3.4), we obtain the bound

$$I_2 \leq \int_U \int_{B_x(5r_n) \cap U} n \exp\{-cn\eta r_n^{d-1}\|y - x\|(1 - \exp\{-n\theta_d r_n^d\})\} \mathbb{E}(W_n(r_n)) \, dx \, dy.$$

Making the change of variable $w = nr_n^{d-1}(y - x)$ and using the fact that for all n sufficiently large $1 - \exp\{-n\theta_d r_n^d\} > \frac{1}{2}$, we get

$$\begin{aligned}
I_2 &\leq \mathbb{E}(W_n(r_n)) \int_U dx \int_{B_x(5nr_n^d) \cap U} (nr_n^d)^{1-d} \exp\{-c\eta\|w\|(1 - \exp\{-n\theta_d r_n^d\})\} dw \\
&\leq (nr_n^d)^{1-d} \mathbb{E}(W_n(r_n)) \int_U \exp\{-c\eta\|w\|/2\} dw \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Thus $I_i \rightarrow 0, i = 1, 2$, and therefore $d_{TV}(W_n(r_n), Po(\mathbb{E}(W_n(r_n)))) \rightarrow 0$ as $n \rightarrow \infty$. Since $\mathbb{E}(W_n(r_n)) \rightarrow \beta$, $Po(\mathbb{E}(W_n(r_n))) \xrightarrow{d} Po(\beta)$. As convergence in *total variation distance* implies convergence in distribution, we get (3.6). \square

Proof of Thm 3.2. Let $G(\lambda, r)$ denote the usual random geometric graph, that is a graph with vertex set \mathcal{P}_λ and edge set $\{(X_i, X_j) : X_i, X_j \in \mathcal{P}_\lambda, d(X_i, X_j) \leq r\}$, where d is the toroidal metric defined in (3.1). We shall use the following fact about the usual random geometric graphs $G(\lambda, r)$ (see [10, Chapter 13]) :

Fact 5.1. For $R_n = \left(\frac{c_0 \log n}{n \theta_d}\right)^{1/d}$, almost surely, the sequence of graphs $G(n, R_n)$ is connected eventually (and hence whp) if and only if $c_0 > 1$.

If $X_1, X_2 \in \mathcal{P}_\lambda$ share an edge in $G(\lambda, \mu, r)$, then they also share an edge in $G(\lambda, 2r)$. Thus $G(n, cn, a^{1/d}r_n)$ can be coupled as a subset of $G(n, 2a^{1/d}r_n)$. From Fact 5.1 it follows easily that for $G(n, cn, a^{1/d}r_n)$ to be connected asymptotically we must have $2^d a/c > 1$.

To complete the proof, it is enough to show the following :

$$\forall a < 1, \quad G(n, cn, a^{1/d}r_n) \text{ is not connected whp.} \quad (5.7)$$

$$\forall a > a(c), \quad \mathbb{P}\left(G(n, cn, a^{1/d}r_n) \text{ is not connected i.o.}\right) = 0, \quad (5.8)$$

where i.o. stands for infinitely often. To show (5.7) note that $ar_n^d = \frac{\log(\frac{n}{n^{1-a}})}{cn\pi} < \frac{\log(\frac{n}{\beta})}{cn\pi}$, for any $\beta > 0$ and sufficiently large n . Hence from Theorem 3.1 it follows that the largest nearest neighbour radius is asymptotically greater than $a^{1/d}r_n$, when $a < 1$ whp. Hence, asymptotically the graph is disconnected for large n with radius $a^{1/d}r_n$ whenever $a < 1$ whp. Thus we have shown (5.7) and the lower limit.

To prove (5.8), we first calculate the probability that every point of \mathcal{P}_n is connected to all points of \mathcal{P}_n within a radius of R_n for some $c_0 > 1$ in $G(n, cn, ar_n)$. Then using a subsequence argument we bound the sum in (5.8) for any $a > a(c)$.

By assumption since $a > a(c)$, by continuity, we can choose a $c_0 > 1$ such that $a\eta(a, c_0c) > 1$. For $X_i \in \mathcal{P}_\lambda$, define the event

$$A_i(\lambda, \mu, r, R) := \{X_i \text{ connects to all points of } \mathcal{P}_\lambda \cap B_{X_i}(R) \text{ in } G(\lambda, \mu, a^{1/d}r)\},$$

and let $B(\lambda, \mu, r, R) = \cup_{X_i \in \mathcal{P}_\lambda} A_i(\lambda, \mu, r, R)^c$. Then observe that $B(\lambda, \mu, r, R) \subset B(\lambda_1, \mu_1, r_1, R_1)$ provided $\lambda \leq \lambda_1, \mu \geq \mu_1, r \geq r_1, R \leq R_1$. Let $N_n = \mathcal{P}_n([0, 1]^2)$. Let $n_j = j^\alpha$ for some integer $\alpha > 0$ that will be chosen later. Let $r_{n_j}^d = \frac{a \log n_j}{c n_j \theta_d}$ and $R_{n_j}^d = \frac{c_0 \log n_j}{\theta_d n_j}$. Since $B(n_k, c n_k, r_{n_k}, R_{n_k}) \subset B(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j})$ for $n_j \leq k \leq n_{j+1}$, $\cup_{k=n_j}^{n_{j+1}} B(n_k, c n_k, r_{n_k}, R_{n_k}) \subset B(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j})$. Now,

$$\begin{aligned}
\mathbb{P} \left(\cup_{k=n_j}^{n_{j+1}} B(n_k, c n_k, r_{n_k}, R_{n_k}) \right) &\leq \mathbb{P} \left(B(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j}) \right) \leq \mathbb{P} \left(\cup_{i=1}^{N_{n_{j+1}}} A_i(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j})^c \right) \\
&\leq \sum_{i=1}^{n_{j+1} + n_{j+1}^{\frac{3}{4}}} \mathbb{P} \left(A_i(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j})^c \right) + \mathbb{P} \left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}} \right) \\
&\leq 2n_{j+1} p_{n_j} + \mathbb{P} \left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}} \right) \tag{5.9}
\end{aligned}$$

where due to stationarity we write $p_{n_j} = \mathbb{P} \left(A_i(n_{j+1}, c n_j, r_{n_{j+1}}, R_{n_j})^c \right)$ for all i . From [10, Lemma 1.4], we know that the second term on the right in (5.9) is summable. Conditioning on the number of points of $\mathcal{P}_{n_{j+1}}$ in $B_O(R_{n_j})$ and then using the Boole's inequality, we get

$$\begin{aligned}
p_{n_j} &\leq \sum_{k=0}^{\infty} \frac{(n_{j+1} \theta_d R_{n_j}^d)^k e^{-n_{j+1} \theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-c n_j \|B_O(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\|} dx \\
&\leq \sum_{k=0}^{\infty} \frac{(n_{j+1} \theta_d R_{n_j}^d)^k e^{-n_{j+1} \theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-c n_j \|B_O(r_{n_{j+1}}) \cap B_{x_n}(r_{n_{j+1}})\|} dx \\
&\leq \sum_{k=0}^{\infty} \frac{(n_{j+1} \theta_d R_{n_j}^d)^k e^{-n_{j+1} \theta_d R_{n_j}^d}}{k!} \frac{k}{\theta_d R_{n_j}^d} \int_{B_O(R_{n_j})} e^{-c n_j \zeta(r_{n_{j+1}}, R_{n_j}) \theta_d r_{n_{j+1}}^d} dx, \\
&= n_{j+1} \theta_d R_{n_j}^d e^{-c n_j \zeta(r_{n_{j+1}}, R_{n_j}) \theta_d r_{n_{j+1}}^d}
\end{aligned}$$

where $\theta_d r_{n_{j+1}}^d \zeta(r_{n_{j+1}}, R_{n_j}) = \inf \{ \|B_O(r_{n_{j+1}}) \cap B_x(r_{n_{j+1}})\| : x \in B_O(R_{n_j}) \}$ and $x_n \in \partial B_O(R_{n_j})$ as the infimum is attained for a point on the boundary of $B_O(R_{n_j})$. Further, $\zeta(r_{n_{j+1}}, R_{n_j}) = \eta(r_{n_{j+1}}, R_{n_j})$ when $d = 2$, and $\zeta(r_{n_{j+1}}, R_{n_j}) \geq \eta(r_{n_{j+1}}, R_{n_j})$ when $d \geq 3$. We sketch the derivations of $\eta(r_{n_{j+1}}, R_{n_j})$ in the appendix. Given a $\gamma, \epsilon > 0$ choose j large enough such that $(\frac{j}{j+1})^\alpha \geq \gamma$ and

also $\zeta(r_{n_{j+1}}, R_{n_j}) > \eta(a, c_0 c) - \epsilon$. Now simplifying by substituting for R_{n_j} and $r_{n_{j+1}}$, we get

$$\begin{aligned} p_{nj} &\leq \frac{(j+1)^\alpha \alpha c_0 \log j}{j^\alpha} e^{-\frac{j^\alpha}{(j+1)^\alpha} \zeta(r_{n_{j+1}}, R_{n_j}) \alpha \log(j+1)} \\ &\leq \frac{\alpha c_0 \log j}{\gamma} e^{-\gamma(\eta(a, c_0 c) - \epsilon) \alpha \log(j+1)} \\ &= \frac{\alpha c_0 \log j}{\gamma(j+1)^{\gamma(\eta(a, c_0 c) - \epsilon) \alpha}} \end{aligned}$$

Choose $\alpha > 0$ such that $(\gamma(\eta(a, c_0 c) - \epsilon) \alpha - 1) \alpha - 1 = a_0 > 0$. Then $\frac{n_{j+1}}{(j+1)^{\gamma(\eta(a, c_0 c) - \epsilon) \alpha}} = (j+1)^{-a_0 - 1}$ and so

$$\sum_{j \geq 0} \mathbf{P} \left(\bigcup_{k=n_j}^{n_{j+1}} B(n_k, cn_k, r_{n_k}, R_{n_k}) \right) \leq \sum_j \left(\frac{2\alpha c_0 \log j}{\gamma(j+1)^{a_0+1}} + \mathbf{P} \left(|N_{n_{j+1}} - n_{j+1}| > n_{j+1}^{\frac{3}{4}} \right) \right) < \infty.$$

Hence by Borel-Cantelli Lemma, $\mathbf{P}(B(n, cn, r_n, R_n) \text{ occurs i.o.}) = 0$. Hence almost surely, only finitely many of the events $B(n, cn, r_n, R_n)$ occur. This implies that almost surely, there is a path from every vertex in $G(n, cn, a^{1/d} r_n)$ to every other vertex that is at a distance R_n from it, for all large n . Since $c_0 > 1$, it follows from Fact 5.1 that almost surely, $G(n, cn, a^{1/d} r_n)$ is connected eventually. This proves (5.8). \square

6 Appendix

6.1 Derivation of Equation 3.2.

We shall prove that given a bounded random closed set F , we have that $\mathbf{P}(\mathcal{P}_\mu \cap F = \emptyset) = \exp\{-\mu \mathbf{E}(\|F\|)\}$. The derivation now follows easily from the fact that $\mathbf{E}(\|B_O(r) \cap C(\lambda, r)\|) = \theta_d r^d (1 - \exp\{-\lambda \theta_d r^d\})$ (see [6, Chapter 3]). Let $F \subset B_0(t)$ a.s. for some $t > 0$. Now given that exactly N points of \mathcal{P}_μ lie in $B_0(t)$, the probability that none of the points lie in F is given by $(1 - \frac{\mathbf{E}(\|F\|)}{\theta_d t^d})^N$. Since N is Poisson random variable with mean $\mu \theta_d t^d$, we obtain that

$$\mathbf{P}(\mathcal{P}_\mu \cap F = \emptyset) = \mathbf{E} \left(\left(1 - \frac{\mathbf{E}(\|F\|)}{\theta_d t^d} \right)^N \right) = \exp\{-\mu \mathbf{E}(\|F\|)\}.$$

□

6.2 Proof of Lemma 5.1.

For every $m \in \mathbb{N}$, partition U into disjoint cubes of side-length m^{-1} and corners at $m^{-1}\mathbb{Z}^d$. Let the cubes and their centres be denoted by $H_{m,1}, H_{m,2}, \dots$ and $a_{m,1}, a_{m,2}, \dots$ respectively. Let

$$\xi_{m,i} := \mathbf{1}_{\{\mathcal{P}_\lambda(H_{m,i})=1\} \cap \{\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_{a_{m,i}}(r), r) \cap H_{m,i}^c) = 0\}}.$$

We have made use of the representation of the degree of a node as in derivation of equation 3.2. Let $p_{m,i} = \mathbf{E}(\xi_{m,i})$, $p_{m,i,j} = \mathbf{E}(\xi_{m,i}\xi_{m,j})$. The remaining part of the proof is based on the notion of dependency graphs and the Stein-Chen method.

Define $I_m := \{i \in \mathbb{N}; H_{m,i} \subset [0, 1]^d\}$ and $E_m := \{< i, j > \subset I_m^2 : 0 < \|a_{m,i} - a_{m,j}\| < 5r\}$. The graph $G_m = (I_m, E_m)$ forms a dependency graph (see [10, Chapter 2]) for the random variables $\{\xi_{m,i}\}_{i \in I_m}$. The dependency neighbourhood of a vertex i is $N_{m,i} = i \cup \{j : < i, j > \in E_m\}$.

Let $W^m = \sum_{i \in I_m} \xi_{m,i}$. Then a.s., $W(\lambda, \mu, r) = \lim_{m \rightarrow \infty} W^m$. By [10, Theorem 2.1],

$$d_{TV}(W^m, Po(\mathbf{E}(W^m))) \leq \min(3, \frac{1}{\mathbf{E}(W^m)})(b_1(m) + b_2(m)),$$

where $b_1(m) = \sum_{i \in I_m} \sum_{j \in N_{m,i}} p_{m,i} p_{m,j}$ and $b_2(m) = \sum_{i \in I_m} \sum_{j \in N_{m,i}/\{i\}} p_{m,i,j}$. Now we shall show that the LHS and RHS in the above equation converge to the corresponding limits stated in the Lemma 5.1.

Let $w_m(x) = m^d p_{m,i}$ for $x \in H_{m,i}$. Then $\sum_{i \in I_m} p_{m,i} = \int_U w_m(x) dx$. Clearly, $\lim_{m \rightarrow \infty} w_m(x) = \lambda \mathbf{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r)) = 0)$. Since $w_m(x) \leq m^d \mathbf{P}(\mathcal{P}_\lambda(H_{m,i}) = 1) \leq \lambda$,

$$\lim_{m \rightarrow \infty} \mathbf{E}(W^m) = \lambda \int_U \mathbf{P}(\mathcal{P}_\lambda(\mathcal{C}(\mathcal{P}_\mu \cap B_x(r), r)) = 0) dx = \mathbf{E}(W(\lambda, \mu, r)),$$

where the first equality is due to the dominated convergence theorem and the second from the

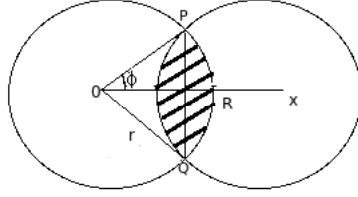


Figure 2: $|x| = R$, $\phi = \phi(r, R)$ and $L(r, R)$ is the area of the lens of intersection, the shaded region.

Palm theory for Poisson point processes. Similarly by letting $u_m(x, y) = m^{2d} p_{m,i} p_{m,j} 1_{[j \in N_{m,i}]}$ and $v_m(x, y) = m^{2d} p_{m,i,j} 1_{[j \in N_{m,i/\{i\}]}$ for $x \in H_{m,i}$, $y \in H_{m,j}$, one can show that $b_i(m) \rightarrow I_i$, $i = 1, 2$. \square

6.3 Proof of (3.9).

We sketch the derivation here first for $d = 2$ and then for $d \geq 3$. We sketch them separately as we obtain tighter bounds for $d = 2$ than in higher dimensions. Refer to Figure 2 for this derivation when $d = 2$. Let $L(r, R) = \|B_O(r) \cap B_x(r)\|$, $\|x\| = R$ be the area of lens of intersection of circles (or spheres) of radius r centered $R (< 2r)$ apart. It is clear that $L(r, R)$ is cut into two equal halves by the line PQ and the area of each of those halves is the area enclosed between the chord in the circle $B_0(r)$ and its circumference. The area of the segment OPQ (with PQ considered as the arc along the circumference of the circle) is easily seen to be $\phi(r, R)r^2$. The area of the triangle OPQ is $r \sin(\phi(r, R)) \times r \cos(\phi(r, R)) = \frac{r^2}{2} \sin(2\phi(r, R))$. And, now it is clear that $L(r, R) = (2\phi(r, R) - \sin(2\phi(r, R)))r^2$ and that $\cos(\phi(r, R)) = \frac{R}{2r}$. Now for $d \geq 3$. The width of the lens of intersection is $2r - R$. Thus the lens of intersection contains a ball of diameter $2r - R$. The volume of such a ball is $\theta_d(r - \frac{R}{2})^d$. Thus $L(r, R) > \theta_d(r - \frac{R}{2})^d$. Since $\zeta(a, c_0c) = L(r_n, R_n)$, the corresponding expressions for $\eta(a, c_0c)$ follows. \square

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