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A completeness theorem for strong normalization in minimal deduction modulo

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Abstract. Deduction modulo is an extension of first-order predicate logic where axioms are replaced by rewrite rules and where many theories, such as arithmetic, simple type theory and some variants of set theory, can be expressed. An important question in deduction modulo is to find a condition of the theories that have the strong normalization property. Dowek and Werner have given a semantic sufficient condition for a theory to have the strong normalization property: they have proved a "soundness" theorem of the form: if a theory has a model (of a particular form) then it has the strong normalization property. In this paper, we refine their notion of model in a way allowing not only to prove soundness, but also completeness: if a theory has the strong normalization property, then it has a model of this form. The key idea of our model construction is a refinement of Girard's notion of reducibility candidates. By providing a sound and complete semantics for theories having the strong normalization property, this paper contributes to explore the idea that strong normalization is not only a proof-theoretic notion, but also a model-theoretic one.

We say that a theory is consistent if it does not contain a contradiction. Syntactically, it means that we cannot prove all the formulae of this theory. In order to prove consistency of a theory, we can use semantic tools: it has been shown, for example, that having a $\{0, 1\}$ -valued model is a sufficient semantic condition for consistency of theories expressed in predicate calculus. It means that all theories of predicate calculus which have a $\{0, 1\}$ -valued model, are consistent. But is this an exhaustive test? Do all consistent theories have a $\{0, 1\}$ -valued model? Gödel proved that having a $\{0, 1\}$ -valued model is also a necessary condition of consistency for theories expressed in predicate calculus [10].

Deduction modulo [4] is a logical framework, based on Natural Deduction where axioms are replaced by rewrite rules, allowing to express proofs of many theories like arithmetic [8], simple type theory [5], some variants of set theory [6], etc... In this framework, β -reduction of proofs is used to represent their cut elimination, as in the Curry-De Bruijn-Howard correspondence. Therefore strong normalization of the β -reduction ensures the cut elimination property of the corresponding theory, and furthermore its consistency. It ensures also the disjunction property, the witness property, soundness of various proof search methods, etc... But strong normalization is a strictly more powerful property than consistency, as there exists consistent theories that don't strongly normalize

[12]. Then we may wonder if there is such a sufficient and necessary semantic condition of theories that are strongly normalizing.

The algebra of reducibility candidates \mathcal{C} was first introduced by J.Y. Girard [9], following the work of Tait [14]. We can see *a posteriori*, their work as proofs of strong normalization, obtained by the existence of a \mathcal{C} -valued model. This work has been extended for two non-trivial (as they contain strongly normalizing and not strongly normalizing theories) logical frameworks: Pure Type Systems by P.A. Melliès and B. Werner [13], and Deduction modulo by G. Dowek and B. Werner [7], for example.

But what about the converse? We may wonder if having a \mathcal{C} -valued model is also a necessary semantic condition for the strong normalization property, *i.e.* if all strongly normalizing theories expressed in Deduction modulo or Pure Type Systems have a \mathcal{C} -valued model.

In this paper, we exhibit a new algebra \mathcal{C}' which is a refinement of \mathcal{C} and we prove that having a \mathcal{C}' -valued model is still a sufficient semantic condition of strongly normalizing theories expressed in minimal deduction modulo. And moreover, that it is also a necessary semantic condition of strongly normalizing theories.

1 Minimal deduction modulo

1.1 Syntax

We consider a set (T) of *sorts*, an infinite set of *variables* of each sort, a set (f) of function symbols, and a set (P) of predicate symbols, that come with their *rank*. The formation rules for objects and propositions are the usual ones.

- Variables of sort T are terms of sort T .
- If f is a function symbol of rank $\langle T_1, \dots, T_n, U \rangle$ and t_1, \dots, t_n are respectively objects of sort T_1, \dots, T_n , then $f(t_1, \dots, t_n)$ is a term of sort U .
- If P is a predicate symbol of rank $\langle T_1, \dots, T_n \rangle$ and t_1, \dots, t_n are respectively objects of sort T_1, \dots, T_n , then $P(t_1, \dots, t_n)$ is an *atomic proposition*.

Propositions are built-up from atomic propositions with the usual connective \Rightarrow and quantifier \forall . Remark that, implicitly, quantification in $\forall x.A$ is restricted over the sort of the variable x .

Definition 1 (Terms and Propositions).

We call \mathcal{O} (as objects), the set of terms: $t ::= x \mid f t \dots t$

We call \mathcal{P} , the set of propositions: $A ::= P t \dots t \mid A \Rightarrow A \mid \forall x.A$

In this language, proof-terms can contain both term variables (written x, y, \dots) and proof variables (written α, β, \dots). Terms are written t, u, \dots while proof-terms are written π, ρ, \dots . We call \mathcal{X} the set of proof variables and \mathcal{Y} the set of term variables.

Definition 2 (Proof-terms).

We call \mathcal{T} , the set of proof-terms: $\pi ::= \alpha \mid \lambda\alpha.\pi \mid \pi \pi' \mid \lambda x.\pi \mid \pi t$

Notice that variables α and x are bound in the constructions $\lambda\alpha.\pi$, and $\lambda x.\pi$. Alphabetic equivalence, free and bound variables are defined as usual.

Each proof-term construction corresponds to a natural deduction rule: terms of the form α express proofs built with the axiom rule, terms of the form $\lambda\alpha.\pi$ and $(\pi \pi')$ express proofs built respectively with the introduction and elimination rules of the implication and terms of the form $\lambda x.\pi$ and (πt) express proofs built with the introduction and elimination rules of the universal quantifier.

We call *neutral* those proof-terms of \mathcal{T} that are not abstractions *i.e.* of the form α , $(\pi\pi')$ or (πt) . A proof-term is called *isolated* if it is neutral and only reduces on neutral terms.

1.2 Typing rules

We call *contexts*, lists of declarations $[\alpha : A]$ where α is a proof-variable and A is a proposition, such that each variable in a declaration is different from all the other variables of the context (in this way, we only consider *well formed* contexts, therefore we have to deal with alphabetic equivalence, when concatening them).

Given a congruence relation on propositions \equiv , we define typing rules as usual, in deduction modulo:

$$\begin{array}{c}
 \frac{}{\Gamma, \alpha : A \vdash_{\equiv} \alpha : B} \quad A \equiv B \quad (\text{axiom}) \\
 \\
 \frac{\Gamma, \alpha : A \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \lambda\alpha \pi : C} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-intro}) \\
 \\
 \frac{\Gamma \vdash_{\equiv} \pi : C \quad \Gamma \vdash_{\equiv} \pi' : A}{\Gamma \vdash_{\equiv} (\pi \pi') : B} \quad C \equiv A \Rightarrow B \quad (\Rightarrow\text{-elim}) \\
 \\
 \frac{\Gamma \vdash_{\equiv} \pi : A}{\Gamma \vdash_{\equiv} \lambda x.\pi : B} \quad B \equiv \forall x.A, \quad x \notin FV(\Gamma) \quad (\forall\text{-intro}) \\
 \\
 \frac{\Gamma \vdash_{\equiv} \pi : B}{\Gamma \vdash_{\equiv} \pi t : C} \quad B \equiv \forall x.A, \quad C \equiv (t/x)A, \quad t \text{ has the sort of } x \quad (\forall\text{-elim})
 \end{array}$$

Fig. 1. Typing rules

1.3 Proof reduction rules and strong normalization

As usual in deduction modulo, the process of cut elimination is modeled by β -reduction. We consider the contextual closure of the reduction rules given figure 2. These rules correspond to proof reduction in natural deduction.

We write $(\pi'/\alpha)\pi$ (resp. $(t/x)\pi$) the substitution of α (resp. x) by π' (resp. t) in π . We write $\pi \rightarrow \pi'$ if π reduces in one step to π' , $\pi \rightarrow^+ \pi'$ if π reduces in at least one step to π' , and $\pi \rightarrow^* \pi'$ if π reduces in an arbitrary number of steps

$$\begin{aligned}
(\lambda\alpha.\pi \pi') &\rightarrow (\pi'/\alpha)\pi \\
(\lambda x.\pi t) &\rightarrow (t/x)\pi
\end{aligned}$$

Fig. 2. Proof reduction rules

to π' .

A proof is said to be *normal* if it contains no redex. It is said to be *weakly normalizing* if it has a normal form and *strongly normalizing* if all reduction sequences issued from this proofs are finite. We write SN for the set of strongly normalizing proofs.

1.4 Theories expressed in minimal deduction modulo

A theory expressed in minimal deduction modulo is defined by a many-sorted language in predicate logic $\mathcal{L} = \langle\langle(T), (F), (P)\rangle\rangle$ and a congruence relation \equiv on propositions of the associated many-sorted logic. We suppose \equiv not *ambiguous*, i.e. there does not exist $x \in \mathcal{Y}$, $A, B, C \in \mathcal{P}$ such that $A \Rightarrow B \equiv \forall x.C$. We will call \mathcal{L}_{\equiv} this theory.

Remark 1. Given a theory \mathcal{L}_{\equiv} , we will write \vdash for \vdash_{\equiv} .

Proposition 1 (confluence and subject-reduction). \rightarrow is confluent.

And for all contexts Γ , proof-terms π, π' and propositions A , if $\Gamma \vdash \pi : A$ and $\pi \rightarrow \pi'$, then $\Gamma \vdash \pi' : A$.

Example As mentioned above deduction modulo permits to express (intentional) simple type theory [1] without any axiom. We show in the following, how minimal deduction modulo permits to express minimal (intentional) simple type theory, without any axiom (see [5] for details).

The *sorts* are *simple types* inductively defined by:

- ι and o are sorts,
- if T and U are sorts then $T \rightarrow U$ is a sort.

The language is composed of the individual symbols

- $S_{T,U,V}$ of sort $(T \rightarrow U \rightarrow V) \rightarrow (T \rightarrow U) \rightarrow T \rightarrow V$,
- $K_{T,U}$ of sort $T \rightarrow U \rightarrow T$,
- \Rightarrow , of sort o ,
- $\check{\forall}_T$ of sort $(T \rightarrow o) \rightarrow o$,

the function symbols $\alpha_{T,U}$ of rank $\langle T \rightarrow U, T, U \rangle$,

and the predicate symbol ε of rank $\langle o \rangle$.

The combinators $S_{T,U,V}$ and $K_{T,U}$ are used to express functions. The objects \Rightarrow , and $\check{\forall}_T$ allow to represent propositions as objects of sort o . Finally, the predicate ε allows to transform such an object t of type o into the actual corresponding proposition $\varepsilon(t)$.

$$\begin{aligned}
\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) &\rightarrow \alpha(\alpha(x, z), \alpha(y, z)) \\
\alpha(\alpha(K_{T,U}, x), y) &\rightarrow x \\
\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) &\rightarrow \varepsilon(x) \Rightarrow \varepsilon(y) \\
\varepsilon(\alpha(\dot{\forall}, x)) &\rightarrow \forall y \varepsilon(\alpha(x, y))
\end{aligned}$$

2 Language-dependent truth values algebras

We introduce in this paper, a new category of algebras : *language-dependent truth values algebras* (LDTVAS) which are both a simplification and a refinement of *truth values algebras* (TVAS) defined by G. Dowek [3]. First a simplification, because as we consider theories with rewrite rules only and no axioms, we don't have to care about what is called *positive* truth values in [3]. The refinement comes from the interpretation of the universal quantifier: $\hat{\forall}$.

In TVAS, $\hat{\forall}$ is defined as a function mapping subsets of the domain of the algebra \mathcal{B} , to \mathcal{B} . Given a substitution φ that maps term-variables to \hat{T} (the domain of T), the interpretation $\llbracket \forall x.A \rrbracket_{\varphi}$ of a proposition $\forall x.A$, is obtained by applying $\hat{\forall}$ to the set $\{\llbracket A \rrbracket_{\varphi+\langle x,v \rangle}, v \in \hat{T}\}$. The function $\hat{\forall}$ is typically a greatest lower bound or an intersection. That is sufficient for building a notion of model valued on \mathcal{C} , such that having such a model is a sufficient condition for a theory to be strongly normalizing. But it creates difficulties when we want to prove that this condition is necessary.

In LDTVAS, we propose another definition of $\hat{\forall}$: as a function mapping functions from \hat{T} to \mathcal{B} , to elements of \mathcal{B} . In this case $\llbracket \forall x.A \rrbracket_{\varphi}$ is obtained by applying $\hat{\forall}$ to the function that maps elements v of \hat{T} to $\llbracket A \rrbracket_{\varphi+\langle x,v \rangle}$.

In the case of the reducibility candidates algebra \mathcal{C} , the domain consists of sets of proof-terms and the domain of a sort T is the set of closed terms of sort T . In [3], $\llbracket \forall x.A \rrbracket_{\varphi}$ is defined as the set of proof-terms π such that for all $t \in \hat{T}$, πt is in $\bigcap \{\llbracket A \rrbracket_{\varphi+\langle x,t' \rangle}, t' \in \hat{T}\}$. The main properties that ensure the fact that having a \mathcal{C} -valued model is a sufficient condition for a theory to be strongly normalizing, are that all elements of the domain contains only strongly normalizing proof-terms, and that all proofs of a proposition are in their interpretation. In order to make this condition be also necessary, a solution is to make interpretations of propositions contain only their proofs.

In TVAS, the definition of $\hat{\forall}$ leads to interpret too strictly propositions of the form $\forall x.A$. For example, if P is an unary atomic proposition, then $\pi = \lambda x. \lambda \alpha. (\alpha x)$ is a proof of $A = \forall x. (\forall y. P(y)) \Rightarrow P(x)$, but π is not in the interpretation of A as for all $t \neq t'$, πt is not a proof of $(\forall y. P(y)) \Rightarrow P(t')$.

In LDTVAS, we choose to always take for each \hat{T} the set of closed terms of sort T . Therefore LDTVAS depend on the language they are built on. And we will see, in the following, how our new definition allows us to define $\llbracket \forall x.A \rrbracket_{\varphi}$ as

a “dependent intersection”, *i.e.* the set of proof-terms π such that for all $t \in \hat{T}$, πt is in $\llbracket A \rrbracket_{\varphi + \langle x, t \rangle}$, and therefore capture exactly proofs of $\forall x.A$.

2.1 Definition

For all sets E , we call $\mathbb{P}(E)$ the set of subsets of E .

For all sorts T of a language \mathcal{L} , we write \hat{T} , the set of closed terms of sort T .

Definition 3 (language-dependent tvas).

Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic.

$\langle \mathcal{B}, \hat{\rightarrow}, (\hat{A}_T), (\hat{\forall}_T) \rangle$ is a LDTVAs for \mathcal{L} if and only if:

- \mathcal{B} is a set,
- $\hat{\rightarrow}$ is a function from $\mathcal{B} \times \mathcal{B}$ to \mathcal{B} ,
- for all sorts T ,
- \hat{A}_T is a set of functions from \hat{T} to \mathcal{B} : $\hat{A}_T \subseteq \hat{T} \mapsto \mathcal{B}$
- $\hat{\forall}_T$ is a function from \hat{A}_T to \mathcal{B} .

Definition 4 (Morphism).

Let $\mathcal{B}^1 = \langle \mathcal{B}^1, \hat{\rightarrow}^1, (\hat{A}_T^1), (\hat{\forall}_T^1) \rangle$ and $\mathcal{B}^2 = \langle \mathcal{B}^2, \hat{\rightarrow}^2, (\hat{A}_T^2), (\hat{\forall}_T^2) \rangle$ be two LDTVAs.

A morphism from \mathcal{B}^1 to \mathcal{B}^2 is a function F from \mathcal{B}^1 to \mathcal{B}^2 such that:

- for all $E, G \in \mathcal{B}^1$, $F(E \hat{\rightarrow}^1 G) = F(E) \hat{\rightarrow}^2 F(G)$,
- for all sorts T , $x \in \hat{T}$ and $f \in \hat{A}_T^1$, $F(\hat{\forall}_T^1 f) = \hat{\forall}_T^2 F \circ f$.

2.2 Models

Definition 5 (Valuation).

Given a LDTVAs for $\mathcal{L} = \langle (T), (f), (P) \rangle$, a valuation φ is a substitution mapping term-variables of a sort to closed terms of the same sort. For all propositions A (resp. terms t), we call $\mathcal{V}(A)$ (resp. $\mathcal{V}(t)$) the set of valuations whose domain contains the set of free variables of A (resp. t).

We write $x \notin \varphi$ for expressing the fact that $\varphi(x)$ is not defined.

Definition 6. For all $A \in \mathcal{P}$, terms t and $\varphi \in \mathcal{V}(A)$, we write $|A|_{\varphi}$ the result of the substitution φ on A .

Definition 7 (Models).

Let $\mathcal{L} = \langle (T), (f), (P) \rangle$ be a many-sorted language in predicate logic,

let \equiv be a congruence relation on propositions of minimal deduction based on \mathcal{L} ,

let $\mathcal{B} = \langle \mathcal{B}, \hat{\rightarrow}, (\hat{A}_T), (\hat{\forall}_T) \rangle$ be a LDTVAs for \mathcal{L} .

1. We call \mathcal{B} -valued interpretations those functions which map every ordered pair of a proposition A and a valuation in $\mathcal{V}(A)$ to an element of \mathcal{B} .
2. A \mathcal{B} -valued interpretation $\llbracket \cdot \rrbracket$ is a \mathcal{B} -valued model if and only if:
 - for all $A, B \in \mathcal{P}$ and $\varphi \in \mathcal{V}(A \Rightarrow B)$, $\llbracket A \Rightarrow B \rrbracket_{\varphi} = \llbracket A \rrbracket_{\varphi} \hat{\rightarrow} \llbracket B \rrbracket_{\varphi}$
 - for all $A \in \mathcal{P}$, x of sort T and $\varphi \in \mathcal{V}(\forall x.A)$ such that $x \notin \varphi$,
 $\llbracket \forall x.A \rrbracket_{\varphi} = \hat{\forall}_T(t \mapsto \llbracket A \rrbracket_{\varphi + \langle x, t \rangle})$

- for all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \mathcal{V}(\forall x.A)$ such that $x \notin \varphi$,
 $\llbracket (t/x)A \rrbracket_\varphi = \llbracket A \rrbracket_{\varphi + \langle x, t \rangle}$.
- 3. A \mathcal{B} -valued model $\llbracket \cdot \rrbracket$ is a model of the theory \mathcal{L}_\equiv if and only if:
for all $A, A' \in \mathcal{P}$, $\varphi \in \mathcal{V}(A)$ and $\psi \in \mathcal{V}(A')$, if $|A|_\varphi \equiv |A'|_\psi$, then $\llbracket A \rrbracket_\varphi = \llbracket A' \rrbracket_\psi$

Remark 2. The previous conditions can be reformulated as: 2. Interpretations of propositions have to be adapted to the connectives to be a model. 3. Models have to be adapted to the congruence to be a model of the associated theory.

The following lemma explains that our definition of morphism is correct for the property of being a model of a theory \mathcal{L}_\equiv .

Lemma 1. For all LDTVAs \mathcal{B}_1 and \mathcal{B}_2 and morphisms F from \mathcal{B}_1 to \mathcal{B}_2 , if $\llbracket \cdot \rrbracket$ is a \mathcal{B}_1 -valued model of a theory \mathcal{L}_\equiv , then $F \circ \llbracket \cdot \rrbracket$ is a \mathcal{B}_2 -valued model of \mathcal{L}_\equiv .

3 About reducibility candidates and typing

In order to build a sufficient and necessary semantic condition for strongly normalizing theories expressed in minimal deduction modulo, we use the following method: we first define, for each theory \mathcal{L}_\equiv a LDTVA \mathcal{C}_\equiv , wich depends on \equiv such that having a \mathcal{C}_\equiv -valued model is a necessary condition to be strongly normalizing. Then we define another LDTVA \mathcal{C}' , wich does not depend on \equiv anymore and a morphism from each \mathcal{C}_\equiv to \mathcal{C}' , therefore having a \mathcal{C}' -valued model is also a necessary condition to be strongly normalizing. Finally, we prove that this condition is also sufficient.

For the following, we set a theory \mathcal{L}_\equiv . As we said previously, we would like to define a LDTVA such that models valued on this LDTVA capture exactly proofs of the interpreted proposition. For this purpose, we first define \mathcal{C}_\equiv , a LDTVA of \equiv well-typed reducibility candidates, and then build a \mathcal{C}_\equiv -valued model of \mathcal{L}_\equiv .

3.1 \mathcal{C}_\equiv , a ldtva of (\equiv) well-typed reducibility candidates

As a typing judgement $\Gamma \vdash \pi : A$ associates an ordered pair formed by a context Γ and a proof π , to a proposition A , we will consider, for the domain of \mathcal{C}_\equiv , the set of sets of such pairs, and interpret propositions by the pairs they are associated to. We write \mathcal{U} the set of such pairs. We will consider, for the domain of \mathcal{C}_\equiv (wich we will also call \mathcal{C}_\equiv), the set of subsets of \mathcal{U} which verify adapted versions of the usual properties (CR₁) and (CR₂) of reducibility candidates, another property (CR _{\equiv}) and a modified version of (CR₃), which express both well-typing: as we want to capture exactly well-typed proof-terms, we first exclude explicitey proof-terms that are not well-typed, *i.e.* proof-terms that are not proofs of the interpreted proposition.

Definition 8 (\mathcal{U}).

$\mathcal{U} = \{(\Gamma, \pi) \text{ such that } \Gamma \text{ is a context and } \pi \text{ is a proof-term}\}$.

Definition 9.

For all $E \subseteq \mathcal{U}$, we define the following properties :

- (CR $_{\equiv}$) $\exists A_E$ such that $\forall(\Gamma, \pi) \in E, \Gamma \vdash \pi : A_E$
(and all propositions which satisfy this property are \equiv -equivalent).
- (CR $_{1\equiv}$) $\forall(\Gamma, \pi) \in E, \pi \in SN$
- (CR $_{2\equiv}$) $\forall(\Gamma, \pi) \in E, \forall\pi' \in \mathcal{T}$ such that $\pi \rightarrow \pi', (\Gamma, \pi') \in E$
- (CR $_{3\equiv}$) For all $(\Gamma, \pi) \in \mathcal{U}$ such that π is neutral and $\Gamma \vdash \pi : A_E$,
if for all one-step reducts τ of $\pi, (\Gamma, \tau) \in E$, then (Γ, π) is in E .

Remark 3. For all $E \subseteq \mathcal{U}$, if E satisfies (CR $_{\equiv}$) and (CR $_{3\equiv}$), then for all proof-variables $\alpha, (\alpha : A_E, \alpha) \in E$.

Definition 10 (domain \mathcal{C}_{\equiv}). We call \mathcal{C}_{\equiv} the set of subsets of \mathcal{U} which satisfy (CR $_{\equiv}$), (CR $_{1\equiv}$), (CR $_{2\equiv}$) and (CR $_{3\equiv}$).

Then we adapt the usual interpretation of \Rightarrow to elements of \mathcal{U} .

Definition 11 ($\overset{\circ}{\rightarrow}$). For all $E, F \subseteq \mathcal{U}$,
 $E \overset{\circ}{\rightarrow} F = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that for all } (\Gamma', \pi') \in E, (\Gamma\Gamma', \pi\pi') \in F\}$

Remark 4. We recall the fact that we only consider well-formed contexts, therefore the only variables Γ and Γ' can share have to be declared proofs of equivalent propositions, otherwise we have to deal with α -equivalence when concatenating Γ and Γ' .

Lemma 2. $\overset{\circ}{\rightarrow}$ is a function from $\mathcal{C}_{\equiv} \times \mathcal{C}_{\equiv}$ to \mathcal{C}_{\equiv} .

Proof. Let $E, F \in \mathcal{C}_{\equiv}$, and $(\Gamma, \pi) \in E \overset{\circ}{\rightarrow} F$,

(CR $_{\equiv}$) Let α be a proof-variable. As E satisfies (CR $_{\equiv}$) and (CR $_{3\equiv}$), $(\alpha : A_E, \alpha) \in E$, therefore $(\Gamma, \alpha : A_E, \pi\alpha) \in F$. As F satisfies (CR $_{\equiv}$), we have $\Gamma, \alpha : A_E \vdash \pi : A_F$. Therefore $\Gamma \vdash \pi : A_E \Rightarrow A_F$ (by case on the last rule used in the derivation of $\Gamma, \alpha : A_E \vdash \pi : A_F$). Finally, $A_{E \overset{\circ}{\rightarrow} F} \equiv A_E \Rightarrow A_F$.

(CR $_{1\equiv}$) Let α be a proof-variable. As E satisfies (CR $_{\equiv}$) and (CR $_{3\equiv}$), $(\alpha : A_E, \alpha) \in E$, therefore $(\Gamma, \alpha : A_E, \pi\alpha) \in F$. As F satisfies (CR $_{1\equiv}$), we have $\pi\alpha \in SN$, therefore $\pi \in SN$.

(CR $_{2\equiv}$) Let τ such that $\pi \rightarrow \tau$. Then, for all $(\Gamma', \pi') \in E, (\Gamma\Gamma', \pi\pi') \in F$ and $\pi\pi' \rightarrow \tau\pi'$, therefore $(\Gamma\Gamma', \tau\pi') \in F$ as F satisfies (CR $_{2\equiv}$). Finally, $(\Gamma, \tau) \in E \overset{\circ}{\rightarrow} F$.

(CR $_{3\equiv}$) Let $(\Gamma, \mu) \in \mathcal{U}$ such that $\Gamma \vdash \mu : A_{E \overset{\circ}{\rightarrow} F}$ and all its one-step reducts are in $E \overset{\circ}{\rightarrow} F$. Then, $\Gamma \vdash \mu : A_E \Rightarrow A_F$. For all $(\Gamma', \pi') \in E, \mu\pi'$ is neutral and we have $\Gamma\Gamma' \vdash \mu\pi' : F$. Let τ be a one-step reduct of $\mu\pi'$. We prove, by induction on the maximal length of a reduction sequence from $\pi' (\in SN)$, that $(\Gamma\Gamma', \tau) \in F$. As μ is neutral, either $\tau = \mu\pi''$ with $\pi' \rightarrow \pi''$, and we conclude by induction hypothesis. Either $\tau = \mu'\pi'$, and in this case, $(\Gamma\Gamma', \tau) \in F$, by hypothesis on μ . Finally, for all $(\Gamma', \pi') \in E, (\Gamma\Gamma', \mu\pi') \in F$, and finally $(\Gamma, \mu) \in E \overset{\circ}{\rightarrow} F$.

Definition 12 (\mathring{A}_T). For all sorts T ,
 $\mathring{A}_T = \{f : \hat{T} \mapsto \mathcal{C}_\equiv, \text{ such that there exists } A_f \in \mathcal{P} \text{ and } x_f \in \mathcal{X} \text{ such that}$
for all $t \in \hat{T}$ and $(\Gamma, \pi) \in f(t)$, $\Gamma \vdash \pi : (t/x_f)A_f\}$

Remark 5. In other words, $f \in \mathring{A}_T$ iff for all $t \in \hat{T}$, $A_{f(t)} = (t/x_f)A_f$.

Now we can define what we called “dependent intersection” previously.

Definition 13 ($\mathring{\forall}_T$). For all sorts T and functions $f \in \mathring{A}_T$,
 $\mathring{\forall}_T.f = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that for all } t \in \hat{T}, (\Gamma, \pi t) \in f(t)\}$

Lemma 3. For all sorts T , $\mathring{\forall}_T$ is a function from \mathring{A}_T to \mathcal{C}_\equiv .

Proof. Let $f \in \mathring{A}_T$, and $(\Gamma, \pi) \in \mathring{\forall}_T.f$

(CR $_{\equiv}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t)$. As $f \in \mathring{A}_T$, we have $\Gamma \vdash \pi t : (t/x_f)A_f$.
Therefore $\Gamma \vdash \pi : \forall x_f.A_f$, by case on the last rule used in the derivation of
 $\Gamma \vdash \pi t : (t/x_f)A_f$. Finally, $A_{\mathring{\forall}_T.f} \equiv \forall x_f.A_f$.

(CR $_{1\equiv}$) Let $t \in \hat{T}$ ($\neq \emptyset$). Then $(\Gamma, \pi t) \in f(t) \in \mathcal{C}_\equiv$ therefore $\pi t \in SN$ and so
does π .

(CR $_{2\equiv}$) Let π' such that $\pi \rightarrow \pi'$. Then, for all $t \in \hat{T}$, $\pi t \rightarrow \pi' t$, therefore
 $\pi' t \in f(t) \in \mathcal{C}_\equiv$.

(CR $_{3\equiv}$) Let $(\Gamma, \mu) \in \mathcal{U}$ such that μ is neutral, $\Gamma \vdash \mu : \forall x.A_f$, and for
all one-step reducts μ' of μ , $(\Gamma, \mu') \in \mathring{\forall}_T.f$. Let $t \in \hat{T}$, then μt is neutral,
 $\Gamma \vdash \mu t : (t/x)A_f$, and, as μ is neutral, all one-step reducts of μt are of the
form $\mu' t$, with $\mu \rightarrow \mu'$, hence $(\Gamma, \mu t) \in f(t)$ as $f(t)$ satisfies (CR $_{3\equiv}$). Finally,
 $(\Gamma, \mu) \in \mathring{\forall}_T.f$.

Definition 14 (\mathcal{C}_\equiv). \mathcal{C}_\equiv is the LDTVVA $\langle \mathcal{C}_\equiv, \overset{\circ}{\rightarrow}, (\mathring{A}_T), (\mathring{\forall}_T) \rangle$.

3.2 Building a \mathcal{C}_\equiv -valued interpretation

Let us now define a first \mathcal{C}_\equiv -valued model, by using directly definitions of $\overset{\circ}{\rightarrow}$ and $\mathring{\forall}_T$,
and well-chosen interpretations of atomic propositions.

Definition 15. Let A be a proposition and $\varphi \in \mathcal{V}(A)$.

We define the subset of \mathcal{U} , $[A]_\varphi$ by induction over the structure of A .

- $[P t_1 \dots t_n]_\varphi = \{(\Gamma, \pi) \in \mathcal{U} \text{ such that } \pi \in SN \text{ and } \Gamma \vdash \pi : P \varphi(t_1) \dots \varphi(t_n)\}$
- $[B \Rightarrow C]_\varphi = [B]_\varphi \overset{\circ}{\rightarrow} [C]_\varphi$
- $[\forall x.B]_\varphi = \mathring{\forall}_T (t \mapsto [B]_{\varphi+(x,t)})$

Lemma 4. For all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \mathcal{V}(\forall x.A)$ such that $x \notin \varphi$,
we have $[(t/x)A]_\varphi = [A]_{\varphi+(x,t)}$.

Proof. By induction on A .

Lemma 5. For all $A \in \mathcal{P}$, and $\varphi \in \mathcal{V}(A)$,

$[A]_\varphi \in \mathcal{C}_\equiv$ with $A_{[A]_\varphi} = |A|_\varphi$ (i.e., $\forall (\Gamma, \pi) \in [A]_\varphi$, $\Gamma \vdash \pi : |A|_\varphi$).

Proof. By induction on A .

- If A is an atomic proposition $P t_1 \dots t_n$,
 - (CR $_{\equiv}$) By definition. (with $A_{[P t_1 \dots t_n]_\varphi} \equiv P \varphi(t_1) \dots \varphi(t_n)$).
 - (CR $_{1\equiv}$) By definition.
 - (CR $_{2\equiv}$) By subject-reduction and the fact that each reduct of a proof-term in SN is also in SN .
 - (CR $_{3\equiv}$) If all one-step reducts of a proof-term are in SN , then it is also in SN .
- If $A = B \Rightarrow C$, as $\overset{\circ}{\rightarrow} : \mathcal{C}_{\equiv} \times \mathcal{C}_{\equiv} \mapsto \mathcal{C}_{\equiv}$, we conclude by induction hypothesis (with $A_{[B \Rightarrow C]_\varphi} = A_{[B]_\varphi \overset{\circ}{\rightarrow} [C]_\varphi} \equiv A_{[B]_\varphi} \Rightarrow A_{[C]_\varphi} \equiv |B|_\varphi \Rightarrow |C|_\varphi = |B \Rightarrow C|_\varphi$).
- If $A = \forall x.B$, let T be the sort of x and $f = t \mapsto [B]_{\varphi+\langle x,t \rangle}$. Then f is a function from \hat{T} to \mathcal{C}_{\equiv} , by induction hypothesis and the fact that t is closed. Moreover, for all $t \in \hat{T}$, $A_{f(t)} = |B|_{\varphi+\langle x,t \rangle} = (t/x)|B|_\varphi$, by induction hypothesis. Therefore $f \in \mathring{A}_T$ and $\forall_T f \in \mathcal{C}_{\equiv}$ (with $A_{[\forall x.B]_\varphi} = \forall x.A_f = \forall x.|B|_\varphi$).

At this point, we have \mathcal{C}_{\equiv} -valued model which is adapted to typing but not necessarily \equiv -adapted. Indeed, in a theory where we have two atomic proposition symbols P and Q such that $P \equiv (Q \Rightarrow Q)$ (notice that such a theory can be strongly normalizing), then for all valuations $\varphi \in \mathcal{V}(P) \cap \mathcal{V}(Q)$, $[P]_\varphi \neq [Q]_\varphi \overset{\circ}{\rightarrow} [Q]_\varphi$. We have then to modify this interpretation to make it a \mathcal{C}_{\equiv} -valued model of \mathcal{L}_{\equiv} .

3.3 Adapting this interpretation to the congruence

Definition 16. We define a second interpretation $[\cdot]_\cdot$, as follows : for all $A \in \mathcal{P}$ and $\varphi \in \mathcal{V}(A)$,

$$[A]_\varphi = \bigcap_{|A|_\varphi \equiv |A'|_\psi} [A']_\psi$$

Remark 6. For all $A, A' \in \mathcal{P}$, $\varphi \in \mathcal{V}(A)$ and $\psi \in \mathcal{V}(A')$ such that $|A|_\varphi \equiv |A'|_\psi$, we have $[A]_\varphi = [A']_\psi$, by definition.

Then we prove that $[\cdot]_\cdot$ is also a \mathcal{C}_{\equiv} -valued interpretation adapted to typing.

Lemma 6. For all $A \in \mathcal{P}$, and $\varphi \in \mathcal{V}(A)$,

$$[A]_\varphi \in \mathcal{C}_{\equiv} \text{ with } A_{[A]_\varphi} = |A|_\varphi \text{ (i.e., } \forall (\Gamma, \pi) \in [A]_\varphi, \Gamma \vdash \pi : |A|_\varphi).$$

Proof. Let $A \in \mathcal{P}$, and $\varphi \in \mathcal{V}(A)$,

By lemma 5 and the fact that $[A]_\varphi \subseteq [A']_\psi$, for all $|A'|_\psi \equiv |A|_\varphi$.

Lemma 7. For all $A \in \mathcal{P}$, x of sort T , $t \in \hat{T}$ and $\varphi \in \mathcal{V}(\forall x.A)$ such that $x \notin \varphi$, we have $[(t/x)A]_\varphi = [A]_{\varphi+\langle x,t \rangle}$.

Proof. By lemma 7.

Finally, we proved, that $[\cdot]_\cdot$ is a \mathcal{C}_{\equiv} -valued interpretation of propositions adapted to typing and to the congruence relation \equiv . Let us now show that if the theory \mathcal{L}_{\equiv} is strongly normalizing, then $[\cdot]_\cdot$ is a \mathcal{C}_{\equiv} -valued model of \mathcal{L}_{\equiv} , i.e. it is also adapted to connectives.

3.4 $[\cdot]$ is a \mathcal{C}_{\equiv} -valued model of strongly normalizing theories \mathcal{L}_{\equiv}

In order to prove that $[\cdot]$ is a \mathcal{C}_{\equiv} -valued model of \mathcal{L}_{\equiv} , if it is strongly normalizing, we proceed by contraposition, showing that if $[\cdot]$ is not connectives-adapted, then we can exhibit a typing judgement $\Gamma \vdash \pi : A$ such that $\pi \notin SN$.

Lemma 8.

If there exists $A, B \in \mathcal{P}$ and $\varphi \in \mathcal{V}(A \Rightarrow B)$, such that $[A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\rightarrow} [B]_{\varphi}$ then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \mathcal{V}(C)$ such that $\Gamma \vdash \pi : |C|_{\psi}$ and $(\Gamma, \pi) \notin [C]_{\psi}$.

Proof. – If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin [A \Rightarrow B]_{\varphi}$ and $(\Gamma, \pi) \in [A]_{\varphi} \overset{\circ}{\rightarrow} [B]_{\varphi}$. Then $\Gamma \vdash \pi |A|_{\varphi} \Rightarrow |B|_{\varphi} = |A \Rightarrow B|_{\varphi}$.

We take $C = A \Rightarrow B$ and $\psi = \varphi$.

– If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in [A \Rightarrow B]_{\varphi}$ and $(\Gamma, \pi) \notin [A]_{\varphi} \overset{\circ}{\rightarrow} [B]_{\varphi}$.

Then there exists $(\Gamma', \pi') \in [A]_{\varphi}$ such that $(\Gamma\Gamma', \pi\pi') \notin [B]_{\varphi}$. As $(\Gamma, \pi) \in [A \Rightarrow B]_{\varphi}$, and $(\Gamma', \pi') \in [A]_{\varphi}$, we have $\Gamma \vdash \pi : |A \Rightarrow B|_{\varphi} = |A|_{\varphi} \Rightarrow |B|_{\varphi}$ and $\Gamma' \vdash \pi' : |A|_{\varphi}$. Therefore $\Gamma\Gamma' \vdash \pi\pi' : |B|_{\varphi}$.

We take $C = A \Rightarrow B$ and $\psi = \varphi$.

Lemma 9.

If there exists $A \in \mathcal{P}$, $\varphi \in \mathcal{V}(A)$, and x of sort T such that $x \notin \varphi$, and

$$[\forall x.A]_{\varphi} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi + \langle x, t \rangle})$$

then there exists $\pi \in \mathcal{T}$, $C \in \mathcal{P}$, $\psi \in \mathcal{V}(C)$ such that $\Gamma \vdash \pi : |C|_{\psi}$ and $(\Gamma, \pi) \notin [C]_{\psi}$.

Proof. – If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \notin [\forall x.A]_{\varphi}$ and

$$(\Gamma, \pi) \in \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi + \langle x, t \rangle}).$$

Then $\Gamma \vdash \pi : |\forall x.A|_{\varphi}$.

We take $C = \forall x.A$ and $\psi = \varphi$.

– If there exists $(\Gamma, \pi) \in \mathcal{U}$ such that $(\Gamma, \pi) \in [\forall x.A]_{\varphi}$ and $(\Gamma, \pi) \notin \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi + \langle x, t \rangle})$.

Then there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [A]_{\varphi + \langle x, t \rangle}$. As $(\Gamma, \pi) \in [\forall x.A]_{\varphi}$, we have $\Gamma \vdash \pi : |\forall x.A|_{\varphi}$, therefore $\Gamma \vdash \pi t : (t/x)|A|_{\varphi} = |A|_{\varphi + \langle x, t \rangle}$. We take $C = A$ and $\psi = \varphi + \langle x, t \rangle$.

Lemma 10.

If there exists $A, B \in \mathcal{P}$, $\varphi \in \mathcal{V}(A \Rightarrow B)$ or $\varphi' \in \mathcal{V}(\forall x.A)$ with x of sort T , $x \notin \varphi'$ and

$$[A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi' + \langle x, t \rangle})$$

then there exists $D \in \mathcal{P}$, $\pi \in \mathcal{T}$, $\psi \in \mathcal{V}(D)$ such that $\Gamma \vdash \pi : |D|_{\psi}$ and $(\Gamma, \pi) \notin [D]_{\psi}$.

Proof. By lemmas 8 and 9, there exists C , Γ , π and ψ such that $\Gamma \vdash \pi : |C|_{\psi}$ and $(\Gamma, \pi) \notin [C]_{\psi}$. Therefore, there exists a proposition D and $\psi' \in \mathcal{V}(D)$ such that $|D|_{\psi'} \equiv |C|_{\psi}$ and $(\Gamma, \pi) \notin [D]_{\psi'}$. And $\Gamma \vdash \pi : |D|_{\psi'}$, by equivalence of $|C|_{\psi}$ and $|D|_{\psi'}$.

Lemma 11.

If there exists $A, B \in \mathcal{P}$, $\varphi \in \mathcal{V}(A \Rightarrow B)$ or $\varphi' \in \mathcal{V}(\forall x.A)$ with x of sort T , $x \notin \varphi'$

$$\text{and} \quad [A \Rightarrow B]_{\varphi} \neq [A]_{\varphi} \overset{\circ}{\rightarrow} [B]_{\varphi} \quad \text{or} \quad [\forall x.A]_{\varphi'} \neq \overset{\circ}{\forall}_T(t \mapsto [A]_{\varphi' + \langle x, t \rangle})$$

then there exists a (term-closed) proposition E , $\pi \in \mathcal{T}$ and a context Γ such that

$$\Gamma \vdash \pi : E \text{ and } \pi \notin SN.$$

Proof. By lemma 10, there exists a proposition D , a context Γ , a proof π and $\varphi \in \mathcal{V}(D)$ such that $\Gamma \vdash \pi : |D|_\varphi$ and $(\Gamma, \pi) \notin [D]_\varphi$. By induction on D .

- if D is atomic, then as $\Gamma \vdash \pi : |D|_\varphi$, we have $\pi \notin SN$.
- if $D = |F \Rightarrow G|_\varphi$,
then $\Gamma \vdash \pi : |F \Rightarrow G|_\varphi$ and $(\Gamma, \pi) \notin [F \Rightarrow G]_\varphi = [F]_\varphi \overset{\circ}{\rightarrow} [G]_\varphi$. Then there exists $(\Gamma', \pi') \in [F]_\varphi$ such that $(\Gamma\Gamma', \pi\pi') \notin [G]_\varphi$. As $(\Gamma, \pi) \in [F \Rightarrow G]_\varphi$, and $(\Gamma', \pi') \in [F]_\varphi$, we have $\Gamma \vdash \pi : |F \Rightarrow G|_\varphi = |F|_\varphi \Rightarrow |G|_\varphi$ and $\Gamma' \vdash \pi' : |F|_\varphi$. Therefore $\Gamma\Gamma' \vdash \pi\pi' : |G|_\varphi$. We conclude by induction hypothesis.
- if $D = \forall x.F$,
then $\Gamma \vdash \pi : |\forall x.F|_\varphi$ and $(\Gamma, \pi) \notin [\forall x.F]_\varphi$. Then there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \notin [F]_{\varphi+\langle x, t \rangle}$. As $(\Gamma, \pi) \in [\forall x.F]_\varphi$, we have $\Gamma \vdash \pi : |\forall x.F|_\varphi$, therefore $\Gamma \vdash \pi t : (t/x)|F|_\varphi = |F|_{\varphi+\langle x, t \rangle}$. We conclude by induction hypothesis.

Proposition 2 (Completeness). *If the theory \mathcal{L}_\equiv is strongly normalizing, then $[\cdot]_\cdot = \langle A, \varphi \rangle \mapsto [A]_\varphi$ is a \mathcal{C}_\equiv -model of this theory.*

Proof. By remark 6 and lemmas 6 and 11.

3.5 The sheep property

We finally prove one more property concerning $[\cdot]_\cdot$, about well-typed substitution. A property we will need in section 4.

Definition 17 (sheep property).

- For all $F \in \mathcal{C}_\equiv$, we say that F satisfies $\overset{\circ}{\rightarrow}$ -sheep property iff for all $E \in \mathcal{C}_\equiv$, $(\Gamma, \pi) \in \mathcal{U}$, $\alpha \in \mathcal{X}$, if $(\Gamma, \alpha) \in E$ and $(\Gamma, \pi\alpha) \in F$, then $(\Gamma, \pi) \in E \overset{\circ}{\rightarrow} F$.
- For all x of sort T and $f \in \mathring{A}_T$, we say that f satisfies $\overset{\circ}{\forall}$ -sheep property iff as soon as there exists $t \in \hat{T}$ such that $(\Gamma, \pi t) \in f(t)$, we have $(\Gamma, \pi) \in \forall_T f$.
- For all LDTVAs \mathcal{B} , we say that a \mathcal{B} -valued interpretation $\llbracket \cdot \rrbracket_\cdot$ of \mathcal{L}_\equiv is sheep-adapted iff for all $A \in \mathcal{P}$ and $\varphi \in \mathcal{V}(A)$, $\llbracket A \rrbracket_\varphi$ satisfies $\overset{\circ}{\rightarrow}$ -sheep property; and for all x of sort T , $A \in \mathcal{P}$, and $\varphi \in \mathcal{V}(\forall x.A)$ such that $x \notin \varphi$, the function $t \mapsto \llbracket A \rrbracket_{\varphi+\langle x, t \rangle}$ satisfies $\overset{\circ}{\forall}$ -sheep property.

Then we use the following lemmas to prove that $[\cdot]_\cdot$ is sheep-adapted.

Lemma 12. *For all sorts T , x of sort T , $t, t' \in \hat{T}$, $\pi \in \mathcal{T}$, $A \in \mathcal{P}$, $\varphi \in \mathcal{V}(\forall x.A)$, if $(\Gamma, (t/x)\pi) \in [(t/x)A]_\varphi$ (resp. $\llbracket (t/x)A \rrbracket_\varphi$), then $(\Gamma, (t'/x)\pi) \in [(t'/x)A]_\varphi$ (resp. $\llbracket (t'/x)A \rrbracket_\varphi$).*

Proof. Notice that if the $[\cdot]$ version of this lemma is true, then also is the $\llbracket \cdot \rrbracket_\cdot$ version. Let us prove the $[\cdot]$ version by induction on $(t/x)A$. We just have to notice that if $(t/x)A$ is atomic then so does $(t'/x)A$, that if $(t/x)\pi \in SN$ then so does $(t'/x)\pi$, and that if $\Gamma \vdash (t/x)\pi : |(t/x)A|_\varphi$, then $\Gamma \vdash (t'/x)\pi : |(t'/x)A|_\varphi$. Induction cases are obvious.

Lemma 13. *If \mathcal{L}_{\equiv} is strongly normalising, then for all $E \in \mathcal{C}_{\equiv}$, $\alpha \in \mathcal{X}$, $\pi, \pi' \in \mathcal{T}$, $A \in \mathcal{P}$, $\varphi \in \mathcal{V}(A)$, and Γ, Γ' contexts such that $(\Gamma, \alpha) \in E$, $(\Gamma', \pi') \in E$ and $(\Gamma, \pi) \in [A]_{\varphi}$ (resp. $[A]_{\varphi}$) then $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [A]_{\varphi}$ (resp. $[A]_{\varphi}$).*

Proof. Notice that if the $[\cdot]$ version of this lemma is true, then also is the $[\cdot]$ version. Let us prove the $[\cdot]$ version by induction on A .

- If A is atomic, we have $\Gamma \vdash \pi : |A|_{\varphi}$, $\Gamma \vdash \alpha : A_E$ and $\Gamma' \vdash \pi' : A_E$, therefore, $\Gamma\Gamma' \vdash (\pi'/\alpha)\pi : |A|_{\varphi}$. Moreover \mathcal{L}_{\equiv} is strongly normalizing, then $(\pi'/\alpha)\pi \in SN$ and $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [A]_{\varphi}$.
- If $A = C \Rightarrow D$, then $(\Gamma, \pi) \in [C]_{\varphi} \overset{\circ}{\rightarrow} [D]_{\varphi}$. Let $(\Gamma'', \tau) \in [C]_{\varphi}$ such that u doesn't contain α (by α -equivalence). Then $(\Gamma\Gamma'', \pi\tau) \in [D]_{\varphi}$ therefore $(\Gamma\Gamma'\Gamma'', (\pi'/\alpha)(\pi\tau)) = (\Gamma\Gamma'\Gamma'', (\pi'/\alpha)\pi\tau) \in [D]_{\varphi}$, by induction hypothesis. Finally, $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [C]_{\varphi} \overset{\circ}{\rightarrow} [D]_{\varphi} = [A]_{\varphi}$.
- If $A = \forall x.C$, then $(\Gamma, \pi) \in \overset{\circ}{\forall}_T(t \mapsto [C]_{\varphi+\langle x, t \rangle})$. Let t be term of same sort T as x , then $(\Gamma, \pi t) \in [C]_{\varphi+\langle x, t \rangle}$. Therefore $(\Gamma\Gamma', (\pi'/\alpha)(\pi t)) = (\Gamma\Gamma', (\pi'/\alpha)\pi t) \in [C]_{\varphi+\langle x, t \rangle}$, by induction hypothesis. Finally $(\Gamma\Gamma', (\pi'/\alpha)\pi) \in [A]_{\varphi}$.

Proposition 3. *If \mathcal{L}_{\equiv} is strongly normalising, then $[\cdot]$ is sheep-adapted.*

Proof. By lemmas 12 and 13.

4 From \mathcal{C}_{\equiv} to \mathcal{C}'

We have introduced, for each theory \mathcal{L}_{\equiv} a LDTVA \mathcal{C}_{\equiv} such that having a \mathcal{C}_{\equiv} -valued model is a complete semantic condition for strong normalization property. But each LDTVA \mathcal{C}_{\equiv} depends on the congruence relation \equiv , while we want to define a sufficient and necessary semantics for *all* strongly normalizing theories expressed in minimal deduction modulo.

Let us now introduce \mathcal{C}' , an algebra which does not depend anymore on \equiv , and a morphism of algebras from each \mathcal{C}_{\equiv} of strongly normalizing \equiv to \mathcal{C}' , in order to prove that having a \mathcal{C}' -valued model is also a complete semantic condition for strong normalization property. Actually, we build a more general morphism: from each sheep-adapted sub-LDTVA of each \mathcal{C}_{\equiv} to \mathcal{C}' .

We will consider for the domain of \mathcal{C}' (wich we will also call \mathcal{C}'), the subsets of \mathcal{T} which verify the usual properties (CR_1) , (CR_2) of reducibility candidates, and a modified version of (CR_3) (which gives a solution to avoid not well-typed proof-terms without talking about typing).

4.1 \mathcal{C}' , yet another algebra of candidates.

Definition 18.

For all sets E of proof-terms, we define the following properties :

$$(CR_1) \quad \text{For all } \pi \in E, \pi \in SN.$$

(CR₂) For all $\pi \in E$, for all $\pi' \in \mathcal{T}$ such that $\pi \rightarrow \pi'$, then $\pi' \in E$.

(CR'₃) for all $n \in \mathbb{N}$, for all $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, if

- for all $i \leq n$, μ_i is neutral and not normal,

- $\forall \rho_1, \dots, \rho_n \in \mathcal{T}$ such that for all $i \leq n$, $\mu_i \rightarrow \rho_i$, $(\rho_i/\alpha_i)_i \nu \in E$

then $(\mu_i/\alpha_i)_i \nu \in E$.

Remark 7. If E satisfies (CR'₃) then, in particular, all neutral *not normal* terms whose all one-steps reducts are in E , is in E . That is slightly different from the usual (CR₃) of reducibility candidates, where the neutral term can be normal, therefore all neutral normal terms are in all reducibility candidates. We will see that (CR'₃) is still a sufficient property to prove soundness in section 5. And we conjecture that the belonging of these normal neutral terms to all reducibility candidates is the fact which prevents from proving that having a model valued on usual reducibility candidates is a complete condition of strong normalization property.

Definition 19 (\rightsquigarrow).

For all $E, F \subseteq \mathcal{T}$, $E \rightsquigarrow F = \{\pi \in SN \text{ such that for all } \pi' \in E, \pi\pi' \in F\}$.

Lemma 14. \rightsquigarrow is a function from $\mathcal{C}' \times \mathcal{C}'$ to \mathcal{C}' .

Proof. Let $E, F \in \mathcal{C}'$ and $\pi \in E \rightsquigarrow F$,

(CR₁) $\pi \in SN$, by definition.

(CR₂) If ρ is a one-step reduct of π , then for all $\pi' \in E$, $\rho\pi'$ is a one-step reduct of $\pi\pi'$.

(CR'₃) If there exists $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_i \nu$ and for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \rightarrow \rho_i$, then $(\rho_i/\alpha_i)_i \nu \in E \rightsquigarrow F$. Then, for all $\pi' \in E$, $\tau\pi' = (\mu_i/\alpha_i)_i \nu\pi' = (\mu_i/\alpha_i)_i \nu'$ with $\nu' = \nu\pi'$. And for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \rightarrow \rho_i$, we have $(\rho_i/\alpha_i)_i \nu' = (\rho_i/\alpha_i)_i \nu \pi' \in F$ by hypothesis, therefore $\tau\pi' \in F$ as it satisfies (CR'₃). And finally, $\tau \in E \rightsquigarrow F$.

Definition 20 ($\tilde{\mathcal{A}}_T$).

For all sorts T , $\tilde{\mathcal{A}}_T = \hat{T} \mapsto \mathcal{C}'$.

Definition 21 ($\tilde{\forall}_T$). For all sorts T and function $f \in \tilde{\mathcal{A}}_T$,

$\tilde{\forall}_T.f = \{\pi \in \mathcal{T} \text{ such that for all } t \in \hat{T}, \pi t \in f(t)\}$

Lemma 15. For all sorts T , $\tilde{\forall}_T$ is a function from $\tilde{\mathcal{A}}_T$ to \mathcal{C}' .

Proof. Let T be a sort, $f \in \tilde{\mathcal{A}}_T$ and $\pi \in \tilde{\forall}_T.f$.

(CR₁) Let $t \in \hat{T}$ ($\neq \emptyset$), then $\pi t \in f(t) \in \mathcal{C}'$, therefore $\pi t \in SN$ and so does π .

(CR₂) Let π' such that $\pi \rightarrow \pi'$. Then for all $t \in \hat{T}$, $\pi't$ is a one-step reduct of πt .

(CR₃') If there exists $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, such that each μ_i is neutral not normal, $\tau = (\mu_i/\alpha_i)_i \nu$ and for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \rightarrow \rho_i$, then $(\rho_i/\alpha_i)_i \nu \in \forall_T.f$. Then, for all $t \in \hat{T}$, $\tau t = (\mu_i/\alpha_i)_i \nu t = (\mu_i/\alpha_i)_i \nu'$ with $\nu' = \nu t$. And for all $(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\mu_i \rightarrow \rho_i$, we have $(\rho_i/\alpha_i)_i \nu' = (\rho_i/\alpha_i)_i \nu t \in \underline{f}(t)$ by hypothesis, therefore $\tau t \in f(t)$ as it satisfies (CR₃'). And finally, $\tau \in \forall_T.f$.

Definition 22 (\mathcal{C}'). \mathcal{C}' is the LDTVA $\langle \mathcal{C}', \dot{\rightarrow}, (\tilde{A}_T), (\tilde{\forall}_T) \rangle$.

4.2 Building a function from \mathcal{C}_{\equiv} to \mathcal{C}'

We build a function $Cl(\cdot)$ from \mathcal{C}_{\equiv} to \mathcal{C}' , mapping \mathcal{C}_{\equiv} -valued sheep-adapted models to \mathcal{C}' -valued models of the theory \mathcal{L}_{\equiv} . We first set a (big enough) context Δ , in order to make elements of the image of an element of \mathcal{C}_{\equiv} under $Cl(\cdot)$ be well-typed (by a same proposition), in the same context. Then, we extend these image sets to make them satisfy (CR₃'). Finally, given an image set, there exists a proposition such that for each reduction sequence from every element of this set, there exists a step of the reduction such that the reduct is typable by the proposition in Δ .

Definition 23 (Δ). We consider a context which contains an infinite number of variables for each proposition. $\Delta = (\beta_i^A : A)_{A \in \mathcal{P}, i \in \mathbb{N}}$.

Definition 24 (leaves).

The leaves of a proof-term π are its first reducts which are normal or not neutral. (ρ is a leaf of π if and only if it is normal or not neutral and there exists $n \geq 0$ and $\pi_1 \dots \pi_{n-1}$ normal not neutral terms such that $\pi = \pi_1 \rightarrow \dots \rightarrow \pi_{n-1} \rightarrow \rho$). We call $\mathcal{L}(\pi)$ the set of leaves of π .

Remark 8. The only leaf of a normal or not neutral proof-term is itself.

If π is a neutral non-normal proof-term, then $\rho \in \mathcal{L}(\pi)$ if and only if there exists a one-step reduct π' of π such that $\rho \in \mathcal{L}(\pi')$.

Definition 25 (Cl). For all $E \subseteq \mathcal{U}$, we define $Cl(E)$ as follows :
for all $k \in \mathbb{N}$,

- $Cl^0(E) = \{\pi \in \mathcal{T} \text{ such that } (\Delta, \pi) \in E\}$
- $Cl^{k+1}(E) = Cl^k(E) \cup \{\pi \in \mathcal{T}, \text{ such that:}$
 $\exists \nu_\pi \in \mathcal{T}, \exists (\mu_i)_i \subseteq SN, \text{ each neutral not normal s.t.}$
 $\pi = (\mu_i/\alpha_i)_i \nu_\pi \text{ and } \forall (\rho_i)_i \subseteq \mathcal{T}, \text{ s.t. } \forall i \leq n, \rho_i \in \mathcal{L}(\mu_i),$
 $\text{we have } (\rho_i/\alpha_i)_i \nu_\pi \in Cl^k(E)\}$
- $Cl(E) = \cup_{n \in \mathbb{N}} Cl^n(E)$

Let us now prove that $Cl(\cdot)$ maps each element of \mathcal{C}_{\equiv} to an element of \mathcal{C}' .

Lemma 16.

For all $\pi \in SN$, if π is not isolated then there exists an abstraction τ such that for all abstractions τ' such that $\pi \rightarrow^* \tau'$, we have $\tau \rightarrow^* \tau'$.

We call τ the primary leaf of π .

Proof. If π is not isolated, then all reductions sequences from π reach a not neutral proof-term, by confluency. Notice that the not-neutral reducts of π are either all proof-abstractions, either all term-abstractions, by confluency. In particular, the head-reduction sequence from π reach a not-neutral proof-term. Let τ be the first non-neutral proof-term reached in this reductions sequence. If π is not neutral, then $\pi = \tau$, therefore all non-neutral reducts of π are obviously reducts of $\pi = \tau$. Otherwise, π is neutral and has the form $\rho \theta_1 \dots \theta_n$, with each θ_i a term or a proof-term, $n > 0$ (as π is neutral), and ρ is not neutral (as π is not isolated). We suppose, in the following that ρ is a proof-abstraction $\lambda\alpha.\rho'$, then θ_1 is a proof-term (the proof is the same in the case of a term-abstraction). Let us prove by induction on the maximal length of a reductions sequence from π ($\in SN$), that each not neutral reduct of π is a reduct of τ . If this maximal length is equal to zero, then π cannot be neutral. Otherwise, let $\pi', \pi'' \in \mathcal{T}$ such that π'' is not neutral and $\pi = (\lambda\alpha.\rho') \theta_1 \dots \theta_n \rightarrow \pi' \rightarrow^* \pi''$.

- If $\pi' = (\theta_1/\alpha)\rho' \theta_2 \dots \theta_n$, then π' is the head-one-step-reduct of π , therefore π'' is a reduct of τ , by induction hypothesis.
- If $\pi' = (\lambda\alpha.\rho'') \theta_1 \dots \theta_n$, with $\rho' \rightarrow \rho''$, let τ' be the first not neutral term reached in the head-reduction of $(\theta_1/\alpha)\rho'' \theta_2 \dots \theta_n$ (then τ' is also the first not neutral term reached in the head-reduction of π'). By induction hypothesis, τ' is a reduct of τ . Moreover, π'' is a reduct of τ' by induction hypothesis (as τ' is the first not neutral head-reduct of $(\theta_1/\alpha)\rho' \theta_2 \dots \theta_n$). Finally, π'' is a reduct of τ .
- If $\pi' = (\lambda\alpha.\rho') \theta_1 \dots \theta_{i-1} \theta'_i \theta_{i+1} \dots \theta_n$, we use the same sort of argument than in the previous point.

Definition 26 ($\pi \parallel \alpha$). For all $\alpha \in \mathcal{X}$ and $\pi \in \mathcal{T}$, we write $\pi \parallel \alpha$ the number of occurrences of α in π .

Definition 27 (\mathcal{K}).

$\mathcal{K} = \{ \langle \nu, n, (\mu_1, \dots, \mu_n) \rangle \text{ such that}$
 $\cdot n \in \mathbb{N}, \nu \in \mathcal{T}, \mu_1 \dots \mu_n \in SN$
 $\cdot \text{for all } i \leq n, \nu \parallel \alpha_i \leq 1$
 $\cdot \text{for all } (\rho_i)_i \text{ each respectively in } \mathcal{L}(\mu_i), (\rho_i/\alpha_i)_i \nu \in SN \}$

Definition 28 (\rightarrow).

Let $\delta = \langle \nu, n, (\mu_1, \dots, \mu_n) \rangle$ and $\delta' = \langle \nu', n', (\mu'_1, \dots, \mu'_n) \rangle$ in \mathcal{K} .

We say that $\delta \rightarrow \delta'$ if and only if:

- (a) $\nu = \nu'$ and there exists $i_0 \leq n$ such that for all $i \neq i_0, \mu_i = \mu'_i, \mu_{i_0}$ is neutral and $\mu_{i_0} \rightarrow \mu'_{i_0}$ or
- (b) there exists $i_0 \leq n$ such that μ_{i_0} is not neutral, $\nu' = (\mu_{i_0}/\alpha_{i_0})\nu, n' = n - 1$ and $\mu'_1 \dots \mu'_{n'} = \mu_1 \dots \mu_{i_0-1} \mu_{i_0+1} \dots \mu_n$ or
- (c) $\nu \rightarrow \nu'$ and the μ'_i are copies of the μ_i resulting of the linearization of the occurrences of the variables α_i in ν' (with ...).

Definition 29 ($\ell(\pi)$). For all $\pi \in SN$, we define $\ell(\pi)$ as follows: if π is isolated, $\ell(\pi) = \alpha_0$ (a special variable), otherwise, $\ell(\pi)$ is the primary leaf of π .

Definition 30 ($\|\delta\|$). For all $\delta = \langle \nu, n, (\mu_1, \dots, \mu_n) \rangle \in \mathcal{K}$, $\|\delta\| = (\ell(\mu_i)/\alpha_i)_i \nu$.

Remark 9. For all $\delta \in \mathcal{K}$, $\|\delta\| \in SN$, by definition.

Lemma 17. For all $\delta, \delta' \in \mathcal{K}$, if $\delta \rightarrow \delta'$ then $\|\delta\| \rightarrow^* \|\delta'\|$.

Proof. By case on $\delta \rightarrow \delta'$.

- (a) If the μ_{i_0} which is reduced (on μ'_{i_0}) is isolated then $\ell(\mu_{i_0}) = \ell(\mu'_{i_0}) = \alpha_0$, therefore $\|\delta\| = \|\delta'\|$. Otherwise $\ell(\mu_{i_0}) \rightarrow^* \ell(\mu'_{i_0})$, by lemma 16, therefore $\|\delta\| \rightarrow^* \|\delta'\|$.
- (b) In this case, $\|\delta\| = \|\delta'\|$.
- (c) In this case, $\|\delta\| \rightarrow \|\delta'\|$ (as $\nu \rightarrow \nu'$).

Lemma 18. All \rightarrow -reductions sequences from an element δ of \mathcal{K} are finite.

Proof. As all the μ_i are in SN (and n is finite), there can only be a finite number of consecutive (a) and (b) reductions. As $\|\delta\| \in SN$, there can only be a finite number of (c) reductions from δ , by lemma 17. Hence there cannot be an infinite \rightarrow -reductions sequence from δ .

Proposition 4.

For all $E \in \mathcal{C}_{\equiv}$, $Cl(E) \in \mathcal{C}'$.

Proof. Let $E \in \mathcal{C}_{\equiv}$.

(CR₂) Let $\pi \in Cl(E)$ and $\pi' \in \mathcal{T}$ such that $\pi \rightarrow \pi'$. Then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(E)$. By induction on k .

- If $k = 0$, then $(\Delta, \pi) \in E$, therefore $(\Delta, \pi') \in E$ as it satisfies (CR_{2 \equiv}).
- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i \nu$ with each μ_i in SN , neutral and not normal, and such that $\forall i \leq n$, and for all $(\rho_i)_i$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1}(E)$. We suppose that for all $i \leq n$, $\nu \parallel \alpha_i \leq 1$.

As each μ_i is neutral:

- . Either $\pi' = (\mu'_{i_0}/\alpha_{i_0})(\mu_i/\alpha_i)_{i \neq i_0} \nu$, with $\mu_{i_0} \rightarrow \mu'_{i_0}$. In this case,
 - . if $\mu'_{i_0} \in \mathcal{L}(\mu_{i_0})$ then $\pi' = (\mu_i/\alpha_i)_{i \neq i_0} \nu''$, with $\nu'' = (\mu'_{i_0}/\alpha_{i_0}) \nu$, and for all $(\rho_i)_i$ such that $\forall i \neq i_0$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i \nu'' \in Cl^{k-1}(E)$, hence $\pi' \in Cl^k(E)$.
 - . Otherwise, μ'_{i_0} is neutral, not normal and all its leaves are leaves of μ_{i_0} , hence $\pi' \in Cl^k(E)$.
- . Either $\pi' = (\mu_i/\alpha_i)_i \nu'$ with $\nu \rightarrow \nu'$. In this case, we conclude by the fact that $Cl^{k-1}(E)$ satisfies (CR₂) by induction hypothesis.

(CR₁) Let $\pi \in Cl(E)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(E)$. By induction on k .

- If $k = 0$, then $(\Delta, \pi) \in E$, therefore $\pi \in SN$ as E satisfies (CR_{1 \equiv}).
- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i \nu$ with each μ_i in SN , neutral and not normal, and such that $\forall i \leq n$, and $\forall (\rho_i)_i$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1}(E) \subseteq SN$, by induction hypothesis. Then, if we suppose that for all $i \leq n$, $\nu \parallel \alpha_i \leq 1$, if $\pi \rightarrow \pi'$, and if we write $\delta_\pi = \langle \nu, n, (\mu_1, \dots, \mu_n) \rangle$ (and the same for $\delta_{\pi'}$, as $\pi' \in Cl^k(E)$ as explained in the previous point), then $\delta_\pi \rightarrow \delta_{\pi'}$. Hence all reductions sequences from π are finite, by lemma 18.

(CR₃') Let $n \in \mathbb{N}$, $\nu, \mu_1, \dots, \mu_n \in \mathcal{T}$, such that for all $i \leq n$, μ_i is neutral and not normal, and for all $\mu'_1, \dots, \mu'_n \in \mathcal{T}$ such that $\forall i \leq n$, $\mu_i \rightarrow \mu'_i$, $(\mu'_i/\alpha_i)_i \nu \in Cl(E)$. As the number of one-step reducts of a term is finite, there exists $k \in \mathbb{N}$, such that for all $\mu'_1, \dots, \mu'_n \in \mathcal{T}$ such that $\forall i \leq n$, $\mu_i \rightarrow \mu'_i$, we have $(\mu'_i/\alpha_i)_i \nu \in Cl^k(E)$. Therefore, for all $\rho_1 \dots \rho_n$ each respectively a leaf of $\mu_1 \dots \mu_n$, $(\rho_i/\alpha_i)_i \nu \in Cl^k(E)$ as it satisfies (CR₂) and each μ_i is neutral not normal. Finally, $(\mu_i/\alpha_i)_i \nu \in Cl^{k+1}(E)$.

4.3 Proving that this function is a morphism

\Rightarrow -morphism

We prove now that for all $E, F \in \mathcal{C}_{\equiv}$ such that F satisfies $\overset{\circ}{\rightarrow}$ -sheep property, we have $Cl(E \overset{\circ}{\rightarrow} F) = Cl(E) \overset{\circ}{\rightarrow} Cl(F)$.

Lemma 19. *For all $E \subseteq \mathcal{T}$ and $\pi \in \mathcal{T}$, If $\pi \in SN$, π is neutral not normal and $\forall \rho \in \mathcal{L}(\pi)$, $\rho \in Cl(E)$, then $\pi \in Cl(E)$*

Proof. As $\pi \in SN$, $\mathcal{L}(\pi)$ is defined and finite. And, if we call $k_m = \max\{\min\{k, \rho \in Cl^k(E)\}, \rho \in \mathcal{L}(\pi)\}$, then $\pi \in Cl^{k_m+1}(E) \subseteq Cl(E)$.

Remark 10. In the same way, if there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_i \nu_\pi$ and $\forall (\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i \nu_\pi \in Cl(E)$, we have $\pi \in Cl(E)$.

Lemma 20. *For all $E, F \in \mathcal{C}_{\equiv}$, if F satisfies $\overset{\circ}{\rightarrow}$ -sheep property, $(\Delta, \alpha) \in E$, and $(\alpha/\beta)\pi \in Cl(F)$ then $\lambda\beta.\pi \in Cl(E \overset{\circ}{\rightarrow} F)$.*

Proof. There exists a minimal k such that $\pi \in Cl^k(F)$. By induction on k .

- if $k = 0$ then $(\Delta, \pi) \in F$. As $(\Delta, \alpha) \in E$, we have $(\Delta, \lambda\beta.\pi\alpha) \in F$, by (CR_{3 \equiv}). Therefore $(\Delta, \lambda\beta.\pi) \in E \overset{\circ}{\rightarrow} F$, by $\overset{\circ}{\rightarrow}$ -sheep-property. And $\lambda\beta.\pi \in Cl^0(E \overset{\circ}{\rightarrow} F)$.
- if $k > 0$, then $(\alpha/\beta)\pi = (\mu_i/\alpha_i)_{i \leq n} \nu$ with each μ_i in SN, neutral and not normal, and such that $\forall i \leq n$, and $\forall (\rho_i)_{i \leq n}$ such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, we have $(\rho_i/\alpha_i)_{i \leq n} \nu \in Cl^{k-1}(F)$, therefore $\lambda\beta.(\rho_i/\alpha_i)_{i \leq n} \nu = (\rho_i/\alpha_i)_{i \leq n} \lambda\alpha.\nu \in Cl(E \overset{\circ}{\rightarrow} F)$, by induction hypothesis. Therefore $\lambda\beta.\pi \in Cl(E \overset{\circ}{\rightarrow} F)$ by remark 10.

Lemma 21. *For all $E, F \in \mathcal{C}_{\equiv}$ and $\pi \in Cl(E) \overset{\circ}{\rightarrow} Cl(F)$, π cannot reduce to a term-abstraction.*

Proof. Let $\pi \in Cl(E) \overset{\circ}{\rightarrow} Cl(F)$, then $\pi \in SN$. If π reduces to a term-abstraction, then its normal form is a term-abstraction $\lambda x.\rho$, by confluency. Let $\alpha \in \mathcal{X}$ such that $(\Delta, \alpha) \in E$, then $\alpha \in Cl(E)$ and $\pi\alpha \in Cl(F)$, therefore $(\lambda x.\rho)\alpha \in Cl(F)$, as it satisfies (CR₂). Moreover, $(\lambda x.\rho)\alpha$ is normal, therefore $(\lambda x.\rho)\alpha \in Cl^0(F)$. Hence $\Delta \vdash (\lambda x.\rho)\alpha : A_F$. That's absurd.

Proposition 5. For all $E, F \in \mathcal{C}_{\equiv}$,
if F satisfies $\overset{\circ}{\rightarrow}$ -sheep property, then $Cl(E \overset{\circ}{\rightarrow} F) = Cl(E) \overset{\sim}{\rightarrow} Cl(F)$.

Proof. \subseteq Let $\pi \in Cl(E \overset{\circ}{\rightarrow} F)$,

then $\pi \in SN$ by (CR₁). Moreover there exists (a minimal) $k \in \mathbb{N}$, such that $\pi \in Cl^k(E \overset{\sim}{\rightarrow} F)$. Let $\pi' \in Cl(E)$, then there exists (a minimal) $j \in \mathbb{N}$, such that $\pi' \in Cl^j(E)$. Let us show that $\pi\pi' \in Cl(F)$ by induction on $k + j$.

- If $k + j = 0$ then $(\Delta, \pi) \in E \overset{\circ}{\rightarrow} F$ and $(\Delta, \pi') \in E$ therefore $(\Delta, \pi\pi') \in F$. And $\pi\pi' \in Cl^0(F)$.
- If $k > 0$, then there exists $\nu_\pi \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi = (\mu_i/\alpha_i)_i \nu_\pi$ and $\forall(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i \nu_\pi \in Cl^{k-1}(E \overset{\circ}{\rightarrow} F)$. Therefore $(\rho_i/\alpha_i)_i (\nu_\pi \pi') = (\rho_i/\alpha_i)_i \nu_\pi \pi' \in Cl(F)$ by induction hypothesis. Hence $\pi\pi' \in Cl(F)$, as it satisfies (CR'₃).
- If $j > 0$, then there exists $\nu_{\pi'} \in \mathcal{T}$, and $(\mu_i)_i \subseteq SN$, each neutral not normal such that $\pi' = (\mu_i/\alpha_i)_i \nu_{\pi'}$ and $\forall(\rho_i)_i \subseteq \mathcal{T}$, such that for all $i \leq n$, $\rho_i \in \mathcal{L}(\mu_i)$, then $(\rho_i/\alpha_i)_i \nu_{\pi'} \in Cl^{j-1}(E)$. Therefore $(\rho_i/\alpha_i)_i (\pi \nu_{\pi'}) = (\rho_i/\alpha_i)_i \pi \nu_{\pi'} \in Cl(F)$ by induction hypothesis. Hence $\pi\pi' \in Cl(F)$, as it satisfies (CR'₃).

\supseteq Let $\pi \in Cl(E) \overset{\sim}{\rightarrow} Cl(F)$. then $\pi \in SN$ and for all $\pi' \in Cl(E)$, $\pi\pi' \in Cl(F)$. By lemma 21, π cannot reduce to a term-abstraction.

- If π is a proof-abstraction $\lambda\alpha.\pi'$, if we choose α such that $\Delta \vdash \alpha : A_E$, by α -equivalence, then we have $\pi' \in Cl(F)$ as it satisfies (CR₂), therefore $\pi \in Cl(E \overset{\circ}{\rightarrow} F)$ by lemma 20.
- If π is neutral and normal, let $\alpha \in \mathcal{X}$ such that $\Delta \vdash \alpha : A_E$, then $\pi\alpha \in Cl(F)$. Moreover π is neutral and normal, therefore $\pi\alpha$ is normal, hence $\pi\alpha \in Cl^0(F)$, *i.e.* $(\Delta, \pi\alpha) \in F$, with $(\Delta, \alpha) \in E$, therefore $(\Delta, \pi) \in E \overset{\circ}{\rightarrow} F$, as F satisfies $\overset{\circ}{\rightarrow}$ -sheep property. Finally, $\pi \in Cl^0(E \overset{\circ}{\rightarrow} F) \subseteq Cl(E \overset{\circ}{\rightarrow} F)$.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either proof-abstractions, by lemma 21. And all these leaves are in $Cl(E) \overset{\sim}{\rightarrow} Cl(F)$, as it satisfies (CR₂), therefore they also are in $Cl(E \overset{\circ}{\rightarrow} F)$, as we saw in the previous points. Finally, $\pi \in Cl(E \overset{\circ}{\rightarrow} F)$, by lemma 19.

\forall -morphism

We prove now that for all sorts T and $f \in \mathring{A}_T$, satisfying $\mathring{\forall}$ -sheep property, $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$. Notice that for all functions $f \in \mathring{A}_T$, $Cl \circ f \in \mathring{A}_T$.

Lemma 22. For all $E \in \mathcal{C}_{\equiv}$, $k \in \mathbb{N}$, terms t , term-variables x , proof-terms π' , if $(t/x)\pi \in Cl^k(E)$, then $(\lambda x.\pi)t \in Cl^k(E)$.

Proof. By induction on k .

- If $k = 0$, by (CR_{3 \equiv}).
- If $k > 0$, by induction hypothesis.

Proposition 6. For all sorts T and $f \in \mathring{A}_T$,
if f satisfies $\mathring{\forall}$ -sheep property, then $Cl(\mathring{\forall}_T f) = \mathring{\forall}_T Cl \circ f$.

Proof. \subseteq Let $\pi \in Cl(\mathring{\forall}_T f)$, then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi \in Cl^k(\mathring{\forall}_T f)$. By induction on k .

- If $k = 0$, $(\Delta, \pi) \in \mathring{\forall}_T f$ then for all $t \in \hat{T}$, $(\Delta, \pi t) \in f(t)$, therefore $\pi t \in Cl^0(f(t))$. Finally, $\pi \in \mathring{\forall}_T Cl^0 \circ f \subseteq \mathring{\forall}_T Cl \circ f$.
- If $k > 0$, then $\pi = (\mu_i/\alpha_i)_i \nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_i$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1}(\mathring{\forall}_T f) \subseteq \mathring{\forall}_T Cl \circ f$, by induction hypothesis. Let $t \in \hat{T}$, then if we write $\nu' = \nu t$, we have $\pi t = (\mu_i/\alpha_i)_i \nu'$ and for all $(\rho_i)_i$ each respectively a leaf of μ_i , $(\rho_i/\alpha_i)_i \nu' = (\rho_i/\alpha_i)_i \nu t \in Cl \circ f(t)$. Therefore $\pi t \in Cl \circ f(t)$ by remark 10. Finally, $\pi \in \mathring{\forall}_T Cl \circ f$.

\supseteq Let $\pi \in \mathring{\forall}_T Cl \circ f$. Let $t \in \hat{T}$. Then there exists (a minimal) $k \in \mathbb{N}$ such that $\pi t \in Cl^k \circ f(t)$. Let us first remark that π is therefore in SN , and that it cannot reduce to a proof-abstraction (using the same sort of argument as in lemma 21).

- If π is neutral and normal, then πt is normal, therefore $\pi t \in Cl^0 \circ f(t)$. Hence $(\Delta, \pi t) \in f(t)$, and $(\Delta, \pi) \in \mathring{\forall}_T f$, as f satisfies $\mathring{\forall}$ -sheep property. Finally, $\pi \in Cl^0(\mathring{\forall}_T f)$.
- If $\pi = \lambda x. \pi'$, by induction on k .
 - * If $k = 0$, then $(\Delta, \pi t) \in f(t)$, therefore $(\Delta, \pi) \in \mathring{\forall}_T f$, as f satisfies $\mathring{\forall}$ -sheep property. Hence $\pi \in Cl^0(\mathring{\forall}_T f)$.
 - * If $k > 0$, then $\pi t = (\mu_i/\alpha_i)_i \nu$, with each μ_i neutral not normal and such that for all $(\rho_i)_i$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i \nu \in Cl^{k-1} \circ f(t)$.
 - If $\nu \neq \alpha_1$, then $\nu = \nu' t$, with $\pi = (\mu_i/\alpha_i)_i \nu'$, and for all $(\rho_i)_i$ each respectively a leaf of μ_i , we have $(\rho_i/\alpha_i)_i \nu' \in Cl(\mathring{\forall}_T f)$, by induction hypothesis. We conclude by lemma 19.
 - Otherwise, we only know that each leaf of $(\lambda x. \pi') t$ is in $Cl^{k-1} \circ f(t)$. In particular, $(t/x) \pi' \in Cl^{k-1} \circ f(t)$, as it satisfies (CR_2) . Therefore $(t/x) \pi' = (\tau_i/\alpha_i)_i \nu'$, with each τ_i neutral not normal and such that for all $(\rho_i)_i$ each respectively a leaf of τ_i , we have $(\rho_i/\alpha_i)_i \nu' \in Cl^{k-1} \circ f(t)$. If we write $\tau_i = (t/x) \tau'_i$, for each i , and $\nu' = (t/x) \nu''$, then we have $\pi = (\tau'_i/\alpha_i)_i (\lambda x. \nu'')$, and for all $(\rho'_i)_i$ each respectively a leaf of τ'_i , $(t/x) \rho'_i$ is a leaf of $(t/x) \tau'_i = \tau_i$. Therefore $((t/x) \rho'_i/\alpha_i) (t/x) \nu'' \in Cl^{k-1} \circ f(t)$, hence $((\rho'_i/\alpha_i)(\lambda x. \nu'')) \in Cl^{k-1} \circ f(t)$ by lemma 22. Finally, $((\rho'_i/\alpha_i)(\lambda x. \nu'')) \in Cl(\mathring{\forall}_T f)$, by induction hypothesis, and $\pi = \lambda x. \pi' \in Cl(\mathring{\forall}_T f)$, by lemma 19.
- Otherwise, $\pi \in SN$, is neutral and not normal. All its leaves are either neutral, either term-abstractions. And all these leaves are in $\mathring{\forall}_T Cl \circ f$, as it satisfies (CR_2) , therefore they also are in $Cl(\mathring{\forall}_T f)$, as we saw in the previous points. Finally, $\pi \in Cl(\mathring{\forall}_T f)$, by lemma 19.

We finally get the following (second) completeness result:

Theorem 1.

If \mathcal{L}_{\equiv} is strongly normalizing, then $Cl \circ [\cdot]$ is a \mathcal{C}' -valued model of \mathcal{L}_{\equiv} .

Proof. By lemma 1 and propositions 2, 3, 4, 5, 6.

5 Soundness

We finally prove in this section, that having a \mathcal{C}' -valued model is also a sound condition of strongly normalizing theories \mathcal{L}_{\equiv} .

Lemma 23. *If $[\cdot]$ is a \mathcal{C}' -valued model of a theory \mathcal{L}_{\equiv} , then for all $A \in \mathcal{P}$, contexts Γ , $\varphi \in \mathcal{V}(A) \cap \mathcal{V}(\Gamma)$, $\pi \in \mathcal{T}$ and σ substitutions such that for all declarations $\alpha : B$ in Γ , $\sigma\alpha \in \mathcal{F}(B, \varphi)$, we have:*

$$\text{if } \Gamma \vdash \pi : A \text{ then } \sigma\varphi\pi \in \llbracket A \rrbracket_{\varphi}.$$

Proof. By induction on the length of the derivation of $\Gamma \vdash \pi : A$. By case on the last rule used. If the last rule used is :

- axiom: in this case, π is a variable α , and Γ contains a declaration $\alpha : B$ with $A \equiv B$ (therefore $|A|_{\varphi} \equiv |B|_{\varphi}$). Then $\sigma\varphi\pi = \sigma\alpha \in \llbracket B \rrbracket_{\varphi} = \llbracket A \rrbracket_{\varphi}$.
- \Rightarrow -intro: in this case, π is an abstraction $\lambda\alpha.\tau$, and we have $\Gamma, \alpha : B \vdash \tau : C$ with $A \equiv B \Rightarrow C$. Let σ' such that for all variables β declared in Γ , $\sigma'\beta = \sigma\beta$ and $\sigma'\alpha$ is an element of $\llbracket B \rrbracket_{\varphi}$. Then $\sigma'\varphi\tau \in \llbracket C \rrbracket_{\varphi}$ by induction hypothesis (and $\sigma'\varphi\tau$ is in SN , therefore $\sigma\varphi\pi$ is also in SN). Let $\pi' \in \llbracket B \rrbracket_{\varphi}$, we prove by induction on the sum of both maximal lengths of a reductions sequence from $\sigma\varphi(\lambda\alpha.\tau)$ and π' (each in SN) that every one-step reduct of the neutral not normal proof-term $\sigma\varphi(\lambda\alpha.\tau) \pi'$ is in $\llbracket C \rrbracket_{\varphi}$. If the one-step reduct is $\sigma\varphi(\pi'/\alpha)\tau$, we conclude by induction hypothesis (on the length of the derivation) as $\pi' \in \llbracket B \rrbracket_{\varphi}$. Otherwise, the reduction takes place either in $\sigma\varphi(\lambda\alpha.\tau)$, either in π' . We conclude by induction hypothesis on the sum of the maximal lengths of reductions sequence from $\sigma\varphi(\lambda\alpha.\tau)$ and π' . And the fact that both $\llbracket B \rrbracket_{\varphi}$ and $\llbracket B \Rightarrow C \rrbracket_{\varphi}$ satisfy (CR_2) . Finally, $\sigma\varphi(\lambda\alpha.\tau) \pi' \in \llbracket C \rrbracket_{\varphi}$, as it satisfies (CR_3') and $\sigma\varphi(\lambda\alpha.\tau) \pi'$ is neutral, not normal. Hence $\sigma\varphi(\lambda\alpha.\tau) \pi' \in \llbracket B \rrbracket_{\varphi} \xrightarrow{\sim} \llbracket C \rrbracket_{\varphi} = \llbracket B \Rightarrow C \rrbracket_{\varphi} = \llbracket A \rrbracket_{\varphi}$.
- \Rightarrow -elim: in this case, π is an application $\rho\tau$, and we have $\Gamma \vdash \rho : C \equiv B \Rightarrow A$ and $\Gamma \vdash \tau : B$. Therefore, by induction hypothesis, $\sigma\varphi\rho \in \llbracket B \Rightarrow A \rrbracket_{\varphi} = \llbracket B \rrbracket_{\varphi} \xrightarrow{\sim} \llbracket A \rrbracket_{\varphi}$ and $\sigma\varphi\tau \in \llbracket B \rrbracket_{\varphi}$. Therefore $\sigma\varphi(\rho\tau) \in \llbracket A \rrbracket_{\varphi}$.
- \forall -intro: in this case, π is a term abstraction $\lambda x.\pi'$ and we have $\Gamma \vdash \pi' : B$ with $A \equiv \forall x.B$. Let $t \in \hat{T}$ (with T the sort of x), and $\varphi' = \varphi + \langle x, t \rangle$. Then $\sigma\varphi'\pi' = \sigma\varphi(t/x)\pi' \in \llbracket B \rrbracket_{\varphi'}$, by induction hypothesis. Therefore, $\sigma\varphi(\lambda x.\pi') \in \tilde{\forall}_T(t \mapsto \llbracket B \rrbracket_{\varphi + \langle x, t \rangle}) = \llbracket A \rrbracket_{\varphi}$ (by induction on the maximal length of a reductions sequence from πt , with $t \in \hat{T}$, using the fact that for all $t \in \hat{T}$, $\llbracket B \rrbracket_{\varphi + \langle x, t \rangle}$ satisfies (CR_2) and (CR_3')).
- \forall -elim: in this case, π is an application ρt , and we have $\Gamma \vdash \rho : \forall x.B$ with $A = (t/x)B$ and $x \notin FV(\Gamma)$. By induction hypothesis, we have $\sigma\varphi\rho \in \llbracket \forall x.B, \varphi \rrbracket = \tilde{\forall}_T(t \mapsto \llbracket B \rrbracket_{\varphi + \langle x, t \rangle})$. Therefore $\sigma\varphi(\rho t) = \sigma\varphi\rho(\varphi t) \in \llbracket B \rrbracket_{\varphi + \langle x, \varphi t \rangle} = \llbracket (t/x)B \rrbracket_{\varphi} = \llbracket A \rrbracket_{\varphi}$.

Theorem 2. *If \mathcal{L}_{\equiv} has a \mathcal{C}' -valued model, then \mathcal{L}_{\equiv} is strongly normalizing.*

Proof. If \mathcal{F} is a \mathcal{C}' -valued model of \equiv then for all typing judgement $\Gamma \vdash \pi : A$ and σ and φ as in the previous proposition, we have $\sigma\varphi\pi \in \mathcal{F}(A, \varphi)$ hence $\sigma\varphi\pi \in SN$, therefore $\pi \in SN$.

Conclusion

We have defined a refinement of truth values algebras, which is not completely syntax-independent anymore, as it depends on the language of terms used, but seems to allow to build more precise models on it. Then we exhibited one of these truth values algebras \mathcal{C}' such that having a \mathcal{C}' -valued model is still a sound condition of strongly normalizing theories, and even more, is also a complete condition. We proved this completeness theorem in an original way : build a structure adapted to the congruence relation and then show that it is also adapted to connectives when the theory is strongly normalizing. This way, we are able to build an interpretation of propositions adapted to the congruence relation, even if the theory is not strongly normalizing.

In future work, we wish to extend this result to other logical frameworks with or without rewriting. For logical frameworks with rewriting, we want to extend first this result to (complete) Deduction modulo, and to $\lambda\Pi$ -calculus modulo [2]. We want to study how these language-dependent truth values algebras can help us in building models of logical frameworks with dependent types, as $\lambda\Pi$ -calculus modulo, or Pure Type Systems.

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